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THE EVEN CYCLE PROBLEM FOR DIRECTED GRAPHS

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1. INTRODUCTION

The problem of deciding if a given digraph (directed graph) has an even length dicycle (i.e., directed cycle of even length) has come up in various connection. It is a well-known hard problem to decide if a hypergraph is bipartite. Seymour [11] (see also [15]) showed that a minimally nonbipartite hypergraph has at least as many hyperedges as vertices. He characterized those with the same number of hyperedges and vertices in terms of digraphs with no even length dicycle.

Problems in qualitative linear algebra have motivated the concept of a sign-nonsingular matrix. This is a real matrix A such that each matrix A' with the same sign pattern as A (i.e., corresponding entries in A and A' either have the same sign or both equal 0) has linearly independent columns. Klee et al [4] showed that it is hard to decide if a given matrix is sign-nonsingular. However, they left an important special case open: They showed that the problem of deciding if a square matrix is sign-nonsingular is equivalent with the even length dicycle problem for digraphs.

Although the concepts of the determinant $\det A$ and the permanant $\operatorname{per} A$ of a real square matrix A are analogous, they are not equally easy to compute. Computing $\operatorname{per} A$ is hard even for the 0-1 case (see [17]). That special case amounts to finding the number of perfect matchings in bipartite graphs, a problem that plays a role in models in physics [3] and chemistry [10]. Polya [8] suggested that one might try to multiply some entries in a matrix A by -1 and thereby obtain a matrix A' such that $\operatorname{per} A = \det A'$. Vazirani and Yannakakis [17] showed that the problem of finding such a matrix A' is equivalent to the even length dicycle problem.

In 1975 Lovász [6] raised two fundamental questions on the even length dicycle problem.

L(1) Does there exist a natural number k such that any digraph in which there are at least k arcs leaving each vertex has an even length dicycle?

L(2) Does there exist a natural number k such that any strongly k-connected digraph has an even length dicycle? ("Strongly k-connected" means that the removal of any vertex set of cardinality $< k$ leaves a digraph in which each vertex can be reached by a directed path from each other vertex.)

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For undirected graphs there is a host of results on configurations that are guaranteed by large degrees or large connectivity. For a survey, see [15]. Moreover, questions involving large connectivity or large minimum degree are equivalent in the undirected case since large connectivity implies large minimum degree and large minimum degree implies, by a result of W. Mader, the presence of a subgraph of large connectivity (see [15]). This changes dramatically when we go to digraphs: L(1) was answered in the negative by the present author [13], and the present paper provides an affirmative answer to L(2). The best constant in L(2) is 3. Interesting partial results on L(1), L(2) were found by Friedland [2] who used (the validity of) the van der Waerden conjecture to prove the following: If m is a natural number, $m \geq 7$, and D is a digraph in which there are precisely m arcs leaving and entering each vertex, then D has an even length dicycle. A related result with an equally interesting proof, based on the so-called Lovász Local Lemma, was obtained by Alon and Lineal [1]. Friedland conjectured that his result also holds for $m \geq 3$. A stronger result was suggested by the author [13, Question 1]. Both conjectures follow from the result of the present paper. Vazirani and Yannakakis [17] suggested that one might obtain an affirmative answer to L(2) from [2] by proving that, for r sufficiently large, every strongly r-connected digraph contains a digraph satisfying the assumption of [2]. We show in this paper that no such r exists.

2. Terminology and preliminaries

A digraph D consists of a finite set $V(D)$ of *vertices* and a set $E(D)$ of ordered pairs xy of distinct vertices called *arcs*. If the arc $e = xy$ is present, we say that x *dominates* y and that e *leaves* x and *enters* y. More generally, if A and B are disjoint vertex sets such that $x \in A$, $y \in B$, then e *leaves* A and *enters* B. The number of arcs leaving x is the *outdegree* of x and is denoted $d^+(x, D)$. The *indegree* $d^-(x, D)$ is defined analogously.

A *dipath* is a digraph with distinct vertices $x_1, x_2, \dots, x_m$ and arcs $x_1 x_{i+1}$, $i = 1, 2, \dots, m-1$. We call this an $x_1 - x_m$ dipath. More generally, if A and B are disjoint vertex sets in a digraph D, then an $A - B$ dipath is an $x - y$ dipath P such that $x \in A$, $y \in B$ and $V(P) \cap (A \cup B) = \{x, y\}$. We say that y can be *reached* from x in D if such a P exists. If each vertex of D can be reached from each other vertex of D, then D is *strong*. If $A \subseteq V(D) \cup E(D)$, then $D - A$ is obtained from D by deleting A and all arcs leaving or entering vertices in A. We write $D - x$ instead of $D - \{x\}$ if $x \in V(D) \cup E(D)$. If $A \subseteq V(D)$, then the subdigraph $D(A)$ *induced* by A is defined as $D - (V(D) \backslash A)$. D is *strongly k-connected* if $|V(D)| \geq k + 1$ and $D - A$ is strong for each vertex set A of cardinality $< k$. A *strong component* of D is a maximal strong subdigraph. An *initial component* (respectively *terminal component*) of D is a component H of D such that no arcs of D enters (respectively leaves) H. It is easy to see that every digraph has at least one initial component and at least one terminal component.

If D is a digraph, then *splitting a vertex* v of outdegree or indegree at least 2 in D means that we replace v by two vertices x and y where x dominates y. All arcs vz (respectively uv) in D are replaced by yz (respectively ux) in the new digraph. *Subdividing* an arc xy means that we replace xy by an

$x - y$ dipath that has no intermediate vertices in common with D.

A *dicycle* (more precisely, an m-dicycle) is a digraph with vertex set x_1, $x_2, \ldots, x_m$ and arc set $x_1x_2, x_2x_3, \ldots, x_{m-1}x_m, x_mx_1$. If we add the arcs $x_1x_m, x_mx_{m-1}, \ldots, x_2x_1$ we obtain a *double-cycle* (more precisely an m-double-cycle). A *weak m-double-cycle* is obtained from an m-double-cycle by splitting vertices and subdividing arcs. A *weak odd double-cycle* is a weak m-double cycle for some odd m. A digraph D is *even* if every subdivision of D contains a dicycle of even length. Equivalently, if the arcs of D are assigned weights 0 or 1, then there is a dicycle of even total weight. A weak odd double-cycle has an odd number of dicycles and every arc is in an even number of dicycles. This implies that every weak odd double-cycle is even. Conversely, we have

Theorem 2.1 [12]. *A digraph is even if and only if it contains a weak odd double-cycle.*

(For the main result in this paper we only use the trivial part of Theorem 2.1. Only in the last section is Theorem 2.1 used in its full strength.)

Contracting an arc xy of D means that we replace x, y by a single vertex z and all arcs leaving or entering x or y will leave or enter z in the new digraph. (No loops are introduced.)

Lemma 2.2. *Let xy be an arc of D such that either $d^+(x, D) = 1$ or $d^-(y, D) = 1$. Let D' be obtained from D by contracting xy into a vertex z. Then D' contains a weak k-double-cycle if and only if D contains a weak k-double cycle.*

Proof. Assume that $d^+(x, D) = 1$. Let M' be a weak k-double-cycle in D'. If M' contains no arc that in D enters x, then $M' \subseteq D$. So assume that M' contains an arc that enters x in D. Let M be the subdigraph in D containing x, y, xy, all arcs of M', and all vertices of M' (except z). Then M is a weak k-double-cycle in D. Conversely, the contraction of xy transforms any weak k-double-cycle in D into a weak k-double-cycle in D'. □

Consider a digraph D that is strong but not strongly 2-connected. Let v be a vertex such that $D-v$ is not strong. Let H be an initial or terminal component of D. We define the *H-reduction* of D at v as follows: If H is a terminal component, then the H-reduction at v is obtained from $D(V(H) \cup \{v\})$ by adding all the arcs vz for all vertices z in H for which there is an arc in D of the form uz where $u \notin V(H)$. If H is an initial component, then the definition is analogous.

Lemma 2.3. *Let D be a strong digraph such that $D - v$ is not strong. Let H be a terminal component of $D - v$. Let D' be the H-reduction of D at v. If D' has a weak k-double-cycle, then D has a weak k-double-cycle.*

Proof. Let M' be a weak k-double-cycle in D'. If vz' is an arc in M', then D has an arc zz' where $z \notin V(H)$. Let P be a dipath to z from v. If there is another arc vy' in M', then we consider an arc yy' (where $y \notin V(H)$) and a dipath P' to y from v. We walk along P' backwards from y to v and we stop when we hit P. The subdipath of P' which we traverse in this way is called P''. Now we replace vz' in M' by P, and, if vy' is present in M', we replace vy' by P''. This transforms M' into a weak k-double-cycle in D.

Finally we shall use *Menger's Theorem* (for digraphs), which is presented in almost all books on graph theory.

3. SUFFICIENT CONDITIONS FOR WEAK 3-DOUBLE-CYCLES

Lemma 3.1. *Let v be a vertex in a strongly 2-connected digraph D. If $D - v$ has a dicycle whose vertices all are dominated by v (in D) or a dicycle whose vertices all dominate v, then D contains a weak 3-double-cycle.*

Proof. Suppose C is a dicycle whose vertices all dominate v. By Menger's Theorem, let P_1, P_2 be two $v - V(C)$ dipaths such that $P_1 \cap P_2 = \{v\}$. Then $P_1 \cup P_2 \cup C$ union two arcs from C to v form a subdivision of a 3-double-cycle. □

Lemma 3.2. *Let v_1, v_2, v_3, v_4 be vertices in a strongly 2-connected digraph D such that D contains the arcs $v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4$. Then D contains a weak 3-double-cycle.*

Proof. Let P_1, P_2 be two dipaths from v_4 to v_1 and v_2, respectively, such that $P_1 \cap P_2 = \{v_4\}$. If $v_3 \notin V(P_1) \cup V(P_2)$, then $D - v_4$ has a $v_3 - (V(P_1) \cup V(P_2))$ dipath P_3. Assume without loss of generality that P_3 intersects P_1. Now $P_1 \cup P_2 \cup P_3 \cup \{v_1v_3, v_1v_4, v_2v_3, v_3v_4\}$ is a weak 3-double-cycle. So we can assume that $v_3 \in V(P_1)$. Then v_3 partitions P_1 into two dipaths R_1 and R_2, say. Let P_3 be a $(V(R_1) \cup V(P_2)) - V(R_2)$ dipath in $D - v_3$.

Then $P_1 \cup P_2 \cup P_3 \cup \{v_1v_3, v_1v_4, v_3v_4\}$ contains a weak 3-double-cycle. □

We are now ready for the main results of this paper.

Theorem 3.3. *Let D be a strong digraph such that each vertex has outdegree at least 2. Let v_1, v_2, v_3 be vertices such that all other vertices of D have outdegree at least 3. Assume further that, for each vertex $x \neq v_1$, every vertex $\neq x$ can be reached from v_1 in $D - x$. Then D contains a weak 3-double-cycle. In particular, D is even.*

Proof. We assume that Theorem 3.3 is false and let D be a counterexample with as few vertices as possible and (subject to that condition) with as few arcs as possible. We shall establish a number of properties of D that will finally result in a contradiction.

(1) D is strongly 2-connected.

Proof. Suppose u is a vertex such that $D - u$ is not strong. Let D' be a terminal component of $D - u$. The assumption of Theorem 3.3 implies that $v_1 \notin V(D')$. Let D'' be the D'-reduction at u. For each vertex x in D', $d^+(x, D'') = d^+(x, D) \geq 2$. Hence $|V(D'')| \geq 3$. If $d^+(u, D'') < 2$, then D' has a vertex x (namely, the vertex dominated by u in D'') such that $D - x$ has no dipath from v_1 to $D' - x$. This contradiction shows that $d^+(u, D'') \geq 2$. Now it is easy to see that D'' satisfies the assumption of Theorem 3.3 with u playing the role of v_1. But D'' contains no weak 3-double-cycle by Lemma 2.3. This contradiction proves (1).

(2) $d^+(v_1) = 2$.

Proof. If all vertices of D have outdegree at least 3, then we let z be any vertex dominating v_1. Then $D - zv_1$ satisfies the assumption of Theorem 3.3 with $v_2 = z$ contrary to the minimality of D. So some v_i has outdegree 2. As D is strongly 2-connected each v_i $(i = 1, 2, 3)$ can play the role of v_1.

Let u_1 and u_2 be the two vertices dominated by v_1. We show that the notation can be chosen such that the following holds.

(3) If we delete the arc v_1u_2 and contract v_1u_1, then the resulting digraph D_1 has minimum outdegree at least 2.

Proof. If (3) is false, then either

(i) $d^+(u_1, D) = 2$ and u_1 dominates v_1 or
(ii) some vertex z_1 of outdegree 2 (in D) dominates both v_1 and u_1.

As we may interchange between u_1 and u_2 we can also assume that either

(iii) $d^+(u_2, D) = 2$ and u_2 dominates v_1 or
(iv) some vertex z_2 of outdegree 2 (in D) dominates both v_1 and u_2.

If z_1 exists (i.e., (ii) holds), then $z_1 \neq u_2$ by Lemma 3.1. Similarly, $z_2 \neq u_1$ if (iv) holds. If both (ii) and (iv) hold, then $z_1 \neq z_2$. If (ii) holds, then (3) holds with the ordered triple z_1, v_1, u_1 or z_1, u_1, v_1 playing the role of v_1, u_1, u_2. (To see this we note that the notation can be chosen such that $v_2 = z_1$ and v_3 is either u_2 (if (iii) holds) or z_2 (if (iv) holds). So if z_1, v_1, u_1 cannot play the role of v_1, u_1, u_2, then (iii) must hold and u_2 dominates z_1. But then z_1, u_1, v_1 can play the role of v_1, u_1, u_2.) So we can assume that (ii) does not hold. Hence (i) holds. Similarly, (iii) holds. So v_1 dominates and is dominated by both of u_1, u_2. By Lemma 3.1, there is no arc between u_1 and u_2. Let y be the vertex $\neq v_1$ dominated by u_1. Then u_1, y, v_1 can play the role of v_1, u_1, u_2 in (3).

We are going to investigate the digraph D_1 defined in (3). The new vertex obtained by identifying v_1 and u_1 is denoted u_1'. If the deletion of vu_1 and contradiction of vu_2 results in a digraph D_2 of minimum outdegree at least 2, then all statements below related to D_1 have counterparts related to D_2.

By Lemma 2.2, D_1 contains no weak 3-double-cycle. We claim that D_1 has at most three vertices of outdegree 2. For when we contract v_1u_1 we "loose" a vertex of outdegree 2. We may create a new vertex of outdegree 2 if u_1 dominates v_1 and $d^+(u_1, D) = 3$, or if some vertex w of outdegree 3 in D dominates both v_1 and u_1. But only one such vertex w can exist (by Lemma 3.2) and w cannot exist if u_1 dominates v_1 (by Lemma 3.1). This proves the claim that D_1 has at most three vertices of outdegree 2. Therefore D_1 cannot be strongly 2-connected. (If D_1 were strongly 2-connected it would be a counterexample to Theorem 3.3 contradicting the minimality of D.) So D_1 contains a vertex z_1 such that $D_1 - z_1$ is not strong. We choose z_1 such that $D_1 - z_1$ is not strong, and we choose a terminal component H_1 of $D_1 - z_1$ such that H_1 is relatively minimal. (That is, if z' is a vertex of D_1 and H' is a terminal component of $D_1 - z'$ such that $V(H') \subseteq V(H_1)$, then $H' = H_1$.) Put $I_1 = D_1 - (V(H_1) \cup \{z_1\})$.

(4) $u_1' \in \{z_1\} \cup V(H_1)$ and $u_2 \in V(I_1)$.

Proof. If $u_1' \in V(I_1)$, then $D - z_1$ would not be strong, contrary to (1). So $u_1' \in \{z_1\} \cup V(H_1)$. If $u_2 \notin V(I_1)$, then $D - z_1$ would fail to be strong (if $z_1 \neq u_1'$) and $D - u_1$ would fail to be strong (if $z_1 = u_1'$).

Now let D_1' be the H_1-reduction of D_1 at z_1. As all vertices of D_1 have outdegree ≥ 2, D_1' has at least three vertices. Now we prove

(5) D_1' is strongly 2-connected.

Proof. We let t denote any vertex of D_1' and show that $D_1' - t$ is strong. This is clear if $t = z_1$ so assume $t \neq z_1$. The minimality of H_1 implies that each vertex of $D_1' - t$ can reach z_1. So it only remains to show that, for each vertex s in $D_1' - t$, $D_1' - t$ has a $z_1 - s$ dipath. If $u_1' \neq t$ and $u_1' \neq z_1$, then we consider a dipath P in $D - t$ from z_1 to s (or to v_1 if $s = u_1'$). If P contains some arc leaving I_1, then we let $w_1 w_2$ be the last such arc on P. The subdipath of P from w_2 to s contains no vertex of I_1. In particular, it does not contain the arc $v_1 u_2$. Moreover, z_1 dominates w_2 in D_1'. This shows that $D_1' - t$ has a dipath from z_1 to s. If $u_1' = z_1$ we argue similarly except that now P denotes a $u_1 - s$ dipath in $D - t$. If $u_1' = t$ we also argue similarly except that now P denotes a $z_1 - s$ dipath in $D - u_1$.

As D_1' has fewer vertices than D, D_1' cannot be a counterexample to Theorem 3.3. By Lemma 2.3, D_1' contains no weak 3-double-cycles. Therefore, D_1' has at least four vertices of outdegree 2. We now investigate the possibilities of where those four vertices can be. One possible vertex of outdegree 2 in D_1' is z_1. Other possibilities are v_2 and v_3. (Possibly $u_1 = v_2$ or v_3.) Also there may be a vertex in H_1, that in D has outdegree 3 and dominates both v_1 and u_1. Lemma 3.2 shows that there cannot be two such vertices. Finally, u_1 may have outdegree 3 in D and dominate v_1. But if this is the case, then D cannot have a vertex dominating both v_1 and u_1 by Lemma 3.1. As there are no other possibilities for vertices of outdegree 2 in D_1' we conclude

(6) D_1' has precisely four vertices of outdegree 2. Three of them are z_1, v_2, v_3 (or z_1, u_1', v_i in case $v_{5-i} = u_1$ where $i = 2$ or 3). The fourth vertex of outdegree 2 is either u_1' (if $d^+(u_1, D) = 3$ and u_1 dominates v_1) or a vertex that in D has outdegree 3 and dominates both v_1 and u_1.

(6) shows that both of v_2, v_3 are in H_1 (except that one of them may equal u_1). It is possible that $u_1 = v_2$ say. It is also possible that $z_1 = u_1'$. But both of these possibilities cannot occur since there are four vertices of outdegree 2 in D_1'. By (4), $u_2 \in V(I_1)$. So $d^+(u_2, D) \geq 3$ and none of v_2, v_3 dominates u_2. Hence the deletion of $v_1 u_1$ and contraction of $v_1 u_2$ results in a digraph D_2 of minimum outdegree at least 2. We noted after (3) that the statements concerning D_1 have counterparts for D_2. The counterparts to statements (3), (4), (5), and (6) are denoted (3′), (4′), (5′), and (6′), respectively. As (6) implies that $d^+(u_2, D) \geq 3$, (6′) implies that $d^+(u_1, D) \geq 3$. In particular, $\{u_1, u_2\} \cap \{v_1, v_2, v_3\} = \varnothing$ and $\{v_2, v_3\} \subseteq V(H_1) \setminus \{u_1'\}$. After having proved (3) we

introduced D_1, z_1, H_1, I_1, D'_1. We define similarly D_2, z_2, H_2, I_2, D'_2 (by interchanging u_1 and u_2). Now (6′) implies

(7) I_2 contains u_1 but not u'_2. $H_2 - u'_2$ contains v_2, v_3.

(6′) also implies that either u_2 dominates v_1 in D or else some vertex of outdegree 3 in D dominates both u_2 and v_1. Such a vertex must be in $I_1 \cup \{z_1\}$. (Note that possibly $z_1 = u'_1$. But a vertex dominating u_2 and v_1 must be distinct from u_1 by Lemma 3.1.) Hence

(8) Some vertex of $I_1 \cup \{z_1\}$ dominates v_1 in D.

(9) Either $z_1 \neq u'_1$ or $z_2 \neq u'_2$.

Proof. Suppose $z_1 = u'_1$ and $z_2 = u'_2$. The equality $z_1 = u'_1$ implies that every $v_2 - u_2$ dipath in $D - v_1$ contains u_1. Similarly, the equality $z_2 = u'_2$ implies that every $v_2 - u_1$ dipath in $D - v_1$ contains u_2. But $D - v_1$ contains a $v_2 - \{u_1, u_2\}$ dipath. That dipath contradicts one of the two preceding statements.

By (9), the notation can be chosen such that $z_1 \neq u'_1$. Possibly $z_2 = u'_2$ and we can no longer interchange between u_1 and u_2. Below, therefore, we investigate I_2 and H_2.

First observe that, by (8), z_1 dominates u'_1 in D'_1. By (6) there is precisely one more vertex, say r, that in D'_1 is dominated by z_1.

(10) If $z_2 = u'_2$ or $z_2 \in V(I_1)\backslash\{u_2\}$, then $z_1 \in V(H_2)$.

Proof. By (7), $v_2 \in V(H_2)$. As D is strongly 2-connected, $D - v_1$ has a $v_2 - z_1$ dipath P. By the assumption of (10), P does not contain z_2. Hence P is in H_2 (because H_2 is a terminal component of $D_2 - z_2$ and P starts in that terminal component). In particular, the end z_1 of P is in H_2.

(11) If $z_2 = u'_2$ or $z_2 \in (V(H_1)\backslash\{u'_1\}) \cup \{z_1\}$, then $I_1 - u_2 \subseteq H_2$.

Proof. Suppose first $z_2 = u'_2$. By (10), $z_1 \in V(H_2)$. Every vertex in $I_1 - u_2$ can be reached from z_1 in $D - u_2$. Any such dipath P avoids u_1, v_1 and is in $D_2 - z_2$. As P starts in the terminal component H_2 of $D_2 - z_2$ it also ends in H_2. Hence $I_1 - u_2 \subseteq H_2$.

Suppose next that $z_2 \in (V(H_1)\backslash\{u'_1\}) \cup \{z_1\}$. As $z_2 \neq u'_2$ we can apply (6′), which implies that $u'_2 \in V(H_2)$. As D is strongly 2-connected, $D - z_1$ has dipaths from u_2 to all other vertices of I_1. As these dipaths are in $D_2 - z_2$ (because $z_2 \neq u'_2$) and they start in the terminal component H_2 of $D_2 - z_2$, we conclude that $I_1 - u_2 \in H_2$.

(12) If $z_2 \in V(I_1)\backslash\{u_2\}$, then $(V(I_1)\backslash\{u_2, z_2\}) \cup \{z_1, u'_2\} \subseteq V(H_2)$.

Proof. By (10), $z_1 \in V(H_2)$. By (6′), $u'_2 \in V(H_2)$. As $D - z_2$ has shortest dipaths from $\{z_1, u_2\}$ to all vertices of $I_1 - \{u_2, z_2\}$ and these (shortest) dipaths are in $D_2 - z_2$, we conclude that $I_1 - \{u_2, z_2\} \subseteq H_2$.

By the definition of H_2 and I_2, there is no arc in D_2 from H_2 to I_2. As $u_1 \in V(I_2)$ and H_2 contains almost all of I_1 (by (11) and (12)), there are not many arcs from I_1 to u_1. More precisely, (10), (11), and (12) imply

(13) There is at most one vertex (namely, one of z_1, z_2, u_2) in $I_1 \cup \{z_1\}$ that in D dominates u_1.

Proof. If $z_2 = u_2'$, we apply (10), (11). If $z_2 \in V(I_1)\backslash\{u_2\}$ we apply (12). if $z_2 \in (V(H_1)\backslash\{u_1'\})\cup\{z_1\}$ we apply (11). Note that if $z_2 = z_1$, then $u_2' \in V(H_2)$ and hence u_2 does not dominate u_1 in that case.

Let G be the digraph obtained from the subdigraph of D induced by $V(I_1)\cup\{r, v_1, z_1\}$ by adding the arcs rv_1, rz_1 if they are not already present. As D is strongly 2-connected D contains, by Menger's Theorem, two $r - \{v_1, z_1\}$ dipaths that have only r in common. These dipaths together with the subdigraph of D induced by $V(I_1) \cup \{r, v_1, z_1\}$ is a subdivision of G. Hence G contains no weak 3-double-cycle. Clearly, $d^+(v_1, G) = 1$ and, therefore, we contract v_1u_2 into u_2' and denote the resulting digraph by G'. By Lemma 2.2,

(14) G' contains no weak 3-double-cycle.

We shall prove that G' satisfies the assumption of Theorem 3.3 with r playing the role of v_1.

(15) All vertices of G' have outdegree at least 2 in G'.

Proof. Each vertex of G (except r and v_1) has the same outdegree in G as in D (where it has outdegree at least 3) unless it dominates u_1. So if a vertex has outdegree 1 in G', it would either have to dominate (in D) each of v_1, u_1, u_2 (which is impossible by Lemma 3.2), or it would have to equal u_2' (and u_2 would dominate both v_1 and u_1, which is impossible by Lemma 3.1), or it would equal r (which clearly has outdegree 2 in G and G').

(16) For any two vertices z, z' in $G' - r$, $G' - z'$ has an $r - z$ dipath.

Proof. We can assume that $z \neq u_2', z_1$. As $D - z'$ has an $r - z$ dipath (if $z' \neq u_2'$) and $D - u_2$ has an $r - z$ dipath, we easily get an $r - z$ dipath in $G' - z'$.

As r can be reached from each vertex in $D - u_1$, we conclude that r can be reached (in G and hence also in G') from each vertex in G'. (Note that any dipath in $D - u_1$ from $I_1 \cup \{z_1\}$ to r is in G because $d^+(v_1, G) = 1$.) Combining this with (16) we conclude that

(17) G' is strong.

We finally investigate the vertices of outdegree 2 in G'. In G', r has outdegree 2. In G all vertices (except r, v_1, and possibly one more) have outdegree ≥ 3 by (13). When we form G' we may create a new vertex of outdegree 2, namely, u_2' or a vertex dominating both v_1 and u_2. By Lemmas 3.1, 3.2 only one new vertex of outdegree 2 is created in this way. So

(18) G' has at most three vertices of outdegree 2.

(14)–(18) imply that G' is a counterexample to Theorem 3.3. This contradiction to the minimality of D completes the proof. □

4. Even digraphs

Theorem 3.3 immediately implies

Theorem 4.1. *If all vertices (except possibly three) in a strongly* 2*-connected digraph* D *have outdegree at least* 3*, then* D *is even.*

We shall show that Theorem 4.1 is best possible in a strong sense.

The 4-double-cycle is strongly 2-connected and is not even. It has 4 vertices of outdegree 2. Another such example is the digraph D_1 consisting of a dicycle $x_1x_2x_3x_4x_5x_1$ and the additional arcs x_2x_4, x_2x_5, x_5x_2, x_4x_1, x_3x_1, x_1x_3.

D_1 is strongly 2-connected and has precisely four vertices of outdegree 2. It contains no weak odd double-cycle and therefore is not even.

In Theorem 4.1 it is also important that D is strongly 2-connected. Indeed, there are infinitely many strong digraphs of minimum indegree 2 and minimum outdegree 3 that are not even. We give here just one example. Let D_2 be obtained from D_1 above by adding a new vertex y such that y dominates x_1 and is dominated by x_1, x_3, x_4, x_5. Then $d^+(y, D_2) = 1$ and, if we contract yx_1, then we obtain D_1. As D_1 is not even it follows by Theorem 2.1 and Lemma 2.2 that D_2 is not even. Now take three disjoint copies of D_2 and identify the three y-vertices into one vertex. Then the resulting digraph is strong, noneven, and has minimum in- and outdegree 2 and 3, respectively. However, if the minimum in- and outdegree are both at least 3, the situation changes.

Theorem 4.2. *If* D *is a strong digraph of minimum in- and outdegree at least* 3*, then* D *is even.*

Proof. If D is strongly 2-connected we apply Theorem 4.1. So assume D has a vertex v such that $D-v$ is not strong. Let H be either an initial or terminal component of $D-v$. We chose v and H such that $|V(H)|$ is minimum. Suppose H is a terminal component. Let D' be the H-reduction of D at v. We claim that D' is strongly 2-connected. So we let v' denote any vertex of D' and we shall prove that $D'-v'$ is strong. This is clear if $v'=v$. So assume that $v' \neq v$. If $D'-v'$ is not strong we let H' be a terminal or initial component of $D'-v'$ not containing v. But then H' is also a terminal or initial component of $D-v'$ contradicting the minimality of H. So D' is strongly 2-connected. Also D' has at most one vertex of outdegree < 3, namely, v.

By Theorem 4.1, D' has a weak 3-double-cycle. By Lemma 2.3, D has a weak 3-double-cycle. Hence D is even. □

Corollary 4.3. *Every strongly* 3*-connected digraph contains a dicycle of even length.*

In [14] it is pointed out that there exists a strongly 2-connected digraph D^7 on 7 vertices that has no dicycle of even length (namely, the one that is the union of the two dicycles $x_1x_2x_3x_4x_5x_6x_7x_1$ and $x_1x_4x_7x_3x_6x_2x_5x_1$). It was asked if there are infinitely many such digraphs. This is still open. However, we get the following weaker statement.

Proposition 4.4. *There are infinitely many strongly* 2*-connected digraphs that are not even.*

Proof. Let G be a graph drawn in the Euclidean plane such that all edges are straight line segments and such that no two edges cross. (Graph and edge are defined in the next section.) Let p be a point not on G such that no half line starting at p contains an edge of G. Orient every edge in the clockwise direction around p. The resulting digraph D is not even. (Fix a half line L starting at p and assign weight 1 to an arc of D if and only if the arc intersects L.) It is easy to describe G and p such that D is strongly 2-connected. □

A digraph is *k-diregular* if all vertices have indegree and outdegree k. Friedland [2] proved that every k-diregular digraph is even for $k \geq 7$. He conjectured that this also holds for $k \geq 3$. This conjecture follows from Theorem 4.2 since every terminal component of a k-diregular digraph is k-diregular. Vazirani and Yannakakis [17] pointed out that a proof of Lovász's conjecture would follow from [2] if one could prove that, for k sufficiently large, every strongly k-connected digraph contains a 7-diregular subgraph. If so, it would also contain a 3-diregular subgraph (since every m-diregular digraph is the union of m 1-diregular subgraphs). But this last statement is incorrect. By a result of Pyber and Szemeredi (see [9]) there exists, for each k, a graph G such that all vertices have degree $\geq k$ and G has no 3-regular subgraph. (Graph and degree are defined in the next section. 3-regular means that all vertices have degree 3.) It is no loss of generality to assume that G is bipartite (as every graph of minimum degree $2k-1$ contains a bipartite graph of minimum degree k, see [9, 15]. Let V_1, V_2 be the bipartition of $V(G)$. Let G' be another copy of G where V_1', V_2' correspond to V_1 and V_2, respectively. Now form the disjoint union $G \cup G'$. Direct all edges from $V_1 \cup V_2'$ to $V_2 \cup V_1'$. Add all arcs from $V_2 \cup V_1'$ to $V_1 \cup V_2'$. The resulting digraph is strongly k-connected and has no 3-diregular subgraph.

5. Applications to colourings of hypergraphs, sign-nonsingular matrices, and permanents

In this section we apply Theorem 4.2 to the problems mentioned in the introduction.

A *hypergraph* H is a pair V, E where V is a finite set of vertices and E is a collection of subsets called *hyperedges* of V each of cardinality at least 2. The number of hyperedges containing the vertex v is the *degree* of v. If all hyperedges have cardinality 2 they are called *edges* and the hypergraph is called a *graph.* The hypergraph H is *bipartite* if there exists a partition $V = V_1 \cup V_2$ such that each hyperedge intersects both V_1 and V_2. H is *minimally nonbipartite* if H is not bipartite but every proper subhypergraph is bipartite. Now let D be a digraph on n vertices. We now define a hypergraph H_D as follows. The vertex set of H_D is $V(D)$. For every vertex v in D we let E_v consist of v and the vertices dominated by v. Now the hyperedge set of H_D is the collection of the sets E_v, $v \in V(D)$. Seymour [11] proved that H_D is minimally nonbipartite if and only if D is strong and has no dicycle of even length. He also proved that every minimally nonbipartite hypergraph (V, E) with $|V| = |E|$ is of the form H_D. From the results in [13] it follows that the cardinalities of all hyperedges in such a hypergraph may be greater than any (fixed) natural number. But

Theorem 4.2 implies

Theorem 5.1. *If $H = (V, E)$ is a minimally nonbipartite hypergraph with $|V| = |E|$, then either some hyperedge of H has cardinality ≤ 3 or some vertex has degree ≤ 3.*

An n by n real matrix $A = [a_{ij}]$ is *sign-nonsingular* if A is nonsingular and all nonzero terms in the standard expression of the determinant $\det A$ have the same sign. By permuting rows and multiplying some rows by -1, if necessary, we may assume that all entries a_{ii} in the main diagonal are positive. Now we form a digraph D with vertex set $\{v_1, v_2, \dots, v_n\}$ such that v_i dominates v_j if $a_{ij} \neq 0$. If $a_{ij} > 0$ (respectively $a_{ij} < 0$) we assign the weight 1 (respectively 0) to the arc $v_i v_j$. Now one can easily show (see [4]) that A is sign-nonsingular if and only if D_A has no dicycle of even total weight. Let us say that an entry a_{ij} is *redundant* if $a_{ij} \neq 0$ but a_{ij} is not a factor of any nonzero term in the standard expression of $\det A$ (or, equivalently, the arc $v_i v_j$ is in no dicycle of D_A).

By the results of [13] there are sign-nonsingular matrices whose rows all have many nonzero entries. But Theorem 4.2 implies

Theorem 5.2. *If A is a sign-nonsingular square matrix with no redundant entries, then some row or some column of A has at most 3 nonzero entries.*

We now turn to Polya's problem [8]. For simplicity we consider an n by n $0-1$ matrix $A = [a_{ij}]$ with $1-s$ in the main diagonal. A *modification* of A is a matrix A' obtained from A by replacing some $1-s$ by -1. We say that A' is a *good modification* if $\operatorname{per} A = \det A'$. We let G_A be the bipartite graph with vertex set $V_1 \cup V_2$ when $V_1 = \{r_1, r_2, \dots, r_n\}$, $V_2 = \{c_1, c_2, \dots, c_n\}$ and G_A has the edge $\{r_i, c_j\}$ if $a_{ij} = 1$. Then $\operatorname{per} A$ is the number of perfect matchings (i.e., subgraphs in which all vertices have degree 1) in G_A. A *Pfaffian orientation* of a graph G is an assignment of orientations to every edge such that the resulting digraph D has the following property: If M_1, M_2 are two perfect matchings in G, then each cycle in $M_1 \cup M_2$ has an odd number of edges oriented in both directions (of the cycle). Now the following three statements below are equivalent:

(i) A has a good modification.
(ii) G_A has a Pfaffian orientation.
(iii) The digraph D'_A obtained from G by directing the edges $\{r_i, c_i\}$ from V_1 to V_2 and all other edges from V_2 to V_1 is not even.

The equivalence of (i) and (ii) is due to Kasteleyn [3], and the equivalence of (ii) and (iii) is due to Vazirani and Yannakakis [17].

Note that, by Lemma 2.2, the digraph D'_A is even iff the digraph D''_A obtained by contracting all arcs $r_i c_i$ $(i = 1, 2, \dots, n)$ is even. Thus Theorem 4.2 implies

Theorem 5.3. *If A is a nonnegative real n by n matrix and A has no redundant entry and each row and column has at least 4 positive elements, then A has no good modification.*

Theorem 5.4. *If G is a bipartite graph of minimum degree at least 4 such that each edge is contained in a perfect matching, then G has no Pfaffian orientation.*

Little [5] showed that a graph has a Pfaffian orientation if it contains no $K_{3,3}$-subdivision. ($K_{3,3}$ is the graph with six vertices $v_1, v_2, v_3, u_1, u_2, u_3$ and nine edges $\{v_i, u_j\}$, $i, j = 1, 2, 3$.) Using Theorem 2.1 and the equivalence of (ii) and (iii) above, one can prove the following:

Theorem 5.5. *Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$. Then G has no Pfaffian orientation if and only if G contains a subgraph H such that H is a $K_{3,3}$-subdivision with $\{v_1, v_2, v_3\} \subseteq V_1$ and $\{u_1, u_2, u_3\} \subseteq V_2$ and $G - V(H)$ has a perfect matching.*

It would be interesting to extend Theorem 5.5 to nonbipartite graphs. The Petersen Graph shows that this is not immediately possible. But the Petersen graph might somehow be the only obstacle as it is the matching theorem of Lovász in [7].

Another interesting problem is to decide if there exists a polynomial time algorithm for deciding if a given digraph has an even length dicycle. For planar digraphs a polynomial time algorithm is described in [16].

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ABSTRACT. If each arc in a strongly connected directed graph of minimum indegree and outdegree at least 3 is assigned a weight 0 or 1, then the resulting weighted directed graph has a directed cycle of even total weight. This proves a conjecture made by L. Lovász in 1975 and has applications to colour-critical hypergraphs, sign-nonsingular matrices, and permanents of matrices.

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