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THE EVOLUTION OF LOW MASS STARS

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SUMMARY

This paper presents a method of computing the evolution of low mass stars from the main sequence to the helium flash. Many such computations are handicapped by the fact that the hydrogen burning shell source becomes very thin at a relatively early stage of evolution, and this means that either a very short time step is needed on the red giant branch, or else that some special method must be introduced. The present method has neither handicap: the same procedure is used whether there is a thin shell or not, and the time step can be chosen so that the overall structure of the star changes by a fixed small amount per step. About 100 or 150 steps are necessary to evolve a star to the helium flash. The basic features of the method are that the structure equations are solved using a non-Lagrangian mesh, so that a thin shell if it exists can move slowly through the mesh, even if it moves rapidly with respect to the mass coordinate; and that the composition and the mesh are solved simultaneously with the structure equations, giving greater stability and self-consistency. Although the work described here does not take account of convective mixing of composition, it is suggested that a simple generalization of the equations used here would give a reasonable treatment of both convective and semi-convective mixing, if they occur.

1. INTRODUCTION

Since the work of Hoyle & Schwarzschild (1955) many people have been concerned with the computational study of the evolution of low mass, usually Population II, stars. Such work is often handicapped to a greater or lesser extent by the principal feature of low mass stars once they become giants, the appearance of a particularly thin shell source of nuclear energy surrounding the inert, nearly isothermal, largely degenerate helium core. In contrast to more massive stars, where the thickness of burning shells is seldom much less than $10^{-2}\,M_\odot$, low mass stars on the giant branch may have shells whose thickness approaches $10^{-4}\,M_\odot$ as the core approaches the helium flash, i.e. as the core mass approaches 0·4 to 0·5 $M_\odot$. In many types of evolutionary computation the choice of time step is determined, directly or indirectly, by the consideration that the shell source should not consume more than a fraction, say 10 per cent, of its mass in one time step. This clearly implies a large number of time steps before the core can grow to 0·4 or 0·5 $M_\odot$. However, it is remarkable that as the shell becomes thinner its structure, and the structure of the star as a whole, becomes in many ways simpler to study analytically (Weigert 1966; Eggleton 1967; Paczynski 1970). Indeed, it is possible to determine quite accurately the structure of the shell and neighbouring regions and in fact of the whole star without following the evolution step by step. The only aspects of the evolution of low mass red giants which depend appreciably though not greatly on earlier stages of evolution are:

1. The temperature distribution in the helium core (and even this can largely be determined by an analytic procedure, Eggleton 1968).
2. The composition of material exterior to the shell, which may have been processed by nuclear reactions in the early life of the star, and by the deep convective envelope (Iben 1968).

It is therefore clear that a method must exist which will enable one to follow the evolution of a low mass red giant with comparatively few time steps, say fifty or a hundred between the bottom of the giant branch and the onset of the helium flash. The method we describe does this, but it has the desirable feature that it works independently of whether there is a thin shell or not—it is not necessary to take special action, or introduce special approximations, when the shell source becomes very thin.

For the time being, this method ignores a feature which is important in more massive stars, though it is not important in stars of $\sim 0{\cdot}9 M_{\odot}$, at least until they reach the horizontal branch. This feature is mixing of composition due to convection. Although an outer convection zone is an important feature of low mass stars, the interior where nuclear reactions occur is normally radiative. There may be convective cores of ~ 1 or 2 per cent of the star's mass on or near the main sequence, but these have little effect on the star's structure, and disappear very quickly once the star leaves the main sequence (Demarque, Mengel & Aizenman 1970 to be published). The only convective mixing that affects composition occurs at the base of the giant track, when the outer convective envelope penetrates down to a region where the composition was previously affected by nuclear reactions. The effect of this is to decrease the hydrogen content of the envelope by a few per cent. The small discontinuity in composition which results from this will affect the star's evolution somewhat later, when the burning shell reaches it. We do not include the effect of convection on composition in this paper, although we believe it can be included by means of a slight modification of equation (22) below. This modification is briefly outlined in the last section of this paper.

2. THE COMPUTATIONAL METHOD

The equations of structure, with respect to mass as independent space variable, for a star with no nuclear reactions except hydrogen burning, are

$$\frac{d \log P}{dm} = -\frac{Gm}{4\pi r^4 P}, \tag{1}$$

$$\frac{d \log T}{dm} = \frac{d \log P}{dm} \cdot \nabla, \tag{2}$$

$$\frac{d \log r}{dm} = \frac{1}{4\pi r^3 \rho}, \tag{3}$$

$$\frac{dL}{dm} = \epsilon - \epsilon_{\nu} - \frac{Du}{Dt} + \frac{P}{\rho^2}\frac{D\rho}{Dt}, \tag{4}$$

where the symbols have their usual meaning. ϵ is the energy generation rate of hydrogen burning alone, and ∇ is the logarithmic gradient of temperature against pressure. The recipe used for calculating ∇ in convective or radiative regions is

given in the Appendix. Throughout this paper ordinary derivatives represent differentiation with respect to a space variable at constant time; D/Dt represents the Lagrangian time derivative, i.e. at constant mass coordinate; and dots will be used later to represent non-Lagrangian time derivatives. Boundary conditions at the centre are

$$r = L = 0 \quad \text{when } m = 0; \tag{5}$$

and at the surface we use the simple forms

$$L = \pi acr^2T^4 \quad \text{and} \quad P\chi = g \quad \text{when} \quad m = m_*, \tag{6}$$

where χ is the opacity and g the gravity. In a radiative region the equation which determines the change of composition, and hence the evolution of the star, is

$$0 = XR + \frac{DX}{Dt}, \tag{7}$$

where R is the reaction rate, related to ϵ by

$$\epsilon = EXR. \tag{8}$$

E is the energy per gram available from hydrogen burning ($6.3.10^{18}$ erg g^{-1}).

To integrate these equations by a finite difference technique we must first decide on a way of distributing mesh points which will ensure that within each interval no quantity of physical importance varies by a large amount. A good choice of mesh would be one which minimizes an expression of the form

$$\sum_{k=2}^{N} \sum_{i} \{f_k^{(i)} - f_{k-1}^{(i)}\}^2, \tag{9}$$

where $f^{(i)}$ is the ith function whose variation from step to step we would like to be small, and k is a suffix labelling the mesh points ($k = 1, 2, \ldots N$). If we introduce a variable q which varies from 0 to 1 through the star, with equal increments between mesh points, and if we approximate expression (9) by an integral, we have to solve a variational problem

$$\delta \int_0^{m_*} \sum_i \left(\frac{df^{(i)}}{dm}\right)^2 \frac{dm}{dq}\, dm = 0. \tag{10}$$

Since the quantities $df^{(i)}/dm$ can be supposed to be known functions of m, this variational problem has a simple solution which can be written as a pair of differential equations

$$\frac{dq}{dm} = \phi \left\{ \sum \left(\frac{df^{(i)}}{dm}\right)^2 \right\}^{1/2}, \tag{11}$$

$$\frac{d\phi}{dm} = 0, \tag{12}$$

with boundary conditions

$$q = 0 \quad \text{at} \quad m = 0, \quad q = 1 \quad \text{at} \quad m = m_*. \tag{13}$$

We thus have six differential equations to solve, with three boundary conditions at each end. Numerically, it is easiest to solve them if they are written with q as the independent variable, since we take equal intervals of q.

We have to choose the functions $f^{(i)}$ so as to give the most suitable mesh for a star. It is clear from equations (11) and (12) that in a region where $f^{(1)}$ say is changing rapidly while the other functions are not, this recipe gives a mesh at equal intervals of $f^{(1)}$. In the outer layers of a star, equal intervals of $\log P$ are usually convenient; in the energy generating region, whether it is a shell or not, equal intervals of L would be convenient; and in an inert core, where neither of these quantities changes rapidly, equal intervals of mass are convenient. This suggests that a suitable choice for equation (11) would be

$$\frac{dq}{dm} = \phi\left\{c_1^2\left(\frac{d\log P}{dm}\right)^2 + c_2^2\left(\frac{dL}{dm}\right)^2 + c_3^2\right\}^{1/2}, \tag{14}$$

where the c_i are coefficients determining the relative importance of the three terms. Such a choice works, but we have found from experience that a somewhat better choice is the simpler but similar one

$$\frac{dq}{dm} = \phi\left\{-c_1\frac{d\log P}{dm} + c_2\epsilon + c_3\right\}, \tag{15}$$

which is the form of equation (11) if

$$f^{(1)} = -c_1\log P + c_2 L_H + c_3 m; \quad f^{(i)} = 0 \quad \text{for} \quad i \neq 1. \tag{16}$$

L_H is the luminosity due to hydrogen burning alone. This specially simple case, with only one non-zero function, is sufficiently trivial that it could be solved more easily than by using the equations (11) and (12), but we prefer to keep the generality that these equations possess. Convenient choices of the coefficients c_i are

$$c_1 = \alpha_1/\log(P_c/P_s), \tag{17}$$

$$c_2 = \alpha_2/L_s, \tag{18}$$

$$c_3 = \alpha_3/m_*, \tag{19}$$

where suffices c, s represent the centre and surface. The α_i are constants of order unity which determine how much of the mesh is devoted to different regions of the star. It is simplest to use surface and central values from the previous model in an evolutionary sequence rather than from the current model. The structure of a model ought to be independent of the choice of mesh for a sufficiently refined mesh, and we find that changes of α_i by factors of 2 or 3 produce changes in the model of 2 or 3 per cent, for a 100-point mesh. Figs 1 and 2 illustrate the structure of slightly-evolved and highly-evolved models using a mesh chosen in this way (except that equation (16) was slightly modified to put mesh points closer together in the ionization zones).

Clearly with such a choice of mesh the mass coordinate at a given mesh point will vary with time as the star evolves, so that we need to write

$$\frac{Df}{Dt} = \dot{f} - \frac{df}{dm}\dot{m}, \tag{20}$$

where dots represent differentiation at constant q. This recipe is easily applied in equation (4) since the derivatives du/dm, $d\rho/dm$ are in principle known functions of the physical variables at a point. Some care must be taken in applying it to

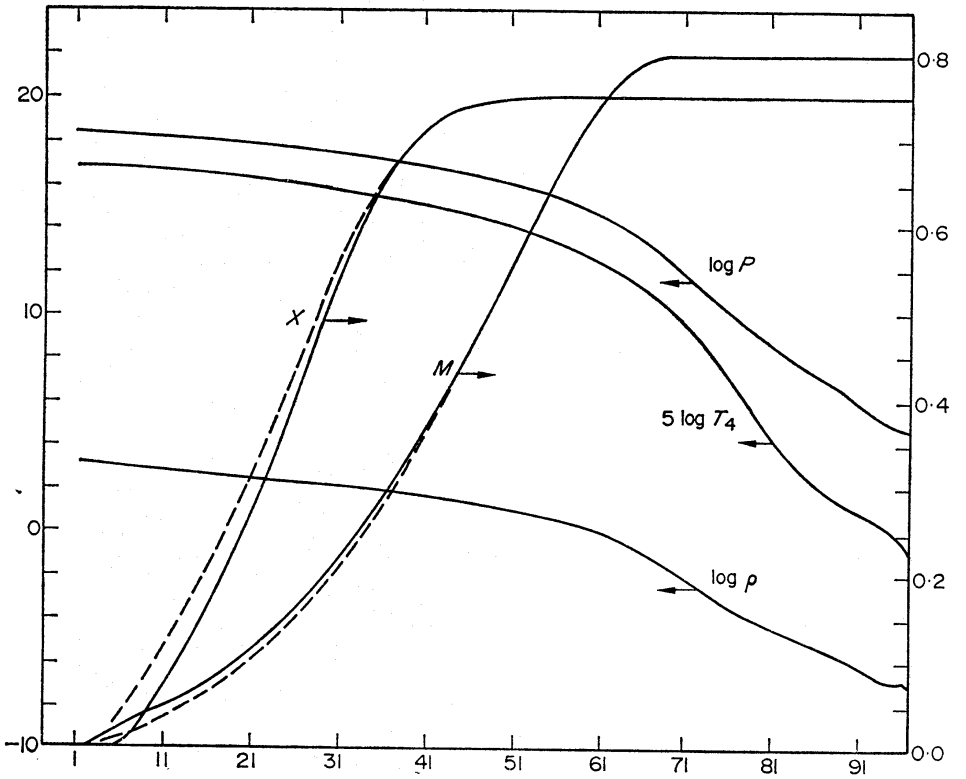

FIG. 1. *The structure of a Population II star of* 0·8 $M_\odot$ *when hydrogen is just becoming exhausted at the centre. Broken lines refer to model* 38 *of our sequence, solid lines to model* 43. *The changes in P, ρ, and T are small and are not shown. Mass is in solar units, P and ρ in c.g.s. units, and* T_4 *is in units of* 10^4 *degrees.*

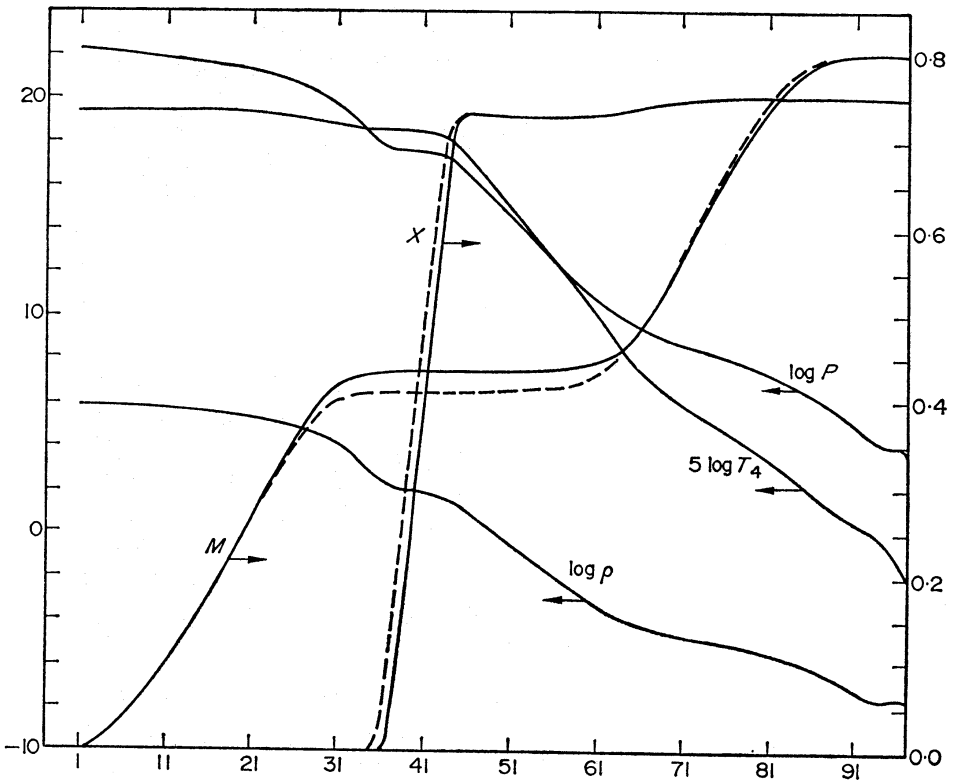

FIG. 2. *The structure of the same star as in Fig.* 1, *once it approaches the helium flash. Broken lines refer to model* 105 *of our sequence, solid lines to model* 110.

equation (7). We find it convenient, and believe it to be necessary, to write equation (7) as two equations

$$\frac{dX}{dm} = \xi, \tag{21}$$

$$0 = XR + \dot{X} - \xi\dot{m}. \tag{22}$$

These equations have no boundary conditions but there are two extra equations derived in the next section which have the effect of boundary conditions. The total problem is the solution of eight simultaneous equations at each mesh interval, with six boundary conditions and two extra conditions.

3. THE DIFFERENCE EQUATIONS

At each mesh point, including the centre and surface, eight quantities are 'guessed'. In principle, these could be log P, log T, log r, L, m, ϕ, X, ξ, although in practice it is more convenient to guess quantities of which these are functions (see Appendix). The four structure equations (1)–(4) and the two mesh equations (11) and (12) can be replaced by difference approximations of the form (using q as independent variable)

$$f_k - f_{k-1} = \frac{h}{2}\left\{\left(\frac{df}{dm}\frac{dm}{dq}\right)_k + \left(\frac{df}{dm}\frac{dm}{dq}\right)_{k-1}\right\}, \tag{23}$$

with f replaced by log P, log T, log r, L, m or ϕ, and with dm/dq given by equation (15). The time derivatives in equation (4) are given by equation (20) with

$$\dot{f}_k = \frac{f_k - f_k^{(0)}}{\Delta t}, \tag{24}$$

where f is replaced by u, ρ or m. $f_k^{(0)}$ is the value of f_k in the previous model. The differences are not properly centred in time, and so are less accurate than equation (23), but this type of differencing contributes greatly to stability.

The remaining two structure equations (21) and (22) cannot be replaced by the difference approximation (23) because they become unstable, once a thin shell develops, as we explain shortly. It appears to be necessary to write them in the form

$$X_k - X_{k-1} = h\left(\xi\frac{dm}{dq}\right)_{k-1}, \tag{25}$$

$$0 = h(\dot{X}_k + X_k R_k - \xi_k \dot{m}_k)\left(\frac{dm}{dq}\right)_k, \tag{26}$$

where $k = 1$ at the centre (the extra factors in (26) are of course not necessary, but are convenient in making this equation similar to the others). The following argument shows why this form can be both stable and accurate, whether or not there is a thin shell source.

For stars with only one energy source there are two time-scales of importance to equations (25) and (26), the nuclear and shell time-scales:

$$t_n = EXm_*/L_H; \quad t_s = R_m^{-1}. \tag{27}$$

R_m is the peak value of the reaction rate R. On or not far off the main sequence $t_n \sim 10t_s$, but near the top of the giant branch $t_n \sim 10^4 t_s$. The star as a whole evolves on the nuclear time scale t_n, while t_s can be interpreted (when a shell source is

present) as the time the source takes to consume its own mass of fuel. Our problem is that we wish to use time steps which are determined by t_n (say 1 or 2 per cent of t_n), but such steps can be very large compared to t_s and this often implies instability in equations such as (26) unless care is taken.

Equations (25) and (26) are equivalent to

$$\dot{X}_k + R_k X_k - \frac{\dot{m}_k(X_{k+1} - X_k)}{h(dm/dq)k} = 0. \tag{28}$$

Putting

$$\dot{X}_k = (X_k - X_k^{(0)})/\Delta t, \tag{29}$$

$$\Delta m_k \equiv \dot{m}_k \Delta t, \tag{30}$$

and

$$\delta m_k \equiv h(dm/dq)_k, \tag{31}$$

we can solve equation (26) for X_k in the form

$$X_k = \frac{X_k^{(0)} + X_{k+1} S_k \Delta t}{1 + R_k \Delta t + S_k \Delta t}, \tag{32}$$

where

$$S_k \Delta t = \Delta m_k / \delta m_k. \tag{33}$$

The order of magnitude of the terms $R_k \Delta t$, $S_k \Delta t$ can be easily estimated using the definition (15) of the mesh, and the definitions (27) of the two time scales t_n, t_s. We see that $R_k \Delta t \lesssim \Delta t/t_s$; while $|S_k \Delta t| \lesssim \Delta t/t_s$ near the main sequence, and is $\lesssim \Delta t/ht_s$ once a thin shell develops. In the latter case S_k will clearly be positive in the shell source, although it may be negative and much smaller in the envelope. We take $h \sim 0{\cdot}01$ and $\Delta t \sim ht_n$, the latter being true throughout the evolution except during the phase of rapid core contraction when Δt may have to be somewhat smaller. Consequently we see that near the main sequence the denominator of equation (32) is nearly unity while on the giant branch it is large and positive. The important point is that the denominator never vanishes or drops near to zero. When $S_k \Delta t$ is negative it is small compared to unity: when $S_k \Delta t$ is large compared to unity it is positive. Consequently equation (32) can always give physically sensible values of X_k.

By contrast, suppose that instead of equation (25) we use the apparently similar difference formula

$$X_k - X_{k-1} = h\left(\xi \frac{dm}{dq}\right)_k. \tag{34}$$

Then equation (32) is replaced by

$$X_k = \frac{X_k^{(0)} - X_{k-1} S_k \Delta t}{1 + R_k \Delta t - S_k \Delta t}. \tag{35}$$

The denominator of this expression will go through zero at two points in the shell source, once it is thin (unless $\Delta t \sim ht_s$ rather than ht_n). This is clearly a numerically unstable situation. The case that the difference equation is properly centred like equation (23) is harder to analyse, but it is still numerically unstable though not so badly.

We must of course consider the accuracy of the approximation as well as its stability. Obviously any first-order approximation such as equation (32) will be accurate so far as time dependence is concerned if Δt is so small that both $R_k \Delta t$

and $|S_k\Delta t|$ are substantially less than unity. This is clearly true on or near the main sequence, with $\Delta t \sim ht_n$. Once a thin shell develops we have $R_k\Delta t \gg 1$, $S_k\Delta t \gg 1$ in the shell, and equation (32) takes the limiting form

$$X_k = \frac{X_{k+1}S_k}{R_k + S_k}. \tag{36}$$

This can be seen after a little manipulation to be a difference approximation of order h to the differential equation

$$\dot{m}\frac{dX}{dm} = \epsilon/E. \tag{37}$$

But this is just the approximation which on physical grounds (Eggleton 1967; Paczynski 1970) we expect to hold in a thin shell. So we expect the equations (25) and (26) to be more or less equally accurate (to order h) whether the nuclear source occupies the interior of the star or a thin shell.

Note that when equations (25) and (26) are written down for each mesh interval ($k = 2, 3, \ldots N$) the value of ξ_N does not appear anywhere, since its coefficient $\dot{m}_N$ in equation (26) is zero, taking the star's mass to be constant. Thus we can arbitrarily take $\xi_N = 0$. Also the $2N-2$ equations do not include equation (26) evaluated at $k = 1$. Hence two ' boundary conditions ' are

$$\xi = 0 \quad \text{at} \quad m = m_*, \tag{38}$$

$$\dot{X} + XR = 0 \quad \text{at} \quad m = 0. \tag{39}$$

We now have the right number of equations to close the system.

It is convenient to solve all the equations by means of a general procedure, which will solve a set of equations of the form

$$H_i(f_1^j) = 0, \quad (i = 1, 4) \tag{40}$$

$$F_i(f_k^j) - F_i(f_{k-1}^j) - h\{\beta_i G(f_k^j) + (1-\beta_i)G(f_{k-1}^j)\} = 0, (i = 1, 8, k = 2, N) \tag{41}$$

and

$$H_i(f_N^j) = 0. \quad (i = 5, 8) \tag{42}$$

The suffix j runs from 1 to 8 in all these equations.

One of the functions F_i, appropriate to equation (26), happens to be identically zero. The weights β_i appropriate to equations (25) and (26) are 0 and 1 respectively, and can be 0·5 for the other equations. These equations can be solved by the standard procedure of guessing the f_k^j, evaluating the LHS's of equations (40), (41), and (42), evaluating the derivatives of the LHS's with respect to each guessed quantity (the derivatives can be evaluated analytically, but it is more convenient though slower and less accurate to do it numerically), and then solving for the errors in the guesses by the Newton–Raphson formula. Using a general procedure like this results in a very compact programme, which is a desirable feature. A more detailed description of how the structure equations are actually formulated in the programme used here is given in the Appendix. There are presumably a large number of equivalent ways in which they could be written, within the general framework described above.

4. RESULTS AND COMMENTS

Because this method involves solving eight simultaneous equations rather than four, it is naturally somewhat cumbersome when applied to the early stages

of stellar evolution when, as is well known, the ordinary treatment involving only four equations is quite adequate. This inefficiency is exaggerated by the fact that the whole star, including the convective envelope in which the complications of the mixing length theory and partial ionization are important, is solved simultaneously. It would be more efficient if the complications of the atmosphere at temperatures lower than say 10^5 degrees were avoided by integrating the atmospheres separately and tabulating boundary conditions at 10^5 degrees as functions of true surface quantities. Nevertheless, by including the whole star on the same basis great programming simplification is achieved, and there is the slight advantage that the structure of the atmosphere can be displayed simultaneously with the rest of the star. Solving for the entire structure slows down the calculations not only because of more complicated formulae in the outer layers, but also because the convergence may become rather poor, if the surface convection zone is narrow and strongly superadiabatic.

Once the star reaches the Hayashi track the method of Section 2 comes into its own. We find that typically 60 models are needed between the main sequence and the base of the giant branch, and a further 60 from there to the top of the giant branch, when helium burning sets in. This is dictated by the fact that roughly

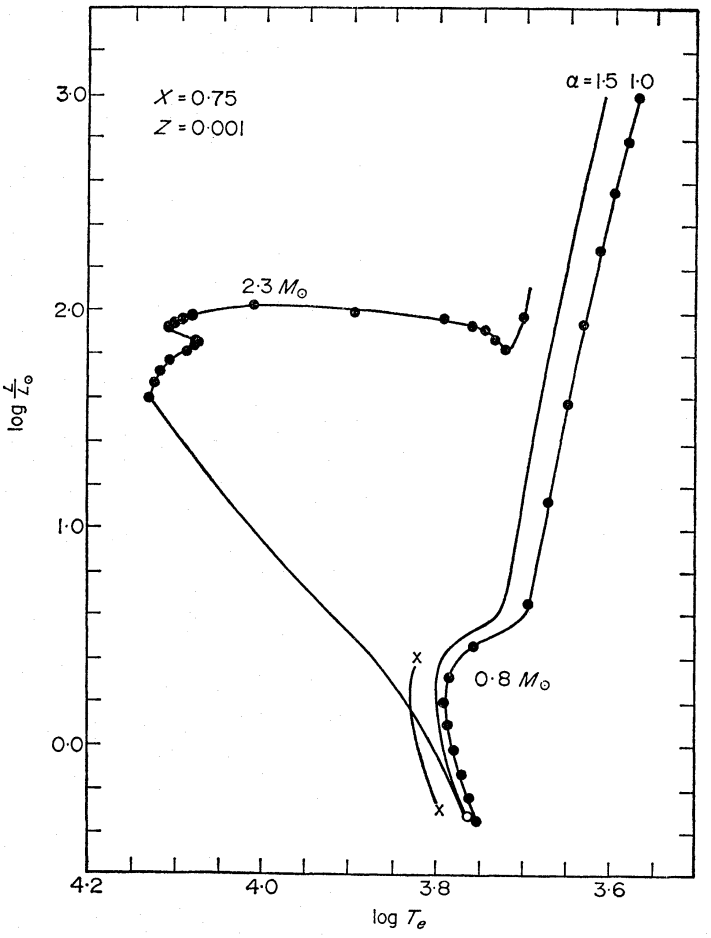

FIG. 3. *Evolutionary tracks for a model of* 0·8 $M_\odot$, *with values for* $\alpha \equiv l/H$ *of* 1·0 *and* 1·5. *A zero-age model interpolated from Iben & Rood* (1970) *is shown by the open circle. An early evolutionary sequence from Demarque* et al. (1970 *to be published*) *is indicated by the crosses. The zero-age main sequence for our Population II composition is shown, and also a preliminary computation of a* 2·3 $M_\odot$ *star. Heavy dots indicate every tenth model of the sequence.*

equal amounts of fuel are consumed in these two stages. Each model demands one or two iterations, depending on the accuracy specified, and for a 100-point mesh each iteration requires ~25 s on an IBM 360/44 machine. Thus an evolutionary sequence requires about 60 min from the main sequence to the helium flash, if fairly low accuracy (~1 per cent) is demanded for each model. Two evolutionary sequences for a star of Population II type are given in Fig. 3, and one of them is tabulated in Table I.

TABLE I

Evolution of a $0{\cdot}8\,M_\odot$ *star with* $X = 0{\cdot}75$, $Z = 0{\cdot}001$, $l/H = 1{\cdot}5$

(1) Model	(2) log L	(3) log T_e	(4) log r	(5) age	(6) X_c	(7) M_{core}	(8) log ρ_c	(9) log T_c	(10) M_{ce}	(11) ΔM
1	−0·342	3·765	−0·332	0·0	0·75	—	2·007	7·096	0·794	0·09
10	−0·216	3·777	−0·293	0·540	0·45	—	2·224	7·148	0·798	0·11
20	−0·065	3·789	−0·242	0·970	0·23	—	2·491	7·212	0·799	0·18
30	0·096	3·800	−0·182	1·270	0·08	—	2·780	7·293	0·800	0·20
40	0·253	3·801	−0·107	1·470	0·00	—	3·124	7·347	0·800	0·15
50	0·400	3·789	−0·009	1·570	—	0·095	3·554	7·368	0·800	0·08
60	0·652	3·723	0·249	1·638	—	0·164	4·584	7·483	0·619	0·011
70	1·293	3·695	0·625	1·670	—	0·231	5·137	7·556	0·331	0·0025
80	1·852	3·671	0·953	1·679	—	0·288	5·396	7·646	0·328	0·0012
90	2·304	3·649	1·224	1·682	—	0·339	5·574	7·731	0·365	0·0007
100	2·665	3·629	1·443	1·683	—	0·387	5·714	7·810	0·402	0·0005
110	2·954	3·613	1·683	1·683	—	0·432	5·832	7·883	0·445	0·00035

Column (1) is the model number, (2) the luminosity in solar units, (3) the effective temperature, (4) the radius in 10^{11} cm, (5) the age in 10^{10} year, (6) the fraction of hydrogen at the centre, (7) the helium core mass in solar units, (8) the central density, (9) the central temperature, (10) the mass fraction at the base of the convective envelope, and (11) the mass in the burning zone in solar units (nuclear luminosity ÷ peak energy generation rate).

Since this paper is intended to present a numerical method, we do not intend to discuss the astrophysical significance of the results obtained so far. It is sufficient to say that the procedure seems capable of following the evolution of low mass stars to the helium flash with considerable efficiency, and thus of providing with comparatively little trouble theoretical results with which observations of globular cluster stars can be compared. However, the results presented here for a star of $0{\cdot}8\,M_\odot$ with $X = 0{\cdot}75$ and $Z = 0{\cdot}001$, and $l/H = 1$ can be compared with the results of Demarque *et al.* (1970 to be published), which are also shown on Fig. 3. A comparison can also be made with the work of Iben & Rood (1970); we deduce values for the same composition and mass from their work by interpolating linearly in X and logarithmically in stellar mass. Their zero-age model is also plotted in Fig. 3. Our values differ from Demarque *et al.* by about 10 per cent, and from Iben and Rood by about 3 per cent.

To follow the evolution of low mass stars beyond the helium flash, and of more massive stars beyond the main sequence, it is necessary to include the mixing effect of convection on composition, an effect which has not been included in the present work. We suggest that the following diffusion-type equation may represent quite well the effect of convective mixing:

$$\frac{d}{dm}\left(\sigma\frac{dX}{dm}\right) = \frac{DX}{Dt} + RX. \qquad (43)$$

The diffusion parameter σ should of course be zero in a radiative region; while in a convective region we would expect from a dimensional argument that

$$\sigma \sim vl\left(\frac{dm}{dr}\right)^2, \tag{44}$$

where v is the velocity of convective motion, which can be obtained from the mixing length theory, and l is the mixing length. The mixing length theory gives the result

$$\rho v^3 \sim F(\nabla_r - \nabla_a)/\nabla_r, \quad \Delta_r \geqslant \Delta_a, \tag{45}$$

where F is the energy flux per unit area. ∇_r and ∇_a are the radiative and adiabatic gradiants, defined in the Appendix. In an ordinary convective region, where $\nabla_r - \nabla_a \sim 1$ the quantity σ is very large, and this implies that dX/dm is very small, and can be taken to be zero. However, equation (43) also admits a semi-convective type of solution, since if $0 < \nabla_r - \nabla_a \ll 1$ we may obtain solutions where dX/dm is not small. Although equation (43) appears as an equation determining X, it may equally be thought of as helping to determine σ, in view of the fact that it is solved simultaneously with the other structure equations. It is clear that equation (43) conserves nuclei, however it distributes them, so that it may well be a reasonably satisfactory means of dealing with semi-convection, at least until we have a better understanding of the instability which is interpreted for the time being as giving rise to semi-convection.

Clearly equation (43) represents only a slight modification to the equations of structure as we treat them in this paper. In equation (22) for composition changes, the LHS, which is zero in a radiative region, has to be replaced by $d(\sigma\xi)/dm$; but the equation is still of the form which is represented by the general difference equations (40) to (42), and so can be solved by exactly the same procedure.

We have only made two applications so far of equation (43). In one of these we followed the evolution of a low mass star including the effect of mixing when the convective envelope eats down into the deep interior. For the $0{\cdot}8\ M_\odot$ star shown in Fig. 3 we found that the surface hydrogen abundance was reduced from 0·75 to 0·735. The effect on the evolutionary track is not noticeable, although there is a temporary slowing down of evolution as the burning shell reaches the region where the composition was mixed by the envelope convection. Curiously enough it appears that as the convection zone penetrates down into the region of varying composition a semi-convective zone should appear, for much the same reason that semi-convection is required in massive stars (Schwarzschild & Harm 1958) when evolving off the main sequence; that is, the opacity depends on composition in such a way that the radiative gradient wants to be larger just inside the radiative zone than just inside the convective zone, leading to a contradiction unless the discontinuity of composition is smoothed out over a finite region.

The second application was a preliminary attempt at a star with a convective hydrogen burning core. A model of 2·3 solar masses and the same composition ($X = 0{\cdot}75$, $Z = 0{\cdot}001$) as the low mass star was evolved until helium burning began. The result is also shown in Fig. 3. The computations took nearly twice as long as for the low mass star. This can be explained partly by the fact that the star's structure changes much more during evolution, particularly in crossing the Hertzsprung gap, and partly by the fact that the time steps were sometimes unnecessarily small, due to our inexperience in applying this method to such a star.

Very few modifications were required to the method described in Sections 2 and 3, apart from the modification implied by the use of equation (43) as described above. This lends weight to the belief that equation (43) will also be able to deal with the potentially important semi-convection regions which can be expected during the helium burning phase (Paczynski 1970). A fuller discussion of the possibility of treating convection and semi-convection by using a diffusion equation will be given in a later paper.

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APPENDIX

SOME DETAILED COMPUTATIONAL CONSIDERATIONS

1. Because $d \log P/dm$, $d \log r/dm$ are singular at $m = 0$ it is convenient to use variables $m^{2/3}$, r^2 instead of m, $\log r$, since near $m = 0$ functions such as $\log P$, $\log T$ can in principle be expanded as Taylor series in one or other of these variables. It is not necessary to expand these series, since the above choice of variables automatically ensures this behaviour. Consequently we can approximate the differential equations by difference equations right down to $m = 0$, and apply boundary conditions exactly at this point. This is rather simpler than the common technique of expanding the solution in a small sphere and applying matching conditions at its surface. Writing $m^{2/3} = \mu$ and $r^2 = x$, we have

$$\frac{d \log P}{d\mu} = -\frac{3G}{8\pi}\left(\frac{\mu}{x}\right)^2 \frac{1}{P}, \tag{50}$$

and

$$\frac{dx}{d\mu} = \frac{3}{4\pi}\left(\frac{\mu}{x}\right)^{1/2} \frac{1}{\rho}, \tag{51}$$

for $\mu > 0$, with

$$\frac{\mu}{x} \to \left(\frac{4\pi\rho}{3}\right)^{2/3}, \tag{52}$$

as $\mu \to 0$. There is however some difficulty over the best choice of luminosity variable to go with this. The obvious choice, $L^{2/3}$, is not satisfactory because L may be negative in some regions. We tried a number of possibilities, and found that the most satisfactory choice was simply L itself. We have

$$\frac{dL}{d\mu} = \frac{3}{2}\epsilon_{\text{total}}\,\mu^{1/2}. \tag{53}$$

This is quite well behaved at the origin, but it is not very accurately represented there by a difference equation of the form of equation (22). However, the more general expression (41), with the appropriate β equal to $\frac{2}{3}$ instead of $\frac{1}{2}$, gives a reasonably accurate difference approximation to equation (53) at the innermost zone, and remains quite accurate for zones further out.

2. The equation of state for a gas, in which some constituents may be partially ionized and in which the electrons may be partially degenerate or relativistic or both, is rather more easily written in terms of temperature and the electron degeneracy parameter ψ (chemical potential $\div kT$) than in terms of say density and temperature, or pressure and temperature. Not only are pressure, density and internal energy explicit functions of ψ, T (and composition) via the Fermi–Dirac integrals when the gas is wholly ionized; when there is partial ionization near the stellar surface the populations of different states of ionization are explicit functions of ψ, T via formulae of the type

$$\frac{n^+}{n} = \frac{\omega^+}{\omega}\,\mathrm{e}^{-\psi-\chi/kT}. \tag{54}$$

In the present work pressure ionization, and its effect on the partition functions ω and the ionization potential χ, was disregarded, except in so far as it was assumed that at temperatures above 10^6 degrees all matter is ionized, whatever the density. Only the ionizations of hydrogen and helium were included—other elements were assumed to be fully ionized throughout. Furthermore, only the ground states were included in the partition functions. The quantities P, ρ, U and ∇_a, and their derivatives where necessary, were calculated explicitly as functions of ψ, T and composition. The Fermi–Dirac integrals were calculated by fairly simple analytic formulae which approximate the integrals and their derivatives to about 0·1 per cent over the whole range of ψ, T.

3. Opacities were taken from a program kindly lent by Dr Paczynski. This uses tables of radiative opacities from Cox & Stewart (1970) and conductive opacities from Hubbard & Lampe (1969). It also includes opacity from molecules at the lowest temperatures. Neutrino loss rates, and energy generation rates for the PP and CNO, triple α and N^{14}, α reactions were also obtained in tabular form from this programme.

4. In convective regions the value of $\nabla \equiv d\log T/d\log P$ was obtained from the formulation of the mixing length theory given by Baker & Temesvary (1966), which can be written thus:

$$\nabla = \nabla_a + (\nabla_r - \nabla_a)Y(Y+A), \tag{55}$$

where

$$\nabla_a = (\partial \log T/\partial \log P)_s,$$

$$\nabla_r = \frac{3\chi PL}{16\pi acGT^4 m}, \tag{56}$$

$$A^{-1} = \frac{\alpha^2}{12\sqrt{2}} \frac{c_p P \rho \chi}{acT^3} \frac{r^2}{Gm} \left\{ \frac{P}{\rho} (\nabla_r - \nabla_a) \right\}^{1/2}, \tag{57}$$

and

$$\frac{9}{4A} Y^3 + Y^2 + AY = 1. \tag{58}$$

For $A \gg 1$ and $A \ll 1$ second order approximations for Y as a function of A were used. In the intermediate regime the unique positive root of equation (58) was calculated explicitly. The mixing length ratio $\alpha \equiv l/H_p$ was normally taken as 1·5, and occasionally as 1·0. In radiative regions ($\nabla_r < \nabla_a$)∇ is given simply by

$$\nabla = \nabla_r. \tag{59}$$