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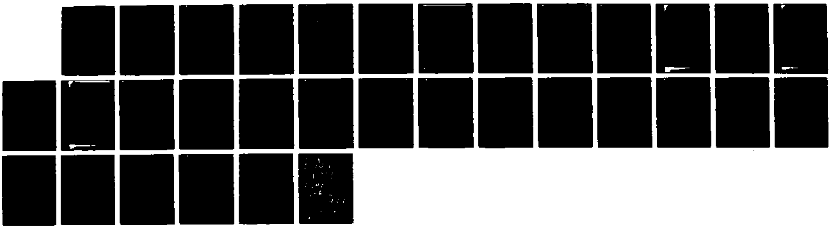

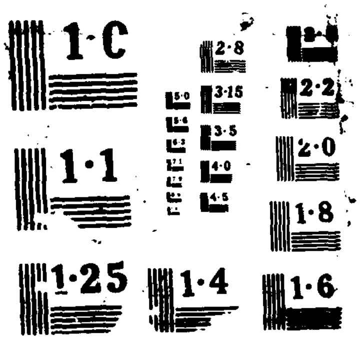
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Many problems in structural dynamics involve stabilizing the Euler-Bernoulli beam equation by proportional dissipative control at the end of the beam. The COFS experiment by NASA is one of such examples. Exponential stability is a very desirable property for such elastic systems. Existing methods such as the energy multiplier method have been successful to establish rigorous proofs for certain types of stabilizing boundary conditions. But there are many other types of boundary conditions for which such methods are not effective.

A recent theorem due to F.L. Huang introduces a frequency domain method to study the exponential stability problems. Here we apply Huang's theorem to prove an exponential decay result for an Euler-Bernoulli beam with rate control of the bending moment only. We are also able to derive its asymptotic stability margin. Finally, we indicate the realizability of various types of boundary feedback stabilization schemes by illustrating the corresponding mechanical designs of passive damping devices.

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The Euler-Bernoulli Beam Equation with Boundary Energy Dissipation

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INTRODUCTION

Many problems in structural dynamics involve stabilizing the elastic energy of partial differential equations such as the Euler-Bernoulli beam equation by boundary conditions. Exponential stability is a very desirable property for such elastic systems. The energy multiplier method ~~[1], [2], [7]~~ has been successfully applied by several people to establish exponential stability for various PDEs and boundary conditions. However, it has also been found [2] that for certain boundary conditions *the energy multiplier method is not effective in proving the exponential* stability property.

A recent theorem of F.L. Huang [4] introduces a frequency domain method to study such exponential decay problems. In this paper, we derive estimates of the resolvent operator on the imaginary axis and apply Huang's theorem to establish an exponential decay result for an Euler-Bernoulli beam with rate control of the bending moment only. We also derive asymptotic limits of eigenfrequencies, which was also done earlier by P. Rideau.[8]. Finally, we indicate the realizability of these boundary feedback stabilization schemes by illustrating some mechanical designs of passive damping devices.

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§1. BACKGROUND AND MOTIVATION

In this paper, we consider the following uniform Euler-Bernoulli beam equation with dissipative boundary conditions

$$\left.\begin{array}{ll} my_{tt}(x,t)+EIy_{xxxx}(x,t) = 0, & 0 < x < 1, \\ y(0,t) = 0, & \\ y_x(0,t) = 0, & \\ -EIy_{xxx}(1,t) = -k_1^2 y_t(1,t)\ , & k_1 \in \mathbb{R}, \\ -EIy_{xx}(1,t) = k_2^2 y_{xt}(1,t)\ , & k_2 \in \mathbb{R}, \\ (y(x,0),y_t(x,0)) = (y_0(x),y_1(x)), & 0 \le x \le 1. \end{array}\right\} \quad (1.1)$$

where m denotes the mass density per unit length, EI is the flexural rigidity coefficient, and the following variables have engineering meanings:

y = vertical displacement, y_t = velocity

y_x = rotation, y_{xt} = angular velocity

$-EIy_{xx}$ = bending moment

$-EIy_{xxx}$ = shear

at a point x, at time t.

From now on, when we write equation (1.1.j), for example, we mean the jth equation in (1.1).

The above equation and conditions are intended to serve as a simple mathematical model for the mast control system in NASA's COFS (Control of Flexible Structures) Program. See Figure 1. A long flexible mast 60 meters in length is clamped at its base on a space shuttle. The mast is formed with 54 bays but can be idealized as a continuous uniform beam. At the very end of the mast, a CMG (control moment gyro) is placed which can apply bending and torsion rate control to the mast according to sensor feedback.

Boundary conditions (1.1.2) and (1.1.3) signify that the beam is clamped, at the left end, $x = 0$, while boundary conditions (1.1.4) and (1.1.5) at the right end, $x = 1$, respectively, signify

$$\left\{\begin{array}{l} \text{shear } (-EIy_{xxx}) \text{ is proportional to velocity } (y_t) \\ \text{bending moment } (-EIy_{xx}) \text{ is negatively proportional to angular} \\ \text{velocity } (y_{xt}) \end{array}\right.$$

Thus the rate feedback laws (1.1.4) and (1.1.5) reflect some basic features of the CMG mast control system in COFS.

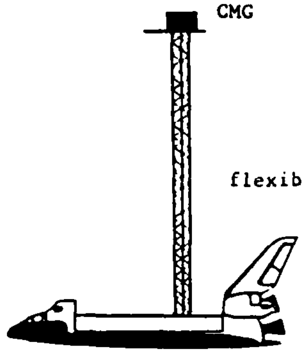


Figure 1 Spacecraft mast control experiment

The elastic energy of vibration, $E(t)$, at time t, for system (1.1) is given by

$$E(t) = \frac{1}{2}\int_0^1 [my_t^2(x,t) + EIy_{xx}^2(x,t)]dx.$$

Note that in (1.1) we have already normalized the beam length to 1.

The qualitative behavior of (1.1) has been studied in an earlier paper [2]. There it is shown that if $\underline{k_1^2 > 0}$, $\underline{k_2^2 \geq 0}$ in (1.1.4) and (1.1.5), respectively, then the energy of vibration of the beam decays uniformly exponentially:

$$E(t) \leq Ke^{-\mu t}E(0) \tag{1.2}$$

for some $K, \mu > 0$ uniformly for all initial conditions $(y_0(x), y_1(x))$. Therefore the flexible mast system can be controlled and stabilized.

The proof of the above in [2] was accomplished by the use of energy multipliers and the construction of a Liapounov functional. Nevertheless, a major mathematical question remained unresolved in [2]:

[Q] "Does the uniform exponential decay property (1.2) hold under the assumption of $k_1^2 = 0$, $k_2^2 > 0$?"

This question is of considerable mathematical interest because the feedback scheme using bending moment only is simple and attractive.

For a long time, we have conjectured that the answer to [Q] is affirmative, as asymptotic eigenfrequency estimates obtained in [8] (and §3) have so suggested: Let A denote the infinitesimal generator of the C_0-semigroup corresponding to (1.1) with $k_1 = 0$ and $k_2^2 > 0$, and let $\sigma(A)$ denote the spectrum of A. Then there exists $\beta > 0$ such that

$$\operatorname{Re}\lambda \le -\beta < 0 \quad \text{for all } \lambda \in \sigma(A). \tag{1.3}$$

Nevertheless, it is well known [4] that the following "theorem" is false.

"Let A generate a C_0-semigroup and

$$\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\} \le -\beta < 0 \tag{1.4}$$

for some $\beta > 0$. Then the C_0-semigroup is exponentially stable:

$$\|\exp(tA)\| \le Me^{-\mu t} \quad \text{for some } M \ge 1,\ \mu > 0\text{''}. \tag{1.5}$$

Therefore, knowing (1.4) alone is not sufficient to confirm (1.5). This statement remains false even if we assume additionally that A has a compact resolvent.

We have repeatedly tried to refine the energy multiplier technique used in [2] to establish (1.5) without much success, no matter how many different and elaborate multipliers were constructed. There always are boundary terms which cannot be absorbed by terms in the dissipative boundary condition.

A recent theorem by F.L. Huang offers an important direct method for proving exponential stability:

THEOREM 1 (F.L. Huang [4])

Let $\exp(tA)$ be a C_0-semigroup in a Hilbert space satisfying

$$\|\exp(tA)\| \le B_0, \quad t \ge 0, \quad \text{for some } B_0 > 0. \tag{1.6}$$

Then $\exp(tA)$ is exponentially stable if and only if

$$\{i\omega \mid \omega \in \mathbb{R}\} \subset \rho(A), \text{ the resolvent set of } A; \text{ and} \tag{1.7}$$

$$B_1 \equiv \sup\{\|(i\omega - A)^{-1}\| \mid \omega \in \mathbb{R}\} < \infty \tag{1.8}$$

are satisfied. $\square$

Huang's theorem effects a frequency domain method to proving exponential decay properties. As mentioned earlier, the energy multiplier method, which corresponds to a time domain method, has not been successful for the case $k_1^2 = 0$, $k_2^2 > 0$.

Therefore the work is to obtain bounds on the resolvent operator $(i\omega - A)^{-1}$. Here we accomplish this by carrying out a careful analysis on the eigenfunctions and eigenfrequencies of the operator A. This is done in §2.

Associated with [Q] is the question of the asymptotic distribution pattern of eigenfrequencies, as numerical study in [2] suggests that a "structural damping" phenomenon is present at low frequencies. Does it also appear at high frequencies? This is answered in §3. (We must state that the work and numerical verification was done ahead of us by P. Rideau in his recent thesis [8]).

In §4, we present mechanical designs of devices satisfying damping boundary conditions (1.1.3) and (1.1.4) to indicate the realizability of the feedback stabilization scheme using passive dampers.

Notations: We use $\| \ \|$ to denote the $\mathcal{L}^2(0,1)$ norm. We define the Sobolev space

$$H^k \equiv H^k(0,1) = \{f:[0,1]\to\mathbb{R} \mid \|f\|^2_{H^k(0,1)} \equiv \sum_{j=0}^{k} \int_0^1 |f^{(j)}(x)|^2 dx < \infty\},\ k \in \mathbb{N}.$$

Also, we let

$$H_0^2 \equiv H_0^2(0,1) \equiv \{f \mid f \in H^2(0,1),\ f(0) = f'(0) = 0\}.$$

The underlying Hilbert space $\mathcal{H}$ for the PDE (1.1) is

$$\mathcal{H} = H_0^2(0,1) \times \mathcal{L}^2(0,1) := \{(f,g) \mid \|(f,g)\|^2_{\mathcal{H}} \equiv \int_0^1 [EI|f''(x)|^2 + m|g(x)|^2] dx < \infty\},$$

whose norm square is the elastic energy.

The unbounded linear operator A associated with (1.1) is given by

$$A = \begin{bmatrix} 0 & I \\ -\alpha^4 \partial_x^4 & 0 \end{bmatrix}, \quad \alpha^4 \equiv \frac{EI}{m}$$

with domain

$$D(A) = \{(f,g) \in H^4 \times H^2 \mid -EIf'''(1) = -k_1^2 g(1), -EIf''(1) = k_2^2 g'(1), f(0) = f'(0) = 0\}.$$

§2. ESTIMATION OF THE RESOLVENT OPERATOR ON THE IMAGINARY AXIS. EXPONENTIAL DECAY OF SOLUTIONS.

Consider the resolvent equation: Given $(f,g) \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, find $(w_\lambda, v_\lambda) \in D(A)$ such that

$$(A-\lambda I)\begin{bmatrix} w_\lambda \\ v_\lambda \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -\left(\alpha\frac{\partial}{\partial x}\right)^4 & 0 \end{bmatrix} - \lambda I_2\right)\begin{bmatrix} w_\lambda \\ v_\lambda \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{2.1}$$

This amounts to solving the following boundary value problem for w_λ:

$$\left.\begin{aligned} &\alpha^4 w_\lambda^{(4)}(x) + \lambda^2 w_\lambda(x) = -[\lambda f(x) + g(x)], \quad x \in (0,1) \\ &w_\lambda(0) = 0 \\ &w_\lambda'(0) = 0 \\ &w_\lambda'''(1) - \lambda \tilde{k}_1^2 w_\lambda(1) = \tilde{k}_1^2 f(1) \\ &w_\lambda''(1) + \lambda \tilde{k}_2^2 w_\lambda'(1) = -\tilde{k}_2^2 f'(1) \end{aligned}\right\} \tag{2.2}$$

where

$$\tilde{k}_1^2 \equiv k_1^2 (EI)^{-1}, \quad \tilde{k}_2^2 \equiv k_2^2 (EI)^{-1}. \tag{2.3}$$

Once w_λ is found we obtain

$$v_\lambda(x) = \lambda w_\lambda(x) + f(x) \tag{2.4}$$

To simplify notation, from now on, unless otherwise specifically mentioned, we set $\alpha^4 = 1$ in (2.2.1) and write k_1, k_2 for $\tilde{k}_1, \tilde{k}_2$, respectively.

The main work in this section is to prove estimate (1.8), i.e., to show the existence of some $B_1 > 0$ such that

$$\int_0^1[|w_\lambda''(x)|^2+|v_\lambda(x)|^2]dx \le B_1\int_0^1[|f''(x)|^2+|g(x)|^2]dx \qquad (2.5)$$

for all $\lambda = i\omega$, $\omega \in \mathbb{R}$ and all $(f,g) \in \mathcal{H}$.

<u>LEMMA 2</u> A^{-1} exists and is a compact operator on $\mathcal{H}$. Furthermore, $\sigma(A)$ consists entirely of isolated eigenvalues.

<u>Proof</u>: Let $\lambda = 0$ in (2.2). We see that w_0 in (2.2) is obtained by integrating four times:

$$\begin{aligned} w_0(x) = &-\int_0^x\int_0^{\xi_3}\int_0^{\xi_4}\int_0^{\xi_2} g(\xi_1)d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\ &+ \frac{x^2}{2}\left[\int_0^1\int_0^{\xi_2} g(\xi_1)d\xi_1 d\xi_2 - k_2^2 f'(1)\right] \\ &+ \left(\frac{x^3}{6} - \frac{x^2}{2}\right)\left[\int_0^1 g(\xi)d\xi + k_1^2 f(1)\right] \end{aligned}$$

and

$$v_0(x) = f(x).$$

Thus A^{-1} exists and maps $\mathcal{H}$ into $H^4(0,1) \times H_0^2(0,1)$. Therefore A^{-1} is compact. The rest of the lemma follows from Theorem 6.29 in [6, Chapter 3]. $\square$

<u>LEMMA 3</u> The resolvent estimate (2.5) holds for $\lambda = i\omega$, $\omega \in \mathbb{R}$, provided that $|\lambda|$ is sufficiently large.

<u>Proof</u>: For simplicity, let us write (w,v) for (w_λ, v_λ) when no ambiguities will occur.

Let

$$\lambda = i\omega = i\eta^2, \quad \eta \neq 0.$$

We need only consider $\omega = \eta^2 > 0$. The estimates for $\omega < 0$ are similar. First, we find a particular solution $w_p(x)$ of (2.2.1).

$$w_p(x) = -\frac{1}{2}\int_0^x \eta^{-3}[\sinh \eta(x-\xi) - \sin \eta(x-\xi)][i\eta^2 f(\xi) + g(\xi)]d\xi \qquad (2.6)$$

Then $w_p(x)$ satisfies

$$\left.\begin{aligned} & w_p^{(4)}(x) - \eta^4 w_p(x) = -[i\eta^2 f(x) + g(x)], \quad x \in (0,1) \\ & w_p(0) = 0 \\ & w_p'(0) = 0. \end{aligned}\right\} \qquad (2.7)$$

Consider the solution $\tilde{w}(x)$ of

$$\left.\begin{aligned} & \tilde{w}^{(4)}(x) - \eta^4 \tilde{w}(x) = 0 \\ & \tilde{w}(0) = 0 \\ & \tilde{w}'(0) = 0 \\ & \tilde{w}'''(1) - i\eta^2 k_1^2 \tilde{w}(1) = h_1, \quad h_1 = -w_p'''(1) + i\eta^2 k_1^2 w_p(1) + k_1^2 f(1) \\ & \tilde{w}''(1) + i\eta^2 k_2^2 \tilde{w}'(1) = h_2, \quad h_2 = -w_p''(1) - i\eta^2 k_2^2 w_p'(1) - k_2^2 f'(1) \end{aligned}\right\} \qquad (2.8)$$

If we can find $\tilde{w}(x)$, then the solution $w(x)$ of (2.2) is obtainable:

$$w(x) = w_p(x) + \tilde{w}(x). \qquad (2.9)$$

But we can solve for $\tilde{w}(x)$ as follows. Since $\lambda^2 = -\eta^4 \neq 0$, we have

$$\tilde{w}(x) = A_1 e^{\eta x} + A_2 e^{i\eta x} + A_3 e^{-\eta x} + A_4 e^{-i\eta x}. \qquad (2.10)$$

The coefficients A_i, $1 \le i \le 4$, satisfy

$$M\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ h_1 \\ h_2 \end{bmatrix} \qquad (2.11)$$

where

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \eta & i\eta & -\eta & -i\eta \\ (\eta^3 - i\eta^2 k_1^2)e^{\eta} & -(i\eta^3 + i\eta^2 k_1^2)e^{i\eta} & -(\eta^3 + i\eta^2 k_1^2)e^{-\eta} & (i\eta^3 - i\eta k_1^2)e^{-i\eta} \\ (\eta^2 + i\eta^3 k_2^2)e^{\eta} & (-\eta^2 - \eta^3 k_2^2)e^{i\eta} & (\eta^2 - i\eta^3 k_2^2)e^{-\eta} & (-\eta^2 + \eta^3 k_2^2)e^{-i\eta} \end{bmatrix} \tag{2.12}$$

If M^{-1} exists for η sufficiently large, then

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ 0 \\ h_1 \\ h_2 \end{bmatrix}. \tag{2.13}$$

We now begin the estimation of $\int_0^1 |w''(x)|^2 dx$. The work below may seem tedious, but the idea is rather simple. The main observation is that the dominant terms in $w_p(x)$ and $\tilde{w}(x)$ do not satisfy the bounds

$$\int_0^1 |w_p''(x)|^2 dx \le C\int_0^1 [|f''(x)|^2 + |g(x)|^2]dx$$

$$\int_0^1 |\tilde{w}''(x)|^2 dx \le C\int_0^1 [|f''(x)|^2 + |g(x)|^2]dx$$

for $|\lambda|$ large. However, in (2.9), <u>those dominant terms cancel</u>, leaving $w(x)$ with smaller terms which are bounded by $O(\|f''\| + \|g\|)$.

<u>1st Step</u> <u>Estimation of $w_p''(x)$.</u>

From (2.6),

$$w_p''(x) = -\frac{1}{2}\int_0^x \eta^{-1}[\sinh \eta(x-\xi) + \sin \eta(x-\xi)][i\eta^2 f(\xi) + g(\xi)]d\xi \tag{2.14}$$

$$= -\frac{1}{2}\int_0^x \eta^{-1}[\sinh \eta(x-\xi) + \sin \eta(x-\xi)]g(\xi)d\xi \qquad \text{(integration by parts)}$$

$$-\frac{1}{2}\int_0^x \eta^{-1}[\sinh \eta(x-\xi) - \sin \eta(x-\xi)]if''(\xi)d\xi$$

$$= -\frac{1}{2}\,\eta^{-1}\int_0^x \sinh\,\eta(x-\xi)[if''(\xi)+g(\xi)]d\xi \;+\; O(\eta^{-1}[\|f''\|+\|g\|)$$

$$= -\frac{1}{4}\,\eta^{-1}\int_0^1 e^{\eta(x-\xi)}[if''(\xi)+g(\xi)]d\xi \;+\; O(\eta^{-1}[\|f''\|+\|g\|])$$

$$= -\frac{1}{4}\,\eta^{-1}\,e^{\eta x}\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi \;+\; O(\eta^{-1}[\|f''\|+\|g\|])$$

2nd Step Estimation of h_1 and h_2.

From (2.6), (2.8) and (2.14),

$$h_1 = \int_0^1 K_{1\eta}(\xi)[i\eta^2 f(\xi)+g(\xi)]d\xi + k_1^2 f(1),$$

where

$$K_{1\eta}(\xi) = \tfrac{1}{2}[\cosh\,\eta(1-\xi)+\cos\,\eta(1-\xi)] - \tfrac{1}{2}k_1^2\eta^{-1}[\sinh\,\eta(1-\xi)-\sin\,\eta(1-\xi)]$$

Integration by parts twice for f yields

$$\int_0^1 K_{1\eta}\,(\xi)i\eta^2 f(\xi)d\xi = \int_0^1 \tilde{K}_{1\eta}(\xi)if''(\xi)d\xi - k_1^2 f(1),$$

where

$$\tilde{K}_{1\eta}(\xi) = \tfrac{1}{2}[\cosh\,\eta(1-\xi)-\cos\,\eta(1-\xi)] - \tfrac{1}{2}\,k_1^2\eta^{-1}[\sinh\,\eta(1-\xi)+\sin\,\eta(1-\xi)].$$

We get

$$K_{1\eta}(\xi) = \tfrac{1}{4}\eta^{-1}e^{\eta}(\eta-ik_1^2)e^{-\eta\xi} + O(1),$$

$$\tilde{K}_{1\eta}(\xi) = \tfrac{1}{4}\eta^{-1}e^{\eta}(\eta-ik_1^2)e^{-\eta\xi} + O(1).$$

Therefore

$$h_1 = \tfrac{1}{4}\eta^{-1}e^{\eta}(\eta-ik_1^2)\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi + O(\|f''\|+\|g\|). \quad (2.15)$$

Similarly, from (2.6) and (2.8),

$$h_2 = \int_0^1 K_{2\eta}(\xi)[i\eta^2 f(\xi)+g(\xi)]d\xi - k_2^2 f'(1)$$

where

$$K_{2\eta}(\xi) = \frac{1}{2}\eta^{-1}[\sinh \eta(1-\xi)+\sin \eta(1-\xi)] + \frac{i}{2}k_2^2[\cosh \eta(1-\xi)-\cos \eta(1-\xi)].$$

Repeating this same integration by parts procedure twice more for f, we get

$$\int_0^1 K_{2\eta}(\xi)i\eta^2 f(\xi)d\xi = \int_0^1 \tilde{K}_{2\eta}(\xi)if''(\xi)d\xi + k_2^2 f'(1).$$

As

$$K_{2\eta}(\xi) = \frac{1}{4}\eta^{-1}e^{\eta}(1+ik_2^2\eta)e^{-\eta\xi} + O(1)$$

$$\tilde{K}_{2\eta}(\xi) = \frac{1}{4}\eta^{-1}e^{\eta}(1+ik_2^2\eta)e^{-\eta\xi} + O(1),$$

we get

$$h_2 = \frac{1}{4}\eta^{-1}e^{\eta}(1+ik_2^2\eta)\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi + O(\|f''\|+\|g\|). \qquad (2.16)$$

<u>3rd Step</u> <u>Estimation of A_1, A_2, A_3 and A_4.</u>

We first write

$$M = \begin{bmatrix} 1 & & & 0 \\ & \eta & & \\ & & \eta^2 & \\ 0 & & & \eta^2 \end{bmatrix} M_1$$

where

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ (\eta-ik_1^2)e^{\eta} & (-i\eta-ik_1^2)e^{i\eta} & (-\eta-ik_1^2)e^{-\eta} & (i\eta-ik_1^2)e^{-i\eta} \\ (1+i\eta k_2^2)e^{\eta} & (-1-\eta k_2^2)e^{i\eta} & (1-i\eta k_2^2)e^{-\eta} & (-1+\eta k_2^2)e^{-i\eta} \end{bmatrix}.$$

So

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = M_1^{-1} \begin{bmatrix} 0 \\ 0 \\ \eta^{-2}h_1 \\ \eta^{-2}h_2 \end{bmatrix}.$$

Further, write

$$M_1^{-1} = (\det M_1)^{-1} \begin{bmatrix} * & * & \mu_{11} & \mu_{12} \\ * & * & \mu_{21} & \mu_{22} \\ * & * & \mu_{31} & \mu_{32} \\ * & * & \mu_{41} & \mu_{42} \end{bmatrix}.$$

From the evaluation of cofactors,

$$\mu_{11} = (1+i)[(1-\eta k_2^2)e^{-i\eta} + i(1+\eta k_2^2)e^{i\eta} + (1+i)(1-i\eta k_2^2)e^{-\eta}],$$
$$-\mu_{12} = (1-i)[(\eta-k_1^2)e^{-i\eta} - i(\eta+k_1^2)e^{i\eta} + (1-i)(\eta+ik_1^2)e^{-\eta}].$$

Let

$$\begin{aligned} \Sigma(\eta) &= (\eta-ik_1^2)\mu_{11} + (1+i\eta k_2^2)\mu_{12} \\ &= (1+i)\eta\{[-2k_2^2\eta+(1+i)(1+k_1^2k_2^2)-2ik_1^2\eta^{-1}]e^{-i\eta} \\ &\quad +[2ik_2^2\eta+(1+i)(1+k_1^2k_2^2)+2k_1^2\eta^{-1}]e^{i\eta}\} \\ &\quad + O(\eta^2 e^{-\eta}) \end{aligned}$$

The term in braces above satisfies

$$|\{\quad\}| \geq |\text{Bracket } 2| - |\text{Bracket } 1|$$

$$\longrightarrow 2(1+k_1^2k_2^2), \quad \text{as} \quad \eta \to \infty$$

Thus

$$|(\quad)| \geq \sqrt{2} \quad \text{for } \eta \text{ sufficiently large,}$$

hence

$$|\mathfrak{L}(\eta)| \geq 2\eta, \quad \mathfrak{L}(\eta) = O(\eta^2). \tag{2.17}$$

Also

$$D_1 \equiv \det M_1 = (\eta - ik_1^2)e^{\eta}\mu_{11} + (1+i\eta k_2^2)e^{\eta}\mu_{12} + O(\eta^2)$$

$$= \mathfrak{L}(\eta)e^{\eta} + O(\eta^2), \quad \text{therefore}$$

$$D_1^{-1} = [\mathfrak{L}(\eta)]^{-1}e^{-\eta} + O(e^{-2\eta}) \tag{2.18}$$

and by (2.17)

$$D_1^{-1} = O(\eta^{-1}e^{-\eta}). \tag{2.19}$$

From (2.15), (2.16) and (2.19),

$$A_1\eta^2 = D_1^{-1}(\mu_{11}h_1 + \mu_{12}h_2)$$

$$= (D_1^{-1}) \cdot \frac{1}{4}\eta^{-1}e^{\eta}\mathfrak{L}(\eta)\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi + O(D_1^{-1}\eta[\|f''\|+\|g\|])$$

as μ_{11}, μ_{12} are $O(\eta)$. Continuing from the above:

$$= [\mathfrak{L}^{-1}(\eta)e^{-\eta} + O(e^{-2\eta})] \cdot \frac{1}{4}\eta^{-1}e^{\eta}\mathfrak{L}(\eta)\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi$$

$$+ O(e^{-\eta}[\|f''\|+\|g\|]) \qquad \text{(by (2.17), (2.18))}$$

$$= \frac{1}{4}\eta^{-1}\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi + O(\eta e^{-\eta}|\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi|)$$

$$+ O(e^{-\eta}[\|f''\|+\|g\|]).$$

But

$$\left|\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi\right| = O(\eta^{-\frac{1}{2}}[\|f''\|+\|g\|]).$$

Thus

$$A_1\eta^2 = \frac{1}{4}\eta^{-1}\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi + O(\eta^{\frac{1}{2}}e^{-\eta}[\|f''\|+\|g\|]). \quad (2.20)$$

As for A_2, we have

$$A_2\eta^2 = D_1^{-1}(\mu_{21}h_1+\mu_{22}h_2) \quad (2.21)$$

where

$$-\mu_{21} = (1-i)(1+i\eta k_2^2)e^{\eta} + 2(1-\eta k_2^2)e^{-i\eta} + (1+i)(1-i\eta k_2^2)e^{-\eta} \quad (2.22)$$

$$\mu_{22} = (1-i)(\eta-ik_1^2)e^{\eta} - 2i(\eta-k_1^2)e^{-i\eta} - (1+i)(\eta-ik_1^2)e^{-\eta} \quad (2.23)$$

By (2.15), (2.16), (2.22) and (2.23), the dominant terms in $\mu_{21}h_1$ and $\mu_{22}h_2$ are $O(\eta e^{2\eta})$. But their coefficients in $\mu_{21}h_1 + \mu_{22}h_2$ are such that they cancel out. We get

$$A_2\eta^2 = D_1^{-1} \cdot O(\eta e^{\eta}[\|f''\|+\|g\|]) = O(\|f''\|+\|g\|) \quad (2.24)$$

Similarly, we can show that

$$A_3\eta^2 = O(\|f''\|+\|g\|) \quad (2.25)$$

$$A_4\eta^2 = O(\|f''\|+\|g\|) \quad (2.26)$$

Final Step Estimation of $\|w''\|+\|v\|$.

By (2.14) and (2.20), we have

$$\begin{aligned} w''(x) &= w_p''(x) + \tilde{w}''(x) \\ &= \left(-\tfrac{1}{4}\eta^{-1}e^{\eta x}\int_0^1 e^{-\eta\xi}[if''(\xi)+g(\xi)]d\xi + \mathcal{O}(\eta^{-1}[\|f''\|+\|g\|])\right) \\ &\quad + A_1\eta^2 e^{\eta x} + A_2(-\eta^2)e^{i\eta x} + A_3(-\eta)^2 e^{-\eta x} + A_4 \cdot (-\eta^2)e^{-i\eta x} \\ &= \{\mathcal{O}(\eta^{-1}[\|f''\|+\|g\|]) + e^{\eta x} \cdot \mathcal{O}(\eta^{\frac{1}{2}}e^{-\eta}[\|f''\|+\|g\|])\} \\ &\quad + (-A_2\eta^2 e^{i\eta x} + A_3\eta^2 e^{-\eta x} - A_4\eta^2 e^{-i\eta x}). \end{aligned}$$

In the first parenthesized term, the $\mathcal{L}^2$-norm of $e^{\eta x}$ is of order of magnitude

$$\left[\int_0^1 (e^{\eta x})^2 dx\right]^{1/2} = \left[\frac{1}{2\eta}(e^{2\eta}-1)\right]^{1/2} = \mathcal{O}(\eta^{-1/2}e^{\eta}),$$

hence

$$\left[\int_0^1 |\text{first parenthesized term}|^2 dx\right]^{1/2} = \mathcal{O}(\|f''\|+\|g\|).$$

The second parenthesized term also satisfies the above bound, because of (2.24), (2.25), and (2.26).

Hence

$$\|w''\| \le C(\|f''\|+\|g\|) = \mathcal{O}(\|f''\|+\|g\|), \quad \text{for } \eta \text{ large}. \tag{2.27}$$

For v, by (2.4) we have

$$\|v\| \le |\lambda|\,\|w\| + \|f\|. \tag{2.28}$$

We want to show that

$$|\lambda|^2\|w\|^2 \le C(\|w''\|^2+\|f''\|^2+\|f\|^2+\|g\|^2), \tag{2.29}$$

for some constant $C > 0$ independent of λ.

Consider (2.2.1), with $\alpha^4 = 1$ and $\lambda = i\eta^2$, and use k_1^2, k_2^2 for $\tilde{k}_1^2$ and $\tilde{k}_2^2$. Multiply (2.2.1) by $\bar{w}(x)$ and integrate by parts twice. We get

$$w'''(1)\bar{w}(1)-w''(1)\bar{w}'(1) + \|w''\|^2 - \eta^4\|w\|^2 = -\langle\lambda f+g,w\rangle$$
$$= -\langle f,\lambda w\rangle-\langle g,w\rangle.$$

From (2.2.3) and (2.2.4), we get

$$i\eta^2k_1^2|w(1)|^2+k_1^2f(1)\bar{w}(1)+i\eta k_2^2|w'(1)|^2+k_2^2f'(1)\bar{w}'(1)+\|w''\|^2-\eta^4\|w\|^2$$
$$= -\langle f,\lambda w\rangle-\langle g,w\rangle.$$

Hence

$$\eta^4\|w\|^2 = \mathrm{Re}\{\|w''\|^2+\langle f,\lambda w\rangle+\langle g,w\rangle+k_1^2f(1)\bar{w}(1)+k_2^2f'(1)\bar{w}'(1)\}$$
$$\le \|w''\|^2 + \frac{|\eta|^4}{4}\|w\|^2+2\|f\|^2+ \frac{1}{2}\|g\|^2 + \frac{1}{2}\|w\|^2 + C[\|f''\|^2+\|w''\|^2],$$

where we have applied the Poincaré inequality and the trace theorem:

$$|f(1)|^2 + |f'(1)|^2 \le C\|f''\|^2$$
$$|w(1)|^2 + |w'(1)|^2 \le C\|w''\|^2.$$

Therefore (2.29) follows for η sufficiently large.

By (2.27) and (2.29), we have

$$\|v\| \le C(\|f''\|+\|f\|+\|w''\|+\|g\|) \tag{2.30}$$
$$\le C(\|f''\|+\|g\|)$$

as

$$\|f\| \le C\|f''\|$$

and (2.27) holds.

Combining (2.27) and (2.30), we have proved (2.5) for $|\lambda|$ sufficiently large, $\lambda = i\omega$, $\omega \in \mathbb{R}$. So Lemma 3 has been proved. □

<u>THEOREM 4</u>. Let $k_1^2 \ge 0$ and $k_2^2 > 0$ in (1.1). Then the uniform exponential decay of energy (1.2) holds.

<u>Proof</u>: In order to apply Theorem 1, we need to verify that assumptions (1.6), (1.7) and (1.8) are satisfied.

We note that (1.6) is satisfied, because A is dissipative and

$$\|exp(tA)\| \le 1.$$

The verification of (1.7) and (1.8) is accomplished if we can verify merely (1.7):

$$(\lambda I-A)^{-1} \text{ exists for all } \lambda = i\omega, \quad \omega \in \mathbb{R}. \tag{2.31}$$

because by (2.31) and Lemma 3,

$$\|w''_\lambda\| + \|v_\lambda\| \le C'(\|f''\|+\|g\|), \quad \forall\lambda = i\omega, \quad \omega \in \mathbb{R},$$

where

$$C' = \max(C,C''), \quad C \text{ as in (2.27) and (2.30)}$$

$$C'' = \max_{|\lambda|\le B_2} \|(\lambda I-A)^{-1}\|, \quad \text{for some } B_2 \text{ sufficiently large,}$$

$$\lambda = i\omega, \quad \omega \in \mathbb{R}.$$

To show (2.31), we assume the contrary that $\sigma(A) \cap \{i\omega|\omega \in \mathbb{R}\} \ne \emptyset$. By Lemma 2, $\sigma(A)$ consists solely of isolated nonzero eigenvalues. Without loss of generality, let

$$\lambda_0 \in \sigma(A), \quad \lambda_0 = i\eta_0^2, \quad \eta_0 \in \mathbb{R}, \quad \eta_0 \ne 0.$$

Then

$$(\lambda_0 I-A)\begin{bmatrix} w_0 \\ v_0 \end{bmatrix} = 0$$

has a nontrivial solution $(w_0,v_0) \in D(A)$. Explicitly, (w_0,v_0) satisfies

$$\begin{cases} i\eta_0^2 w_0 - v_0 = 0 & \text{on } [0,1] \\ w_0^{(4)} + i\eta_0^2 v_0 = 0 & \text{on } [0,1] \\ w_0(0) = 0 \\ w_0'(0) = 0 \\ w_0''(1) - k_1^2 v_0(1) = 0 \\ w_0''(1) + k_2^2 v_0'(1) = 0 \end{cases}$$

Letting

$$w(x,t) = e^{i\eta_0^2 t} w_0(x),$$

we easily check that w satisfies

$$w_{tt} + w_{xxxx} = 0.$$

Also, the energy

$$\int_0^1 [|w_{xx}(x,t)|^2 + |w_t(x,t)|^2]dx$$

is constant, thus

$$\frac{d}{dt}\int_0^1 [|w_{xx}(x,t)|^2 + |w_t(x,t)|^2]dx$$

$$= 0$$

$$= 2\mathrm{Re}[w_{xx}(x,t)\bar{w}_{xt}(x,t) - w_{xxx}(x,t)\bar{w}_t(x,t)]\Big|_{x=0}^{x=1}$$

$$= -2[k_1^2|w_t(1,t)|^2 + k_2^2|w_{xt}(1,t)|^2].$$

Because $k_2^2 > 0$ and $k_1^2 \ge 0$, we deduce

$$|w_{xt}(1,t)| = \eta_0^2|w_0'(1)| = 0,$$

$$\begin{cases} |w_{xxx}(1,t)| = |w_0'''(1)| = 0, & \text{if } k_1^2 = 0; \\ |w_{xxx}(1,t)| = |w_0'''(1)| = 0, \ |w_t(1,t)| = \eta_0^2|w_0(1)| = 0, & \text{if } k_1^2 > 0. \end{cases}$$

Thus $w_0(x)$ is a solution to the boundary value problem

$$\left.\begin{array}{ll} w_0^{(4)} - \eta_0^4 w_0 = 0 & \text{on } [0,1] \\ w_0(0) = 0 & \\ w_0'(0) = 0 & \\ w_0'(1) = 0 & \\ w_0''(1) = 0 & \\ w_0'''(1) = 0 & \end{array}\right\} \qquad (2.32)$$

Write

$$w_0(x) = A_{01}\cos \eta_0(x_0-1)+A_{02}\sin \eta_0(x-1)+A_{03}\cosh \eta_0(x-1)+A_{04}\sinh \eta_0(x-1)$$

Then the five boundary conditions in (2.32) require that

$$\begin{bmatrix} \cos\eta_0 & -\sin\eta_0 & \cosh\eta_0 & -\sinh\eta_0 \\ \sin\eta_0 & \cos\eta_0 & -\sinh\eta_0 & \cosh\eta_0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{01} \\ A_{02} \\ A_{03} \\ A_{04} \end{bmatrix} = 0 \qquad (2.33)$$

has a nontrivial solution $(A_{01}, A_{02}, A_{03}, A_{04})$. However, it is easy to check that the matrix in (2.33) has rank 4 for any $\eta_0 \in \mathbb{R}$, $\eta_0 \neq 0$, a contradiction.

Therefore the proof of Theorem 4 is complete. □

3. ASYMPTOTIC ESTIMATION OF EIGENFREQUENCIES

From the graphs in [2], *one notices that* at low frequencies eigenvalues of the damped operator A seem to exhibit a "structural damping" [3] pattern. Does the structural damping pattern continue into high frequencies, or is it only a low frequency phenomenon, for beams with boundary dissipation? To answer this one must examine the asymptotics of eigenfrequencies.

The work of asymptotic analysis was first done by P. Rideau in his thesis [8] (cf. the acknowledgement at the end of the paper). Unaware of his results, we had carried out the analysis independently. We feel that it is of significant interest to include the work here as it will make the study in this paper more complete, and only a minor effort is required.

Let $(\phi(x),\psi(x))$ be an eigenfunction of A belonging to the eigenvalue $\lambda(\neq 0)$. Then by (2.2), setting $f(x) = g(x) \equiv 0$ and $w_\lambda = \phi$, we see that ϕ satisfies

$$\left.\begin{aligned} &\alpha^4\phi^{(4)}(x) + \lambda^2\phi(x) = 0\\ &\phi(0) = \phi'(0) = 0\\ &\phi'''(1) - \lambda\tilde{k}_1^2\phi(1) = 0\\ &\phi''(1) + \lambda\tilde{k}_2^2\phi'(1) = 0 \end{aligned}\right\} \tag{3.1}$$

To simplify notations, we consider the following eigenvalue problem

$$\left.\begin{aligned} &\phi^{(4)}(x) + \lambda^2\phi(x) = 0\\ &\phi(0) = 0\\ &\phi'(0) = 0\\ &\phi'''(1) - \lambda k_1^2\phi(1) = 0\\ &\phi''(1) + \lambda k_2^2\phi'(1) = 0 \end{aligned}\right\} \tag{3.2}$$

Noting that the following correspondence

$$\left.\begin{matrix} \frac{\lambda}{\alpha^2}\\ \alpha^2\tilde{k}_1^2\\ \alpha^2\tilde{k}_2 \end{matrix}\ \text{in (3.1)}\right\} \longleftrightarrow \left\{\begin{matrix} \lambda\\ k_1^2\\ k_2^2 \end{matrix}\ \text{in (3.2)}\right. \tag{3.3}$$

is in effect.

The boundary value problem (3.1) has a nontrivial solution if and only if

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 \\ \mu\sqrt{\lambda} & \mu^3\sqrt{\lambda} & \mu^5\sqrt{\lambda} & \mu^7\sqrt{\lambda} \\ (\mu^3\lambda^{\frac{3}{2}}-k_1^2\lambda)\cdot e^{\mu\sqrt{\lambda}} & (\mu^9\lambda^{\frac{3}{2}}-\lambda k_1^2)\cdot e^{\mu^3\sqrt{\lambda}} & (\mu^{15}\lambda^{\frac{3}{2}}-k_1^2\lambda)\cdot e^{\mu^5\sqrt{\lambda}} & (\mu^{21}\lambda^{\frac{3}{2}}-k_1^2\lambda)\cdot e^{\mu^7\sqrt{\lambda}} \\ (\mu^2\lambda+k_2^2\mu\lambda^{\frac{3}{2}})\cdot e^{\mu\sqrt{\lambda}} & (\mu^6\lambda+\mu^3k_2^2\lambda^{\frac{3}{2}})\cdot e^{\mu^3\sqrt{\lambda}} & (\mu^{10}\lambda+\mu^5k_2^2\lambda^{\frac{3}{2}})\cdot e^{\mu^5\sqrt{\lambda}} & (\mu^{14}\lambda+\mu^7k_2^2\lambda^{\frac{3}{2}})\cdot e^{\mu^7\sqrt{\lambda}} \end{bmatrix} = 0, \tag{3.4}$$

where μ is the eighth root of unity, $\exp(i\pi/4)$. The derivation of the above is identical to (2.10)-(2.12).

Evaluating this determinant yields the transcendental equation

$$\begin{aligned} &2\sqrt{2}k_2^2\lambda\left\{ie^{-i\sqrt{2\lambda}}-e^{-\sqrt{2\lambda}}-ie^{i\sqrt{2\lambda}}+e^{\sqrt{2\lambda}}\right\} \\ &+\sqrt{\lambda}\Big\{8(1-k_1^2k_2^2)+2(1+k_1^2k_2^2)e^{-\sqrt{2\lambda}}+2(1+k_1^2k_2^2)e^{-i\sqrt{2\lambda}} \\ &\qquad +2(1+k_1^2k_2^2)e^{\sqrt{2\lambda}}+2(1+k_1^2k_2^2)e^{i\sqrt{2\lambda}}\Big\} \\ &+2\sqrt{2}k_1^2\left\{-ie^{-i\sqrt{2\lambda}}-e^{-\sqrt{2\lambda}}+ie^{i\sqrt{2\lambda}}+e^{\sqrt{2\lambda}}\right\} = 0. \end{aligned} \tag{3.5}$$

Now, write

$$\lambda = |\lambda|e^{i\theta}. \tag{3.6}$$

As the closed right half plane does not contain any eigenvalues, and because in (3.5), λ is symmetric with respect to the real axis, we need only consider $\frac{\pi}{2} < \theta \le \pi$ in (3.6). We will actually first consider

$$\frac{\pi}{2} < \theta \le \pi - \delta, \quad \text{for any } \delta > 0 \text{ sufficiently small.} \tag{3.7}$$

The case of $\theta \to \pi$ will be considered in (3.11)-(3.12).

Since

$$\sqrt{\lambda} = |\lambda|^{1/2}\exp(i\theta/2) = |\lambda|^{1/2}[\cos(\theta/2)+i\sin(\theta/2)].$$

we see that

$$e^{-\sqrt{2\lambda}} = e^{-\sqrt{2|\lambda|}\cos(\theta/2)}e^{-i\sqrt{2|\lambda|}\sin(\theta/2)}$$

$$e^{i\sqrt{2\lambda}} = e^{-\sqrt{2|\lambda|}\sin(\theta/2)}e^{i\sqrt{2|\lambda|}\cos(\theta/2)}$$

are $O(e^{-\gamma\sqrt{|\lambda|}})$ for some $\gamma > 0$. Thus, from (3.5)

$$8\sqrt{\lambda}(1-k_1^2k_2^2) + e^{-i\sqrt{2\lambda}}[2i\sqrt{2}k_2^2\lambda+2(1+k_1^2k_2^2)\sqrt{\lambda} - i2\sqrt{2}k_1^2]$$

$$+ e^{\sqrt{2\lambda}}[2\sqrt{2}k_2^2\lambda+2\sqrt{\lambda}(1+k_1^2k_2^2)+2\sqrt{2}k_1^2] = O(|\lambda|\exp(-\gamma\sqrt{|\lambda|})),$$

which implies

$$e^{\sqrt{2|\lambda|}}[-\sin(\theta/2)+\cos(\theta/2)] = -e^{-i\sqrt{2|\lambda|}}[\cos(\theta/2)+\sin(\theta/2)] \times$$

$$\times \left\{\frac{2i\sqrt{2}k_2^2\lambda+2(1+k_1^2k_2^2)\sqrt{\lambda}-i2\sqrt{2}k_1^2}{2\sqrt{2}k_2^2\lambda+2(1+k_1^2k_2^2)\sqrt{\lambda}+2\sqrt{2}k_1^2}\right\} + O(|\lambda|\exp(-\gamma\sqrt{|\lambda|}))$$

But (assuming $k_2^2 > 0$) the term in braces equals

$$i + \frac{(1+i)}{\sqrt{2}}\,\frac{(1+k_1^2k_2^2)}{k_2^2\sqrt{|\lambda|}}\,e^{-i\theta/2} + O(|\lambda|^{-1}).$$

Thus we seek λ's satisfying

$$e^{\sqrt{2|\lambda|}}[-\sin(\theta/2)+\cos(\theta/2)] \tag{3.8}$$

$$= \left[-i - \frac{(1+i)}{\sqrt{2}}\,\frac{(1+k_1^2k_2^2)}{k_2^2\sqrt{|\lambda|}}\,e^{-i\theta/2}\right]e^{-i\sqrt{2|\lambda|}}[\cos(\theta/2)+\sin(\theta/2)]$$

$$+ O(|\lambda|^{-1}).$$

We observe immediately that $\theta \to \pi/2$ as $|\lambda| \to \infty$, since the LHS of this equation would decrease to zero otherwise. Furthermore, the equality can be satisfied (up to higher order terms) only when the first term on the RHS is a <u>positive real number</u>. Thus

$$e^{-i\sqrt{2|\lambda|}[\cos(\theta/2)+\sin(\theta/2)]} \approx e^{-i\sqrt{2|\lambda|}} \cdot \sqrt{2} \approx 1$$

$$2\sqrt{|\lambda|} \approx (2n - \tfrac{1}{2})\pi,$$

or

$$|\lambda| \approx \left[\frac{(2n - \frac{1}{2})}{2}\pi\right]^2, \quad n \text{ is a large positive integer.} \tag{3.9}$$

The above gap of $O(n^2)$ for eigenvalues is common for Euler-Bernoulli beams with energy conserving boundary conditions. Now we see that Euler-Bernoulli beams with boundary energy dissipation also have this property.

One checks that when the RHS of (3.8) is real, its modulus is

$$1 - \frac{(1+k_1^2k_2^2)[\cos(\theta/2)+\sin(\theta/2)]}{\sqrt{2}k_2^2\sqrt{|\lambda|}} + O(|\lambda|^{-1}).$$

This in turn implies that the exponent on the LHS of (3.8) must satisfy

$$\sqrt{2|\lambda|}\left[-\sin(\tfrac{\theta}{2})+\cos(\tfrac{\theta}{2})\right] \approx -\frac{(1+k_1^2k_2^2)\left[\cos(\frac{\theta}{2})+\sin(\frac{\theta}{2})\right]}{\sqrt{2}k_2^2\sqrt{|\lambda|}}.$$

If we now write $\theta = \frac{\pi}{2} + \varepsilon$, $\varepsilon > 0$, and expand to lowest order in ε, we have

$$\varepsilon \approx \frac{(1+k_1^2k_2^2)}{k_2^2|\lambda|}.$$

Now suppose $\lambda = \xi + i\eta$. Then $\theta = \tan^{-1}(\eta/\xi)$ and $|\lambda| = (\xi^2+\eta^2)^{1/2}$. Expanding $\tan^{-1}$ about $\eta/\xi = \infty$, we have

$$\varepsilon \approx -\frac{\xi}{\eta},$$

or

$$-\frac{\xi}{\eta} \approx \frac{(1+k_1^2k_2^2)}{k_2^2(\xi^2+\eta^2)^{1/2}}.$$

Hence

$$\xi \approx -\frac{1}{k_2^2}(1+k_1^2k_2^2), \quad \text{as} \quad |\lambda| \to \infty. \tag{3.10}$$

By (3.7), the only remaining case to be considered is when $\theta \to \pi$, i.e. when λ approaches the negative real axis. We write

$$\sqrt{\lambda} = |\lambda|^{1/2}(\cos(\theta/2)+i\sin(\theta/2))$$

as before, but this time was assume $|\pi-\theta| < \delta$, with δ small -- say $0 \leq \delta < \pi/8$. Then one easily checks that

$$\begin{aligned} |e^{-\sqrt{2\lambda}}| &\leq c, \\ |e^{i\sqrt{2\lambda}}| &\leq c, \end{aligned} \tag{3.11}$$

for some $c > 0$ so that (3.5) can be rewritten as:

$$\begin{aligned} &(2\sqrt{2}\ ik_2^2\lambda+2\sqrt{2}(1+k_1^2k_2^2)-2\sqrt{2}ik_1^2)e^{\sqrt{2|\lambda|}(\sin(\theta/2)-i\cos(\theta/2))} \\ &= (2\sqrt{2}k_2^2\lambda+2\sqrt{\lambda}(1+k_1^2k_2^2)+2\sqrt{2}k_1^2)e^{\sqrt{2|\lambda|}(\cos(\theta/2)+i\sin(\theta/2))} \\ &\qquad + O(\lambda). \end{aligned} \tag{3.12}$$

However, for θ in the range of interest, $\sin(\theta/2) > 2\cos(\theta/2)$ and in particular, $\sin(\theta/2) > 0.5$. Thus, the modulus of the L.H.S. of (3.12) will be much larger than that of the R.H.S. (for $|\lambda|$ large) so this equation has no solutions if $|\lambda|$ is large.

<u>THEOREM 5</u> Let $\lambda = \xi + i\eta$ be an eigenfrequency of vibration of the beam equation (1.1). Then for $|\lambda|$ large,

$$|\lambda| \sim \left[\frac{(2n-\frac{1}{2})\pi}{2}\right]^2\left(\frac{EI}{m}\right)^{1/2}, \quad \text{n's are large positive integers.}$$

$$\xi \to -\frac{[m(EI)]^{1/2}\left\{1+k_1^2k_2^2[m(EI)]^{-1}\right\}}{k_2^2} \quad \text{as} \quad |\lambda| \to \infty \qquad (3.13)$$

Proof: Just use (2.3), (3.3), (3.9) and (3.10).

By (3.13), the eigenvalues will be distributed nearly parallel to the imaginary axis at high frequencies. Therefore there is no"structural damping" when the frequencies are high. This has also been confirmed numerically in Rideau's thesis [8].

We note that when $k_2^2 = 0$ and $k_1^2 > 0$, asymptotic limits can be obtained in the similar way.

4. DESIGN OF PASSIVE DAMPING MECHANISMS

The following is a (more or less exhaustive) list of combinations of dissipative boundary conditions for an Euler-Bernoulli beam:

$$\left.\begin{aligned} -EIy_{xxx}(1,t) &= -k_1^2y_t(1,t), \quad k_1^2 > 0 \\ -EIy_{xx}(1,t) &= 0 \end{aligned}\right\} \qquad (4.1)$$

$$\left.\begin{aligned} -EIy_{xxx}(1,t) &= 0 \\ -EIy_{xx}(1,t) &= k_2^2y_{xt}(1,t), \quad k_2^2 > 0 \end{aligned}\right\} \qquad (4.2)$$

$$\left.\begin{aligned} y_x(1,t) &= 0 \\ -EIy_{xxx}(1,t) &= -k_1^2y_t(1,t), \quad k_1^2 > 0 \end{aligned}\right\} \qquad (4.3)$$

$$\left.\begin{aligned} y(1,t) &= 0 \\ -EIy_{xx}(1,t) &= k_2^2y_{xt}(1,t), \quad k_2^2 > 0 \end{aligned}\right\} \qquad (4.4)$$

$$\left.\begin{aligned} -EIy_{xxx}(1,t) &= -k_1^2y_t(1,t) + c_1y_{xt}(1,t), \quad k_1^2 > 0 \\ -EIy_{xx}(1,t) &= k_2^2y_{xt}(1,t) + c_2y_t(1,t), \quad k_2^2 > 0 \end{aligned}\right\} \qquad (4.5)$$

where in (4.5), c_1 and c_2 are real constants satisfying

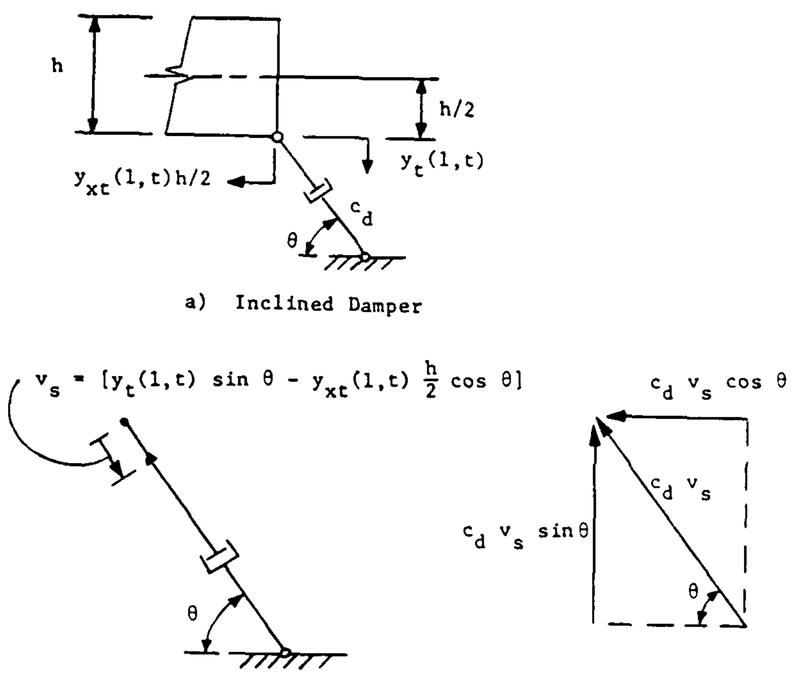
$$(c_1-c_2)\alpha\beta - k_1^2\alpha^2 - k_2^2\beta^2 \le 0 \quad \forall\ \alpha,\beta \in \mathbb{R}. \qquad (4.6)$$

Note that the boundary conditions (1.1.4) and (1.1.5) correspond to $c_1 = c_2 = 0$ in (4.5). Obviously, (4.6) is satisfied in this case.

We want to show that all stabilization schemes (4.1)-(4.5) can be realized in practice, at least by designing passive dampers.

As (4.5) seems to represent the most complicated case among (4.1)-(4.5), we treat it here, at least for certain special values of c_1 and c_2 (cf. (4.7) later). The other cases can be studied similarly.

The following damper arrangement gives a design which effects the coupling of shear (resp. bending moment) with velocity and angular velocity:



a) Inclined Damper

b) Shortening Velocity of Damper

c) Damper Forces on Beam

$$\text{Shear}(1,t) = -c_d\ v_s \sin\theta$$

$$\text{Moment}(1,t) = c_d\ v_s \cos\theta\ h/2$$

Figure 2 Damper arrangement for (4.5)

A single damper (cf. Figure 2a) is attached to the lower end of the beam at an inclination angle θ with respect to the horizontal. Using the velocity at the end of the damper, v_s, and the associated forces shown in Figure 2b and 2c, we obtain

$$\mathrm{Shear}(1,t) = -c_d v_s \sin\theta$$
$$\mathrm{Moment}(1,t) = c_d v_s (\cos\theta)\frac{h}{2}\ ,$$

where c_d represents the damping coefficient associated with the damper in use. As

$$v_s = y_t(1,t)\sin\theta - y_{xt}(1,t)\frac{h}{2}\cos\theta,$$

we get

$$\mathrm{Shear}(1,t) = -EIy_{xxx}(1,t) = -(c_d\sin^2\theta)y_t + (\frac{h}{2}\cdot c_d\sin\theta\cos\theta)y_{xt}$$
$$\mathrm{Moment}(1,t) = -EIy_{xx}(1,t) = (\frac{h^2}{4}c_d\cos^2\theta)y_{xt} + (-\frac{h}{2}c_d\sin\theta\cos\theta)y_t.$$

A comparison of the above with (4.5) shows that

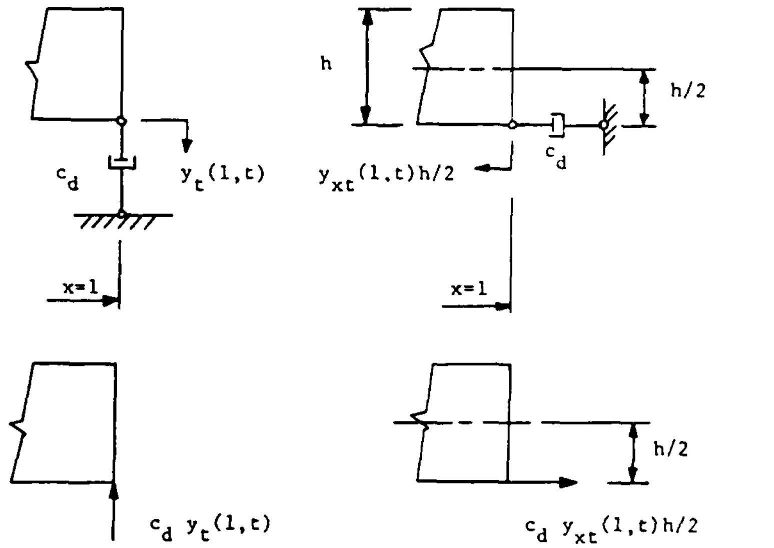
$$k_1^2 = c_d\sin^2\theta;\ k_2^2 = \frac{h^2}{4}c_d\cos^2\theta$$
$$c_1 = -c_2 = \frac{h}{2}c_d\sin\theta\cos\theta, \qquad (4.7)$$

thus

$$(c_1-c_2)\alpha\beta - k_1^2\alpha^2 - k_2^2\beta^2 = -(k_1\alpha-k_2\beta)^2 \le 0 \quad \forall\ \alpha,\beta \in \mathbb{R}$$

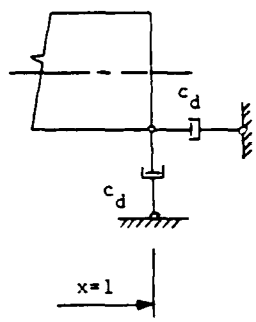
so (4.6) is satisfied and the boundary conditions (4.5) are dissipative.

It is noted that when $\theta = \pi/2$ (vertical damper), the above gain constants reduce to $k_1^2 = c_{d,1}$ and $k_2^2 = c_1 = c_2 = 0$, cf. Figure 3a. Similarly, for $\theta = 0$ (horizontal damper), the gain constants become $k_2^2 = c_{d,2}h^2/4$ and $k_1^2 = c_1 = c_2 = 0$, as shown in Figure 3b. Consequently, the boundary conditions (1.1.4)-(1.1.5) can be realized as in Figure 3c, with $k_1^2 = c_{d,1}$ and $k_2^2 = c_{d,2}h^2/4$.



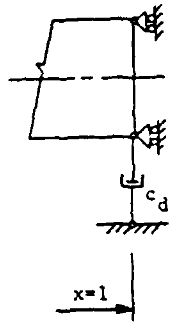
$\text{Shear}(1,t) = -c_d\ y_t(1,t)$ $\text{Moment}(1,t) = c_d\ y_{xt}(1,t)\ h^2/4$

a) Vertical Damper b) Horizontal Damper



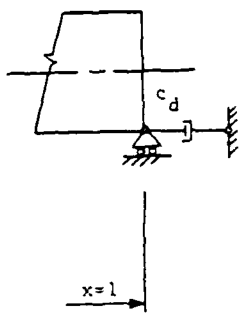
c) $\text{Shear}(1,t) = -c_d y_t(1,t)$
$\text{Moment}(1,t) = c_d y_{xt}(1,t)\ h^2/4$

Figure 3 Damper arrangement for (4.1), (4.2) and (1.1.4)+(1.1.5)



$\text{Shear}(1,t) = -c_d y_t(1,t)$
$\text{Slope}(1,t) = y_x(1,t) = 0$

Figure 4 Damper arrangement for (4.3)



$\text{Displ}(1,t) = y(1,t) = 0$
$\text{Moment}(1,t) = c_d y_{xt}(1,t)\ h^2/4$

Figure 5 Damper arrangement for (4.4)

The other boundary conditions (4.1)-(4.4) can be realized and designed, respectively, as in Figures 3a, 3b, 4 and 5.

The method of estimation which we have developed in this paper and Huang's theorem (Thm. 1) can be applied to study exponential stability for all of these boundary stabilization schemes.

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