The Exact Bias of the Banzhaf Measure of Power when Votes are Not Equiprobable and Independent

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Abstract

I discuss a numerical scheme for computing the Banzhaf swing probability when votes are not equiprobable and independent. Examples indicate a substantial bias in the Banzhaf measure of voting power if either assumption is not met. The analytical part derives the exact magnitude of the bias due to the common probability of an affirmative vote deviating from one half and due to common correlation in unweighted simple-majority games. The former bias is polynomial, whereas the latter is linear. I derive a modified square-root rule for two-tier voting systems which takes into account both the homogeneity and the size of constituencies. The numerical scheme can be used to calibrate an accurate empirical model of a heterogeneous voting body, or to estimate such a model from ballot data.

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1 Introduction

Despite their respectable age, power indices by Banzhaf (1965) and Shapley and Shubik (1954), henceforth Bz and SSI, remain a popular choice in empirical work. Both indices measure the distribution of *a priori* voting power, the distribution of power that follows from the constitution and rules of a voting body alone. Despite this similarity there exist voting situations, hypothetical and real, in which the two indices yield markedly different results. Which index to use therefore becomes a question of practical importance in the empirical work.

To answer this question, Straffin (1977) derives probabilistic models consistent with each of the two indices. He shows that, depending on the distribution of the voting poll, the expected individual effect of each member of a voting body on the outcome of voting numerically coincides with either the SSI or Bz measure. Straffin's prescription for empirical work is as follows: "If we believe that voters in a certain body have such common standards, the Shapley-Shubik index might be most appropriate; if we believe voters behave independently, the Banzhaf index is the instrument of choice" (Straffin 1994, ch. 32, p. 1137). The question explored in this paper is: What is the error of an empirical researcher who, following Straffin's prescription, applies the Bz measure to a voting body in which Straffin's Independence Assumption is not met?

To answer this question, I compute the bias of the Bz absolute measure of power, which results from the votes not being equiprobable and independent – the bias being the numerical inaccuracy in reflecting a voter's probability of being decisive. I use a numerical scheme to construct a probability distribution on the set of coalitions for given probabilities and correlation coefficients, and compare the Bz measure for this distribution to its equivalent in the case of independent votes.[1] Section 2 argues that pairwise correlation as a simple model of stochastic

[1]The numerical scheme has been introduced in Kaniovski and Pflug (2005) for modeling financial default risk. In this paper I provide an analytical solution to a slightly less general version of the scheme, and use it to model dependent voting.

dependence is sufficiently general for most empirical applications including voting by blocs. Section 3 discusses a numerical scheme for computing the Bz swing probability when the votes are not equally probable and correlated, and shows how to estimate the probabilities and correlation coefficients from ballot data. Section 4 presents an analytical derivation of the exact magnitude of the bias due to the common probability of a YES vote deviating from one half and due to common correlation in unweighted simple-majority games, and derives a modified Penrose's square-root rule in the case of correlated votes.

2 Probabilistic voting assumptions

Let p_i be the probability of the i-th member voting YES. Straffin (1977) introduces two probabilistic assumptions: "*Independence Assumption*: The p_i's are selected independently from the uniform distribution on $[0, 1]$. or: *Homogeneity Assumption*: A number p is selected from the uniform distribution on $[0, 1]$, and $p_i = p$ for all i" (p. 112). He then proceeds to prove two well-known characterization theorems. Theorem 1 states that under the Independence Assumption the probability of the i-th member's vote being decisive, or the i-th expected individual effect on the outcome of voting, coincides with the Banzhaf measure of voting power for i

$$\beta_i = \frac{\eta_i}{2^{n-1}} \ . \tag{1}$$

Here η_i is the number of coalitions in which i is decisive, and n the total number of members. The Banzhaf index is obtained by normalization of β_i's to add up to unity, which unfortunately destroys its probabilistic meaning. Theorem 2 makes a similar statement for the Homogeneity Assumption and the SSI.

The crucial assumption in both models is that each member votes independently. This is evident from the proofs, both of which rely on multilinear extensions of a game introduced by Owen (1972). A multilinear extension of a game played by $N = \{1, 2, \ldots, n\}$ members is

$$f(x_1, \ldots, x_n) = \sum_{S \subseteq N} \prod_{j \in S} x_j \prod_{j \in N \setminus S} (1 - x_j) v(S) , \quad \text{where } 0 \leq x_j \leq 1 \text{ for all } j . \tag{2}$$

The characteristic function, $v(S)$, takes the value of 1 if S is a winning coalition and the value of 0 otherwise. It is completely defined by the voting rule (quota) and the weights assigned to each member. The increment in the multilinear extension incurred by the addition of the i-th member's vote to the voting poll gives the effect of the i-th member on the outcome

$$\Delta_i f(x_1, \ldots, x_n) = \sum_{S \subseteq W_i} \prod_{j \in S \setminus \{i\}} x_j \prod_{j \in N \setminus S} (1 - x_j) , \tag{3}$$

where W_i is the set of winning coalitions in which member i is decisive (critical).

Let x_i be the probability that member i votes YES. The assumption of independent votes endows an increment in the multilinear extension with a unique probabilistic interpretation. Then and only then does $\Delta_i f(x_1, \ldots, x_n)$ become the probability that the i-th vote is decisive. Taking this fact as a point of departure, Straffin shows that the Independence Assumption leads to the Bz measure, whereas the Homogeneity Assumption leads to the SSI. In the general case of possibly dependent votes this probability takes the form $P_i = \sum_{S \subseteq W_i} \pi_S$, where π_S is the probability of the occurrence of coalition S. It is given by a joint probability distribution function on the set of all coalitions. While summation remains valid due to the coalitions being mutually exclusive, the product only applies to independent votes.

It is important to note that while assigning different weights to different members of a voting body, or changing the quota, may change the characteristic function of the game, *stochastic* *properties of the votes have no effect on the characteristic function.* Coalitions that have been winning under equally probable and independent votes continue to do so when the votes lose either property – what changes are the probabilities of their occurrence. Straffin's Independence Assumption implies that all voting outcomes have an equal probability of occurrence. Computing the probabilities if one departs from this assumption is the focus of the present paper.

For all empirical purposes Straffin's Independence Assumption is equivalent to the "equiprobability of each member voting either way; and independence between members" (Felsenthal and Machover 1998, p. 37). Note that "equiprobability either way" means two things: First, all members vote YES with equal probability and, second, this probability equals one half. The Independence Assumption thus leads to a binomial distribution with one half as the probability of success.

As argued in Felsenthal and Machover (1998), Straffin's Independence Assumption can be defended on the Principle of Insufficient Reason. As an assumption it is rational in the absence of prior knowledge about the future issues on the ballot and how divided over these issues the voting body will be. It suits the intended purpose of measuring the *a priori* distribution of voting power, the distribution that follows from the constitution and rules of the voting body, provided that all coalitions are equiprobable.

In Straffin's Homogeneity Assumption, equal probability of acceptance may be interpreted as reflecting the fact that members of a voting body have common standards when evaluating a proposal on ballot. The Homogeneity Assumption thus seems to abandon the *a priori* approach in favor of a more realistic model. The implied individual voting behavior is nevertheless very rigid. In the words of Felsenthal and Machover: "the model . . . is appropriate if we assume that all the voters are identical clones, with the same interests and identical [probabilistic] propensities, formalized by the common random variable P, which in each division produces the same probability p for all of them" (p. 201). To an external observer who does not know the true value of a common p, decisions by voting bodies with p close to zero or one would appear highly correlated, as near unanimous outcomes would be frequent in either case.

One possibility is to combine the two models (Widgrén 1995). As Kirman and Widgrén (1995) define it: "the voters are said to be 'partially homogeneous' when they can be partitioned into groups within which voters are homogeneous, whereas the groups vote independently of each other" (p. 430). However, partial homogeneity suffers from all the limitations of both probabilistic models. In the next section I argue that working directly with correlated votes is a more satisfactory way of modeling truly heterogeneous voting bodies.

2.1 Correlated votes

The crucial assumption in both models is that each member votes independently of all other members. Unfortunately, this assumption is untenable in most voting situations. First, as noted by many authors, including Straffin, members of a voting body may follow common standards when evaluating a proposal on ballot, to the effect that the votes in favor any one such proposal will positively correlate. One example of a common standard is common information or the lack thereof. The more the members communicate with each other, the less their votes are likely to be independent. Second, voting may be strategic. Strategic voting is contingent on how other members are expected to vote and is thus, by definition, not independent. Third, and closely related, there may be tacit collusion between certain members of a voting body, so that an outsider to the group will in effect be facing a voting bloc. Secret negotiations among a group of members prior to voting may lead to bloc formation. The existence and behavior of tacit voting blocs may appear probabilistic to an outsider. Fourth, members may have similar or different preferences, which could lead to correlated voting patterns. All of the above factors suggest that dependent voting must be the norm rather than the exception, and that correlations may either be positive, reflecting a degree of commonality or conformism, or negative, reflecting a degree of rivalry. It is therefore only natural to expect a member's *a priori* power to differ from her actual ability to change the outcome of the voting at any point after the constitutional stage. This expectation is all the more applicable when one considers that the former does not change as long as the rules stay the same, while the latter may change from one issue to another. A realistic model of a voting body should therefore be able to accommodate varying probabilities and correlations between votes.

Correlation between votes provides a general way of taking voters' preferences into account, and the need to do so has been repeatedly stressed in the literature.[2] It is common to represent voter's preferences as points in Euclidean space.[3] Clearly, spatial representations are deterministic, whereas correlations suggest only a probabilistic tendency of a member toward certain positions. Also, correlations can easily be estimated from ballot data, whereas choosing ideal points for members of a real voting body is largely *ad hoc*.

I shall assume that the votes of $n-1$ ($n \geq 3$) members of a voting body are correlated, whereas the n-th member votes independently of all others. The independent member is arbitrarily selected. She is independent because she has already made her choice. Her vote is assumed to be deterministic. This is necessary in order to calculate the n-th swing probability and the bias resulting from the application of the Bz measure. The assumption of pairwise correlation implies the existence of a degree of commonality (positive correlation) or a degree of rivalry (negative correlation) between $n-1$ members of a voting body, including their mutual independence

[2] For a recent debate see Napel and Widgrén (2004) and a critique of Napel and Widgrén in Braham and Holler (2005), as well as a reply and a rejoinder in the same issue of the *Journal of Theoretical Politics*. For particularly ardent criticism of preference free measures of voting power in the context of the European Union see Garrett and Tsebelis (1999), and Garrett and Tsebelis (2001).

[3] As in Steunenberg, Schmidtchen and Koboldt (1999), Napel and Widgrén (2004), the veto player theory of Tsebelis (1995), and in a general theory of voting of Merrill III and Grofman (1999), among others.

as a special case.[4] Note that pairwise correlations cannot capture correlations between an individual member and a bloc of members, but this entails no loss of generality if voting blocs are deterministic, in the sense that each insider votes in unison with all other insiders with probability one. In this case, pairwise correlation between an outsider and a bloc is equivalent to pairwise correlation between the outsider and a hypothetical member holding the total weight of the bloc in votes. The voting blocs typically discussed in the literature are deterministic (e.g., Leech and Leech (2004)).

However, the above is not the only way to model probabilistic dependence between votes. Several alternatives have been proposed in the literature, including the urn model by Berg (1985) and the branching process model by Gelman, Katz and Tuerlinckx (2002). The urn approach has been most extensively developed in the generalizations of Condorcet's Jury Theorem found in Boland (1989) and Berg (1993). The proposed approach has the advantage of extending the probabilistic setting of Straffin's theorem to correlated votes without making explicit or implicit assumptions about the dynamics of a voting procedure or the nature of probabilistic dependence. On the contrary, by virtue of an urn process the voting in Berg's model is sequential. The sequential nature of events follows by construction of an urn scheme, in which colored balls are drawn one at a time and are then replaced by one or several balls of a given color. A model based on an urn process implicitly assumes that the probability of being correct changes every time a vote is cast. Such a model would imply state-dependence in the process of reaching a decision, with the possibility of a lock-in on an alternative (Page 2006). Gelman et al.'s (2002) approach is based on the Ising model from statistical mechanics. In this model correlations are not explicitly defined, but follow implicitly follow from a parameter of spatial proximity. Moreover, the model assumes equiprobable votes. They derive a specification for the variance

[4] With some abuse of terminology, as zero correlation does not imply stochastic independence in general.

of proportional vote differentials between constituencies of different sizes and test it, as well as Penrose's (1946) square-root rule of equal representation in two-tier voting systems which follows from the Bernoulli model in Straffin's Independence Assumption using the data from U.S. Presidential elections. Both specification find little empirical support.

I show that positive correlation between some members of a voting body is likely to reduce the voting power of an independent member, while negative correlation due to contrarian strategies applied by some members is likely to increase her power. The intuition behind this result is that by increasing the probability of ties or near-ties, negative correlation increases probabilities of those voting outcomes in which the independent member is decisive, while positive correlation decreases these probabilities. In any case, the distribution of voting power will change once the independence assumption is relaxed.

3 A numerical scheme for computing the swing probability

3.1 A voting body of two

To fix the ideas, consider a voting body comprised of two members, i and j. Independently of each other, i would vote YES with probability p_i, and j would vote YES with probability p_j. Suppose that i and j do not vote independently, but rather that their votes are correlated with a coefficient of correlation c_{ij}. Define the probabilities of the four possible voting outcomes as: $P\{X_i = 1, X_j = 1\} = \pi_1$, $P\{X_i = 1, X_j = 0\} = \pi_2$, $P\{X_i = 0, X_j = 1\} = \pi_3$, $P\{X_i = 0, X_j = 0\} = \pi_4$, where 1 and 0 respectively indicate the YES and NO vote. We have: $\pi_1 + \pi_2 = p_i$, $\pi_1 + \pi_3 = p_j$ and $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$. As the covariance $cov[X_i, X_j]$ between the two Bernoulli random variables X_i and X_j is $E[X_iX_j] - E[X_i]E[X_j] = \pi_1 - p_ip_j$, the coefficient of correlation $c_{ij} = cov[X_i, X_j]/\sqrt{var[X_i]var[X_j]}$ must satisfy $\pi_1 = p_ip_j + c_{ij}\sqrt{p_i(1-p_i)p_j(1-p_j)}$.

Combining these four equalities recovers the sought distribution subject to two constraints: first, $\pi_1 \in [0, 1]$ as a probability and, second, $\pi_1 \leq \min(p_i, p_j)$ as a probability of an intersection of two events. The inputs must satisfy $p_i,\ p_j \in [0, 1]$ and $c_{ij} \in [-1, 1]$.

3.2 The general case

3.2.1 Notation

With $n \geq 3$ members the aim is to compute the n-th member swing probability and the bias resulting from the application of the Bz measure to n, assuming that n votes independently but the remaining $m = n - 1$ votes correlate. In the general case, the Bz measure can be written as $Bz_n(m, \mathbf{p}, \mathbf{c})$, where $\mathbf{p}$ is the vector of m probabilities and $\mathbf{c}$ the vector of $\binom{m}{2}$ possibly distinct correlation coefficients. If $p_i = p$ and $c_{ij} = c$, we would write $Bz_n(m, p, c)$. This case will be studied analytically. In the above notation $Bz_n(m, 0.5, 0) = \beta_n$, the original Bz measure.

A voting outcome in the subset of $m = n - 1$ members can be represented by binary vector $\mathbf{s} = (v_1, v_2, \ldots, v_m)$ of length m, whose i-th coordinate $v_i = 1$ if member i votes YES, and $v_i = 0$ otherwise. Define the following sets: $\mathbf{S}$ the set of all voting outcomes; $\mathbf{S}(i)$ the set of voting outcomes in which member i votes YES, that is the set of all binary vectors $\mathbf{s}$ such that $v_i = 1$; $\mathbf{S}(i, j) = \mathbf{S}(i) \cap \mathbf{S}(j)$ the set of voting outcomes in which members i and j both vote YES, that is the set of all binary vectors $\mathbf{s}$ such that $v_i = v_j = 1$. Sets $\mathbf{S}$, $\mathbf{S}(i)$ and $\mathbf{S}(i, j)$ respectively contain 2^m, 2^{m-1} and 2^{m-2} elements. For example, for $m = 3$ there will be eight voting outcomes 1:(1,1,1), 2:(1,1,0), 3:(1,0,1), 4:(1,0,0), 5:(0,1,1), 6:(0,1,0), 7:(0,0,1), and 8:(0,0,0). The set $\mathbf{S}$ contains all eight vectors. The set $\mathbf{S}(2)$ contains the four vectors 1, 2, 5 and 6, as only they have 1 in the second coordinate. The set $\mathbf{S}(2, 3)$ contains two vectors 1 and 5, as only they have 1 in the second and third coordinates. It will be convenient to index the voting outcomes in the descending order of the decimals represented by the corresponding binary vectors, starting from the vector of m 1's.

3.2.2 Optimization problem

In the general case we have the following set of equations involving the probabilities

$$\sum_{\mathbf{s} \subseteq \mathbf{S}} \pi_{\mathbf{s}} = 1 \quad \text{and} \quad \sum_{\mathbf{s} \subseteq \mathbf{S}(i)} \pi_{\mathbf{s}} = p_i \ , \tag{4}$$

and correlation coefficients

$$\sum_{\mathbf{s} \subseteq \mathbf{S}(ij)} \pi_{\mathbf{s}} = p_i p_j + c_{ij} \sqrt{p_i(1-p_i)p_j(1-p_j)} \quad \text{for} \quad 1 \le i < j \le m \ . \tag{5}$$

The inputs must satisfy $p_i \in [0,1]$, $c_{ij} \in [-1,1]$, the correlation matrix constructed from c_{ij}'s must be non-negative definite, and $\pi_{\mathbf{s}} \in [0,1]$ for the solution to define a probability distribution.

Given m probabilities and $\binom{m}{2}$ coefficients of correlation, the above system comprises $1 + m + \binom{m}{2}$ equations with 2^m unknowns and hence may not have a unique solution for $m \ge 3$. For a particular solution, Kaniovski and Pflug (2005) propose to choose the one which is closest in the sense of least squares to the probability distribution in the case of independent votes. This solution can be obtained by solving the following quadratic optimization problem

$$\min_{\pi_{\mathbf{s}}} \quad \frac{1}{2} \sum_{\pi_{\mathbf{s}}} \left[\pi_{\mathbf{s}} - \prod_{i=1}^{m} p_i^{v_i} (1-p_i)^{(1-v_i)} \right]^2 \quad \text{for} \quad \mathbf{s} \subseteq \mathbf{S} \ , \tag{6}$$

subject to all above constraints. The values of v_i's are tied to $\mathbf{s}$ and $\pi_{\mathbf{s}}$ via the index function (7), without which the above definition would be incomplete.

The strict convexity of the objective function ensures a *unique solution*. In principle, any probability vector of length 2^m can be used as a criterion for computing the smallest sum of squared deviations. This vector is chosen because it corresponds to the probability distribution in the case of independent votes, so that the resulting optimization problem can be used to compute the numerical bias in the vicinity of the input vector corresponding to the Bz ideal case, if the votes are equiprobable.

The formulation of the numerical scheme is essentially independent of the assignment of probabilities. Defining p_i, the probability of i voting YES, and p_j, the probabilities of j voting NO, leads to a similar system of equations. This is clear with respect to constraints involving the probabilities, while the following simple Lemma shows it also to be true with respect to constraints involving the correlation coefficients.

Lemma 1. *Let x and y be the indicators of events X and Y, and $\bar{x}$, $\bar{y}$ the indicators of the complementary events $\bar{X}$ and $\bar{Y}$. The following equalities on the correlation coefficients hold*

$$c_{x,y} = -c_{\bar{x},y} = -c_{x,\bar{y}} = c_{\bar{x},\bar{y}} \ .$$

Consequently, each of the four alternative assignments of probabilities leads to systems of equations identical except, perhaps, for the sign on the correlation coefficient. I will use this fact in estimating the probabilities and correlation coefficients from ballot data (Section 3.3).

A numerical solution of the general problem is feasible but can be computationally intensive for a large m.[5] In Appendix A, I analytically solve a slightly less general problem, in which all the probabilities are identical but the correlation coefficients may vary.

Proposition 1. *Let $p_i = p \in [0,1]$ for all $i = 1, 2, \ldots, m$ be the probability of i-th member voting YES and $c_{ij} \in [-1,1]$, $1 \leq i < j \leq m$, the correlation coefficient between any two such votes.*

[5] An R script by the author is available upon request.

Setting $q = 1 - p$, the probability of occurrence of a voting outcome defined by the index function

$$i(\boldsymbol{s}) = \sum_{i=1}^{m} 2^{m-i}(1 - v_i) + 1 \ , \quad \textit{where} \quad \boldsymbol{s} = (v_1, v_2, \ldots, v_m) \ , \tag{7}$$

is given by

$$\begin{aligned} \pi_{\boldsymbol{s}} \; &= \; p^{\sum_{i=1}^{m} v_i} q^{m-\sum_{i=1}^{m} v_i} + 2^{2-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{ij} - 2^{3-m} pq \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{i-1} c_{ji} + \sum_{j=i+1}^{m} c_{ij} \right) + \\ &+ \; 2^{4-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{ij} v_i v_j \ , \end{aligned} \tag{8}$$

provided $\pi_{\boldsymbol{s}} \in [0, 1]$.

Proposition 1 can be used to compute the probability of occurrence of any of the 2^m voting outcomes indexed by (7). The requirement $\pi_{\mathbf{s}} \in [0, 1]$ makes the numerical scheme best applicable to moderate values of m, moderate and positive correlation coefficients, and marginal probabilities close to 0.5. Although the optimization problem admits negative correlations, negative coefficients of high absolute value cause the constraints involving the correlation coefficients to be very tight, as their right-hand sides must remain non-negative. When $c_{ij} = c$, substituting $c_{ij} = c$ into $\pi_{\mathbf{s}}$ yields

$$\pi_{\mathbf{s}} = p^{\sum_{i=1}^{m} v_i} q^{m-\sum_{i=1}^{m} v_i} + 2^{2-m} pqc \left(\frac{m(m-1)}{2} - 2(m-1) \sum_{i=1}^{m} v_i + 4 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_i v_j \right) . \tag{9}$$

3.3 Estimating the probabilities and correlation coefficients

The proposed methodology allows calibrating an accurate model of the voting body given one's prior beliefs about the preferences of the members and the degree of commonality or rivalry among them. Expressed in terms of the probabilities and correlation coefficients, these beliefs could be used to forecast the probabilities of different voting outcomes. Or one can estimate probabilities and correlation coefficients based on ballot data. For any pair of members i and j there are four equations connecting p_i, p_j and c_{ij}

$$P\{X_i = 1, X_j = 1\} = p_i p_j + c_{ij}\sqrt{p_i(1-p_i)p_j(1-p_j)}\ ;$$

$$P\{X_i = 1, X_j = 0\} = (1-p_i)p_j - c_{ij}\sqrt{p_i(1-p_i)p_j(1-p_j)}\ ;$$

$$P\{X_i = 0, X_j = 1\} = p_i(1-p_j) - c_{ij}\sqrt{p_i(1-p_i)p_j(1-p_j)}\ ;$$

$$P\{X_i = 0, X_j = 0\} = (1-p_i)(1-p_j) + c_{ij}\sqrt{p_i(1-p_i)p_j(1-p_j)}\ .$$

An estimate of $\theta = (p_i, p_j, c_{ij})$ can be obtained by minimizing the goodness of fit statistic

$$GF_T(\theta) = T\sum_{k=1}^{4}\frac{(f_k - h_k(\theta))^2}{f_k}\ , \tag{10}$$

where f_1, f_2, f_3, f_4, the relative frequencies of the four possible voting outcomes in a sample of size T, and $h_1(\theta), h_2(\theta), h_3(\theta), h_4(\theta)$, the four equations above. The value that minimizes (10) is the Minimum χ^2 estimator of Nayman and Pearson (1928). In a voting body of n members there will be $\binom{n}{2}$ distinct pairs of members and hence that many minimization problems to solve. The independence assumption can be tested using Fisher's exact test based on a hypogeometric distribution (Everitt 1992, chs. 2.4 and 3.6.1).

3.4 Examples

The following three examples illustrate the effect of the probabilities and correlation coefficients on the Bz measure. In all examples, it is assumed that the independent member votes YES.

Example 1 (Table 1): Consider an unweighted simple-majority game with four members, or $\{2.5; 1, 1, 1, 1\}$. Fixing an independent member, if all other members also vote independently, each of the $2^3 = 8$ possible coalitions that can be joined by the independent member would occur with the probability $0.5^3 = 0.125$. The independent member is decisive in 3 of the 8 coalitions. The Bz measure of voting power is equal to $3 \cdot 0.5^3 = 0.375$ (Case 1).

Fixing a member, let any two of the remaining three votes correlate with $c = 0.2$ (Case 2). Positive correlation makes broad coalitions more probable, tight coalitions less probable. The opposite is true of negative correlation (Case 3). Increasing p shifts the probabilities of occurrence toward coalitions with a high percentage of 1's (Case 4). Introducing positive correlation negates some of this shift due to an increase in the probability of occurrence of all high consensus outcomes, including those with a high percentage of 0's (Case 5).

Case 4 documents an increase in the voting power of the independent member due to other members being more likely to vote in concordance with her. In the next section I show that a departure from equiprobability can increase or decrease the power of the independent member. Cases 2 and 3 show that positive correlation between members of a voting body will reduce the voting power of the independent member; negative correlation will have the opposite effect. By increasing the probability of ties or near-ties, negative correlation increases probabilities of those voting outcomes in which the independent member is decisive, while positive correlation decreases these probabilities.

In sum, the above examples show that the application of the Bz measure to these voting situations will result in substantial biases when compared to the case of equiprobable and independent votes. The absolute and the relative biases are computed as:

$$Bz(m, 0.5, 0) - Bz(m, p, c) \quad \text{and} \quad \frac{Bz(m, 0.5, 0) - Bz(m, p, c)}{Bz(m, 0.5, 0)} \,. \tag{11}$$

Before moving on to a weighted voting game, note that the distribution of voting power in an unweighted simple majority game ceases to be trivial when the votes are not equiprobable and independent, and that even small departures from either assumption may generate a substantial discrepancy between the Bz measure and the probability of casting a decisive vote. The following example of a weighted voting game shows the versatility of the numerical scheme.

TABLE 1 ABOUT HERE

Example 2 (Table 2): Consider the weighted simple majority game $\{4.5; 4, 2, 2, 1\}$. When all members vote independently, the Bz vector reads (0.75, 0.25, 0.25, 0.00).

Let $c_{12} = c_{13} = c_{14} = 0.1$, $c_{23} = 0.3$, $c_{24} = c_{34} = 0.5$. This is a situation in which small members are more likely to cooperate with each other than with the large member. Now the Bz vector reads (0.425, 0.325, 0.325, 0.000), allocating considerably less power to the large member, and more power to medium members. The smallest member is a dummy regardless of the stochastic properties of the votes, as the characteristic function is independent of them.

TABLE 2 ABOUT HERE

Example 3 (Figure 1): The final example illustrates the effect of a change in p and c on the Bz measure of voting power an in unweighted simple-majority game with $m = 3$ and $m = 4$. Figure 1 shows that the bias incurred by p deviating from 0.5 is larger than that incurred by c deviating from 0, which appears to vary linearly with the magnitude of the correlation coefficient. This is established rigorously in the next section.

FIGURE 1 ABOUT HERE

4 Assessing the bias of the Bz measure

The examples of the previous section show the Bz measure to be biased when the votes are not independent and identically distributed. This section presents a proposition and two corollaries on the magnitude and the direction of the probability and correlation biases in unweighted simple-majority games. The model studied will be that of a *homogeneous voting body* in which each vote has an equal probability of being affirmative, and each pair of votes is correlated with the same coefficient of correlation; formally, $p_i = p$ for all $i = 1, 2, \ldots, m$, and $c_{ij} = c$ for all $1 \leq i < j \leq m$, as in the last example of the previous section.

In an unweighted simple majority game a vote is decisive when it breaks or creates an exact tie. Assuming that the independent member votes YES, the Banzhaf absolute measure of voting power for the independent member is given by

$$Bz_n(m,p,c) = \sum_{\mathbf{s} \text{ s.t. } \sum_{i=1}^{m} v_i = \frac{m+1}{2}} \pi_{\mathbf{s}} \quad \text{for} \quad \mathbf{s} \subseteq \mathbf{S} \quad \text{when } n \text{ is even, } m \text{ is odd ,} \tag{12}$$

and

$$Bz_n(m,p,c) = \sum_{\mathbf{s} \text{ s.t. } \sum_{i=1}^{m} v_i = \frac{m}{2}} \pi_{\mathbf{s}} \quad \text{for} \quad \mathbf{s} \subseteq \mathbf{S} \quad \text{when } n \text{ is odd, } m \text{ is even .} \tag{13}$$

Proposition 2. *In a simple-majority game with $m+1$ members, in which: (1) the probabilities of a YES vote equal p for all members, $q = 1-p$, and (2) the correlation coefficients equal c for any pair of members, Banzhaf absolute measure of voting power is given by*

$$\binom{m}{\frac{m+1}{2}} \left[p^{\frac{m+1}{2}} q^{\frac{m-1}{2}} - 2^{1-m} pqc(m-1) \right] \quad \textit{when } m \textit{ is odd ;} \tag{14}$$

$$\binom{m}{\frac{m}{2}} \left[(pq)^{\frac{m}{2}} - 2^{1-m} pqcm \right] \quad \textit{when } m \textit{ is even .} \tag{15}$$

Proof. When m is odd, there will be $\binom{m}{\frac{m+1}{2}}$ voting outcomes in which the independent member is decisive by voting YES. The expression for $Bz_n(m,p,c)$ is obtained by adding the probabilities of the relevant voting outcomes, whose general expression is given by Proposition 1 and indexed by (7). For each voting outcome we have $\sum_{i=1}^{m} v_i = \frac{(m-1)}{2}$ and $\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} v_i v_j = \binom{\frac{m-1}{2}}{2}$. Since all relevant voting outcomes have equal probabilities of occurrence,

$$Bz_n(m,p,c) = \binom{m}{\frac{m+1}{2}} \left[p^{\frac{m+1}{2}} q^{\frac{m-1}{2}} - 2^{1-m} pqc(m-1)\right] \; . \tag{16}$$

When m is even, there are $\binom{m}{\frac{m}{2}}$ voting outcomes in which the independent member, n, is a tie-breaker. We have $\sum_{i=1}^{m} v_i = \frac{m}{2}$ and $\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} v_i v_j = \binom{\frac{m}{2}}{2}$. Consequently,

$$Bz_n(m,p,c) = \binom{m}{\frac{m}{2}} \left[(pq)^{\frac{m}{2}} - 2^{1-m} pqcm\right] \; , \tag{17}$$

which completes the proof. □

Proposition 2 can be adapted to fit any weighted supermajority game by replacing the above combinatorial analysis with a listing of coalitions in which the independent member is decisive, such as the one in Table 2. The number of such coalitions may differ from $\binom{m}{\frac{m+1}{2}}$ and $\binom{m}{\frac{m}{2}}$.

Corollary 1. *In a simple-majority game with $m+1$ members, in which: (1) the probabilities of a YES vote equal p for all members, and (2) the votes are uncorrelated, the relative bias equals*

$$\frac{Bz_n(m,0.5,0) - Bz_n(m,p,0)}{Bz_n(m,0.5,0)} = 1 - 2^m p^{\frac{m+1}{2}} q^{\frac{m-1}{2}} \quad \textit{when } m \textit{ is odd}\,; \tag{18}$$

$$\frac{Bz_n(m,0.5,0) - Bz_n(m,p,0)}{Bz_n(m,0.5,0)} = 1 - 2^m p^{\frac{m}{2}} q^{\frac{m}{2}} \quad \textit{when } m \textit{ is even}\,, \tag{19}$$

In a simple-majority game with $m+1$ members, in which: (1) the probabilities of a YES vote *equal* $p = 0.5$ *for all members, and (2) the correlation coefficients equal* c *for any pair of members, the relative bias equals*

$$\frac{Bz_n(m, 0.5, 0) - Bz_n(m, 0.5, c)}{Bz_n(m, 0.5, 0)} = \frac{c(m-1)}{2} \quad when\, m\, is\, odd\,; \tag{20}$$

$$\frac{Bz_n(m, 0.5, 0) - Bz_n(m, 0.5, c)}{Bz_n(m, 0.5, 0)} = c\frac{m}{2} \quad when\, m\, is\, even\,. \tag{21}$$

The above corollary furnishes the relative bias due to p deviating from 0.5 when $c = 0$ is maintained, and due to c deviating from 0 when $p = 0.5$ is maintained.

Whether the probability bias is positive or negative depends on p and the parity of m. When m is even, the bias is always positive, as $2^m p^{\frac{m}{2}} q^{\frac{m}{2}} < 1$ for all $p \neq 0.5$. Let $x = p - 0.5$, then $2^m p^{\frac{m}{2}} q^{\frac{m}{2}} = 2^m[(x+0.5)(0.5-x)]^{\frac{m}{2}}$ is polynomial. Although the bias is also polynomial when m is odd, it can be positive or negative, as $2^m p^{\frac{m+1}{2}} q^{\frac{m-1}{2}}$ can be smaller or larger than 1 for $p \neq 0.5$. To see this, note that for $x \in [0, 1]$ and $m = 2k$, $k = 1, 2, \ldots$, the function $f(x) = 2^m x^{\frac{m+1}{2}}(1-x)^{\frac{m-1}{2}}$ attains a unique maximum at $x^* = \frac{m+1}{2m}$ and $f(x^*) = (1 - \frac{1}{m^2})^{\frac{m}{2}} \sqrt{1 + \frac{2}{m-1}}$. As $m \to \infty$, $(1 - \frac{1}{m^2})^{\frac{m}{2}} \to 1$ from below, while $\sqrt{1 + \frac{2}{m-1}} \to 1$ from above. However, the probability bias can be negative only if $p > 0.5$.

Turning to the correlation bias, as $2^{1-m}pq(m-1) > 0$, positive correlation will bias the Bz measure upwards, negative correlation will have the opposite effect. The absolute and relative biases increase linearly in c. The relative bias increases linearly in m. The bias incurred by the common correlation coefficient deviating from 0 is less severe than that incurred by the common probability deviating from 0.5.

4.1 An application to Penrose's square-root rule (SRR)

SRR gives an approximate answer to the following question: How can voting power be distributed in a council of elected delegates so that each citizen – regardless of the size of her constituency – has an equal *a priori* power in the sense of Banzhaf? The following assumptions lead to a two-stage Bernoulli model: (i) each citizen has one vote, (ii) all citizens' and all delegates' votes are independent and equiprobable, and (iii) the universal voting rule is simple majority. The probability of a citizen being decisive in bringing about her preferred outcome in the council equals the probability that the delegate is decisive in bringing about this outcome, times the probability that the citizen is decisive in electing the delegate. In light of Straffin's characterization, this model has a direct analogy in terms of Banzhaf absolute measures.

Let N be the number of constituencies, each having n_i citizens. Let i and d_i denote, respectively, a citizen and the delegate of the i-th constituency. Then, $\hat{\beta}_i = \beta_{d_i}(N)\beta_i(n_i)$, where $\beta_i(n_i)$ is the voting power of the citizen i in her constituency, $\beta_{di}(N)$ is the voting power of the delegate d_i in the council, and $\hat{\beta}_i$ is the indirect voting power of the citizen i. To find the ratio of delegate powers that will equilibrate the citizens' indirect powers, set the left side to unity and apply Stirling's approximation to the exact probability of casting a decisive vote under the binomial model.[6] This leads to the well-known result that the citizens' indirect powers are approximately equal if the powers of the delegates in the council are proportional to the square root of the size of their constituencies, or $\beta_{d_i}(N)/\beta_{d_j}(N) \approx \sqrt{n_i/n_j} = (\sqrt{h})^{-1}$. The last step assumes, without any loss of generality, that the constituencies differ in size by the fraction $h > 0$ so that $n_j = hn_i$. Leech (2003) shows how to implement a SRR by solving the inverse problem of finding weights

[6]Depending on whether the number of voters is even or odd, the appropriate distribution will be given by $\text{Bin}(0.5n_i, n_i - 1, 0.5)$ or $\text{Bin}(0.5(n_i - 1), n_i - 1, 0.5)$. By Stirling's approximation $x! \approx \sqrt{2\pi}(x^{x+0.5}e^{-x})$. For background information and an exposition of Penrose's SRR see Felsenthal and Machover (1998, ch. 3.4). Gelman et al. (2002) offer a critical discussion and an empirical test of the SRR in U.S. Presidential Elections. See also, Gelman, Katz and Bafumi (2004).

which produce a desired power ratio.

To obtain a SRR when the citizens' votes are correlated, I assume that in each constituency i the votes are *equiprobable but correlated*, with the coefficient of correlation c_i.[7] A high positive coefficient of correlation implies a more homogeneous constituency. The larger and the more homogeneous a constituency is, the less power its citizens have. Differences of opinion with respect to the candidates on the ballot should lead to closer outcomes, thus increasing the efficacy of a vote. Construct the ratio of Bz measures for citizens i and j of two different constituencies according to Proposition 2.[8] Setting $n_j = hn_i$ and dropping the subscript on n_i

$$\frac{\beta_i}{\beta_j} = \frac{2^{-1-n}\binom{n}{\frac{n}{2}}(2 - c_i n)}{2^{-1-[hn]}\binom{[hn]}{\frac{[hn]}{2}}(2 - c_j[hn])} \, . \tag{22}$$

where $[x]$ denotes the integer part of x. Then, by Stirling's approximation

$$\frac{\beta_{d_i}}{\beta_{d_j}} \approx (\sqrt{h})^{-1}\frac{2 - c_j hn}{2 - c_i n} \, . \tag{23}$$

I thus arrive at a modification of SRR which takes into account both the homogeneity and the size of constituencies. All other things being equal, the more homogeneous the constituency is, the lower the voting power of its citizens will be, and the higher the voting power of their delegate ought to be if all citizens were to have equal powers. Setting $c_i = c_j = 0$ leads to the original SRR in Penrose (1946). The above SRR can be applied to moderate constituencies with moderate correlations, in which $c_i n$ and $c_j hn$ are small. This is because β_{d_i} and β_{d_j} are probabilities, so that $2 - c_i n \in (0, 1]$ and $2 - c_j hn \in [0, 1]$.

[7]Proposition 2 allows relaxing both assumptions. The consequences of relaxing the equiprobability assumption have been discussed before, so I focus on correlation. Chamberlain and Rothschild (1981) show that the probability of being decisive falls sharply when the votes are not equiprobable. See also, Good and Mayer (1975).

[8]Whether n_i is even or odd does not substantially alter the analysis that follows; hence, the simpler expression in Proposition 2 is taken.

5 Summary

The crucial assumption underlying the classical measures of voting power in probabilistic models is that each member of the voting body votes independently of all other members. In the case of the Banzhaf measure, this assumption is supplemented by that of equal probabilities of YES and NO votes.

By means of a numerical scheme for computing the Banzhaf swing probability when the votes are not equiprobable and independent, this paper studies the magnitude of numerical error or bias in the Banzhaf absolute measure, which occurs if either assumption is not met. The model is general in that it admits varying probabilities and correlation coefficients. An analytical solution provided for a voting body in which the former are identical while the latter can vary. The generality of the model makes it suitable for empirical implementation, such as the calibration of an accurate model of a voting body based on beliefs about the preferences of individual members and the degree of commonality or rivalry between them, or the estimation of such a model from ballot data.

The analytical part derives the exact magnitude of the bias for an unweighted simple-majority game in which the probability of an affirmative vote is the same for all members and the correlation coefficients are the same for any pair of members. The bias incurred by the common probability deviating from one half can be positive or negative depending on the probability and the size of the voting body, although it is always positive when the number of members is odd. The probability bias is stronger than that incurred by the common coefficient of correlation deviating from zero. The former is a polynomial function and the latter is a linear function of the deviation. Positive correlation between members of a voting body will reduce the voting power of the independent member, negative correlation will have the opposite effect. In any case, the magnitude of the bias increases with the size of the voting body.

The magnitude of the bias in a weighted voting game cannot be studied analytically due to the characteristic function of such a game not being amenable to combinatorial methods, despite it being independent of the stochastic properties of the votes. The approach to general weighted voting games has to remain that of listing all voting outcomes in which the independent voter is decisive and summing their probabilities of occurrence. However, the proposed method allows the bias in any weighted voting game to be computed numerically.

As a further result I derive a modified square-root rule for the representation in two-tier voting systems that takes into account the sizes of the constituencies and the heterogeneity of their electorates. Since in a homogeneous electorate the votes are positively correlated, the larger and the more homogeneous the electorate, the less power a vote has.

The main conclusion of this paper is that, despite the Banzhaf measure being a valid measure of *a priori* voting power and thus useful for evaluating the rules at the constitutional stage of a voting body, it is a poor measure of the actual probability of being decisive at any time past that stage. The Banzhaf measure cannot be used to forecast how frequent a voter will be decisive.

A Appendix: Solution to the optimization problem

Write the Lagrangian $\mathcal{L}(\mathbf{x})$ as

$$\Phi(\mathbf{x}) + \lambda \left[\sum_{\mathbf{s} \subseteq \mathbf{S}} x_{i(\mathbf{s})} - 1 \right] + \sum_{i=1}^{m} \mu_i \left[\sum_{\mathbf{s} \subseteq \mathbf{S}(i)} x_{i(\mathbf{s})} - p \right] + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \kappa_{ij} \left[\sum_{\mathbf{s} \subseteq \mathbf{S}(i,j)} x_{i(\mathbf{s})} - (p^2 + pqc_{ij}) \right] \tag{24}$$

where the objective function is defined as

$$\Phi(\mathbf{x}) = \frac{1}{2} \sum_{\mathbf{s} \subseteq \mathbf{S}} \left[x_{i(\mathbf{s})} - p^{\sum_{i=1}^m v_i} q^{m - \sum_{i=1}^m v_i} \right]^2 . \tag{25}$$

Vector $\mathbf{x}$ is a probability vector of length 2^m. The subscript $i(\mathbf{s}) = \sum_{i=1}^{m} 2^{m-i}(1 - v_i) + 1$ indicates the coordinate of $\mathbf{x}$ that corresponds to the probability of the voting outcome $\mathbf{s} = (v_1, v_2, \dots, v_m)$, so that the coordinates of $\mathbf{x}$ are indexed in the descending order of the decimals represented by the corresponding binary vectors of voting outcomes, starting from the vector of m ones.

Setting $\partial \mathcal{L}(\mathbf{x}) / \partial \mathbf{x} = 0$ implies for every $\mathbf{s} \subseteq \mathbf{S}$

$$x_{i(\mathbf{s})} = p^{\sum_{i=1}^m v_i} q^{m - \sum_{i=1}^m v_i} - \lambda - \sum_{i=1}^{m} \mu_i v_i - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \kappa_{ij} v_i v_j , \tag{26}$$

where v_i and v_j are the i-th and j-th coordinates of $\mathbf{s}$.

Next substitute (26) into each of the three sets of constraints

$$\sum_{\mathbf{s} \subseteq \mathbf{S}} x_{i(\mathbf{s})} = 1, \quad \sum_{\mathbf{s} \subseteq \mathbf{S}(i)} x_{i(\mathbf{s})} = p, \quad \sum_{\mathbf{s} \subseteq \mathbf{S}(i,j)} x_{i(\mathbf{s})} = p^2 + pqc_{ij} . \tag{27}$$

When evaluating the sums use the fact that sets $\mathbf{S}$, $\mathbf{S}(i)$ and $\mathbf{S}(i,j)$ respectively contain 2^m, 2^{m-1} and 2^{m-2} elements, whence

$$\sum_{\mathbf{s} \subseteq \mathbf{S}} p^m = 1, \quad \sum_{\mathbf{s} \subseteq \mathbf{S}(i)} p^m = p, \quad \sum_{\mathbf{s} \subseteq \mathbf{S}(i,j)} p^m = p^2 . \tag{28}$$

Substitution into the first constraint yields

$$4\lambda + 2 \sum_{i=1}^{m} \mu_i + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \kappa_{ij} = 0 . \tag{29}$$

When substituting (26) into the second set of constraints note that the sum is now taken over the set of all vectors having 1 as their i-th coordinate. We need to distinguish between coordinates to the left and the right of the i-th coordinate. Upon the substitution of (26) we have

$$4(\lambda+\mu_i)+2\left(\sum_{\substack{j=1\\ j\neq i}}^{m}\mu_j+\sum_{j=1}^{i-1}\kappa_{ji}+\sum_{j=i+1}^{m}\kappa_{ij}\right)+\sum_{\substack{k=1\\ k\neq i}}^{m-1}\sum_{\substack{l=k+1\\ l\neq i}}^{m}\kappa_{kl}=0\ , \tag{30}$$

which in view of (29) simplifies to

$$2\mu_i+\sum_{j=1}^{i-1}\kappa_{ji}+\sum_{j=i+1}^{m}\kappa_{ij}=0\ . \tag{31}$$

Similarly, the sum in the third set of constraints is taken over the set of all vectors having 1 as their i-th and j-th coordinates. Now we need to distinguish between coordinates to the left of the i-th coordinate, to the right of the j-th coordinate, and in between the two. Thus,

$$\begin{aligned}&2^{4-m}pqc_{ij}+4(\lambda+\mu_i+\mu_j+\kappa_{ij})+\\ &+2\left(\sum_{\substack{k=1\\ k\neq i,j}}^{m}\mu_k+\sum_{\substack{k=i+1\\ k\neq j}}^{m}\kappa_{ik}+\sum_{k=1}^{i-1}\kappa_{ki}+\sum_{l=j+1}^{m}\kappa_{jl}+\sum_{\substack{l=1\\ l\neq i}}^{j-1}\kappa_{lj}\right)+\sum_{\substack{k=1\\ k\neq i,j}}^{m-1}\sum_{\substack{l=k+1\\ l\neq i,j}}^{m}\kappa_{kl}=0\ .\end{aligned} \tag{32}$$

In view of (29) and (31) the above expression simplifies to

$$\kappa^*_{ij}=-2^{4-m}pqc_{ij}\ . \tag{33}$$

Plugging (33) into (29) and (31) yields all other Lagrangian multipliers and the solution $x^*_{i(\mathbf{s})}$

$$
\begin{aligned}
\mu_i^* &= 2^{3-m} pq \left(\sum_{j=1}^{i-1} c_{ji} + \sum_{j=i+1}^{m} c_{ij} \right) \; ; && (34)\\
\lambda_i^* &= -2^{2-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{ij} \quad \text{(after some algebraic manipulations)} \; ; && (35)\\
x^*_{i(\mathbf{s})} &= p^{\sum_{i=1}^m v_i} q^{m-\sum_{i=1}^m v_i} + 2^{2-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{ij} - 2^{3-m} pq \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{i-1} c_{ji} + \sum_{j=i+1}^{m} c_{ij} \right) + \\
&+ \; 2^{4-m} pq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c_{ij} v_i v_j \quad \text{for} \quad \mathbf{s} \subseteq \mathbf{S} \quad \text{and} \quad 1 \le i < j \le m \, . && (36)
\end{aligned}
$$

The $i(\mathbf{s})$-th coordinate of $x^*_{i(\mathbf{s})}$ represents the probability of occurrence of voting outcome $\mathbf{s}$.

When $c_{ij} = c$ for all $1 \le i < j \le m$, substituting $c_{ij} = c$ into $x^*_{i(\mathbf{s})}$ and simplifying it using

$$
\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c = c \frac{m(m-1)}{2} \quad \text{and} \quad \sum_{j=1}^{i-1} c + \sum_{j=i+1}^{m} c = c(m-1) \tag{37}
$$

yields

$$
x^*_{i(\mathbf{s})} = p^{\sum_{i=1}^m v_i} q^{m-\sum_{i=1}^m v_i} + 2^{2-m} pqc \left(\frac{m(m-1)}{2} - 2(m-1) \sum_{i=1}^{m} v_i + 4 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_i v_j \right) . \tag{38}
$$

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Table 1: Game: $\{2.5; 1, 1, 1, 1\}$

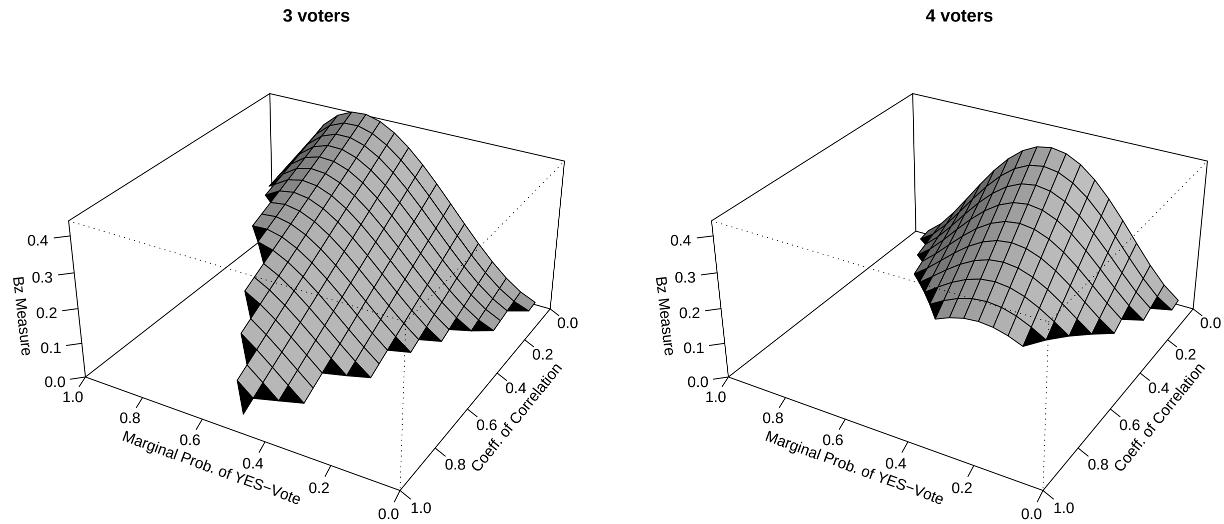
Case No.				1	2	3	4	5
Coalitions			Decisive	$p = 0.5$	$p = 0.5$	$p = 0.5$	$p = 0.75$	$p = 0.75$
				$c = 0$	$c = 0.2$	$c = -0.2$	$c = 0$	$c = 0.2$
1	1	1	-	0.125	0.200	0.050	0.422	0.478
1	1	0	$\surd$	0.125	0.100	0.150	0.141	0.122
1	0	1	$\surd$	0.125	0.100	0.150	0.141	0.122
1	0	0	-	0.125	0.100	0.150	0.047	0.028
0	1	1	$\surd$	0.125	0.100	0.150	0.141	0.122
0	1	0	-	0.125	0.100	0.150	0.047	0.028
0	0	1	-	0.125	0.100	0.150	0.047	0.028
0	0	0	-	0.125	0.200	0.050	0.016	0.072
Bz				0.375	0.300	0.450	0.422	0.366
Absolute bias				0.000	0.075	-0.075	-0.047	0.009
Relative bias				-	0.200	-0.200	-0.125	0.025

Table 2: Game: $\{4.5; 4, 2, 2, 1\}$

Coalitions			Decisive	$c = 0$	c^*
			VOTER 1		
1	1	1	-	0.125	0.2875
1	1	0	$\surd$	0.125	0.0375
1	0	1	$\surd$	0.125	0.0875
1	0	0	$\surd$	0.125	0.0875
0	1	1	$\surd$	0.125	0.0875
0	1	0	$\surd$	0.125	0.0875
0	0	1	$\surd$	0.125	0.0375
0	0	0	-	0.125	0.2875
				Bz_1=0.750	Bz_1=0.425
			VOTERS 2 AND 3		
1	1	1	-	0.125	0.2125
1	1	0	-	0.125	0.0625
1	0	1	-	0.125	0.0625
1	0	0	$\surd$	0.125	0.1625
0	1	1	$\surd$	0.125	0.1625
0	1	0	-	0.125	0.0625
0	0	1	-	0.125	0.0625
0	0	0	-	0.125	0.2125
				$Bz_2 = Bz_3$=0.250	$Bz_2 = Bz_3$=0.325
			VOTER 4		
1	1	1	-	0.125	0.1875
1	1	0	-	0.125	0.0875
1	0	1	-	0.125	0.0875
1	0	0	-	0.125	0.1375
0	1	1	-	0.125	0.1375
0	1	0	-	0.125	0.0875
0	0	1	-	0.125	0.0875
0	0	0	-	0.125	0.1875
				Bz_4=0.000	Bz_4=0.000

$^*c_{12} = c_{13} = c_{14} = 0.1,\ c_{23} = 0.3,\ c_{24} = c_{34} = 0.5$

Figure 1: The absolute Bz measure of voting power in unweighted simple-majority games



The probability of a YES and coefficients of correlation are positive and identical for all voters. Only in the case $p = 0.5$ and $c = 0$ is the Bz measure unbiased. The bias incurred by p deviating from 0.5 is larger than that incurred by c deviating from 0. The former is polynomial, whereas the latter is linear. Note how successively larger portions of the surface disappear due to solutions violating the constraints involved in the optimization problem.