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Abstract

Using a p-adic analogue of the convolution method of Rankin–Selberg and Shimura, we construct the two-variable p-adic L-function of a Hida family of Hilbert modular eigenforms of parallel weight. It is shown that the conditions of Greenberg–Stevens [R. Greenberg and G. Stevens, *p-adic L-functions and p-adic periods of modular forms*, Invent. Math. **111** (1993), 407–447] are satisfied, from which we deduce special cases of the Mazur–Tate–Teitelbaum conjecture in the Hilbert modular setting.

1. Introduction

1.1 This work grew out of an attempt to extend the result of Greenberg–Stevens on the exceptional zero conjecture of Mazur–Tate–Teitelbaum to more general automorphic forms. In the present paper, we establish special cases of the exceptional zero conjecture for Hilbert modular forms. To state our result, we briefly recall the setting of Greenberg–Stevens.

Let E be a (modular) elliptic curve over $\mathbf{Q}$, with $p \geq 5$ a prime. Assume that E has either good ordinary or multiplicative reduction at p (in the following we refer to these two cases as having ordinary reduction). In [MTT86], the p-adic Birch–Swinnerton-Dyer conjecture was proposed, which relates the order of vanishing of the p-adic L-function of E, $L_p(s, E)$, at $s = 1$, to the Mordell–Weil rank of E. The p-adic L-function is constructed by p-adically interpolating the twisted special L-value $L(1, \chi, E)/\Omega(E)$, where χ is a finite-order character of $\mathbf{Z}_p^\times$ and Ω_E a real period of E. One has the formula

$$L_p(1, E) = \left(1 - \frac{1}{a_p}\right)\frac{L(1, E)}{\Omega(E)}$$

with a_p being the unit-root of the local L-factor at p. Now suppose that E is split-multiplicative at p, so that one has Tate's analytic parametrization:

$$E(\overline{\mathbf{Q}}_p) = \overline{\mathbf{Q}}_p^\times / q_E^{\mathbf{Z}}, \quad q_E \in \mathbf{Q}_p^\times.$$

Furthermore we have $a_p = 1$, so we have $L_p(1, E) = 0$. Based on numerical data, Mazur–Tate–Teitelbaum conjectured the relation

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$$\frac{d}{ds}L_p(s,E)\bigg|_{s=1} = \mathcal{L}_p(E)\frac{L(1,E)}{\Omega(E)}.$$

Here the $\mathcal{L}$-invariant of E at p, $\mathcal{L}_p(E)$, is defined to be

$$\mathcal{L}_p(E) = \frac{\log_p q_E}{\mathrm{ord}_p q_E}.$$

Here ord_p is the valuation of $\mathbf{Q}_p^*$ with $\mathrm{ord}_p p = 1$, while $\log_p$ is Iwasawa's p-adic logarithm, normalized so that $\log_p p = 0$.

This conjecture was proved by Greenberg–Stevens [GS93, GS94]. In this proof, there are two important ingredients. The first is Hida's theory of ordinary deformations. The second is the construction of the two-variable p-adic L-function associated to a Hida family. In the construction of Greenberg–Stevens, they used the theory of Λ-adic modular symbols, generalizing those described in [MTT86].

There is another construction, due to Hida [Hid93] and Panchishkin [Pan89, Pan91, Pan03], which is based on the theory of Eisenstein series and the convolution method of Rankin–Shimura. Here, one of the key ingredients is non-vanishing theorems on L-values. This is supplied by Rohrlich [Roh89]. (However, we remark that we need a stronger non-vanishing result to show that the p-adic L-function constructed is not identically zero.)

In this paper, we use the method of Rankin–Shimura to construct the two-variable p-adic L-function, and prove a special case of the conjecture of Mazur–Tate–Teitelbaum in the Hilbert modular setting, as follows.

Let F be a totally real field, with discriminant D_F. Let $p \geq 5$ be a prime unramified in F, i.e. not dividing $6D_F$. Let E/F be a elliptic curve over the totally real field F, such that E has ordinary reduction (i.e. good ordinary or multiplicative) at all places $\mathfrak{p}$ above p. Let $\alpha(\mathfrak{p},E)$ be, as before, the unit-root of the L-factor attached to E/F at the place $\mathfrak{p}$, and let $\beta(\mathfrak{p},E)$ be the non-unit root. Thus, $\beta(\mathfrak{p},E) = \alpha(\mathfrak{p},E)^{-1}\mathcal{N}\mathfrak{p}$ if E has good reduction at $\mathfrak{p}$, and zero otherwise.

Assume that E is modular, in the sense that there is a Hilbert newform $\mathbf{f}_E$ of weight two over F, with trivial character, such that the Galois representation attached to $\mathbf{f}_E$ is isomorphic to that on the p-adic Tate module of E/F. Let $\mathbf{f}$ be the p-stabilization of $\mathbf{f}_E$. Then we can define the p-adic L-function of E/F, $L_p(s,E/F)$, to be the p-adic L-function attached to $\mathbf{f}$. As in the case where $F = \mathbf{Q}$, there is a choice for the transcendental part of the L-value $L(1,E/F)$ in defining $L_p(s,E/F)$. Call this factor $\Omega(E)$.

Assume that for some place $\mathfrak{p}_0$ of F above p, E is split-multiplicative at the place $\mathfrak{p}_0$. Denote by $f_{\mathfrak{p}_0/p}$ the residue field degree of $F_{\mathfrak{p}_0}$ over $\mathbf{Q}_p$, and by $q_{E/F_{\mathfrak{p}_0}}$ the Tate period associated to $E/F_{\mathfrak{p}_0}$. Then our first main result is the following.

Theorem 1.1. *We have*

$$\frac{d}{ds}L_p(s,E/F)\bigg|_{s=1} = f_{\mathfrak{p}_0/p}\frac{\log_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}_0}}}{\mathrm{ord}_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}_0}}} \times \prod_{\mathfrak{p}\neq\mathfrak{p}_0}\left(1-\frac{1}{\alpha(\mathfrak{p},E)}\right)\prod_{\mathfrak{p}\mid p}\left(1-\frac{\beta(\mathfrak{p},E)}{\mathcal{N}\mathfrak{p}}\right)\frac{L(1,E/F)}{\Omega(E)}.$$

More generally, let e be the number of places above p, over which E is split multiplicative. One has the following conjecture of Greenberg and Hida [Gre94, Hid] on exceptional zeros of higher order.

CONJECTURE 1.2. We have

$$L_p(s, E/F) = \mathcal{L}_p(E/F) \prod_{\substack{\mathfrak{p}|p \\ \alpha(\mathfrak{p},E)\neq 1}} \left(1 - \frac{1}{\alpha(\mathfrak{p}, E)}\right)$$
$$\times \prod_{\mathfrak{p}|p}\left(1 - \frac{\beta(\mathfrak{p}, E)}{\mathcal{N}\mathfrak{p}}\right)\frac{L(1, E/F)}{\Omega(E)}(s-1)^e + \text{higher-order terms},$$

where $\mathcal{L}_p(E/F)$ is the $\mathcal{L}$-invariant for E, defined as follows:

$$\mathcal{L}_p(E/F) = \prod_{\substack{\mathfrak{p}|p \\ \alpha(\mathfrak{p},E)=1}} \mathcal{L}_{\mathfrak{p}}(E/F)$$

with

$$\mathcal{L}_{\mathfrak{p}}(E/F) = f_{\mathfrak{p}/p}\frac{\log_p \mathcal{N}_{F_{\mathfrak{p}}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}}}}{\mathrm{ord}_p \mathcal{N}_{F_{\mathfrak{p}}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}}}}$$

for prime $\mathfrak{p}$ where E becomes split multiplicative, with Tate period $q_{E/F_{\mathfrak{p}}} \in F_{\mathfrak{p}}$.

We note that Theorem 1.1 is a consequence of this conjecture. Indeed, assume that Conjecture 1.2 holds. Then for $e \geq 2$, both sides of Theorem 1.1 vanish (the right-hand side vanishes since there is a prime $\mathfrak{p} \neq \mathfrak{p}_0$ with $\alpha(\mathfrak{p}, E) = 1$). For $e = 1$, it follows from the fact that $\alpha(\mathfrak{p}, E) = 1$ if and only if $\mathfrak{p}$ is a prime of split-multiplicative reduction.

In the proof of Theorem 1.1, we follow the method of Greenberg–Stevens. Namely, by using the functional equation for the two-variable p-adic L-function, we obtain a relation between the first derivative with respect to the s variable at $s = 1$, and the first derivative with respect to the weight variable at the weight two. The result of Wiles [Wil88] enables one to evaluate this latter derivative, and hence obtain the right-hand side of Theorem 1.1.

However, the functional equation for the two-variable p-adic L-function does not seem to yield enough relations between the higher derivatives with respect to the s variable and the weight variable, so the method of Greenberg–Stevens is inadequate to establish Conjecture 1.2 in general. A suggestion by Mazur and Hida, is that one should utilize the full Hida family of nearly ordinary deformations of dimension at least $1 + [F : \mathbf{Q}]$ (cf. [Hid89]) to obtain enough such relations. The author hopes to return to this question later.

In the second part of the paper, we investigate the case where $F/\mathbf{Q}$ is abelian, and E is defined over $\mathbf{Q}$. Since $E/\mathbf{Q}$ is modular, so is E/F by base change. We prove the factorization formula relating the p-adic L-function of E/F to that of $E/\mathbf{Q}$ and its twists: let $H = \mathrm{Gal}(F/\mathbf{Q})$, $\widehat{H}$ its character group, then we have the following theorem.

THEOREM 1.3. *We have*

$$L_p(s, E/F) = \langle D_F\rangle_{\mathbf{Q}}^{s-1} \prod_{\phi\in\widehat{H}} L_p(s, E/\mathbf{Q}\otimes\phi),$$

here $\langle\cdot\rangle_{\mathbf{Q}}$ is the projection to the subgroup $1 + p\mathbf{Z}_p$ of one-units in $\mathbf{Z}_p^*$.

From this formula, we deduce as a corollary that E/F satisfies the higher-order exceptional zero conjecture.

The structure of this paper is as follows. In § 2, we recall the general theory of Hilbert modular forms, setting the notation used throughout the paper. In § 3, we recall the construction of certain

Hilbert modular Eisenstein series, which occurs in the theory of Rankin–Shimura. In § 4, we recall Hida's theory for Hilbert modular forms, following Hida [Hid93] and Wiles [Wil88]. In § 5 we then give a construction of the so-called Eisenstein measure, following Panchishkin [Pan03] in the case $F = \mathbf{Q}$. Based on the Eisenstein measure, we give the construction of p-adic L-functions in § 6. The one-variable case was constructed by Dabrowski [Dab94]. In § 7 it is shown, using the method of Rankin–Shimura, that these p-adic L-functions do interpolate the classical L-values. Furthermore, we show that they satisfy the properties listed by Greenberg–Stevens [GS93]. In § 8, we prove a special case of the Mazur–Tate–Teitelbaum conjecture in the Hilbert modular setting, following [GS93]. We also investigate the case of higher-order exceptional zero for base-changed forms in § 9. In the final section, we make further comments concerning the relationship with recent developments.

General notation

As usual, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ denote the field of rational, real and complex numbers. For $z \in \mathbf{C}$, we denote by $\Re(z)$ and $\Im(z)$ the real and imaginary parts of z. For a prime p, $\mathbf{Q}_p$ is the field of p-adic numbers, with the subring of p-adic integers $\mathbf{Z}_p$. We denote by $|\cdot|_p$ the norm on $\mathbf{Q}_p$ such that $|p|_p = p^{-1}$. Fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, with $|\cdot|_p$ extended uniquely to $\overline{\mathbf{Q}}_p$.

We fix, once and for all, an embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$, and an embedding $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$.

For a commutative ring R with one, denote by $R^\times$ the group of units. If R is an integral domain, and $P \subset R$ a prime ideal, we denote by R_P the localization of R at P. In the case where P is the zero ideal, we denote it by Q_R, the field of fractions.

In this paper, F denotes a totally real field. We generally use German Gothic letters, e.g. $\mathfrak{c}$, to denote fractional ideals of F. For p a prime number, we denote by $\mathfrak{c}^{(p)}$ the prime to p-part of $\mathfrak{c}$, and by $\mathfrak{c}_p = (\mathfrak{c}^{(p)})^{-1}\mathfrak{c}$ the part divisible only by primes above p.

2. Generalities on Hilbert modular forms

2.1 We recall the rudiments of the theory of Hilbert modular forms, following Shimura [Shi78]. Let F be a totally real field and let $\mathfrak{r}$ be its ring of integers, and $\mathfrak{d}$ the different of F over $\mathbf{Q}$. Let $d = [F : \mathbf{Q}]$ be the degree of F over $\mathbf{Q}$. For each prime ideal $\mathfrak{p}$ of F, denote by $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$. We denote by $\mathbf{A}_F$ the ring of adèles of F, with F diagonally embedded as the principal adèles. Let $\mathbf{A}_{F,f}$ be the ring of finite adèles, $F_\infty = F \otimes \mathbf{R}$ be the archimedean component of $\mathbf{A}_F$, and let F_∞^+ be the identity component of $F_\infty^\times$. An element $\xi \in F$ is called totally positive, denoted as $\xi \gg 0$, if the archimedean component of ξ lies in F_∞^+. In general, if $z \in \mathbf{A}_F^\times$, we denote by z_∞ the archimedean component, and by z_f the finite adèlic component. Here $\mathbf{A}_F^\times$ is the group of idèles. For $s \in \mathbf{A}_F^\times$, we denote by $s\mathfrak{r}$ the fractional ideal associated to s.

The adèlic norm is denoted as $|\cdot|_{\mathbf{A}_F}$. Furthermore, for $z_\infty = (z_{\infty,1}, \dots, z_{\infty,d}) \in F_\infty$, we use the notation $\mathrm{Tr}(z_\infty) = \sum_{\nu=1}^d z_{\infty,\nu}$, $\mathcal{N}(z_\infty) = \prod_{\nu=1}^d z_{\infty,\nu}$.

To give the adèlic definition, let G be the algebraic group GL_2 defined over $\mathbf{Q}$. Denote by $G(\mathbf{A}_F)$ the group of adèlic points. Under the usual diagonal embedding, we have the subgroup, $G(F)$ of F-rational points of G. We also define $G^+(F)$ as the condition

$$G^+(F) = \{g \in G(F) \mid \det(g)_\infty \in F_\infty^+\}.$$

Furthermore, by abuse of notation, if $z \in \mathbf{A}_F^\times$, we again denote by z the element $\left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right) \in G(F)$.

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For a fractional ideal $\mathfrak{a}$ of F, and $\mathfrak{p}$ a prime ideal of F, let $\mathfrak{a}_\mathfrak{p}$ be the localization of $\mathfrak{a}$ at $\mathfrak{p}$ as a submodule of $F_\mathfrak{p}$, and put $\hat{\mathfrak{a}} = \mathfrak{a} \otimes_\mathbf{Z} \hat{\mathbf{Z}} = \prod_\mathfrak{p} \mathfrak{a}_\mathfrak{p}$. Following Shimura, we define the congruence subgroups $K_\mathfrak{n}$, $K_{1,\mathfrak{n}}$ of $G(\mathbf{A}_{F,f})$, other than the standard subgroups, as

$$K_\mathfrak{n} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(\mathbf{A}_{F,f}), \alpha, \delta \in \hat{\mathfrak{r}}, \beta \in \hat{\mathfrak{d}^{-1}}, \gamma \in \hat{\mathfrak{d}\mathfrak{n}} \right\}, \tag{2.1}$$

$$K_{1,\mathfrak{n}} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in K_\mathfrak{n} \,\middle|\, \delta \equiv 1 \bmod \mathfrak{n} \right\}. \tag{2.2}$$

(Compare with [Shi78, (2.1b) and (2.1c)]. Our $K_\mathfrak{n}$ is what was written as $\prod_\mathfrak{p} W(\mathfrak{n})_\mathfrak{p}$ in Shimura's paper.)

Let $\mathcal{D} \in \mathbf{A}_{F,f}^\times$ be a finite adèle such that $\mathcal{D}\mathfrak{r} = \mathfrak{d}$. Then from (2.1), it can be seen that

$$\begin{pmatrix} \mathcal{D} & 0 \\ 0 & 1 \end{pmatrix} K_\mathfrak{n} \begin{pmatrix} \mathcal{D}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

is the standard Iwahori congruence subgroup of $G(\mathbf{A}_{F,f})$ of level $\mathfrak{n}$.

The reason for conjugating the standard definition of the congruence subgroups by the matrix $\left(\begin{smallmatrix} \mathcal{D} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is not essential, but with this choice, the different would not appear explicitly in the formula for Fourier expansions (compare (2.16) below and the Fourier expansion in Hida [Hid93, pp. 276–277], who employed the standard Iwahori congruence subgroups in the definition of adèlic modular forms).

Finally, at the archimedean place, we put $K_\infty = \prod_{i=1}^d \mathrm{SO}(2)$.

DEFINITION 2.1. Let $k \in \mathbf{Z}_{\geq 0}$ and let $\mathfrak{n}$ be an integral ideal of $\mathfrak{r}$. By a Hilbert modular form of parallel weight k, level $\mathfrak{n}$, we mean a function $\mathbf{f} : G(\mathbf{A}_F) \to \mathbf{C}$, satisfying the following conditions:

(1) $\mathbf{f}$ satisfies the following transformation properties

$$\mathbf{f}(sg) = \mathbf{f}(g) \quad \text{for all } g \in G(\mathbf{A}_F),\, s \in F_\infty^+ G(F), \tag{2.3}$$

$$\mathbf{f}(gr(\theta)) = \mathbf{f}(g) e^{ik\{\theta\}}, \tag{2.4}$$

where $r(\theta) = (r_1(\theta_1), \ldots, r_d(\theta_d)) \in G(F_\infty)$, with

$$r_\nu(\theta_\nu) = \begin{pmatrix} \cos\theta_\nu & \sin\theta_\nu \\ -\sin\theta_\nu & \cos\theta_\nu \end{pmatrix}; \quad \{\theta\} = \theta_1 + \cdots + \theta_d.$$

$$\mathbf{f}(gk) = \mathbf{f}(g) \quad \text{for all } g \in G(\mathbf{A}_F),\, k \in K_{1,\mathfrak{n}}; \tag{2.5}$$

(2) $\mathbf{f}$ restricted to $G(F_\infty)$ is smooth, with Casimir eigenvalue $(k-2)^2/2 + k - 2$ for each archimedean place of F, and is of moderate growth; the complex vector space of such forms is denoted by $M_k(\mathfrak{n})$ (it is a fundamental result that this is finite dimensional).

It would also be convenient at times to have the classical ideal theoretic formulation. This is slightly complicated by the non-triviality of the narrow ideal class group of F in the general case:

$$\mathrm{Cl_F} = F^\times \setminus \mathbf{A}_F^\times / F_\infty^+ \hat{\mathfrak{r}}^\times.$$

Let h be its cardinality, i.e. the strict ideal class number of F. Choose idèle representatives $t_1, \ldots, t_h$. By weak approximation, we may choose the representatives such that $(t_\lambda)_\infty = 1$.

We obtain an h-tuple $(f_1, \ldots, f_h)$ of automorphic functions on the d-fold product of the upper half plane $\mathfrak{H}^d$, as follows.

For $z=(z_1,\ldots,z_d)\in\mathfrak{H}^d$, let $g_\infty\in G(F_\infty^+)$ be such that $z=g_\infty(\mathbf{i})$, here $\mathbf{i}=(i,\ldots,i)$. Then according to [Shi78, (2.15c)],

$$f_\lambda(z)=\mathcal{N}(\det(g_\infty))^{-k/2}j(g_\infty,\mathbf{i})^k\mathbf{f}\left(\begin{pmatrix}t_\lambda^{-1} & 0\\ 0 & 1\end{pmatrix}g_\infty\right). \tag{2.6}$$

Here j is the usual factor of automorphy: for $g_\infty=(g_{\infty,1},\ldots,g_{\infty,d})$, with $g_{\infty,\nu}=\left(\begin{smallmatrix}\alpha_\nu & \beta_\nu\\ \gamma_\nu & \delta_\nu\end{smallmatrix}\right)$, and $z\in\mathfrak{H}^d$,

$$j(g_\infty,z)=\prod(\gamma_\nu z_\nu+\delta_\nu). \tag{2.7}$$

Then f_λ turns out to be holomorphic on $\mathfrak{H}^d$, and at the cusps, and satisfies the following automorphic property: if we define

$$\begin{aligned}\Gamma_{0,\lambda}(\mathfrak{n})&=G^+(F)\cap\begin{pmatrix}t_\lambda^{-1} & 0\\ 0 & 1\end{pmatrix}K_{\mathfrak{n}}\begin{pmatrix}t_\lambda & 0\\ 0 & 1\end{pmatrix}\\ &=\left\{\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G^+(F):\alpha,\delta,\in\mathfrak{r},\beta\in(t_\lambda\mathfrak{d})^{-1},\gamma\in t_\lambda\mathfrak{d}\mathfrak{n},\det\sigma\in\mathfrak{r}^\times\right\},\end{aligned} \tag{2.8}$$

$$\begin{aligned}\Gamma_{1,\lambda}(\mathfrak{n})&=G^+(F)\cap\begin{pmatrix}t_\lambda^{-1} & 0\\ 0 & 1\end{pmatrix}K_{1,\mathfrak{n}}\begin{pmatrix}t_\lambda & 0\\ 0 & 1\end{pmatrix}\\ &=\left\{\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_{0,\lambda}(\mathfrak{n}):\delta\equiv 1 \bmod \mathfrak{n}\right\},\end{aligned} \tag{2.9}$$

then for $\sigma\in\Gamma_{1,\lambda}(\mathfrak{n})$ as above,

$$f_\lambda|_k\sigma=f_\lambda \tag{2.10}$$

with

$$(f_\lambda|_k\sigma)(z):=\mathcal{N}(\det(\sigma))^{k/2}j(\sigma,z)^{-k}f_\lambda(\sigma(z)). \tag{2.11}$$

It is a standard result that the above two descriptions are equivalent.

2.2 Given a Hilbert modular form $\mathbf{f}$, corresponding to a tuple $(f_\lambda)_{\lambda=1}^h$, f_λ is invariant under translation by elements of $(t_\lambda\mathfrak{d})^{-1}$, by (2.10). Hence, there is a Fourier expansion: for $z\in F_\infty$, let $e_F(z)=\exp(2\pi i\mathrm{Tr}(z))$, then

$$f_\lambda=a_\lambda(0)+\sum_{\substack{\mu\in t_\lambda\\ \mu\gg 0}}a_\lambda(\mu)e_F(\mu z). \tag{2.12}$$

If the constant terms are all zero in the expansion of each $f|_k\alpha$ for each $\alpha\in G^+(F)$, then $\mathbf{f}$ is said to be a cusp form. In the adèlic settings, it can be formulated elegantly as

$$\int_{F\backslash\mathbf{A}_F}\mathbf{f}(ng)\,dn=0\quad\text{for all } g\in G(\mathbf{A}_F). \tag{2.13}$$

The subspace of $M_k(\mathfrak{n})$ consisting of cusp forms is denoted as $S_k(\mathfrak{n})$.

We define the normalized Fourier coefficients of $\mathbf{f}$ as follows: for a non-zero fractional ideal $\mathfrak{a}$ of $\mathcal{O}_F$, we can write in a unique way $\mathfrak{a}=(\mu)t_\lambda^{-1}$, where $\mu\in t_\lambda$ is totally positive. Define

$$C(\mathfrak{a},\mathbf{f})=\begin{cases}a_\lambda(\mu)\mathcal{N}(t_\lambda)^{-k/2} & \text{if } \mathfrak{a} \text{ is integral,}\\ 0 & \text{otherwise.}\end{cases} \tag{2.14}$$

$$C_0(\mathfrak{a},\mathbf{f})=a_\lambda(0)\mathcal{N}(t_\lambda)^{-k/2}. \tag{2.15}$$

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Here $C_0(\mathfrak{a},\mathbf{f})$ is the normalized constant term, which is zero for cusp forms. Note that it depends only on the image of $\mathfrak{a}$ in $\mathrm{Cl_F}$. Then we have the adèlic Fourier expansion (at infinity): let $\chi_F:\mathbf{A}_F/F\to\mathbf{C}^\times$ be the standard additive character for which $\chi_F(x_\infty)=e_F(x_\infty)$, then

$$\mathbf{f}\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)=|y|_{\mathbf{A}_F}^{k/2}\sum_{0\ll\xi\in F}C(\xi y\mathfrak{r},\mathbf{f})e_F(\xi\mathbf{i}y_\infty)\chi_F(\xi x)+|y|_{\mathbf{A}_F}^{k/2}C_0(y\mathfrak{r},\mathbf{f}). \tag{2.16}$$

Of course, (2.16) can also be used as the definition of the normalized Fourier coefficients.

2.3 We recall the definition of the diamond and Hecke operators. Following the convention of Shimura, these operators will act on the right of modular forms.

For $\mathbf{f}\in M_k(\mathfrak{n})$, $z\in\mathbf{A}_F^\times$, define

$$\mathbf{f}|[z]_k(g)=\mathbf{f}(zg). \tag{2.17}$$

Note that this action factors through the narrow ray class group of conductor $\mathfrak{n}$:

$$\mathrm{Cl_F}(\mathfrak{n})=F^\times\setminus\mathbf{A}_F^\times/F_\infty^+(\hat{\mathfrak{r}}^\times\cap K_{1,\mathfrak{n}}). \tag{2.18}$$

If $\mathbf{f}$ is an eigenfunction for the diamond operators $[\cdot]_k$, then there exists a Hecke character of F of finite order, whose conductor divides $\mathfrak{n}$, such that

$$\mathbf{f}|[z]_k(g)=\psi(z)\mathbf{f}(g) \tag{2.19}$$

in which case $\mathbf{f}$ is said to have character ψ. The space of Hilbert modular forms of weight k, level $\mathfrak{n}$, character ψ, is denoted as $M_k(\mathfrak{n},\psi)$.

If $\mathfrak{q}$ is a prime ideal of $\mathfrak{r}$, prime to $\mathfrak{n}$, then by choosing a local uniformizer $\pi_\mathfrak{q}$ at the place $\mathfrak{q}$, we can define

$$\mathbf{f}|[\mathfrak{q}]_k=\mathbf{f}|[\pi_\mathfrak{q}]_k \tag{2.20}$$

(this does not depend on the choice of $\pi_\mathfrak{q}$). We extend the definition to any integral ideal $\mathfrak{l}$, by using multiplicativity if $\mathfrak{l}$ is prime to $\mathfrak{n}$, and setting it equal to zero otherwise.

The action of the Hecke operators $T(\mathfrak{l})$ on forms of level $\mathfrak{n}$, can be defined on Fourier coefficients, given by the formula

$$C(\mathfrak{m},\mathbf{f}|T(\mathfrak{l}))=\sum_{\mathfrak{m}+\mathfrak{l}\subset\mathfrak{a}}\mathcal{N}(\mathfrak{a})^{k-1}C(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{l},\mathbf{f}|[\mathfrak{a}]_k). \tag{2.21}$$

See also [Shi78, (2.10)] for their adèlic formulae (in terms of action of the double cosets of $K_\mathfrak{n}$).

These operators satisfy identities which can be written formally as

$$\sum_{\mathfrak{m}}T(\mathfrak{m})\mathcal{N}(\mathfrak{m})^{-s}=\prod_{\mathfrak{p}}[1-T(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s}+[\mathfrak{p}]_k\mathcal{N}(\mathfrak{p})^{k-1-2s}]^{-1}.$$

As usual, if $\mathfrak{p}$ is a prime ideal, and $\mathfrak{p}|\mathfrak{n}$, then we write $U(\mathfrak{p})$ for $T(\mathfrak{p})$:

$$C(\mathfrak{m},\mathbf{f}|U(\mathfrak{p}))=C(\mathfrak{m}\mathfrak{p},\mathbf{f}) \tag{2.22}$$

with the following adèlic formula. Let $\pi_\mathfrak{p}$ be a uniformizer of $F_\mathfrak{p}$. Then

$$(\mathbf{f}|U(\mathfrak{p}))(g)=\mathcal{N}(\mathfrak{p})^{k/2-1}\sum_{v\in\mathfrak{r}_\mathfrak{p}/\pi_\mathfrak{p}\mathfrak{r}_\mathfrak{p}}\mathbf{f}\left(g\begin{pmatrix}\pi_\mathfrak{p} & v\\ 0 & 1\end{pmatrix}\right). \tag{2.23}$$

In addition, we have operators $V(\mathfrak{l})$, satisfying

$$C(\mathfrak{m},\mathbf{f}|V(\mathfrak{l}))=C(\mathfrak{l}^{-1}\mathfrak{m},\mathbf{f}). \tag{2.24}$$

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According to Shimura, this can be defined as follows: let $s = \mathrm{diag}[l, 1]$, with $l \in \mathbf{A}_{F,f}^{\times}$, such that $l\mathfrak{r} = \mathfrak{l}$. Then

$$\mathbf{f}|V(\mathfrak{l})(g) = |l|_{\mathbf{A}_F}^{k/2}\mathbf{f}(gs^{-1})$$

(cf. [Shi78, Proof of Proposition 2.3]).

Finally recall the adèlic Atkin–Lehner operator for Hilbert modular forms: let w be an element with $(w)_\infty = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, while $(w)_f = \left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$, for a finite idèle m such that $m\mathfrak{r} = \mathfrak{n}\mathfrak{d}^2$.

Then for $\mathbf{f} \in M_k(\mathfrak{n})$, the form $\mathbf{f}|J_{\mathfrak{n}} \in M_k(\mathfrak{n})$ is defined by

$$\mathbf{f}|J_{\mathfrak{n}}(g) = \mathbf{f}(\det(g)^{-1}gw) \tag{2.25}$$

here $\det(g)$ is regarded as an element in the centre of $G(\mathbf{A}_F)$ (cf. [Shi78, (2.46)], but note that in [Shi78] the element w is denoted as τ). This does not depend on the choice of τ. If $\mathbf{f}$ has character ψ, then

$$\mathbf{f}|J_{\mathfrak{n}}(g) = \psi(\det(g))^{-1}\mathbf{f}(gw), \tag{2.26}$$

in which case $\mathbf{f}|J_{\mathfrak{n}}$ has central character ψ^{-1}. In any case, we have

$$\mathbf{f}|J_{\mathfrak{n}}^2 = (-1)^{dk}\mathbf{f}. \tag{2.27}$$

This follows from a direct calculation: first, for any $g \in G(\mathbf{A}_F)$,

$$\mathbf{f}|J_{\mathfrak{n}}^2(g) = \mathbf{f}|J_{\mathfrak{n}}(\det(g)^{-1}gw).$$

Noting that $\det(\det(g)^{-1}gw) = (-1)_f m \det(g)^{-1}$, this becomes

$$\begin{aligned} &\mathbf{f}((-1)_f m^{-1} \det(g) \det(g)^{-1} g w^2) \\ &= \mathbf{f}((-1)_f g) \\ &= \mathbf{f}((-1)_\infty g) \\ &= (-1)^{dk}\mathbf{f}(g) \end{aligned}$$

with the last equality follows from (2.4).

2.4 The L-series of $\mathbf{f}$ is defined via the normalized Fourier coefficients. We consider the more general context of twisted L-series as follows.

Let χ be a Hecke character of F of finite order, of conductor $\mathfrak{c}_\chi$. For a prime ideal $\mathfrak{p}$, we define

$$\chi(\mathfrak{p}) = \begin{cases} \chi(\pi_{\mathfrak{p}}), \pi_{\mathfrak{p}} \in F_{\mathfrak{p}}^{\times} & \text{if } \mathfrak{p} \text{ is prime to } \mathfrak{c}_\chi, \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

By multiplicativity, this can be extended to all integral ideals of $\mathfrak{r}$. More generally, if $\mathfrak{c}$ is an ideal divisible by the set of prime divisors of $\mathfrak{c}_\chi$, we define $\chi_{\mathfrak{c}}$ by declaring that $\chi_{\mathfrak{c}}(\mathfrak{p}) = 0$ if $\mathfrak{p}$ is not prime to $\mathfrak{c}$. We also denote by $L^{(\mathfrak{c})}(s, \chi)$ the L-series of χ, with the Euler factors at the places dividing $\mathfrak{c}$ removed:

$$L^{(\mathfrak{c})}(s, \chi) = \prod_{\mathfrak{p}+\mathfrak{c}=\mathfrak{r}} (1 - \chi(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s})^{-1}. \tag{2.29}$$

The L-function of $\mathbf{f}$, twisted by χ, is

$$L(s, \mathbf{f}, \chi) = \sum_{\mathfrak{m}} \chi(\mathfrak{m}) C(\mathfrak{m}, \mathbf{f}) \mathcal{N}(\mathfrak{m})^{-s}.$$

This is the usual L-function of the form $\mathbf{f} \otimes \chi$ (see [Hid91]).

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Suppose that $\mathbf{f} \in S_k(\mathfrak{n}, \psi)$ is a cuspidal, *normalized eigenform*, i.e. $\mathbf{f}$ is an eigenvector for all $T(\mathfrak{l})$, $[\mathfrak{l}]_k$, and satisfies $C(\mathfrak{r}, \mathbf{f}) = 1$. In this case, the eigenvalue for $T(\mathfrak{l})$ is $C(\mathfrak{l}, \mathbf{f})$, and they are algebraic integers. Then the L-series can be given by an Euler product:

$$L(s, \mathbf{f}, \chi) = \prod_{\mathfrak{p}} [1 - \chi(\mathfrak{p}) C(\mathfrak{p}, \mathbf{f}) \mathcal{N}(\mathfrak{p})^{-s} + \chi(\mathfrak{p})^2 \psi_{\mathfrak{n}}(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{k-1-2s}]^{-1}.$$

It can be analytically continued to an entire function on the complex plane. We similarly define $L^{(\mathfrak{c})}(s, \chi, \mathbf{f})$ by removing the Euler factors at the places that divide $\mathfrak{c}$.

We also recall that a normalized eigenform $\mathbf{f} \in S_k(\mathfrak{n}, \psi)$ is said to be a newform if there exists no other eigenform $\mathbf{g} \in S_k(\mathfrak{n}_1)$, with $\mathfrak{n}_1$ strictly divides $\mathfrak{n}$, such that $C(\mathfrak{m}, \mathbf{f}) = C(\mathfrak{m}, \mathbf{g})$, for $\mathfrak{m}$ relatively prime to $\mathfrak{n}$. In this case $\mathfrak{n}$ is called the conductor of $\mathbf{f}$, $\mathrm{cond}(\mathbf{f})$.

2.5 Finally we need Shimura's result on the rationality structure on the space of Hilbert modular forms, which is a consequence of his theory of canonical models.

Shimura defined an action of $\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ on the space of Hilbert modular forms. Given $\mathbf{f} \in M_k(\mathfrak{n}, \psi)$, there is a (unique) $\mathbf{f}^\sigma \in M_k(\mathfrak{n}, \psi^\sigma)$; the action can be described on the Fourier coefficients:

$$\begin{aligned} C(\mathfrak{m}, \mathbf{f}^\sigma) &= C(\mathfrak{m}, \mathbf{f})^\sigma, \\ C_0(\mathfrak{m}, \mathbf{f}^\sigma) &= C_0(\mathfrak{m}, \mathbf{f})^\sigma. \end{aligned} \tag{2.30}$$

The following result follows easily from the existence of this action.

PROPOSITION 2.2 [Shi78]. *Let $\mathbf{f}$ be a Hilbert modular form of weight $k \geq 1$, and let $\mathbf{Q}(\mathbf{f})$ be the field generated by the Fourier coefficients $C(\mathfrak{m}, \mathbf{f})$, then $C_0(\mathfrak{m}, \mathbf{f}) \in \mathbf{Q}(\mathbf{f})$.*

Proof. For any $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q}(\mathbf{f}))$, $C(\mathfrak{m}, \mathbf{f}^\sigma - \mathbf{f}) = C(\mathfrak{m}, \mathbf{f})^\sigma - C(\mathfrak{m}, \mathbf{f}) = 0$. Thus, the non-constant Fourier coefficients of the form $\mathbf{f}^\sigma - \mathbf{f}$ are all zero. Since it is of weight $k \geq 1$, it is in fact zero, i.e. $C_0(\mathfrak{m}, \mathbf{f})^\sigma = C_0(\mathfrak{m}, \mathbf{f})$, for any $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q}(\mathbf{f}))$. □

To state Shimura's result on the rationality structure of Hilbert modular forms, define, for a subring A of $\mathbf{C}$, the A-module

$$M_k(\mathfrak{n}, \psi, A) = \{\mathbf{f} \in M_k(\mathfrak{n}, \psi), C(\mathfrak{m}, \mathbf{f}), C_0(\mathfrak{m}, \mathbf{f}) \in A\}. \tag{2.31}$$

This is the A-submodule of Hilbert modular forms, with (normalized) Fourier coefficients rational over A.

Then the rationality theorem of Shimura states

$$M_k(\mathfrak{n}, \psi) = M_k(\mathfrak{n}, \psi, A) \otimes_A \mathbf{C}. \tag{2.32}$$

Hence, we can define, for any ring A in general,

$$M_k(\mathfrak{n}, \psi, A) = M_k(\mathfrak{n}, \psi, \mathbf{Z}) \otimes_{\mathbf{Z}} A. \tag{2.33}$$

What we have in mind would be the particular case where A is a subring of $\mathbf{C}_p$.

3. Hilbert modular Eisenstein series

3.1 Definition

In this section, we define certain Hilbert modular forms, constructed from Eisenstein series. These play an important role in what follows. We follow the normalizations of

Shimura [Shi78, § 3] (see also Hida [Hid93, ch. 9] for adèlic Hilbert modular Eisenstein series, but we note that Hida's normalizations are different from ours).

Let $P \subset B \subset G$ be the subgroups, defined as follows: for any $\mathbf{Q}$-algebra R,

$$B(R) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \middle| \alpha, \delta \in R^\times, \beta \in R \right\}$$
$$P(R) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \middle| \alpha \in R^\times, \beta \in R \right\}.$$

Here B is the standard Borel subgroup, while P is called the mirabolic subgroup. We have the Iwasawa decomposition:

$$G(\mathbf{A}_{F,f}) = B(\mathbf{A}_{F,f}) K_{\mathfrak{r}},$$
$$G(F_\infty) = B(F_\infty) K_\infty.$$

We define certain functions on $G(\mathbf{A}_F)$. First, given $g \in G(\mathbf{A}_F)$, write $g = bk$, with $b \in B(\mathbf{A}_F)$, $k \in K_\infty K_{\mathfrak{r}}$, define

$$\eta(g) = \Delta(b), \tag{3.1}$$

where

$$\Delta\left(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}\right) = |\alpha/\delta|_{\mathbf{A}_F},$$

the modular character of $B(\mathbf{A}_F)$. Note that

$$\eta(g) = \eta(g_\infty)\eta(g_f) = |\mathcal{N}(\Im(g_\infty(\mathbf{i})))|\eta(g_f). \tag{3.2}$$

From the definition, it is clear that η is left-invariant by $\mathbf{A}_F^\times P(F)$.

Next, let $\varphi = \prod_\nu \varphi_\nu$ be a Hecke character of F of finite order, written as a product of local characters φ_ν, with ν running over all places of F. Let $\mathfrak{c}_\varphi$ be its conductor. For an integral ideal $\mathfrak{c}$ divisible only by the primes dividing $\mathfrak{c}_\varphi$, define the following functions on $G(\mathbf{A}_F)$:

$$\varphi_{\mathfrak{c}}^{\#}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{cases} \displaystyle\prod_{\nu | \mathfrak{c}} \varphi_\nu(\delta_\nu) & \text{if } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in P(\mathbf{A}_F) K_{\mathfrak{c}} G^+(F_\infty), \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We omit the subscript $\mathfrak{c}$ if we take $\mathfrak{c}$ to be $\mathfrak{c}_\varphi$. By its definition, $\varphi_{\mathfrak{c}}^{\#}$ is left-invariant under $P(\mathbf{A}_F)$.

Definition 3.1. With notation as above, suppose that k is a positive integer such that $\varphi|_{F_\infty}(\cdot) = \operatorname{sgn}(\mathcal{N}(\cdot)^k)$. Let $s \in \mathbf{C}$. Define the Eisenstein series $K_k^*(s, \varphi, \mathfrak{c})$:

$$K_k^*(s, \varphi, \mathfrak{c})(g) = |\det(g)|_{\mathbf{A}_F}^{k/2} \sum_{\gamma \in P(F)\mathfrak{r}^\times \backslash G(F)} (\varphi_{\mathfrak{c}}^{-1})^{\#}(\gamma g)\eta(\gamma g)^s j(\gamma g_\infty, \mathbf{i})^{-k}. \tag{3.4}$$

(The condition $\varphi|_{F_\infty}(\cdot) = \operatorname{sgn}(\mathcal{N}(\cdot)^k)$ is needed to ensure that $(\varphi_{\mathfrak{c}}^{-1})^{\#}(g) j(g_\infty, \mathbf{i})^{-k}$ is left-invariant under $\mathfrak{r}^\times$.)

More generally, let ζ_1, ζ_2 be finite-order Hecke characters of F, such that $\zeta_1\zeta_2|_{F_\infty}(\cdot) = \operatorname{sgn}(\mathcal{N}(\cdot)^k)$ for a positive integer k. Choose finite idèles $c_{\zeta_1}, c_{\zeta_2}, \mathcal{D}$ such that $c_{\zeta_1}\mathfrak{r} = \mathfrak{c}_{\zeta_1}$, $c_{\zeta_2}\mathfrak{r} = \mathfrak{c}_{\zeta_2}$, $\mathcal{D}\mathfrak{r} = \mathfrak{d}$,. Set $m = c_{\zeta_1} c_{\zeta_2} \mathcal{D}^2$, and let $w_{c_{\zeta_1} c_{\zeta_2}}$ to be the matrix such that $(w_{c_{\zeta_1} c_{\zeta_2}})_\infty = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, while $(w_{c_{\zeta_1} c_{\zeta_2}})_f = \left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$ (exactly the same matrix used to define the Atkin–Lehner operator).

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DEFINITION 3.2. With the same notation, for $s \in \mathbf{C}$, define the Eisenstein series $E_k^*(s, \zeta_1, \zeta_2)$ as

$$E_k^*(s,\zeta_1,\zeta_2)(g) = |\det(g)|_{\mathbf{A}_F}^{k/2}\zeta_1(\det g)\zeta_1(\mathcal{D}^{-1})\zeta_2^{-1}(c_{\zeta_2}) \times \sum_{\gamma \in P(F)\mathfrak{r}^\times \backslash G(F)} (\zeta_1^{-1})^{\#}(\gamma g \tau_{c_{\zeta_1}c_{\zeta_2}}\mathcal{D}^{-1})\zeta_2^{\#}(\gamma g c_{\zeta_2})\eta(\gamma g)^s j(\gamma g_\infty, \mathbf{i})^{-k}. \quad (3.5)$$

It can be shown that this does not depend on the choices we have made.

The Eisenstein series in Definitions 3.1 and 3.2 are obtained from linear combinations of the 'partial Eisenstein series' of [Shi78] (see (3.5) there). Hence, from [Shi78, § 3], the Eisenstein series $K_k^*(s,\varphi,\mathfrak{c})$, $E_k^*(s,\zeta_1,\zeta_2)$ converge in the region $\Re(2s+k) > 2$, and in this range, define 'non-holomorphic' Hilbert modular forms of weight k, in the sense that they satisfy only condition (1) of Definition 2.1. Furthermore, these series can be analytically continued, as a function of the variable s, to entire functions. Here $K_k^*(s,\varphi,\mathfrak{c})$ is of level $\mathfrak{c}$, while $E_k(s,\zeta_1,\zeta_2)$ is of level $\mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2}$.

From (3.5), which defines $E_k^*(s,\zeta_1,\zeta_2)$, and the condition $\zeta_1\zeta_2|_{F_\infty}(\cdot) = \mathrm{sgn}(\mathcal{N}(\cdot)^k)$, we see that, for $z \in \mathbf{A}_F$, the value

$$\zeta_1^{-1}(z)\zeta_2^{-1}(z)E_k^*(s,\zeta_1,\zeta_2)(zg) \quad (3.6)$$

depends only on the class of z in the usual ideal class group:

$$\overline{\mathrm{Cl}}_{\mathrm{F}} = F^\times \setminus \mathbf{A}_F^\times / F_\infty \hat{\mathfrak{r}}^\times.$$

It follows that

$$\sum_{z \in \overline{\mathrm{Cl}}_{\mathrm{F}}} \zeta_1^{-1}(z)\zeta_2^{-1}(z)E_k^*(s,\zeta_1,\zeta_2)(zg) \quad (3.7)$$

is a form with character $\zeta_1\zeta_2$.

DEFINITION 3.3. Define $K_k(s,\varphi,\mathfrak{c})$, $E_k(s,\zeta_1,\zeta_2)$, by

$$K_k(s,\varphi,\mathfrak{c}) = L^{(\mathfrak{c})}(k+2s,\varphi^{-1})K_k^*(s,\varphi,\mathfrak{c}). \quad (3.8)$$

$$E_k(s,\zeta_1,\zeta_2)(g) = L^{(\mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2})}(k+2s,\zeta_1^{-1}\zeta_2) \sum_{z \in \overline{\mathrm{Cl}}_{\mathrm{F}}} \zeta_1^{-1}(z)\zeta_2^{-1}(z)E_k^*(zg,s,\zeta_1,\zeta_2). \quad (3.9)$$

Recall that we have defined the Atkin–Lehner operator in (2.24). We put

$$G_k(s,\varphi,\mathfrak{c}) := K_k(s,\varphi,\mathfrak{c})|J_{\mathfrak{c}}. \quad (3.10)$$

For $E_k(s,\zeta_1,\zeta_2)$, it enjoys the following symmetry.

PROPOSITION 3.4. *With notation as in Definition 3.2:*

$$E_k(s,\zeta_1,\zeta_2)|J_{\mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2}} = \zeta_1((-1)_\infty)\mathcal{N}(\mathfrak{c}_{\zeta_2}\mathfrak{c}_{\zeta_1}^{-1})^{k/2}\mathcal{N}(\mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2}^{-1})^s E_k(s,\zeta_2^{-1},\zeta_1^{-1}). \quad (3.11)$$

Proof. The key is another expression for $E_k^*(s,\zeta_1,\zeta_2)$:

$$E_k^*(s,\zeta_1,\zeta_2)(g) = |\det(g)|_{\mathbf{A}_F}^{k/2}\zeta_1(\det g)\zeta_1(\mathcal{D}^{-1})\zeta_2^{-1}(c_{\zeta_2}) \times \sum_{\gamma \in P(F)\mathfrak{r}^\times \backslash G(F)} [(\zeta_1^{-1})^{\#}(\gamma g w_{c_{\zeta_1}c_{\zeta_2}}\mathcal{D}^{-1})\zeta_2^{\#}(\gamma g c_{\zeta_2}) \times |\mathcal{N}(\Im(\gamma g_\infty(\mathbf{i})))|^s |\det(\gamma g c_\zeta)_f|_{\mathbf{A}_F}^s j(\gamma g_\infty,\mathbf{i})^{-k}]. \quad (3.12)$$

Indeed, inside the summation sign of (3.5), for $\zeta_2^{\#}(\gamma g c_{\zeta_2})$ to be non-zero, we must have $\gamma g c_{\zeta_2} \in P(\mathbf{A}_F)K_{\mathfrak{c}_{\zeta_2}}G^+(F_\infty)$. In this case, we have, from (3.1)–(3.2),

$$\eta(\gamma g) = \eta(\gamma g c_{\zeta_2}) = |\mathcal{N}(\Im(\gamma g_\infty(\mathbf{i})))||\det(\gamma g c_{\zeta_2})_f|_{\mathbf{A}_F}$$

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(note that $\Delta(p) = |\det(p)|_{\mathbf{A}_F}$ for $p \in P(\mathbf{A}_{F,f})$). Thus (3.12) follows. One also obtains a corresponding expression for $E_k(s, \zeta_2^{-1}, \zeta_1^{-1})$. Using this, and the definition of Atkin–Lehner operator (2.26), the proposition follows from elementary calculation. □

3.2 In this section, we are interested in the Eisenstein series obtained by putting $s = 0$. We denote the resulting series as $K_k(\varphi, \mathfrak{c})$, $G_k(\varphi, \mathfrak{c})$, $E_k(\zeta_1, \zeta_2)$.

Definition 3.5. Let φ be a Hecke character of F of finite order, with conductor $\mathfrak{c}_\varphi$. Define the Gauss sum of φ, $\tau(\varphi)$, to be

$$\tau(\varphi) = \sum_{x \in \mathfrak{c}_\varphi^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \operatorname{sgn}(\varphi(x_\infty))\varphi(x\mathfrak{c}_\varphi\mathfrak{d})e_F(x). \tag{3.13}$$

We state one of the main result concerning these Eisenstein series.

Proposition 3.6. *We normalize the Eisenstein series as follows:*

$$\mathbf{G}_k(\varphi, \mathfrak{c}) = \frac{\mathcal{N}(\mathfrak{c})^{k/2} D_F^{k-1/2}\Gamma(k)^d}{(-2\pi i)^{kd}} G_k(\varphi, \mathfrak{c})$$
$$\mathbf{E}_k(\zeta_1, \zeta_2) = \frac{\zeta_2((-1)_\infty) D_F^{k-1/2}\Gamma(k)^d \tau(\zeta_2)}{\mathcal{N}(\mathfrak{c}_\zeta)(-2\pi i)^{kd}} E_k(\zeta_1, \zeta_2)$$

(here Γ is Euler's Gamma function).

In the case $F = \mathbf{Q}$, $k = 2$, assume $\mathfrak{c}$, respectively ζ_2, is not trivial. With this understood, we have, for $k \geq 2$,

$$\mathbf{G}_k(\varphi, \mathfrak{c}) \in M_k(\mathfrak{c}, \varphi, \overline{\mathbf{Q}})$$
$$\mathbf{E}_k(\zeta_1, \zeta_2) \in M_k(\mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2}, \zeta_1\zeta_2, \overline{\mathbf{Q}})$$

(in particular, these forms are holomorphic. In the exceptional cases mentioned, these fail to be holomorphic). Furthermore, we have the following formula for the Fourier coefficients:

$$C(\xi y\mathfrak{r}, \mathbf{G}_k(\varphi, \mathfrak{c})) = \sum_{\substack{\xi = ed \\ e \in y^{-1}\mathfrak{r}, d \in \mathfrak{r} \\ d \mod \mathfrak{r}^\times}} \mathcal{N}(ey\mathfrak{r})^{k-1}\varphi(d\mathfrak{r}) \quad (\xi \gg 0, y \in \mathbf{A}_F^\times) \tag{3.14}$$

$$C_0(\mathfrak{m}, \mathbf{G}_k(\varphi, \mathfrak{c})) = \begin{cases} 2^{-d}L^{(\mathfrak{c})}(0, \varphi) & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$C(\mathfrak{m}, \mathbf{E}_k(\zeta_1, \zeta_2)) = \sum_{\substack{\mathfrak{a}\mathfrak{b} = \mathfrak{m} \\ \mathfrak{a}, \mathfrak{b} \subset \mathfrak{r}}} \zeta_1(\mathfrak{a})\zeta_2(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{k-1}$$
$$C_0(\mathfrak{m}, \mathbf{E}_k(\zeta_1, \zeta_2)) = \begin{cases} 2^{-d}L^{(\mathfrak{c}_{\zeta_2})}(1-k, \zeta_2) & \text{if } \zeta_1 \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

(Note that, by (3.14), we have $\mathbf{G}_k(\varphi, \mathfrak{c}) = \mathbf{G}_k(\varphi, \mathfrak{c}')$, if $\mathfrak{c}$ and $\mathfrak{c}'$ are divisible by the same set of primes.)

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Proof. This is [Shi78, Proposition 3.4]. In the proof of that result, Shimura explicitly constructed the modular forms whose Fourier expansions are of the form (3.14), (3.15). Our formulae for $G_k(s, \varphi, \mathfrak{c})$, $E_k(s, \zeta_1, \zeta_2)$ (see (3.4), (3.5) and (3.10)) are simply obtained *a posteriori* from Shimura's formulae (written in adèlic form). □

Remark 3.7. By inspection of (3.14) and (3.15), the Fourier coefficients for the non-constant terms are algebraic numbers. By Proposition 2.2, so are the Fourier coefficients of the constant terms, which are given by special values of Hecke L-functions. This algebraicity was first proven by Klingen–Siegel, and later reproved by Shimura in the above manner.

From (3.14), we see that the L-series associated to $\mathbf{E}_k(\zeta_1, \zeta_2)$ is

$$L(s, \mathbf{E}_k(\zeta_1, \zeta_2)) = L^{(\mathfrak{c}_{\zeta_1})}(s, \zeta_1) L^{(\mathfrak{c}_{\zeta_2})}(s + 1 - k, \zeta_2), \tag{3.16}$$

in particular, it has an Euler product. Thus, $\mathbf{E}_k(\zeta_1, \zeta_2)$ is a normalized eigenform. In terms $\mathbf{E}_k(\zeta_1, \zeta_2)$, equation (3.11) can be stated as

$$\mathbf{E}_k(\zeta_1, \zeta_2)|J_{\mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2}} = \frac{\zeta_2((-1)_\infty)}{\mathcal{N}\mathfrak{c}_{\zeta_2}} \mathcal{N}(\mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2})^{k/2} \mathcal{N}(\mathfrak{c}_{\zeta_1})^{1-k} \tau(\zeta_1^{-1})^{-1} \tau(\zeta_2) \mathbf{E}_k(\zeta_2^{-1}, \zeta_1^{-1}). \tag{3.17}$$

3.3 There remains to be added the fact that the space of Eisenstein series provides a complement to the space of cusp forms.

PROPOSITION 3.8. *Let* $\mathrm{Eis}_k(\mathfrak{n})$ *be the* $\mathbf{C}$*-subspace of* $M_k(\mathfrak{n})$ *spanned by the forms*

$$\mathbf{E}_k(\zeta_1, \zeta_2)|V(\mathfrak{m}), \mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2}\mathfrak{m}|\mathfrak{n}, \tag{3.18}$$

then $\mathrm{Eis}_k(\mathfrak{n})$ *is stable under the action of Hecke operators, and we have the decomposition of Hecke modules:*

$$M_k(\mathfrak{n}) = \mathrm{Eis}_k(\mathfrak{n}) \oplus S_k(\mathfrak{n}). \tag{3.19}$$

More generally, for any ring R *that contains the coefficients of the series* $\mathbf{E}_k(\zeta_1, \zeta_2)$*, denote by* $\mathrm{Eis}_k(\mathfrak{n}, R) \subset M_k(\mathfrak{n}, R)$ *the* R*-span of the forms of (3.17). Then if* $R = K$ *is a field, we also have*

$$M_k(\mathfrak{n}, K) = \mathrm{Eis}_k(\mathfrak{n}, K) \oplus S_k(\mathfrak{n}, K). \tag{3.20}$$

We denote by $\mathrm{pr}_{\mathrm{Eis}_k}, \mathrm{pr}_{S_k}$ *the corresponding projection operators on corresponding* $M_k(\mathfrak{n})$.

For a proof, see Shimura [Shi85, § 8].

4. Hida theory of Hilbert modular forms

4.1 In this section, we recall the basics of Hida theory of Hilbert modular forms. In fact, we specialize his results to the parallel weight case. More details can be found in Hida's original papers and his books [Hid88, Hid89, Hid93, Hid06]. Some results hold without restriction on the prime p (as is the case in [Hid88, Hid89]). However, for our applications, we need two results (Theorems 4.2 and 4.4 below), which Hida proved under the assumption that $p \geq 5$, and is unramified in F, i.e. p does not divide $6D_F$ (see [Hid06, Corollaries 4.21 and 4.22]). We therefore make this assumption on p in the rest of the paper.

In Hida's theory, one fixes a tame level $\mathfrak{n}$, with $\mathfrak{n}$ prime to p (thus, p is always assumed to be prime to $6D_F\mathfrak{n}$), and consider Hecke algebras of forms of level $\mathfrak{n}p^\alpha$, $\alpha \geq 1$. Thus, from now on, our notation from previous sections will be modified accordingly.

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First, we recall the structure of the relevant ray class groups.

As in § 2, we let

$$\mathrm{Cl_F}(\mathfrak{n}p^\alpha) = F^\times \setminus \mathbf{A}_F^\times / F_\infty^+ (\hat{\mathfrak{r}}^\times \cap K_{1,\mathfrak{n}p^\alpha})$$

be the narrow ray class group of conductor $\mathfrak{n}p^r$ (and infinity). Define

$$Z_F(\mathfrak{n}) = \varprojlim_\alpha \mathrm{Cl_F}(\mathfrak{n}p^\alpha). \tag{4.1}$$

We choose a decomposition

$$Z_F(\mathfrak{n}) = W_F(\mathfrak{n}) \times Z_F(\mathfrak{n})^{\mathrm{tors}} \tag{4.2}$$

with $W_F(\mathfrak{n})$ free over $\mathbf{Z}_p$, and $Z_F(\mathfrak{n})^{\mathrm{tors}}$ finite. Although the choice of $W_F(\mathfrak{n})$ may not be canonical, it is 'independent of' $\mathfrak{n}$, in the sense that $Z_F(\mathfrak{n})/Z_F(\mathfrak{n})^{\mathrm{tors}}$ is naturally isomorphic to $Z_F(\mathfrak{r})/Z_F(\mathfrak{r})^{\mathrm{tors}}$, under the natural projection $Z_F(\mathfrak{n}) \to Z_F(\mathfrak{r})$. For each integer $\alpha \geq 1$, let

$$\begin{gathered} Z_{F,\alpha}(\mathfrak{n}) = \ker(Z_F(\mathfrak{n}) \to \mathrm{Cl_F}(\mathfrak{n}p^\alpha)), \\ W_{F,\alpha}(\mathfrak{n}) = W_F(\mathfrak{n}) \cap Z_{F,\alpha}(\mathfrak{n}). \end{gathered} \tag{4.3}$$

We can similarly define $\overline{Z}_F(\mathfrak{n}), \overline{Z}_{F,\alpha}(\mathfrak{n})$ by considering the inverse limit of the ray class groups $\overline{\mathrm{Cl}}_{\mathrm{F}}(\mathfrak{n}p^\alpha)$, where

$$\overline{\mathrm{Cl}}_{\mathrm{F}}(\mathfrak{n}p^\alpha) = F^\times \setminus \mathbf{A}_F^\times / F_\infty (\hat{\mathfrak{r}}^\times \cap K_{1,\mathfrak{n}p^\alpha})$$

(under our assumption that $p \geq 5$, we have in fact $Z_{F,\alpha}(\mathfrak{n}) = W_{F,\alpha}(\mathfrak{n}) = \overline{Z}_{F,\alpha}(\mathfrak{n}) = \overline{W}_{F,\alpha}(\mathfrak{n})$).

Let $\mathcal{O}$ be a finite extension of $\mathbf{Z}_p$. We consider the completed group algebras:

$$\begin{aligned} \mathcal{A}_F &= \varprojlim_\alpha \mathcal{O}[Z_F(\mathfrak{n})/Z_{F,\alpha}(\mathfrak{n})] \\ \Lambda_F &= \varprojlim_\alpha \mathcal{O}[W_F(\mathfrak{n})/W_{F,\alpha}(\mathfrak{n})]. \end{aligned} \tag{4.4}$$

Here Λ_F is isomorphic to a power series ring in several variables over $\mathbf{Z}_p$ (if Leopoldt's conjecture holds for F and p, then $\Lambda_F \cong \mathbf{Z}_p[[X]]$, but we do not need this in the following). Note that $\mathcal{A}_F = \Lambda_F[Z_F(\mathfrak{n})^{\mathrm{tors}}]$ under the decomposition $Z_F(\mathfrak{n}) = Z_F(\mathfrak{n})^{\mathrm{tors}} \times W_F(\mathfrak{n})$. If $\mathfrak{l}$ is an ideal prime to $\mathfrak{n}p$, we denote by $[\mathfrak{l}]$ the corresponding group ring element of $\mathcal{A}_F$, and by $\langle[\mathfrak{l}]\rangle$ the element of Λ_F under the above decomposition.

Note that $Z_{\mathbf{Q}}(1) = \mathbf{Z}_p^\times$, $Z_{\mathbf{Q}}(1)^{\mathrm{tors}} \cong \mathbf{F}_p^\times$, and in this case we can naturally choose $W_{\mathbf{Q}}(1) = 1 + p\mathbf{Z}_p$. We use the notation

$$\begin{gathered} \omega_{\mathbf{Q}} : Z_{\mathbf{Q}}(1) \to Z_{\mathbf{Q}}(1)^{\mathrm{tors}} \\ \langle \cdot \rangle_{\mathbf{Q}} : Z_{\mathbf{Q}}(1) \to W_{\mathbf{Q}}(1) = 1 + p\mathbf{Z}_p \end{gathered} \tag{4.5}$$

for the two projection maps. Here $\omega_{\mathbf{Q}}$ is the Teichmüller character, while $\langle \cdot \rangle_{\mathbf{Q}}$ is the projection to the one-units.

We have the norm map $\mathcal{N} : Z_F(\mathfrak{r}) \to Z_{\mathbf{Q}}(1) = \mathbf{Z}_p^\times$. Denote by $\omega_F, \langle \cdot \rangle_F$, the composition of $\mathcal{N}$ with $\omega_{\mathbf{Q}}, \langle \cdot \rangle_{\mathbf{Q}}$, respectively. By abuse of notation, the composition of the projection $Z_F(\mathfrak{n}) \to Z_F(\mathfrak{r})$ with these characters will be denoted by the same symbol.

We now introduce Hecke algebras. Thus, for each weight $k \geq 2$, and $\alpha \geq 1$, consider the Hecke algebra $h_k(\mathfrak{n}p^\alpha, \mathcal{O})$, which is the subalgebra of $\mathrm{End}_{\mathcal{O}}(S_k(\mathfrak{n}p^\alpha, \mathcal{O}))$, generated by the Hecke and diamond operators $T(\mathfrak{q})$, $[\mathfrak{q}]_k$. There is a perfect $\mathcal{O}$-duality:

$$\begin{gathered} (\ ,\ ) : h_k(\mathfrak{n}p^\alpha, \mathcal{O}) \times S_k(\mathfrak{n}p^\alpha, \mathcal{O}) \to \mathcal{O}, \\ (\mathbf{f}, h) = C(\mathfrak{r}, \mathbf{f}|_k h). \end{gathered} \tag{4.6}$$

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In particular, $h_k(\mathfrak{n}p^\alpha, \mathcal{O})$ is a finite $\mathcal{O}$-module. Under this duality, the algebra homomorphisms $\mathrm{Hom}_{\mathcal{O}\text{-alg}}(h_k(\mathfrak{n}p^\alpha, \mathcal{O}), \mathcal{O})$ correspond to the normalized eigenform in $S_k(\mathfrak{n}p^\alpha, \mathcal{O})$.

For $\beta \geq \alpha$, we have, corresponding to the inclusion $S_k(\mathfrak{n}p^\alpha, \mathcal{O}) \subset S_k(\mathfrak{n}p^\beta, \mathcal{O})$, a surjection

$$h_k(\mathfrak{n}p^\beta, \mathcal{O}) \to h_k(\mathfrak{n}p^\alpha, \mathcal{O}). \tag{4.7}$$

Define

$$h_k(\mathfrak{n}p^\infty, \mathcal{O}) = \varprojlim_{\alpha} h_k(\mathfrak{n}p^\alpha, \mathcal{O}). \tag{4.8}$$

We define on $h_k(\mathfrak{n}p^\infty, \mathcal{O})$ an $\mathcal{A}$-algebra structure as follows. For $\alpha \geq 1$, the group homomorphisms

$$\begin{aligned} [\cdot]_k : Z_F(\mathfrak{n})/Z_{F,\alpha}(\mathfrak{n}) \cong \mathrm{Cl_F}(\mathfrak{n}p^\alpha) &\to h_k(\mathfrak{n}p^\alpha, \mathcal{O}) \\ \text{class of } \mathfrak{l} &\to [\mathfrak{l}]_k \end{aligned} \tag{4.9}$$

are compatible, hence extend to the inverse limit:

$$[\cdot]_{k,\infty} : Z_F(\mathfrak{n}) \to h_k(\mathfrak{n}p^\infty, \mathcal{O}). \tag{4.10}$$

Since $[\cdot]_{k,\infty}$ is a continuous character, it can be extended to the completed group algebra $\mathcal{A}_F$. In particular, by restriction to Λ_F, we obtained the structure of a Λ_F-algebra on $h_k(\mathfrak{n}p^\infty, \mathcal{O})$. For the reasons described below (see Theorem 4.1), we introduce a twist, and define the canonical Λ_F-algebra structure on $h_k(\mathfrak{n}p^\infty, \mathcal{O})$ by twisting this algebra structure with the character $P_k : \Lambda_F \to \mathcal{O}$, defined by the condition $P_k(\langle[\mathfrak{l}]\rangle) = \langle\mathfrak{l}\rangle_F^{k-2}$ for $\mathfrak{l}$ prime to $\mathfrak{n}p$.

We have the following result.

THEOREM 4.1 [Hid88, Theorem 3.2]. *For weights $k, k' \geq 2$, we have a canonical isomorphism of Λ_F-algebras:*

$$h_k(\mathfrak{n}p^\infty, \mathcal{O}) \cong h_{k'}(\mathfrak{n}p^\infty, \mathcal{O}) \tag{4.11}$$

sending $T(\mathfrak{l})$ to $T(\mathfrak{l})$.

Hence, we denote by $\mathbf{h}(\mathfrak{n}, \mathcal{O})$ the universal p-adic Hecke algebra of tame level $\mathfrak{n}$.

Dual to this is the description in terms of p-adic modular forms. Let $S_k(\mathfrak{n}p^\infty, \mathcal{O}) = \bigcup_\alpha S_k(\mathfrak{n}p^\alpha, \mathcal{O})$. For $|\cdot|_p$ the norm on $\overline{\mathbf{Q}}_p$ such that $|p|_p = p^{-1}$, define the norm

$$|\mathbf{f}|_p = \sup(|C(\mathfrak{a}, \mathbf{f})|_p) \tag{4.12}$$

on $S_k(\mathfrak{n}p^\infty, \mathcal{O})$, and let $\overline{S}_k(\mathfrak{n}p^\infty, \mathcal{O})$ be the completion with respect to $|\cdot|_p$. This is the space of 'p-adic modular (cusp) forms'. Here $\overline{S}_k(\mathfrak{n}p^\infty, \mathcal{O})$ becomes a module over $\mathcal{A}_F$, and the pairing $(\ ,\ )$ at finite level extends to give a perfect ($\mathcal{O}$-linear) duality:

$$(\ ,\ ) : \mathbf{h}(\mathfrak{n}, \mathcal{O}) \times \overline{S}_k(\mathfrak{n}p^\infty, \mathcal{O}) \to \mathcal{O}. \tag{4.13}$$

In particular, $\overline{S}_k(\mathfrak{n}p^\infty, \mathcal{O})$ does not depend on k, and we denote this as $\overline{S}(\mathfrak{n}, \mathcal{O})$.

To go further, we need to recall Hida's projection operator e to the p-ordinary part. Note that as $h_k(\mathfrak{n}p^\alpha, \mathcal{O})$ is a finite $\mathcal{O}$-algebra, it is semi-local. Let $h_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha, \mathcal{O})$ be the maximal direct summand over which $U(p\mathfrak{r})$ is invertible (equivalently, this is the same as requiring $U(\mathfrak{p})$ to be invertible for each $\mathfrak{p}$ above p). Let e be the corresponding projection. A formula for e can be given

$$e = \lim_{n\to\infty} U(p)^{n!}. \tag{4.14}$$

Dually, we put $S_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha, \mathcal{O}) = S_k(\mathfrak{n}p^\alpha, \mathcal{O})|e$, the space of p-ordinary cusp forms. It is true that, if $\mathbf{f} \in S_k(\mathfrak{n}p^\alpha, \overline{\mathbf{Q}})$, then $\mathbf{f}|e \in S_k(\mathfrak{n}p^\alpha, \overline{\mathbf{Q}})$.

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At this point, it is useful to make some remark about p-stabilization. Suppose that $\tilde{\mathbf{f}}$ is of weight k, and is a newform of conductor $\mathrm{cond}(\tilde{\mathbf{f}})$ divisible by $\mathfrak{n}$, with character ψ. Assume that $|C(\mathfrak{p},\tilde{\mathbf{f}})|_p=1$ for $\mathfrak{p}|p$. Then $\tilde{\mathbf{f}}$, regarded as an element of $S_k(\mathfrak{n}p^\alpha,\mathcal{O})$, is not necessarily an eigenvector for all of the $U(\mathfrak{p})$. However, it can be modified as follows: let $\alpha(\mathfrak{p},\tilde{\mathbf{f}})$, $\beta(\mathfrak{p},\tilde{\mathbf{f}})$ be the unit, respectively non-unit, roots of the equation

$$X^2-C(\mathfrak{p},\tilde{\mathbf{f}})X+\psi_{\mathrm{cond}(\tilde{\mathbf{f}})}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-1}=0.$$

Define the p-stabilization of $\tilde{\mathbf{f}}$ as

$$\tilde{\mathbf{f}}^0=\tilde{\mathbf{f}}\Big|\prod_{\mathfrak{p}|p}(1-\beta(\mathfrak{p},\tilde{\mathbf{f}})V(\mathfrak{p})). \tag{4.15}$$

It is clearly a p-ordinary eigenform in $S_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha,\mathcal{O})$, with $U(\mathfrak{p})$-eigenvalue $\alpha(\mathfrak{p},\tilde{\mathbf{f}})$. we call it the p-ordinary newform attached to $\tilde{\mathbf{f}}$.

Now going back to the formula (4.14) for e, it is clear that these projections are compatible under the surjection $h_k(\mathfrak{n}p^\beta,\mathcal{O})\to h_k(\mathfrak{n}p^\alpha,\mathcal{O})$, hence e extends to $\mathbf{h}(\mathfrak{n},\mathcal{O})$. We put

$$\mathbf{h}^{\mathrm{ord}}(\mathfrak{n},\mathcal{O})=e\mathbf{h}(\mathfrak{n},\mathcal{O})=\varprojlim_{\alpha} h_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha,\mathcal{O}) \tag{4.16}$$

the universal ordinary p-adic Hecke algebra of tame level $\mathfrak{n}$.

The fundamental result is as follows.

Theorem 4.2 [Hid06, Part (1) of Corollary 4.21]. *The algebra $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n},\mathcal{O})$ is a finite free algebra over Λ_F.*

To state Theorem 4.3 below (usually called the control theorem), we need to prepare some notation. Let ϵ be a finite-order character of $W_F(\mathfrak{n})$, factoring through $W_{F,\alpha}(\mathfrak{n})$. Assume that $\mathcal{O}$ contains the values of ϵ (this can be achieved by extension of scalars from $\mathcal{O}$ to $\mathcal{O}[\epsilon]$). Let

$$S_k(\mathfrak{n}p^\alpha,\epsilon,\mathcal{O})=\{\mathbf{f}\in S_k(\mathfrak{n}p^\alpha,\mathcal{O}),\mathbf{f}|[v]_k=\epsilon(v)\mathbf{f}\text{ for all }v\in W_F(\mathfrak{n})\}.$$

(In the case where $F=\mathbf{Q}$, these correspond to forms on the congruence subgroup $\Gamma_1(Np)\cap\Gamma_0(p^\alpha)$ with character ϵ.) These are called forms with wild character ϵ. Furthermore, if ϕ is a character of $Z_F(\mathfrak{n})^{\mathrm{tors}}$, and $\mathbf{f}\in S_k(\mathfrak{n}p^\alpha,\epsilon,\mathcal{O})$ satisfies $\mathbf{f}|[z]_k=\phi(z)\mathbf{f}$ for all $z\in Z_F(\mathfrak{n})^{\mathrm{tors}}$, then we call ϕ the tame character of $\mathbf{f}$. In particular, $\mathbf{f}$ has character $\epsilon\phi$ in the sense of (2.19). We denote this space as $S_k(\mathfrak{n}p^\alpha,\epsilon,\phi,\mathcal{O})$.

Put $S_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha,\epsilon,\mathcal{O})=S_k(\mathfrak{n}p^\alpha,\epsilon,\mathcal{O})|e$, and let $h_k(\mathfrak{n}p^\alpha,\epsilon,\mathcal{O})$ be the corresponding Hecke algebra, and put $h_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha,\epsilon,\mathcal{O})=h_k(\mathfrak{n}p^\alpha,\epsilon,\mathcal{O})$.

For integer $k\geq 2$, and epsilon as above, denote by $P_{k,\epsilon}\in\mathrm{Hom}_{\mathcal{O}\text{-alg}}(\Lambda_F,\overline{\mathbf{Q}}_p)=\mathrm{Spec}_{/\mathcal{O}}(\Lambda_F)(\overline{\mathbf{Q}}_p)$ the algebra homomorphism, defined by the condition $P_{k,\epsilon}(\langle[\mathfrak{l}]\rangle)=\langle\mathfrak{l}\rangle_F^{k-2}\epsilon(\mathfrak{l})$, for $\mathfrak{l}$ prime to $\mathfrak{n}p$. If ϵ is trivial, then this becomes the P_k introduced before. The $P_{k,\epsilon}$ are the 'classical' or 'algebraic' points. When it is not likely to cause confusion, we abuse notation and still denote the prime ideal defined by ker $P_{k,\epsilon}$ by $P_{k,\epsilon}$.

Theorem 4.3 [Hid06, Part (2) of Corollary 4.21]. *For $k\geq 2$, ϵ as above, we have an isomorphism of Λ_F-algebras*

$$\mathbf{h}^{\mathrm{ord}}(\mathfrak{n},\mathcal{O})/P_{k,\epsilon}\mathbf{h}^{\mathrm{ord}}(\mathfrak{n},\mathcal{O})\cong h_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha,\epsilon,\mathcal{O})$$

(we regard $\mathcal{O}$ as a Λ_F-algebra via $P_{k,\epsilon}$). In particular, $h_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha,\epsilon,\mathcal{O})$ is independent of the index α of W_α through which ϵ factors.

4.2 The basic results on ordinary deformations of eigenforms follow readily from Theorems 4.2 and 4.3. Fix an algebraic closure $\overline{Q}_{\Lambda_F}$ of Q_{Λ_F}, the fraction field of Λ_F. Consider a Λ_F-algebra homomorphism:

$$\lambda : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \to \overline{Q}_{\Lambda_F}.$$

By Theorem 4.2, the fraction field of the image of λ, is a finite extension of Q_{Λ_F}. Take $\mathcal{I}$ to be the integral closure of Λ_F in this extension, and $Q_{\mathcal{I}}$ to be the fraction field of $\mathcal{I}$. By Theorem 4.2 again, λ actually takes values in $\mathcal{I}$.

Define $\mathfrak{X}(\mathcal{I}) = \mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathcal{I}, \overline{\mathbf{Q}}_p)$, $\mathfrak{X}(\mathcal{I})_{\mathrm{alg}} = \{P \in \mathfrak{X}(\mathcal{I}), P|_{\Lambda_F} = P_{k,\epsilon}\}$. For each $P \in \mathfrak{X}(\mathcal{I})$, we can form the composition $\lambda_P = P \circ \lambda \in \mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}), \overline{\mathbf{Q}}_p)$, thus defining, by duality, a whole family of 'p-adic cuspidal eigenforms'.

Suppose that $P|_{\Lambda_F} = P_{k,\epsilon}$. Write k_P as k, and α_P as the minimum level α through which ϵ factors. Again assume that ϵ takes values in $\mathcal{O}$. By the control theorem, λ_P factors through $h_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha, \epsilon, \mathcal{O})$, thus defining a classical cuspidal eigenform. We have the following converse theorem.

Theorem 4.4. *Let $\mathbf{f} \in S_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha, \epsilon, \mathcal{O})$ be a p-ordinary cuspidal eigenform. Then there exists a $\lambda : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \to \mathcal{I}$, and $P \in \mathfrak{X}(\mathcal{I})_{\mathrm{alg}}$, with $P|_{\Lambda_F} = P_{k,\epsilon}$, such that λ_P corresponds to $\mathbf{f}$. Furthermore, if, the conductor of $\mathbf{f}$ is divisible by $\mathfrak{n}$ (i.e. $\mathbf{f}$ is a p-ordinary newform), then the localization $\mathcal{I}_P$ is étale over $\Lambda_{F,P_{k,\epsilon}}$.*

Proof. This follows easily from [Hid06, Corollary 4.21], so we just make some brief comments. The eigenform $\mathbf{f}$ correspond to a minimal prime of $h_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha, \epsilon, \mathcal{O})$, which by the control Theorem 4.3, corresponds to a height-one prime P' of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})$ lying over the prime $P_{k,\epsilon}$ of Λ_F. By the flatness of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})$ over Λ_F (Theorem 4.2) and the going down theorem, we can find a minimal prime Q of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})$ contained in P', with $Q \cap \Lambda_F = (0)$. The quotient $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})/Q$ is finite over Λ_F. Let $\mathcal{I}$ be the integral closure of Λ_F in the fraction field of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})/Q$. Then $\mathcal{I}$ is finite over Λ_F. Pick any prime P of $\mathcal{I}$ lying over the prime P' of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})/Q$ (in particular, P lies over the prime $P_{k,\epsilon}$ of Λ_F). The natural map

$$\lambda : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \to \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})/Q \to \mathcal{I}$$

satisfies the requirement that $\lambda_P = P \circ \lambda$ corresponds to the eigenform $\mathbf{f}$.

Finally, from [Hid06, part (3) of Corollary 4.21], the localization $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})_{P_{k,\epsilon}}$ is étale over $\Lambda_{F,P_{k,\epsilon}}$. In particular, with the above notation, the localization $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})_{P'}$ is étale over $\Lambda_{F,P_{k,\epsilon}}$, hence is a regular local ring. It follows that we have the identification $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})_{P'} = (\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})/Q)_{P'} = \mathcal{I}_P$. Hence, $\mathcal{I}_P$ is also étale over $\Lambda_{F,P_{k,\epsilon}}$. □

Remark. Corollary 4.21 of [Hid06] was proved by Hida under the assumption that p does not divide $6D_F$ (in particular $p \geq 5$). It is for this reason that our results depend on this condition.

Hida theory can also be stated more formally in terms of Λ-adic forms. As in [Hid93], we do this in the more general context. For $Q_{\mathcal{I}}$ a finite extension of Q_{Λ_F}, $\mathcal{I}$ the integral closure of Λ_F in $Q_{\mathcal{I}}$ as before, we make the following definition.

Definition 4.5. An $\mathcal{I}$-adic modular form $\mathcal{F}$, of tame level $\mathfrak{n}$, is a set of elements of $\mathcal{I}$ given by the data

$$C(\mathfrak{a}, \mathcal{F}) \quad \text{for } \mathfrak{a} \subset \mathfrak{r}$$
$$C_0(\mathfrak{a}, \mathcal{F})$$

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with $C_0(\mathfrak{a}, \mathcal{F})$ (the 'constant term') depends only on the image of $\mathfrak{a}$ in Cl_{F}, such that for a subset of $P \in \mathfrak{X}(\mathcal{I})_{\mathrm{alg}}$ Zariski dense in $\mathfrak{X}(\mathcal{I})$, there is an element $\mathbf{f}_P \in M_k(\mathfrak{n}p^{\alpha}, \epsilon, \mathcal{O}[\epsilon])$ satisfying

$$C(\mathfrak{a}, \mathbf{f}_P) = P(C(\mathfrak{a}, \mathcal{F})),$$
$$C_0(\mathfrak{a}, \mathbf{f}_P) = P(C_0(\mathfrak{a}, \mathcal{F})).$$

These form an $\mathcal{I}$-module denoted by $\mathcal{M}(\mathfrak{n}, \mathcal{I})$. We let $\mathcal{S}(\mathfrak{n}, \mathcal{I}) \subset \mathcal{M}(\mathfrak{n}, \mathcal{I})$ to be the submodule consisting of those $\mathcal{F}$ such that $P(\mathcal{F})$ is a cusp form for a subset of $P \in \mathfrak{X}(\mathcal{I})_{\mathrm{alg}}$ Zariski dense in $\mathfrak{X}(\mathcal{I})$.

Let ψ be a character of $Z_F(\mathfrak{n})^{\mathrm{tors}}$. We say that an $\mathcal{I}$-adic form $\mathcal{F}$ has character ψ, if for $P_{k,\epsilon} \in \mathfrak{X}(\mathcal{I})_{\mathrm{alg}}$, the specialization at $P_{k,\epsilon}$ has character given by $\epsilon\psi\omega_F^{2-k}$, whenever the specialization is defined. We denote by $\mathcal{M}(\mathfrak{n}, \psi, \mathcal{I}), \mathcal{S}(\mathfrak{n}, \psi, \mathcal{I})$ the module of $\mathcal{I}$-adic forms (respectively, cusp forms) with character ψ.

Again one can define Hecke operators and the ordinary projection operator e on the $\mathcal{I}$-module $\mathcal{M}(\mathfrak{n}, \mathcal{I})$ stable on $\mathcal{S}(\mathfrak{n}, \mathcal{I})$, and compatible with specialization. Put $\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{I}) = \mathcal{M}(\mathfrak{n}, \mathcal{I})|e, \mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{I}) = \mathcal{S}(\mathfrak{n}, \mathcal{I})|e$. Again, $U(p\mathfrak{r})$ is invertible on the ordinary part. We also have the following duality:

$$\mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{I}) \cong \mathrm{Hom}_{\Lambda_F\text{-mod}}(\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}), \mathcal{I}) \tag{4.17}$$

and the ordinary $\mathcal{I}$-adic normalized eigenforms correspond to Λ_F-algebra homomorphism.

Remark 4.6. By the control theorem, if $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{I})$, then in fact $P(\mathcal{I})$ is a classical cusp form for all $P \in \mathfrak{X}(\mathcal{I})_{\mathrm{alg}}$.

4.3 Thus, we fix a $\lambda : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \to \mathcal{I} \hookrightarrow \overline{Q}_{\Lambda_F}$. Assume that $\overline{\mathbf{Q}}_p \cap \mathcal{I} = \mathcal{O}$ (a condition which can be achieved by extending $\mathcal{O}$). Denote by $\psi : Z_F(\mathfrak{n})^{\mathrm{tors}} \to \overline{\mathbf{Q}}_p$ the composition $Z_F(\mathfrak{n})^{\mathrm{tors}} \to \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \xrightarrow{\lambda} \overline{Q}_{\Lambda_F}$. We call ψ the tame character of λ. This is consistent with Definition 4.5: for $P \in \mathfrak{X}(\mathcal{I})_{\mathrm{alg}}$, the eigenform corresponding to λ_P has tame character given by $\psi\omega_F^{2-k_P}$ (the character is given $\epsilon\psi\omega_F^{2-k_P}$).

Analogous with the theory of newforms, we say that λ is primitive if there does not exist a $\lambda' : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}', \mathcal{O}) \to \overline{Q}_{\Lambda_F}$, with $\mathfrak{n}'$ strictly divides $\mathfrak{n}$, such that $\lambda(T(\mathfrak{q})) = \lambda'(T(\mathfrak{q}))$ for all primes $\mathfrak{q}$ not dividing $\mathfrak{n}p$. If λ arises from a newform $\mathbf{f}$ of conductor divisible by $\mathfrak{n}$, as in Theorem 4.4, then clearly λ must be primitive. Conversely, the primitivity of λ implies that [Hid88] the eigenforms corresponding to λ_P, for classical P, has conductor divisible by $\mathfrak{n}$.

We recall the definition of congruence and differential modules.

To motivate, let $\mathbf{f} \in S_k^{\mathrm{ord}}(\mathfrak{n}p^{\infty}, \epsilon, \mathcal{O})$ be a p-ordinary newform of tame level $\mathfrak{n}$. As a consequence of Atkin–Lehner's theory of newforms, it gives a decomposition:

$$h_k^{\mathrm{ord}}(\mathfrak{n}p^{\infty}, \epsilon, Q_{\mathcal{O}}[\epsilon]) = Q_{\mathcal{O}}[\epsilon] \oplus B, \tag{4.18}$$

where projection to $Q_{\mathcal{O}}[\epsilon_0]$ corresponds to the eigenform $\mathbf{f}$. We have an analogue for Hida families.

Theorem 4.7 [Hid88, Corollary 3.7]. *Assume that λ is primitive. Then we have a decomposition*

$$\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \otimes_{\Lambda_F} Q_{\mathcal{I}} = Q_{\mathcal{I}} \oplus \mathcal{B} \tag{4.19}$$

as an algebra direct sum, such that the projection onto the first factor coincides with λ.

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Let $\mathrm{pr}_{\mathcal{B}}$ be the projector to $\mathcal{B}$ in the above decomposition and denote by $\hat{\lambda}: \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \otimes \mathcal{I} \to \mathcal{I}$ the composition of the map $\lambda \otimes 1: \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \otimes \mathcal{I} \to \mathcal{I} \otimes \mathcal{I}$ and the multiplication map $\mathcal{I} \otimes \mathcal{I} \to \mathcal{I}$. Define

$$\delta : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \otimes \mathcal{I} \to \mathcal{I} \oplus \mathrm{pr}_{\mathcal{B}}(\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \otimes \mathcal{I})$$

to be the diagonal map (which is injective by Theorem 4.7). We are interested in how far δ fails to be surjective.

Definition 4.8 [Hid88]. The congruence module is defined by

$$\mathcal{C}_0(\lambda) = \mathrm{Coker}(\delta).$$

The differential module is defined by

$$\mathcal{C}_1(\lambda) = \Omega^1_{\mathbf{h}/\mathcal{I}} \otimes_{\mathbf{h}} \mathcal{I};$$

here we abbreviate $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})$ as $\mathbf{h}$, and we regard $\mathcal{I}$ as an $\mathbf{h}$-module via $\hat{\lambda}$; Ω^1 the module of Kähler differentials.

As the name suggests, $\mathcal{C}_0(\lambda)$ measures congruences between the components of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})$ given by λ and the others. In fact, $P \in \mathfrak{X}(\mathcal{I})$ is the support of the module if and only if there exists two different λ_1, λ_2, with $\lambda_{1,P} = \lambda_{2,P}$. On the other hand, the differential module measures how much the component of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})$ containing $\lambda : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \to \mathcal{I}$ fails to be étale over Λ_F. Thus, these two modules are closely related.

Proposition 4.9 [Hid88, Corollary 3.8]. *The congruence module $\mathcal{C}_0(\lambda)$ and differential module $\mathcal{C}_1(\lambda)$ are torsion $\mathcal{I}$-modules, and we have the equality of their support:*

$$\mathrm{Supp}_{\mathcal{I}}(\mathcal{C}_0(\lambda)) = \mathrm{Supp}_{\mathcal{I}}(\mathcal{C}_1(\lambda)).$$

For $P \in \mathfrak{X}(\mathcal{I})_{\mathrm{alg}}$, we have $(\Omega^1_{\mathbf{h}/\mathcal{I}} \otimes_{\mathbf{h}} \mathcal{I})_P = 0$ by the second part of Theorem 4.4. Thus, we draw the following corollary.

Corollary 4.10. *Let $P \in \mathfrak{X}(\mathcal{I})_{\mathrm{alg}}$. Then under the decomposition $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \otimes Q_{\mathcal{I}} = Q_{\mathcal{I}} \oplus \mathcal{B}$, the idempotent $\mathrm{Id}_{\mathcal{I}} \oplus 0$ lies in $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \otimes \mathcal{I}_P$.*

5. Λ_F-adic Eisenstein measure

5.1 The Eisenstein series $\mathbf{E}_k(\zeta_1, \zeta_2)$ introduced in § 3 can be interpolated to give Λ_F-adic forms.

Recall from § 3, that $\mathbf{E}_k(\zeta_1, \zeta_2)$ is a normalized eigenform. For $\mathfrak{p}|p$, it has $U(\mathfrak{p})$-eigenvalue given by $C(\mathfrak{p}, \mathbf{E}_k(\zeta_1, \zeta_2)) = \zeta_1(\mathfrak{p}) + \zeta_2(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-1}$.

Analogous to the procedure of p-stabilization (cf. (4.15)), we put

$$\mathbf{E}_k(\zeta_1, \zeta_2)^0 = \prod_{\mathfrak{p}|p} \mathbf{E}_k(\zeta_1, \zeta_2)|(1 - \zeta_2(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-1}V(\mathfrak{p})), \tag{5.1}$$

$$\mathbf{E}_k(\zeta_1, \zeta_2)^{00} = \prod_{\mathfrak{p}|p} \mathbf{E}_k(\zeta_1, \zeta_2)^0|(1 - \zeta_1(\mathfrak{p})V(\mathfrak{p})). \tag{5.2}$$

From (3.15) we have

$$C(\mathfrak{m}, \mathbf{E}_k(\zeta_1,\zeta_2)^0) = \sum_{\substack{\mathfrak{a}\mathfrak{b}=\mathfrak{m}\\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{b}+p\mathfrak{r}=\mathfrak{r}}} \zeta_1(\mathfrak{a})\zeta_2(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{k-1}$$

$$C_0(\mathfrak{m}, \mathbf{E}_k(\zeta_1,\zeta_2)^0) = \begin{cases} 2^{-d}L^{(\mathfrak{c}_{\zeta_2}p)}(1-k,\zeta_2) & \text{if } \zeta_1 \text{ is trivial}\\ 0 & \text{otherwise.}\end{cases} \tag{5.3}$$

$$C(\mathfrak{m}, \mathbf{E}_k(\zeta_1,\zeta_2)^{00}) = \sum_{\substack{\mathfrak{a}\mathfrak{b}=\mathfrak{m}\\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{a}\mathfrak{b}+p\mathfrak{r}=\mathfrak{r}}} \zeta_1(\mathfrak{a})\zeta_2(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{k-1}$$

$$C_0(\mathfrak{m}, \mathbf{E}_k(\zeta_1,\zeta_2)^{00}) = 0. \tag{5.4}$$

Both $\mathbf{E}_k(\zeta_1,\zeta_2)^0$ and $\mathbf{E}_k(\zeta_1,\zeta_2)^0$ are eigenforms. For $\mathbf{E}_k(\zeta_1,\zeta_2)^0$, it has $U(\mathfrak{p})$-eigenvalue given by $\zeta_1(\mathfrak{p})$, which is either a p-adic unit or zero, depending on whether $\mathfrak{p}$ is prime to $\mathfrak{c}_{\zeta_1}$. It follows that

$$\mathbf{E}_k(\zeta_1,\zeta_2)^0|e = \begin{cases} \mathbf{E}_k(\zeta_1,\zeta_2)^0 & \text{if } \mathfrak{c}_{\zeta_1} \text{is prime to } p\\ 0 & \text{otherwise.}\end{cases} \tag{5.5}$$

Remark. From (5.4), we see that the $U(\mathfrak{p})$ eigenvalues of $\mathbf{E}_k(\zeta_1,\zeta_2)^{00}$ are all zero. In particular, $\mathbf{E}_k(\zeta_1,\zeta_2)^{00}$ is not ordinary.

Define $\mathrm{Eis}_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha,\mathcal{O}) = \mathrm{Eis}_k(\mathfrak{n}p^\alpha,\mathcal{O})|e$, with Eis_k as in Proposition 3.8; we assume that $\mathcal{O}$ is a large enough to contain all of the Fourier coefficients of the Eisenstein series that appears. Then $\mathrm{Eis}_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha)$ is spanned by the forms

$$\mathbf{E}_k(\zeta_1,\zeta_2)^0|V(\mathfrak{m}), \tag{5.6}$$

with ζ_1,ζ_2 characters F of conductor dividing $\mathfrak{n}p^\alpha$, $\mathfrak{m}$ integral ideal, subject to the condition $\mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2}^{(p)}\mathfrak{m}|\mathfrak{n}$ (here $\mathfrak{c}_{\zeta_2}^{(p)}$ denotes the prime to p part of the ideal $\mathfrak{c}_{\zeta_2}$).

Proposition 5.1. *Let ζ_1,ζ_2 be characters of $Z_F(\mathfrak{n})^{\mathrm{tors}}$, such that $\mathfrak{c}_{\zeta_1}^{(p)}\mathfrak{c}_{\zeta_2}^{(p)}|\mathfrak{n}$, and that $\zeta_1\zeta_2|_{F_\infty} = \mathrm{Id}$. Then there exists an element $\mathcal{E}(\zeta_1,\zeta_2)^0 \in \mathcal{M}(\mathfrak{n},\Lambda_F)\otimes Q_{\Lambda_F}$, such that for any $P_{k,\epsilon}\in\mathfrak{X}(\Lambda_F)_{\mathrm{alg}}$, we have*

$$P_{k,\epsilon}(\mathcal{E}(\zeta_1,\zeta_2)^0) = \mathbf{E}_k(\zeta_1,\epsilon\zeta_2\omega_F^{2-k})^0.$$

Proof. A special case of this was stated in [Wil88, Proposition 1.3.1]. The general case is similar. We define, for an integral ideal $\mathfrak{m}$,

$$C(\mathfrak{m},\mathcal{E}(\zeta_1,\zeta_2)^0) = \sum_{\substack{\mathfrak{a}\mathfrak{b}=\mathfrak{m}\\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{b}+p\mathfrak{r}=\mathfrak{r}}} \zeta_1(\mathfrak{a})\zeta_2(\mathfrak{b})\mathcal{N}(\mathfrak{b})\langle[\mathfrak{b}]\rangle \in \Lambda_F. \tag{5.7}$$

(Recall that the elements $\langle[\mathfrak{b}]\rangle\in\Lambda_F$ were defined in the paragraph following (4.4).)

One checks immediately that

$$P_{k,\epsilon}(C(\mathfrak{m},\mathcal{E}(\zeta_1,\zeta_2)^0)) = C(\mathfrak{m},\mathbf{E}_k(\zeta_1,\epsilon\zeta_2\omega_F^{2-k})^0).$$

One the other hand, by the main theorem of Deligne–Ribet [DR80] on p-adic L-functions of Hecke characters over totally real fields, there exists an element $F_{\zeta_2}\in Q_{\Lambda_F}$ such that

$$P_{k,\epsilon}(F_{\zeta_2}) = L^{(\mathfrak{c}p)}(1-k,\epsilon\zeta_2\omega_F^{2-k}),\quad \mathfrak{c} = \text{conductor of } \epsilon\zeta_2\omega_F^{2-k}.$$

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Thus, it suffices to define

$$C_0(\mathfrak{m}, \mathcal{E}(\zeta_1, \zeta_2)^0) = \begin{cases} 2^{-d} F_{\zeta_2} & \text{if } \zeta_1 \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

□

In general the constant term of $\mathcal{E}(\zeta_1, \zeta_2)^0$, i.e. the p-adic L-function of Deligne–Ribet, may not lie in Λ_F. Thus, $\mathcal{E}(\zeta_1, \zeta_2)^0$ is only an element of $\mathcal{M}(\mathfrak{n}, \Lambda_F) \otimes Q_{\Lambda_F}$.

Put $\mathcal{E}(\zeta_1, \zeta_2)^{00} = \prod_{\mathfrak{p}|p} \mathcal{E}(\zeta_1, \zeta_2)^0 | (1 - \zeta_1(\mathfrak{p}) V(\mathfrak{p}))$. Then we have

$$\begin{aligned} C(\mathfrak{m}, \mathcal{E}(\zeta_1, \zeta_2)^{00}) &= \sum_{\substack{\mathfrak{a}\mathfrak{b} = \mathfrak{m} \\ \mathfrak{a}, \mathfrak{b} \subset \mathfrak{r}, \mathfrak{a}\mathfrak{b} + p\mathfrak{r} = \mathfrak{r}}} \zeta_1(\mathfrak{a}) \zeta_2(\mathfrak{b}) \mathcal{N}(\mathfrak{b}) \langle [\mathfrak{b}] \rangle \\ C_0(\mathfrak{m}, \mathcal{E}(\zeta_1, \zeta_2)^{00}) &= 0. \end{aligned} \tag{5.9}$$

Thus, $\mathcal{E}(\zeta_1, \zeta_2)^{00} \in \mathcal{M}(\mathfrak{n}, \Lambda_F)$. Again $\mathcal{E}(\zeta_1, \zeta_2)^{00}$ is not ordinary.

We have the following Λ_F-adic analogue of Proposition 3.8.

PROPOSITION 5.2. *Fix a character ψ of $Z_F(\mathfrak{n})^{\mathrm{tors}}$. Then $\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F)$ is a finitely generated Λ_F-modules. Define $\mathcal{E}is^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F)$ to be the Λ_F-submodule of $\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F) \otimes Q_{\Lambda_F}$ generated by the forms*

$$\mathcal{E}(\zeta_1, \zeta_2)^0 | V(\mathfrak{m}), \quad \psi = \zeta_1 \zeta_2, \quad \mathfrak{c}_{\zeta_1} \mathfrak{c}_{\zeta_2}^{(p)} \mathfrak{m} | \mathfrak{n}.$$

Then $\mathcal{E}is^{\mathrm{ord}}(\mathfrak{n}, \psi)$ is stable under the Hecke operators.

Assume that $\mathcal{O}$ is a large enough finite extension of $\mathbf{Z}_p$, for example, containing the roots of unity of order equal to the cardinality of $Z_F(\mathfrak{n})^{\mathrm{tors}}$. Then we have the following decomposition of Hecke modules

$$\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F) \otimes Q_{\Lambda_F} = \mathcal{E}is^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F) \otimes Q_{\Lambda_F} \oplus \mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F) \otimes Q_{\Lambda_F}. \tag{5.10}$$

Furthermore, if $P \in \mathfrak{X}(\Lambda_F)_{\mathrm{alg}}$, then such decomposition is valid with Q_{Λ_F} replaced by $\Lambda_{F,P}$, the localization of Λ_F at P. We denote by $\mathrm{pr}_{\mathcal{E}is}, \mathrm{pr}_{\mathcal{S}}$ the corresponding projection operators on $\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi)$.

Proof. The decomposition (5.10) follows from the decomposition (3.20). See for example the discussion in [Wil88, § 1.4, pp. 545–546]. □

Similar to Definition 4.8, we make the following definition.

DEFINITION 5.3. The congruence module $\mathcal{C}(\mathcal{E}is)$ as follows

$$\mathcal{C}(\mathcal{E}is) = \frac{\mathrm{pr}_{\mathcal{E}is}(\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F)) \oplus \mathrm{pr}_{\mathcal{S}}(\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F))}{\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F)} \tag{5.11}$$

which by Proposition 5.2 is a finitely generated torsion Λ_F-module.

From Proposition 5.2, we draw the following corollary.

COROLLARY 5.4. *For $P \in \mathfrak{X}(\Lambda_F)$, denote by $\Lambda_{F,P}$ the localization of Λ_F at P, then the decomposition*

$$\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F) \otimes \Lambda_{F,P} = \mathcal{E}is^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F) \otimes \Lambda_{F,P} \oplus \mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F) \otimes \Lambda_{F,P}$$

holds if and only if P does not lie in the support of $\mathcal{C}(\mathcal{E}is)$. This holds, in particular, if $P \in \mathfrak{X}(\Lambda_F)_{\mathrm{alg}}$.

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5.2 In this section we define the Eisenstein measure on $\overline{Z}_F(\mathfrak{r})$.

We introduce a norm $\|\cdot\|$ on Λ_F-adic forms as follows: Λ_F is complete under the norm [Hid89]

$$\|w\| = \sup_{P \in \mathfrak{X}(\Lambda_F)_{\mathrm{alg}}} |P(w)|_p \quad \text{for } w \in \Lambda_F. \tag{5.12}$$

We extend this norm to the finite Λ_F-module $\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F)$:

$$\|\mathcal{F}\| = \sup(\|C_\lambda(0, \mathcal{F})\|, \|C(\mathfrak{a}, \mathcal{F})\|), \tag{5.13}$$

then $\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \psi, \Lambda_F)$ is complete under $\|\cdot\|$. Furthermore, the operators $U(\mathfrak{p})$, e are bounded operators.

Fix a character ψ of $\overline{Z}_F(\mathfrak{n})^{\mathrm{tors}}$. We now state the main results of this section.

Proposition 5.5. *Let θ, ϕ be Hecke characters of F of finite order, unramified at infinity, and assume that $\mathcal{O}$ contains the values of θ, ϕ. Let $r \geq 0$ be an integer. Then for any character χ of $\overline{Z}_F(\mathfrak{r})$ of finite order, there exists Λ_F-adic forms $\mathcal{H}(\chi, \phi, \theta, r)^0, \mathcal{H}(\chi, \phi, \theta, r)^{00} \in \mathcal{M}^{\mathrm{ord}}(\mathrm{lcm}(\mathfrak{n}, \mathfrak{c}_\phi^{(p)}\mathfrak{c}_\theta^{(p)}), \psi, \Lambda_F) \hat{\otimes}_{\mathcal{O}} \mathcal{O}[\chi]$, such that for any $P_{k,\epsilon}$ with $k \geq r+2$, we have*

$$\begin{aligned} P(\mathcal{H}(\chi, \phi, \theta, r)^0) &= \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^0 \mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|e \\ P(\mathcal{H}(\chi, \phi, \theta, r)^{00}) &= \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00} \mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|e. \end{aligned} \tag{5.14}$$

Proposition 5.6. *Notations as in Proposition 5.5. There exist distributions $\mu_{\phi,\theta,r}$ on $\overline{Z}(\mathfrak{r})$, with values in the Λ_F-module $\mathcal{M}^{\mathrm{ord}}(\mathrm{lcm}(\mathfrak{n}, \mathfrak{c}_\phi^{(p)}\mathfrak{c}_\theta^{(p)}), \psi, \Lambda_F)$, such that, for χ a finite order character of $\overline{Z}_F(\mathfrak{r})$,*

$$\int_{\overline{Z}_F(\mathfrak{r})} \chi \, d\mu_{\phi,\theta,r} = \mathcal{H}(\chi, \phi, \theta, r)^{00}. \tag{5.15}$$

This distribution is bounded with respect to the norm (5.13), i.e., a measure.

Furthermore, we have the integration identity:

$$\int_{\overline{Z}_F(\mathfrak{r})} \chi \langle . \rangle_F^r \, d\mu_{\phi,\theta,0} = (-1)^{rd} \int_{\overline{Z}_F(\mathfrak{r})} \chi \, d\mu_{\phi,\theta,r}. \tag{5.16}$$

When $r = 0$, we write $\mu_{\phi,\theta,0}$ simply as $\mu_{\phi,\theta}$.

The proof of these two propositions is based on computation with Fourier expansion of Eisenstein series. We defer it to Appendix Appendix A.

As an immediate consequence, we obtain the following corollary, already proved in [Dab94],

Corollary 5.7. *Let $P = P_{k,\epsilon} \in \mathfrak{X}(\Lambda_F)_{\mathrm{alg}}$, with $k \geq r+2$. Then the linear form $\mathbf{m}_{P,\phi,\theta,r} := P \circ \mu_{\phi,\theta,r}$ defines a measure on $\overline{Z}(\mathfrak{r})$, taking values in the $\mathcal{O}$-module $M_k^{\mathrm{ord}}(\mathrm{lcm}(\mathfrak{n}, \mathfrak{c}_\phi^{(p)}\mathfrak{c}_\theta^{(p)})p^\infty, \epsilon\psi\omega_F^{2-k}, \mathcal{O}[\epsilon])$.*

For χ a finite-order character of $\overline{Z}_F(\mathfrak{r})$:

$$\int_{\overline{Z}_F(\mathfrak{r})} \chi \, d\mathbf{m}_{P,\phi,\theta,r} = \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00} \mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|e \tag{5.17}$$

and we have the identity

$$\int_{\overline{Z}_F(\mathfrak{r})} \chi \langle \cdot \rangle_F^r \, d\mathbf{m}_{P,\phi,\theta,0} = (-1)^{rd} \int_{\overline{Z}_F(\mathfrak{r})} \chi \, d\mathbf{m}_{P,\phi,\theta,r}. \tag{5.18}$$

When $r = 0$, we write $\mathbf{m}_{P,\phi,\theta,0}$ as $\mathbf{m}_{P,\phi,\theta}$.

Remark 5.8. The expression (5.14) may seem unnatural. However, it occurs in the p-adic analogue of the Rankin–Selberg–Shimura method.

6. p-adic-L-functions

6.1 In this section, we use the Eisenstein measure constructed in § 5 to define the p-adic L-functions attached to Hilbert newforms.

In the definition of the Eisenstein measures, there is a choice for the auxiliary characters θ, ϕ. To simplify subsequent calculations, we make the following assumption from now on.

ASSUMPTION 6.1. *We assume that $\mathfrak{n}$ divides the conductor of θ, and $\mathfrak{c}_\phi$ is prime to $\mathfrak{c}_\theta p$.*

Let $\mathbf{f} \in S_{k_0}^{\mathrm{ord}}(\mathfrak{n}p^\alpha, \epsilon_0\psi\omega_F^{2-k_0}, \mathcal{O}[\epsilon_0])$ be a p-ordinary cuspidal newform. In using the convolution method of Rankin–Selberg–Shimura to construct p-adic L-functions, it is important to choose θ to be *adapted to* $\mathbf{f}$ in the following sense.

DEFINITION 6.2. We say that θ is *adapted to* $\mathbf{f}$, if the following conditions are satisfied.

(1) The conductor of $\epsilon_0\theta\omega_F^{1-k_0}$ is divisible by $\mathfrak{m}_0 = \prod_{\mathfrak{p}|p} \mathfrak{p}$.

(2) We have $L(k_0 - 1, (\epsilon_0\theta\omega_F^{1-k_0})^{-1}, \mathbf{f}) \neq 0$.

Condition (2) of Definition 6.2 is about non-vanishing of twisted L-values, which in general is a difficult problem. For our purpose, we need two results in this direction. The first is a non-vanishing result due to Shimura.

THEOREM 6.3 [Shi78]. *For $\Re(s) \geq (k_0 + 1)/2$, we have $L(s, \eta, \mathbf{f}) \neq 0$ for any Hecke character η of finite order.*

In particular, for $k_0 \geq 3$, $L(k_0 - 1, (\epsilon_0\theta\omega_F^{1-k_0})^{-1}, \mathbf{f}) \neq 0$ for any θ of finite order, and we can easily choose θ to be adapted to $\mathbf{f}$.

For $k_0 = 2$, we are at the centre of the functional equation, and we cannot expect to have such a strong non-vanishing result. In this case, we appeal to a theorem of Rohrlich.

THEOREM 6.4 [Roh89]. *Let S be a finite set of places of F. For any $s_0 \in \mathbf{C}$, there exist infinitely many Hecke characters η of finite order, unramified at the places in S, such that $L(s_0, \eta, \mathbf{f}) \neq 0$.*

In our setting, apply Rohrlich's theorem to the form $\mathbf{f} \otimes (\epsilon_0\theta'\omega_F^{1-k_0})^{-1}$, for some finite-order character θ', unramified at ∞, that satisfies $\mathfrak{n}|\mathfrak{c}_{\theta'}$, $\mathfrak{m}_0|\mathfrak{c}_{\epsilon_0\theta'\omega_F^{1-k_0}}$. Take $s_0 = 1$, S to be the set of places above $\mathfrak{n}, p$ and ∞. The theorem gives a η, unramified at $\mathfrak{n}, p$ and ∞, such that $L(1, \eta, \mathbf{f} \otimes (\epsilon_0\theta'\omega_F^{1-k_0})^{-1}) = L(1, (\epsilon_0\theta'\eta^{-1}\omega_F^{1-k_0})^{-1}, \mathbf{f}) \neq 0$. The character $\theta'\eta^{-1}$ is then adapted to $\mathbf{f}$.

6.2 By Theorem 4.4, we can lift $\mathbf{f}$ to a Hida family, i.e. a primitive algebra homomorphism $\lambda : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \to \mathcal{I}$, such that the specialization λ_{P_0} for some P_0 lying above P_{k_0,ϵ_0} gives back $\mathbf{f}$. Write $\mathcal{F} = \mathcal{F}_\lambda$ for the corresponding $\mathcal{I}$-adic form. To define the p-adic L-functions, we need to introduce certain projectors associated to $\mathbf{f}$, $\mathcal{F}$.

We first define Hida's ordinary cuspidal projectors, $l_\mathbf{f}, l_\mathcal{F}$, associated to $\mathbf{f}$, $\mathcal{F}$, respectively.

As in (4.18), the p-ordinary newform $\mathbf{f}$ gives a decomposition

$$h_{k_0}^{\mathrm{ord}}(\mathfrak{n}p^\infty, \epsilon_0, Q_\mathcal{O}[\epsilon_0]) = Q_\mathcal{O}[\epsilon_0] \oplus B, \tag{6.1}$$

where projection to $Q_{\mathcal{O}}[\epsilon_0]$ corresponds to the eigenform $\mathbf{f}$. Let $t_{\mathbf{f}}$ be the element of $h_{k_0}^{\mathrm{ord}}(\mathfrak{n}p^\infty, \epsilon_0, Q_{\mathcal{O}}[\epsilon_0])$ that corresponds to the element $(\mathrm{Id}_{\mathcal{O}}, 0)$ in the decomposition. Then the form $l_{\mathbf{f}}$ is defined via the perfect pairing $(\cdot,\cdot)$ between $h_{k_0}^{\mathrm{ord}}(\mathfrak{n}p^\infty, \epsilon_0, \mathcal{O}[\epsilon_0])$ and $S_{k_0}^{\mathrm{ord}}(\mathfrak{n}p^\infty, \epsilon_0, \mathcal{O}[\epsilon_0])$:

$$l_{\mathbf{f}}(\mathbf{g}) = (t_{\mathbf{f}}, \mathbf{g}) \in Q_{\mathcal{O}}[\epsilon_0] \quad \text{for } \mathbf{g} \in S_{k_0}^{\mathrm{ord}}(\mathfrak{n}p^\infty, \epsilon_0, Q_{\mathcal{O}}[\epsilon_0]), \tag{6.2}$$

see [Hid91, (9.3 b)]. Furthermore, Shimura's theory [Shi78] implies that

$$l_{\mathbf{f}}(\mathbf{g}) \in \overline{\mathbf{Q}} \quad \text{if } \mathbf{g} \in S_{k_0}^{\mathrm{ord}}(\mathfrak{n}p^\infty, \overline{\mathbf{Q}}).$$

In a similar way, by (4.19), we have the decomposition

$$\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \otimes_{\Lambda_F} Q_{\mathcal{I}} = Q_{\mathcal{I}} \oplus \mathcal{B}. \tag{6.3}$$

Let $t_{\mathcal{F}}$ be the element of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \otimes Q_{\mathcal{I}}$ that corresponds to the element $(1_{\mathcal{I}}, 0)$ in the decomposition. Then the form $l_{\mathcal{F}}$ is again defined via the perfect pairing (,) between $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})$ and $\mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{I})$ (cf. (4.17)):

$$l_{\mathcal{F}}(\mathcal{G}) = (t_{\mathcal{F}}, \mathcal{G}) \in Q_{\mathcal{I}} \quad \text{for } \mathcal{G} \in \mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{I}) \otimes Q_{\mathcal{I}}. \tag{6.4}$$

By Corollary 4.10, we have $l_{\mathcal{F}}(\mathcal{G}) \in \mathcal{I}_{P_0}$ if $\mathcal{G} \in \mathcal{S}(\mathfrak{n}, \mathcal{I}) \otimes \mathcal{I}_{P_0}$, and we have the consistency: $l_{\mathbf{f}}(P_0(\mathcal{G})) = P_0(l_{\mathcal{F}}(\mathcal{G}))$.

6.3 We may extend the definition of Hida's projectors to forms not necessarily cuspidal, and not necessarily of tame level $\mathfrak{n}$. This can be done by applying the cuspidal projectors of Proposition 3.8 and 5.2, followed by the trace operator.

In general, for any integral ideals $\mathfrak{n}$, $\mathfrak{a}$, we have the trace operator

$$\mathrm{Tr}_{\mathfrak{n}}^{\mathfrak{n}\mathfrak{a}} : S_k(\mathfrak{n}\mathfrak{a}, \zeta) \to S_k(\mathfrak{n}, \zeta)$$

defined by

$$\mathbf{f} | \mathrm{Tr}_{\mathfrak{n}}^{\mathfrak{n}\mathfrak{a}}(g) = \sum_{w \in K_{1,\mathfrak{n}}/K_{1,\mathfrak{n}\mathfrak{a}}} \mathbf{f}(gw). \tag{6.5}$$

In our setting, assume that $\mathfrak{a}$ is prime to p, and consider the trace operator $\mathrm{Tr}_{\mathfrak{n}p^\alpha}^{\mathfrak{n}\mathfrak{a}p^\alpha}$, which we abbreviate as $\mathrm{Tr}_{\mathfrak{n}}^{\mathfrak{n}\mathfrak{a}}$. As in Hida [Hid91, § 7], this operator preserves $\mathcal{O}$-integrality, and extends to the space of p-adic forms $\overline{S}(\mathfrak{n}, \mathcal{O})$. The operator Tr preserves p-ordinarity, because Tr commutes with all of the operators $U(\mathfrak{p})$. The proof of this commutativity follows from the following facts: first, we see from (6.5) that, in the definition of Tr, we can take the w to have non-trivial components only at places dividing $\mathfrak{a}$; on the other hand, from (2.23), we see that the adèlic definition of $U(\mathfrak{p})$ involves elements with non-trivial component only at the place $\mathfrak{p}$. Since $\mathfrak{a}$ is prime to $\mathfrak{p}$ by assumption, the commutativity of the operators follows from the commutativity of these two types of elements.

Proposition 6.5. *The trace operator* $\mathrm{Tr}_{\mathfrak{n}p^\alpha}^{\mathfrak{n}\mathfrak{a}p^\alpha}$ *can be lifted to*

$$\mathrm{Tr}_{\mathfrak{n}}^{\mathfrak{n}\mathfrak{a}} : \mathcal{S}(\mathfrak{n}\mathfrak{a}, \Lambda_F) \to \mathcal{S}(\mathfrak{n}, \Lambda_F)$$

compatible with specializations. It sends $\mathcal{S}^{\mathrm{ord}}(\mathfrak{n}\mathfrak{a}, \Lambda_F)$ *to* $\mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \Lambda_F)$.

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Proof. Combining the two dualities (4.13) and (4.17), we obtain

$$\mathcal{S}(\mathfrak{n},\Lambda_F)\cong \mathrm{Hom}_{\Lambda_F}(\mathrm{Hom}_{\mathcal{O}}(\overline{S}(\mathfrak{n},\mathcal{O}),\mathcal{O}),\Lambda_F)$$

and similarly for $\mathcal{S}(\mathfrak{na},\Lambda_F)$. Thus, to lift $\mathrm{Tr}^{\mathfrak{na}p^\alpha}_{\mathfrak{n}p^\alpha}$, it suffices to show that its action on $\overline{S}(\mathfrak{na},\mathcal{O})$ commute with the action of Λ_F, i.e. the diamond operators. However, this is clear. □

Definition 6.6. Take $\mathcal{O}$ to be a sufficiently large finite extension of $\mathbf{Z}_p$. The general projectors $l^{\mathrm{gen}}_{\mathbf{f},\mathfrak{na}}$, $l^{\mathrm{gen}}_{\mathcal{F},\mathfrak{na}}$, defined on p-ordinary forms of tame level $\mathfrak{na}$, are given by

$$\begin{aligned} l^{\mathrm{gen}}_{\mathbf{f},\mathfrak{na}}(\mathbf{g}) &= l_{\mathbf{f}}(\mathbf{g}|\mathrm{pr}_{S_{k_0}}\,\mathrm{Tr}^{\mathfrak{na}}_{\mathfrak{n}})\\ l^{\mathrm{gen}}_{\mathcal{F},\mathfrak{na}}(\mathcal{G}) &= l_{\mathcal{F}}(\mathcal{G}|\mathrm{pr}_{\mathcal{S}}\,\mathrm{Tr}^{\mathfrak{na}}_{\mathfrak{n}}). \end{aligned} \tag{6.6}$$

6.4 We now apply these projectors to the Eisenstein measure to construct the p-adic L-functions. In the next section, we use the method of Rankin–Shimura to relate the special values of these p-adic L-functions in terms of the special values of classical L-functions (see formula (6.9) below). The Rankin–Shimura computations show that, in order to have a formula of the shape of (6.9), we need to modify the Eisenstein measure slightly as follows. The computations in § 7 will explain *a posteriori* why we made such a modification (compare, in particular, (6.7) and (7.36) below).

Take $\mathcal{O}$ to be a sufficiently large finite extension of $\mathbf{Z}_p$. Let $s_{\phi,\theta}$ be the distribution on $\overline{Z}_F(\mathfrak{r})$ defined by

$$\int_{\overline{Z}_F(\mathfrak{r})}\eta\, ds_{\phi,\theta}=\eta\phi(\mathfrak{n}^{-1}\mathfrak{c}^{(p)}_{\theta})\theta\omega_F^{-1}(\mathfrak{c}_\phi)\langle[\mathfrak{c}_\phi]\rangle\in\Lambda_F\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\eta] \tag{6.7}$$

for η a character of $\overline{Z}_F(\mathfrak{r})$. This is easily seen to be a measure. Define $\widehat{\mu}_{\phi,\theta,r}$ to be the convolution of $s_{\phi,\theta}$ with $\mu_{\phi,\theta,r}$, i.e.

$$\int_{\overline{Z}_F(\mathfrak{r})}\eta\, d\widehat{\mu}_{\phi,\theta,r}=\bigg(\int_{\overline{Z}_F(\mathfrak{r})}\eta\, ds_{\phi,\theta}\bigg)\bigg(\int_{\overline{Z}_F(\mathfrak{r})}\eta\, d\mu_{\phi,\theta,r}\bigg).$$

We similarly define $\widehat{\mathbf{m}}_{P,\phi,\theta,r}$ to be the convolution of $s_{P,\phi,\theta}:=P(s_{\phi,\theta})$ with $\mathbf{m}_{P,\phi,\theta,r}$. Again, we write $\widehat{\mu}_{\phi,\theta}$, $\widehat{\mathbf{m}}_{P,\phi,\theta}$ when we take $r=0$.

Definition 6.7. Given a choice of the character θ,ϕ, define the p-adic L-function attached to $\mathbf{f}\in S_{k_0}(\mathfrak{n}p^\alpha,\epsilon_0,\psi\omega_F^{2-k_0})$ as follows: for $s\in\mathbf{Z}_p$, χ a finite-order character of $\overline{Z}_F(\mathfrak{r})$,

$$\begin{aligned} L_p(s,\mathbf{f},\chi,\phi) &= L_p(s,\mathbf{f},\chi,\phi,\theta)\\ &= l^{\mathrm{gen}}_{\mathbf{f},\mathfrak{c}_\phi\mathfrak{c}^{(p)}_\theta}\bigg(\int_{\overline{Z}_F(\mathfrak{r})}\langle . \rangle_F^{s-1}\chi\, d\widehat{\mathbf{m}}_{P,\phi,\theta}\bigg)\quad (P=P_{k_0,\epsilon_0}). \end{aligned}$$

Similarly, we define the p-adic L-function attached to $\mathcal{F}$ by

$$\begin{aligned} L_p(s,\mathcal{F},\chi,\phi) &= L_p(s,\mathcal{F},\chi,\phi,\theta)\\ &= l^{\mathrm{gen}}_{\mathcal{F},\mathfrak{c}_\phi\mathfrak{c}^{(p)}_\theta}\bigg(\int_{\overline{Z}_F(\mathfrak{r})}\langle . \rangle_F^{s-1}\chi\, d\widehat{\mu}_{\phi,\theta}\bigg). \end{aligned}$$

(We omit ϕ from the notation when we take ϕ to be the trivial character.)

By Corollaries 4.10 and 5.4, the only possible poles belong to the support of the congruence modules $\mathcal{C}_0(\lambda)$ and $\mathcal{C}(\mathcal{E}is)$. In particular, we have $L_p(s,\mathcal{F},\chi,\phi,\theta)\in\bigcap_{P\in\mathfrak{X}(\mathcal{I})_{\mathrm{alg}}}\mathcal{I}_P\subset Q_{\mathcal{I}}$. Furthermore, for $P\in\mathfrak{X}(\mathcal{I})_{\mathrm{alg}}$, we have the consistency $L_p(s,\mathbf{f}_P,\chi,\phi,\theta)=P(L_p(s,\mathcal{F},\chi,\phi,\theta))$.

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The fundamental result about these p-adic L-functions is as follows: in the following, we denote by $\alpha(\mathfrak{p},\mathbf{f}), \alpha(\mathfrak{p},\mathcal{F})$ the $U(\mathfrak{p})$ eigenvalues of $\mathbf{f},\mathcal{F}$ for $\mathfrak{p}|p$. We have $\alpha(\mathfrak{p},\mathbf{f}) = P_0(\alpha(\mathfrak{p},\mathcal{F}))$. For an ideal $\mathfrak{m}$ divisible only by primes above p, we extend the definition of α to such $\mathfrak{m}$ by multiplicativity.

Theorem 6.8. *For an integer $r \geq 0$, χ a finite-order character of $\overline{Z}_F(\mathfrak{r})$, we have a factorization*

$$L_p(r+1,\mathcal{F},\chi,\phi,\theta) = \left(\prod_{\mathfrak{p}|p}\left(1-\frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^r}{\alpha(\mathfrak{p},\mathcal{F})}\right)\right)L_p^*(r+1,\mathcal{F},\chi,\phi,\theta),$$

where $L_p^*(r+1,\mathcal{F},\chi) \in \bigcap_{P\in\mathfrak{X}(\mathcal{I})_{\mathrm{alg}}}\mathcal{I}_P$. Furthermore, if an algebraic point P has the property that θ is adapted to $\mathbf{f}_P$, then there is a complex number $\Omega(\mathbf{f}_P,\theta)$, independent of χ, ϕ and r, such that, if $r \leq k_P - 2$, then $P(L_p^*(r+1,\mathcal{F},\chi,\phi,\theta)) \in \overline{\mathbf{Q}}$, given by the value

$$\begin{aligned} P(L_p^*(r+1,\mathcal{F},\chi,\phi,\theta)) &= \frac{1}{\alpha(\mathfrak{c}_{\chi\omega_F^{-r}},\mathbf{f}_P)}D_F^r\Gamma(r+1)^d\mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi)^{r+1} \\ &\quad\times\frac{L(r+1,\mathbf{f}_P,(\chi\phi\omega_F^{-r})^{-1})}{(-2\pi i)^{dr}\tau((\chi\phi\omega_F^{-r})^{-1})\Omega(\mathbf{f}_P,\theta)}; \end{aligned} \tag{6.8}$$

here Γ is Euler's Gamma function and $\tau(\cdot)$ is the Gauss sum (see (3.13)).

Specializing Theorem 6.8 at P_0, we obtain the following.

Corollary 6.9 [Dab94]. *Suppose that θ is adapted to $\mathbf{f}$. Then with the same $\Omega(\mathbf{f},\theta)$ as above, for $r \leq k_0 - 2$ and χ, we have $L_p(r+1,\mathbf{f},\chi,\phi,\theta) \in \overline{\mathbf{Q}}$, given by the value*

$$\begin{aligned} L_p(r+1,\mathbf{f},\chi,\phi,\theta) &= \left(\prod_{\mathfrak{p}|p}\left(1-\frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^r}{\alpha(\mathfrak{p},\mathbf{f})}\right)\right) \\ &\quad\times\frac{1}{\alpha(\mathfrak{c}_{\chi\omega_F^{-r}},\mathbf{f})}D_F^r\Gamma(r+1)^d\mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi)^{r+1} \\ &\quad\times\frac{L(r+1,\mathbf{f},(\chi\phi\omega_F^{-r})^{-1})}{(-2\pi i)^{dr}\tau((\chi\phi\omega_F^{-r})^{-1})\Omega(\mathbf{f},\theta)}. \end{aligned} \tag{6.9}$$

The proof of Theorem 6.8 is the subject of the next section. For the moment, we give the following remarks: the projector $l_{\mathbf{f}}$ can be calculated analytically by means of Peterson inner products. The method of Rankin–Selberg–Shimura expresses this in terms of the L-value of the convolution of $\mathbf{f}$ with a suitable Eisenstein series, which factorizes into a product of two twisted L-values of $\mathbf{f}$. By our choice of θ, one of the twisted L-value of $\mathbf{f}$ is non-zero, and it is involved in the expression for $\Omega(\mathbf{f},\theta)$.

6.5 Here we make some general comments about the p-adic L-function $L_p(s,\mathbf{f},\chi,\phi,\theta)$ of $\mathbf{f}$. If θ is chosen to be adapted to $\mathbf{f}$, then by (6.8), $\Omega(\mathbf{f},\theta)$ gives a transcendental part of the special values $L(r+1,\mathbf{f},(\chi\phi\omega_F^{-r})^{-1})$, in the sense that

$$\frac{L(r+1,\mathbf{f},(\chi\phi\omega_F^{-r})^{-1})}{(-2\pi i)^{dr}\Omega(\mathbf{f},\theta)} \in \overline{\mathbf{Q}}.$$

The use of Rankin's method to prove algebraicity results on special values of L-functions was initiated by Shimura [Shi78]. Note that as far as the algebraicity result is concerned, the factor $\Omega(\mathbf{f},\theta)$ can be modified by multiples of $\overline{\mathbf{Q}}^\times$. In the classical case where $F = \mathbf{Q}$, and $\mathbf{f}$ is of weight

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two, such modification can be made so that the transcendental factor is given by the periods of $\mathbf{f}$. In the general case, such a result is not yet known, and one must be content with arbitrary choices. Thus, if $u \in \overline{\mathbf{Q}}^{\times}$, then the p-adic L-function $uL_p(s, \mathbf{f}, \chi, \phi, \theta)$ satisfies same interpolation property (6.9), with $\Omega(\mathbf{f}, \theta)$ replaced by $u^{-1}\Omega(\mathbf{f}, \theta)$. In the following, we refer to $uL_p(s, \mathbf{f}, \chi, \phi, \theta)$ as the p-adic L-function defined by the transcendental factor (or 'period') $u^{-1}\Omega(\mathbf{f}, \theta)$.

The choice for the character ϕ is included only to allow more flexibility in some arguments (see, for example, the proof of Theorem 8.2 in Appendix Appendix B). In the context of Iwasawa theory of $\mathbf{f}$, one takes ϕ to be trivial, and considers the p-adic L-function $L_p(s, \mathbf{f}, \chi, \mathrm{Id}, \theta)$. Suppose that $k_0 \geq 3$. Then for θ adapted to $\mathbf{f}$, $L_p(s, \mathbf{f}, \chi, \mathrm{Id}, \theta)$ is not identically zero, by Corollary 6.9, and the non-vanishing result of Shimura (Theorem 6.3). For $k_0 = 2$, however, we do not have a definite answer. There is a general conjecture.

CONJECTURE 6.10. For a cuspidal Hilbert eigenform $\mathbf{f}$ of parallel weight two, then for all but finitely many ray class characters χ, ramified only at primes above p, we have $L(1, \mathbf{f}, \chi) \neq 0$.

We have the following result of Rohrlich.

THEOREM 6.11 [Roh84, Roh88]. *Conjecture 6.10 holds in the case $F = \mathbf{Q}$.*

In general, Conjecture 6.10 implies that the p-adic L-function $L_p(s, \mathbf{f}, \chi, \mathrm{Id}, \theta)$ is not identically zero. In any case, by Theorem 6.4, there exists a twist by ϕ, so that $L_p(s, \mathbf{f}, \chi, \phi, \theta)$ is not identically zero.

In the sections to follow, we abbreviate $L_p(s, \mathbf{f}, \chi, \phi, \theta)$ and $L_p(s, \mathcal{F}, \chi, \phi, \theta)$ as $L_p(s, \mathbf{f}, \chi, \phi)$, $L_p(s, \mathcal{F}, \chi, \phi)$, respectively, if a choice of θ is fixed. Furthermore, we abbreviate

$$\begin{aligned} L_p(s, \mathbf{f}, \chi) &= L_p(s, \mathbf{f}, \chi, \mathrm{Id}), \\ L_p(s, \mathbf{f}) &= L_p(s, \mathbf{f}, \mathrm{Id}, \mathrm{Id}). \end{aligned} \tag{6.10}$$

Similarly,

$$\begin{aligned} L_p(s, \mathcal{F}, \chi) &= L_p(s, \mathcal{F}, \chi, \mathrm{Id}), \\ L_p(s, \mathcal{F}) &= L_p(s, \mathcal{F}, \mathrm{Id}, \mathrm{Id}). \end{aligned} \tag{6.11}$$

7. Proof of Theorem 6.8

7.1 In this section we prove Theorem 6.8, based on the method of Rankin–Shimura. The method was already employed by various authors, for example Dabrowski [Dab94], Hida [Hid91] and Panchishkin [Pan89, Pan91, Pan03]. The proof is computational, and the reader is advised to skip it on a first reading.

DEFINITION 7.1. Let dg be the Haar measure on $G(\mathbf{A}_F)$, normalized so that its push-forward to $G(F) \setminus G(\mathbf{A}_F)/F_\infty^+ K_\infty \cong \mathfrak{H}^d$ coincides with the standard measure $\prod_{\nu=1}^d (dx_\nu\, dy_\nu)/y_\nu^2$. Given forms $\mathbf{F}_1, \mathbf{F}_2$, of weight k, level $\widetilde{\mathfrak{n}}$, with at least one of them being a cusp form, denote by $\langle \mathbf{F}_1, \mathbf{F}_2 \rangle_{\widetilde{\mathfrak{n}}}$ the Petersson inner product of level $\widetilde{\mathfrak{n}}$

$$\langle \mathbf{F}_1, \mathbf{F}_2 \rangle_{\widetilde{\mathfrak{n}}} = \int_{G(F)\backslash G(\mathbf{A}_F)/F_\infty^+ K_\infty K_{1,\widetilde{\mathfrak{n}}}} \overline{\mathbf{F}_1(g)} \mathbf{F}_2(g)\, dg$$

(here $\overline{z}$ is the complex conjugate of $z \in \mathbf{C}$).

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Using the definition of the Atkin–Lehner operator (see (2.25)), the following identity can be easily checked:

$$\langle \mathbf{F}_1 | J_{\widetilde{\mathfrak{n}}}, \mathbf{F}_2 | J_{\widetilde{\mathfrak{n}}} \rangle_{\widetilde{\mathfrak{n}}} = \langle \mathbf{F}_1, \mathbf{F}_2 \rangle_{\widetilde{\mathfrak{n}}} \tag{7.1}$$

or, equivalently, by (2.27),

$$\langle \mathbf{F}_1 | J_{\widetilde{\mathfrak{n}}}, \mathbf{F}_2 \rangle_{\widetilde{\mathfrak{n}}} = (-1)^{dk} \langle \mathbf{F}_1, \mathbf{F}_2 | J_{\widetilde{\mathfrak{n}}} \rangle_{\widetilde{\mathfrak{n}}}. \tag{7.2}$$

Similarly, using the definition of the trace operator (6.5), we have the following: if $\mathbf{F}_1$ is of level $\widetilde{\mathfrak{n}}$, $\mathbf{F}_2$ of level $\widetilde{\mathfrak{n}}\mathfrak{m}$, then

$$\langle \mathbf{F}_1, \mathbf{F}_2 \rangle_{\widetilde{\mathfrak{n}}\mathfrak{m}} = \langle \mathbf{F}_1, \mathbf{F}_2 | \mathrm{Tr}_{\widetilde{\mathfrak{n}}}^{\widetilde{\mathfrak{n}}\mathfrak{m}} \rangle_{\widetilde{\mathfrak{n}}}. \tag{7.3}$$

Definition 7.2. The twisted Petersson pairing $\langle , \rangle'_{\widetilde{\mathfrak{n}}}$ for forms of level $\widetilde{\mathfrak{n}}$, is defined by

$$\langle \mathbf{F}_1, \mathbf{F}_2 \rangle'_{\widetilde{\mathfrak{n}}} = \langle \mathbf{F}_1^{\rho} | J_{\widetilde{\mathfrak{n}}}, \mathbf{F}_2 \rangle_{\widetilde{\mathfrak{n}}}$$

here $\rho \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ is complex cojugation, and $\mathbf{F}_1^{\rho}$ is the action of ρ on $\mathbf{F}_1$ as defined in (2.30). Note that this pairing is $\mathbf{C}$-bilinear.

From (7.3), we clearly have

$$\langle \mathbf{F}_1, \mathbf{F}_2 \rangle'_{\widetilde{\mathfrak{n}}\mathfrak{m}} = \langle \mathbf{F}_1, \mathbf{F}_2 | \mathrm{Tr}_{\widetilde{\mathfrak{n}}}^{\widetilde{\mathfrak{n}}\mathfrak{m}} \rangle'_{\widetilde{\mathfrak{n}}}. \tag{7.4}$$

The reason for introducing the twist is as follows.

Proposition 7.3 [Hid91, p. 381]. *The pairing $\langle , \rangle'_{\widetilde{\mathfrak{n}}}$ is self-adjoint with respect to the action of all of the Hecke operators.*

From the self-adjointness with respect to the Hecke operators, it follows that cusp forms are orthogonal to the Eisenstein series, with respect to the twisted Petersson pairing. Similarly one also obtains the following: given

$$\mathbf{F}_1 \in S_k^{\mathrm{ord}}(\widetilde{\mathfrak{n}}p^{\alpha}, \overline{\mathbf{Q}})$$
$$\mathbf{F}_2 \in M_k(\widetilde{\mathfrak{n}}p^{\alpha}, \overline{\mathbf{Q}})$$

$(\alpha \geq 1)$, we have

$$\langle \mathbf{F}_1, \mathbf{F}_2 \rangle'_{\widetilde{\mathfrak{n}}} = \langle \mathbf{F}_1, \mathbf{F}_2 | e \rangle'_{\widetilde{\mathfrak{n}}}. \tag{7.5}$$

Using Proposition 7.3, one can deduce the following result.

Proposition 7.4 [Hid91, Lemma 9.3]. *Let $\mathbf{f} \in S_k^{\mathrm{ord}}(\mathfrak{n}p^{\alpha}, \overline{\mathbf{Q}})$ be a p-ordinary newform. Then for $\mathbf{F} \in S_k^{\mathrm{ord}}(\mathfrak{n}p^{\alpha}, \overline{\mathbf{Q}})$, we have the formula*

$$l_{\mathbf{f}}(\mathbf{F}) = \frac{\langle \mathbf{f}, \mathbf{F} \rangle'_{\mathfrak{n}p^{\alpha}}}{\langle \mathbf{f}, \mathbf{f} \rangle'_{\mathfrak{n}p^{\alpha}}} \tag{7.6}$$

(note that by the remark after (6.2), $l_{\mathbf{f}}(\mathbf{F}) \in \overline{\mathbf{Q}}$).

Corollary 7.5. *With $\mathbf{f}$ as in Proposition 7.4, we have, for $\mathbf{F} \in M_k^{\mathrm{ord}}(\mathfrak{n}\mathfrak{a}p^{\alpha}, \overline{\mathbf{Q}})$ (here $\mathfrak{a}$ is an integral ideal prime to p),*

$$l_{\mathbf{f},\mathfrak{n}\mathfrak{a}}^{\mathrm{gen}}(\mathbf{F}) = \frac{\langle \mathbf{f}, \mathbf{F} \rangle'_{\mathfrak{n}\mathfrak{a}p^{\alpha}}}{\langle \mathbf{f}, \mathbf{f} \rangle'_{\mathfrak{n}p^{\alpha}}}. \tag{7.7}$$

Proof. This follows by combining (7.4) and (7.6), together with the fact that, under the Petersson pairing, cusp forms are orthogonal to Eisenstein series. □

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Now we establish several preliminary results. Anticipating later calculations, consider the following situation:

$$\mathbf{f} \in S_k^{\mathrm{ord}}(\mathfrak{n}p^\alpha, \epsilon\psi\omega_F^{2-k}) \ \ (\alpha \geq 1),$$
$$\mathbf{g} \in M_l(\mathfrak{c}\mathfrak{m}p^\alpha, \zeta).$$

Here we assume that $\mathbf{f}$ is a p-ordinary newform of tame level $\mathfrak{n}$, with $U(\mathfrak{p})$-eigenvalue $\alpha(\mathfrak{p}, \mathbf{f})$, while $k > l$; the ideals $\mathfrak{c}$ and $\mathfrak{m}$ are assumed to satisfy the condition that $\mathfrak{c}$ is prime to p, while $\mathfrak{m}$ is divisible only by primes above p. We also assume that $\mathfrak{n}$ divides $\mathfrak{c}$. Finally, ζ can be any Hecke character of finite order, of conductor dividing $\mathfrak{c}\mathfrak{m}p^\alpha$.

With this setup, we have, as in Proposition 3.6, the Eisenstein series

$$\mathbf{G}_{k-l}(\epsilon\psi\zeta^{-1}\omega_F^{2-k}, \mathfrak{c}\mathfrak{m}p^\alpha) \in M_{k-l}(\mathfrak{c}\mathfrak{m}p^\alpha, \epsilon\psi\zeta^{-1}\omega_F^{2-k}).$$

Note that, by Proposition 3.6, we have $\mathbf{G}_{k-l}(\epsilon\psi\zeta^{-1}\omega_F^{2-k}, \mathfrak{c}\mathfrak{m}p^\alpha) = \mathbf{G}_{k-l}(\epsilon\psi\zeta^{-1}\omega_F^{2-k}, \mathfrak{c}\mathfrak{m}p)$. We abbreviate it as $\mathbf{G}_{k-l}$.

Consider the product

$$\mathbf{g}\mathbf{G}_{k-l} \in M_k(\mathfrak{c}\mathfrak{m}p^\alpha, \epsilon\psi\omega_F^{2-k}).$$

If we apply the ordinary projector e to $\mathbf{g}\mathbf{G}_{k-l}$, then we obtain a p-ordinary form of level $\mathfrak{c}\mathfrak{m}p^\alpha$. Recall that $\mathfrak{m}$ is divisible only by primes above p. Now the character of this form is the same as $\mathbf{f}$, in particular, its conductor divides $\mathfrak{n}p^\alpha$, and hence divides $\mathfrak{c}p^\alpha$ (because we assumed that $\mathfrak{n}$ divides $\mathfrak{c}$). By the control Theorem 4.3, the p-part of the level of a p-ordinary form can be taken to be the p-part of the level of its character (as long as $p\mathfrak{r}$ divides the level). Thus, in our case, we have

$$(\mathbf{g}\mathbf{G}_{k-l})|e \in M_k^{\mathrm{ord}}(\mathfrak{c}p^\alpha, \epsilon\psi\omega_F^{2-k}).$$

In fact, we have a more precise statement.

Proposition 7.6 [Pan89, (4.11)]. *Let $\mathbf{F} \in M_k(\mathfrak{c}\mathfrak{m}p^\alpha)$, where $\alpha \geq 1$, and $\mathfrak{m}$ is divisible only by primes above p. Then*

$$\mathbf{F}|U(\mathfrak{m}) = (-1)^{dk}\mathcal{N}(\mathfrak{m})^{k/2-1}\mathbf{F}|(J_{\mathfrak{c}\mathfrak{m}p^\alpha})(\mathrm{Tr}_{\mathfrak{c}p^\alpha}^{\mathfrak{c}\mathfrak{m}p^\alpha})(J_{\mathfrak{c}p^\alpha}). \tag{7.8}$$

In particular, $\mathbf{F}|U(\mathfrak{m}) \in M_k(\mathfrak{c}p^\alpha)$.

(We note that in [Pan89, (4.11)] the sign $(-1)^{dk}$ is missing.)

By Corollary 7.5, we obtain

$$l_{\mathbf{f},\mathfrak{c}}^{\mathrm{gen}}((\mathbf{g}\mathbf{G}_{k-l})|e) = \frac{\langle \mathbf{f}, (\mathbf{g}\mathbf{G}_{k-l})|e\rangle'_{\mathfrak{c}p^\alpha}}{\langle \mathbf{f}, \mathbf{f}\rangle'_{\mathfrak{n}p^\alpha}}. \tag{7.9}$$

We now unravel the term $\langle \mathbf{f}, (\mathbf{g}\mathbf{G}_{k-l})|e\rangle'_{\mathfrak{c}p^\alpha}$:

$$\begin{aligned}\langle \mathbf{f}, (\mathbf{g}\mathbf{G}_{k-l})|e\rangle'_{\mathfrak{c}p^\alpha} &= \frac{1}{\alpha(\mathfrak{m}, \mathbf{f})}\langle \mathbf{f}|U(\mathfrak{m}), (\mathbf{g}\mathbf{G}_{k-l})|e\rangle'_{\mathfrak{c}p^\alpha} \\ &= \frac{1}{\alpha(\mathfrak{m}, \mathbf{f})}\langle \mathbf{f}, (\mathbf{g}\mathbf{G}_{k-l})|eU(\mathfrak{m})\rangle'_{\mathfrak{c}p^\alpha}\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{\alpha(\mathfrak{m}, \mathbf{f})} \langle \mathbf{f}, (\mathbf{g}\mathbf{G}_{k-l})|U(\mathfrak{m})e\rangle'_{\mathfrak{c}p^\alpha} \quad \text{(since e and $U(\mathfrak{m})$ commute)} \\
&= \frac{1}{\alpha(\mathfrak{m}, \mathbf{f})} \langle \mathbf{f}, (\mathbf{g}\mathbf{G}_{k-l})|U(\mathfrak{m})\rangle'_{\mathfrak{c}p^\alpha} \quad \text{(by (7.5))} \\
&= \frac{1}{\alpha(\mathfrak{m}, \mathbf{f})} \langle \mathbf{f}^\rho|J_{\mathfrak{n}p^\alpha}, (\mathbf{g}\mathbf{G}_{k-l})|U(\mathfrak{m})\rangle_{\mathfrak{c}p^\alpha}.
\end{aligned} \tag{7.10}
$$

Now applying Proposition 7.6, we have

$$
(\mathbf{g}\mathbf{G}_{k-l})|U(\mathfrak{m}) = (-1)^{dk}\mathcal{N}(\mathfrak{m})^{k/2-1}(\mathbf{g}\mathbf{G}_{k-l})|J_{\mathfrak{c}\mathfrak{m}p^\alpha}\mathrm{Tr}^{\mathfrak{c}\mathfrak{m}p^\alpha}_{\mathfrak{c}p^\alpha} J_{\mathfrak{c}p^\alpha}. \tag{7.11}
$$

On the other hand, the following identity is easily verified:

$$
\mathbf{f}^\rho|J_{\mathfrak{n}p^\alpha} = \mathcal{N}(\mathfrak{n}^{-1}\mathfrak{c})^{k/2}\mathbf{f}^\rho|V(\mathfrak{n}^{-1}\mathfrak{c})J_{\mathfrak{c}p^\alpha}. \tag{7.12}
$$

Combining (7.10)–(7.12), with the identities (7.1), (7.4), we obtain

$$
\begin{aligned}
\langle \mathbf{f}, (\mathbf{g}\mathbf{G}_{k-l})|e\rangle'_{\mathfrak{c}p^\alpha} &= \frac{(-1)^{dk}}{\alpha(\mathfrak{m}, \mathbf{f})}\mathcal{N}(\mathfrak{m})^{k/2-1}\mathcal{N}(\mathfrak{n}^{-1}\mathfrak{c})^{k/2}\langle \mathbf{f}^\rho|V(\mathfrak{n}^{-1}\mathfrak{c}), (\mathbf{g}\mathbf{G}_{k-l})|J_{\mathfrak{c}\mathfrak{m}p^\alpha}\mathrm{Tr}^{\mathfrak{c}\mathfrak{m}p^\alpha}_{\mathfrak{c}p^\alpha}\rangle_{\mathfrak{c}p^\alpha} \\
&= \frac{(-1)^{dk}}{\alpha(\mathfrak{m}, \mathbf{f})}\mathcal{N}(\mathfrak{m})^{k/2-1}\mathcal{N}(\mathfrak{n}^{-1}\mathfrak{c})^{k/2}\langle \mathbf{f}^\rho|V(\mathfrak{n}^{-1}\mathfrak{c}), (\mathbf{g}\mathbf{G}_{k-l})|J_{\mathfrak{c}\mathfrak{m}p^\alpha}\rangle_{\mathfrak{c}\mathfrak{m}p^\alpha}.
\end{aligned} \tag{7.13}
$$

Now

$$
(\mathbf{g}\mathbf{G}_{k-l})|J_{\mathfrak{c}\mathfrak{m}p^\alpha} = \mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha}\mathbf{G}_{k-l}|J_{\mathfrak{c}\mathfrak{m}p^\alpha},
$$

and by (3.10) and the definition of $\mathbf{G}_{k-l}$ as in Proposition 3.6, we have

$$
\mathbf{G}_{k-l}|J_{\mathfrak{c}\mathfrak{m}p^\alpha} = \frac{D_F^{k-l-1/2}\mathcal{N}(\mathfrak{c}\mathfrak{m}p^\alpha)^{(k-l)/2}\Gamma(k-l)^d}{(2\pi i)^{(k-l)d}}K_{k-l}(\epsilon\psi\zeta^{-1}\omega_F^{2-k}, \mathfrak{c}\mathfrak{m}p^\alpha). \tag{7.14}
$$

Thus, we obtain (abbreviate $K_{k-l}(\epsilon\psi\zeta^{-1}\omega_F^{2-k}, \mathfrak{c}\mathfrak{m}p^\alpha)$ as K_{k-l}) the following result.

Lemma 7.7. *We have*

$$
\begin{aligned}
l^{\mathrm{gen}}_{\mathbf{f},\mathfrak{c}}((\mathbf{g}\mathbf{G}_{k-l})|e) = (-1)^{dk} &\frac{D_F^{k-l-1/2}\mathcal{N}(\mathfrak{m})^{k/2-1}\mathcal{N}(\mathfrak{n}^{-1}\mathfrak{c})^{k/2}\mathcal{N}(\mathfrak{c}\mathfrak{m}p^\alpha)^{(k-l)/2}\Gamma(k-l)^d}{\alpha(\mathfrak{m}, \mathbf{f})(2\pi i)^{(k-l)d}} \\
&\times \frac{\langle \mathbf{f}^\rho|V(\mathfrak{n}^{-1}\mathfrak{c}), \mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha}K_{k-l}\rangle_{\mathfrak{c}\mathfrak{m}p^\alpha}}{\langle \mathbf{f}, \mathbf{f}\rangle'_{\mathfrak{n}p^\alpha}}.
\end{aligned} \tag{7.15}
$$

The term $\langle \mathbf{f}^\rho|V(\mathfrak{n}^{-1}\mathfrak{c}), \mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha}K_{k-l}\rangle_{\mathfrak{c}\mathfrak{m}p^\alpha}$ is related to a special value of an L-function, by the formula of Rankin–Selberg. We recall the formalism.

Definition 7.8. Given forms $\mathbf{F}_1 \in M_k(\widetilde{\mathfrak{n}}, \psi_1), \mathbf{F}_2 \in M_l(\widetilde{\mathfrak{n}}, \psi_2)$, define the Rankin–Selberg L-function of $\mathbf{F}_1$ and $\mathbf{F}_2$:

$$
L(s, \mathbf{F}_1, \mathbf{F}_2) = \sum_{\mathfrak{m}} C(\mathfrak{m}, \mathbf{F}_1)C(\mathfrak{m}, \mathbf{F}_2)\mathcal{N}(\mathfrak{m})^{-s}. \tag{7.16}
$$

The completed Rankin–Selberg L-function is obtained by multiplying suitable factors ([Pan89, (0.2)]):

$$
\Psi^{(\widetilde{\mathfrak{n}})}(s, \mathbf{F}_1, \mathbf{F}_2) = (2\pi)^{-2ds}\Gamma(s)^d\Gamma(s+1-l)^d L^{(\widetilde{\mathfrak{n}})}(2s+2-k-l, \psi_1\psi_2)L(s, \mathbf{F}_1, \mathbf{F}_2). \tag{7.17}
$$

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From the definition, it is immediate that

$$\Psi^{(\widetilde{\mathfrak{n}})}(s, \mathbf{F}_1|V(\mathfrak{a}), \mathbf{F}_2|V(\mathfrak{b})) = \mathcal{N}(\mathfrak{a}\mathfrak{b})^{-s}\Psi^{(\widetilde{\mathfrak{n}})}(s, \mathbf{F}_1|U(\mathfrak{b}), \mathbf{F}_2|U(\mathfrak{a})) \tag{7.18}$$

for any integral ideal $\mathfrak{a}$, $\mathfrak{b}$ which are relatively prime.

We need the formula of Rankin–Selberg, which expresses this L-function in terms of a Petersson inner product.

Theorem 7.9 [Pan89, (4.7)]. *Assume that $\mathbf{F}_1 \in S_k(\widetilde{\mathfrak{n}}, \psi_1), \mathbf{F}_2 \in M_l(\widetilde{\mathfrak{n}}, \psi_2)$, with $k > l$. Then $\Psi^{(\widetilde{\mathfrak{n}})}(s, \mathbf{F}_1, \mathbf{F}_2)$ extends to a entire function on the complex plane, and we have*

$$\Psi^{(\widetilde{\mathfrak{n}})}(s, \mathbf{F}_1, \mathbf{F}_2) = D_F^{1/2}\pi^{-ds}\Gamma(s+1-l)^d\langle \mathbf{F}_1^\rho, \mathbf{F}_2 K_{k-l}(s+1-k)\rangle_{\widetilde{\mathfrak{n}}} \tag{7.19}$$

here $K_{k-l}(s) = K_{k-l}(s, \psi_1\psi_2, \widetilde{\mathfrak{n}})$ is the Eisenstein series as in (3.8).

Applying Theorem 7.9 to the situation, we have

$$\begin{aligned} \mathbf{F}_1 &= \mathbf{f}|V(\mathfrak{n}^{-1}\mathfrak{c}) \\ \mathbf{F}_2 &= \mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha} \\ \widetilde{\mathfrak{n}} &= \mathfrak{c}\mathfrak{m}p^\alpha \\ s &= k-1. \end{aligned}$$

We then obtain from Lemma 7.7 the following result.

Lemma 7.10. *We have*

$$\begin{aligned} l_{\mathbf{f},\mathfrak{c}}^{\mathrm{gen}}((\mathbf{g}\mathbf{G}_{k-l})|e) = (-1)^{dk}&\frac{D_F^{k-l-1}\pi^{(k-1)d}\mathcal{N}(\mathfrak{m})^{k/2-1}\mathcal{N}(\mathfrak{n}^{-1}\mathfrak{c})^{k/2}\mathcal{N}(\mathfrak{c}\mathfrak{m}p^\alpha)^{(k-l)/2}}{\alpha(\mathfrak{m},\mathbf{f})(2\pi i)^{(k-l)d}} \\ &\times \frac{\Psi^{(\mathfrak{c}p)}(k-1, \mathbf{f}|V(\mathfrak{n}^{-1}\mathfrak{c}), \mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha})}{\langle \mathbf{f},\mathbf{f}\rangle'_{\mathfrak{n}p^\alpha}}. \end{aligned} \tag{7.20}$$

Now we specialize Lemma 7.10. Suppose that $\mathbf{g} \in M_l(\mathfrak{c}\mathfrak{m}, \zeta)$ is an eigenform, with $U(\mathfrak{p})$-eigenvalue $\alpha(\mathfrak{p}, \mathbf{g})$. Put $\mathfrak{m}_0 = \prod_{\mathfrak{p}|p}\mathfrak{p}$. Define, similar to (5.2),

$$\begin{aligned} \mathbf{g}^{00} &= \mathbf{g}\Big|\prod_{\mathfrak{p}|p}(1-\alpha(\mathfrak{p},\mathbf{g})V(\mathfrak{p})) \\ &= \mathbf{g}\Big|\left(\sum_{\mathfrak{h}|\mathfrak{m}_0}\mu(\mathfrak{h})\alpha(\mathfrak{h},\mathbf{g})V(\mathfrak{h})\right) \\ &\in M_l(\mathfrak{c}\mathfrak{m}\mathfrak{m}_0, \zeta); \end{aligned} \tag{7.21}$$

here μ is the Möbius function: it is the multiplicative function on the set of integral ideals, such that for $\mathfrak{q}$ a prime ideal,

$$\mu(\mathfrak{q}^n) = \begin{cases} -1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

We would like to compare

$$l_{\mathbf{f},\mathfrak{c}}^{\mathrm{gen}}((\mathbf{g}^{00}\mathbf{G}_{k-l})|e)$$

and

$$l_{\mathbf{f},\mathfrak{c}}^{\mathrm{gen}}((\mathbf{g}\mathbf{G}_{k-l})|e).$$

By formula (7.10), it suffices to relate $\mathbf{g}^{00}|J_{\mathfrak{c}\mathfrak{m}\mathfrak{m}_0p^\alpha}$ to $\mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha}$.

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The following identity can be easily checked:

$$\mathbf{g}|V(\mathfrak{h})J_{\mathfrak{c}\mathfrak{m}\mathfrak{m}_0 p^\alpha} = \mathcal{N}(\mathfrak{m}_0)^{l/2}\mathcal{N}(\mathfrak{h})^{-l}\mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha}V(\mathfrak{m}_0\mathfrak{h}^{-1}). \tag{7.22}$$

This gives

$$\mathbf{g}^{00}|J_{\mathfrak{c}\mathfrak{m}\mathfrak{m}_0 p^\alpha} = \mathcal{N}(\mathfrak{m}_0)^{l/2}\mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha}|\bigg(\sum_{\mathfrak{h}|\mathfrak{m}_0}\mu(\mathfrak{h})\alpha(\mathfrak{h},\mathbf{g})\mathcal{N}(\mathfrak{h})^{-l}V(\mathfrak{m}_0\mathfrak{h}^{-1})\bigg). \tag{7.23}$$

Hence, we obtain

$$\begin{aligned}
&\Psi^{(\mathfrak{c}p)}(k-1,\mathbf{f}|V(\mathfrak{n}^{-1}\mathfrak{c}),\mathbf{g}^{00}|J_{\mathfrak{c}\mathfrak{m}\mathfrak{m}_0 p^\alpha})\\
&\quad=\mathcal{N}(\mathfrak{m}_0)^{l/2}\bigg(\sum_{\mathfrak{h}|\mathfrak{m}_0}\mu(\mathfrak{h})\alpha(\mathfrak{h},\mathbf{g})\mathcal{N}(\mathfrak{h})^{-l}\Psi^{(\mathfrak{c}p)}(k-1,\mathbf{f}|V(\mathfrak{n}^{-1}\mathfrak{c}),\mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha}V(\mathfrak{m}_0\mathfrak{h}^{-1}))\bigg)\\
&\quad=\mathcal{N}(\mathfrak{m}_0)^{l/2}\bigg(\sum_{\mathfrak{h}|\mathfrak{m}_0}\mu(\mathfrak{h})\alpha(\mathfrak{h},\mathbf{g})\mathcal{N}(\mathfrak{h})^{-l}\mathcal{N}(\mathfrak{m}_0\mathfrak{h}^{-1})^{-(k-1)}\\
&\qquad\times\Psi^{(\mathfrak{c}p)}(k-1,\mathbf{f}|V(\mathfrak{n}^{-1}\mathfrak{c})U(\mathfrak{m}_0\mathfrak{h}^{-1}),\mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha})\bigg).
\end{aligned}\tag{7.24}$$

Now the operators $V(\mathfrak{n}^{-1}\mathfrak{c})$ and $U(\mathfrak{m}_0\mathfrak{h}^{-1})$ commute, since the ideals $\mathfrak{n}^{-1}\mathfrak{c}$ and $\mathfrak{m}_0\mathfrak{h}^{-1}$ are relatively prime. Thus, (7.24) becomes

$$\begin{aligned}
&\mathcal{N}(\mathfrak{m}_0)^{l/2}\bigg(\sum_{\mathfrak{h}|\mathfrak{m}_0}\mu(\mathfrak{h})\alpha(\mathfrak{h},\mathbf{g})\mathcal{N}(\mathfrak{h})^{-l}\mathcal{N}(\mathfrak{m}_0\mathfrak{h}^{-1})^{-(k-1)}\\
&\qquad\times\Psi^{(\mathfrak{c}p)}(k-1,\mathbf{f}|U(\mathfrak{m}_0\mathfrak{h}^{-1})V(\mathfrak{n}^{-1}\mathfrak{c}),\mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha})\bigg)\\
&\quad=\mathcal{N}(\mathfrak{m}_0)^{l/2-(k-1)}\bigg(\sum_{\mathfrak{h}|\mathfrak{m}_0}\mu(\mathfrak{h})\alpha(\mathfrak{h},\mathbf{g})\alpha(\mathfrak{m}_0\mathfrak{h}^{-1},\mathbf{f})\mathcal{N}(\mathfrak{h})^{k-l-1}\bigg)\\
&\qquad\times\Psi^{(\mathfrak{c}p)}(k-1,\mathbf{f}|V(\mathfrak{n}^{-1}\mathfrak{c}),\mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha})\\
&\quad=\mathcal{N}(\mathfrak{m}_0)^{l/2-(k-1)}\alpha(\mathfrak{m}_0,\mathbf{f})\prod_{\mathfrak{p}|p}\bigg(1-\frac{\mathcal{N}(\mathfrak{p})^{k-l-1}\alpha(\mathfrak{p},\mathbf{g})}{\alpha(\mathfrak{p},\mathbf{f})}\bigg)\\
&\qquad\times\Psi^{(\mathfrak{c}p)}(k-1,\mathbf{F}|V(\mathfrak{n}^{-1}\mathfrak{c}),\mathbf{g}|J_{\mathfrak{c}\mathfrak{m}p^\alpha}).
\end{aligned}\tag{7.25}$$

Applying Lemma 7.10 to $\mathbf{g}^{00}$ and $\mathbf{g}$, using the results (7.24)–(7.25), we obtain the following.

LEMMA 7.11. *We have*

$$l^{\mathrm{gen}}_{\mathbf{f},\mathfrak{c}}((\mathbf{g}^{00}\mathbf{G}_{k-l})|e)=\prod_{\mathfrak{p}|p}\bigg(1-\frac{\mathcal{N}(\mathfrak{p})^{k-l-1}\alpha(\mathfrak{p},\mathbf{g})}{\alpha(\mathfrak{p},\mathbf{f})}\bigg)l^{\mathrm{gen}}_{\mathbf{f},\mathfrak{c}}((\mathbf{g}\mathbf{G}_{k-l})|e). \tag{7.26}$$

At last we are ready to give the proof of Theorem 6.8.

Proof of Theorem 6.8. In this proof, the notation are as in §6. Let $\mathcal{F}\in\mathcal{S}(\mathfrak{n},\psi,\mathcal{I})$ be an $\mathcal{I}$-adic form, with specializations $\mathbf{f}_P\in S_{k_P}(\mathfrak{n}p^{\alpha_P},\epsilon_P\psi\omega_F^{2-k_P})$ for $P\in\mathfrak{X}_{\mathrm{alg}}(\mathcal{I})$.

Let $r\geq 0$ be an integer. Assume that P satisfies

$$k_P-r-1\geq 1,\ \text{and condition }(\mathbf{1})\text{ of Definition 6.2 holds.} \tag{7.27}$$

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We now take

$$\begin{aligned}\mathbf{g} &= \mathbf{E}_{k_P-r-1}(\chi\phi\omega_F^{-r}, \epsilon_P\theta\omega_F^{1-k_P}) \\ &= \mathbf{E}_{k_P-r-1}(\chi\phi\omega_F^{-r}, \epsilon_P\theta\omega_F^{1-k_P})^0 \quad (\text{as } P \text{ satisfies } (7.27))\end{aligned} \tag{7.28}$$

which we abbreviate as $\mathbf{E}_{k_P-r-1}$. Recall that $\alpha(\mathfrak{p}, \mathbf{E}_{k_P-r-1}) = \chi\phi\omega_F^{-r}(\mathfrak{p})$. We have

$$\mathbf{E}_{k_P-r-1} \in M_{k_P-r-1}(\mathfrak{c}_\phi\mathfrak{c}_\theta\mathfrak{c}_\chi p^{\alpha_P}, \epsilon_P\chi\phi\theta\omega_F^{1-k_P-r}).$$

In the notation of this section,

$$\begin{aligned} l &= k_P - r - 1 \\ \zeta &= \epsilon_P\chi\phi\theta\omega_F^{1-k_P-r} \\ \mathfrak{c} &= \mathfrak{c}_\phi\mathfrak{c}_\theta^{(p)} \\ \mathfrak{m} &= \mathfrak{c}_\chi \end{aligned} \tag{7.29}$$

and we have

$$\mathbf{G}_{k_P-l} = \mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{c}_\phi\mathfrak{c}_\theta p). \tag{7.30}$$

From Definition 6.7 of p-adic L-functions, we calculate

$$\begin{aligned} L_p(r+1, \mathbf{f}_P, \chi, \phi, \theta) &= l^{\mathrm{gen}}_{\mathbf{f}_P, \mathfrak{c}_\phi\mathfrak{c}_\theta^{(p)}}\left(\int_{\overline{Z}_F(\mathfrak{r})} \langle\cdot\rangle_F^r \chi \, d\hat{\mathbf{m}}_{P,\phi,\theta}\right) \\ &= \left(\int_{\overline{Z}_F(\mathfrak{r})} \langle\cdot\rangle_F^r \chi \, ds_{P,\phi,\theta}\right) l^{\mathrm{gen}}_{\mathbf{f}_P, \mathfrak{c}_\phi\mathfrak{c}_\theta^{(p)}}\left(\int_{\overline{Z}_F(\mathfrak{r})} \langle\cdot\rangle_F^r \chi \, d\mathbf{m}_{P,\phi,\theta}\right) \\ &= (-1)^{rd}\left(\int_{\overline{Z}_F(\mathfrak{r})} \langle\cdot\rangle_F^r \chi \, ds_{P,\phi,\theta}\right) l^{\mathrm{gen}}_{\mathbf{f}_P, \mathfrak{c}_\phi\mathfrak{c}_\theta^{(p)}}\left(\int_{\overline{Z}_F(\mathfrak{r})} \chi \, d\mathbf{m}_{P,\phi,\theta,r}\right) \quad (\text{by } (5.18)) \\ &= (-1)^{rd}\left(\int_{\overline{Z}_F(\mathfrak{r})} \langle\cdot\rangle_F^r \chi \, ds_{P,\phi,\theta}\right) l^{\mathrm{gen}}_{\mathbf{f}_P, \mathfrak{c}_\phi\mathfrak{c}_\theta^{(p)}}\left((\mathbf{E}^{00}_{k_P-r-1}\mathbf{G}_{r+1})|e\right). \end{aligned} \tag{7.31}$$

By Lemma 7.11, we obtain

$$L_p(r+1, \mathbf{f}_P, \chi, \phi, \theta) = \prod_{\mathfrak{p}|p}\left(1 - \frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^r}{\alpha(\mathfrak{p}, \mathbf{f})}\right) l^{\mathrm{gen}}_{\mathbf{f}_P, \mathfrak{c}_\phi\mathfrak{c}_\theta^{(p)}}((\mathbf{E}^0_{k_P-r-1}\mathbf{G}_{r+1})|e). \tag{7.32}$$

It follows that if we define

$$L_p^*(r+1, \mathcal{F}, \chi, \phi) = (-1)^{rd}\left(\int_{\overline{Z}_F(\mathfrak{r})} \langle\cdot\rangle_F^r \chi \, ds_{P,\phi,\theta}\right) l_{\mathcal{F}}((\mathcal{H}(\chi, \phi, \theta, r)^0). \tag{7.33}$$

Then,

$$L_p^*(r+1, \mathcal{F}, \chi, \phi) \in \bigcap_{P \in \mathfrak{X}(\mathcal{I})_{\mathrm{alg}}} \mathcal{I}_P$$

and the equation

$$L_p(r+1, \mathcal{F}, \chi, \phi) = \left(\prod_{\mathfrak{p}|p}\left(1 - \frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^r}{\alpha(\mathfrak{p}, \mathcal{F})}\right)\right) L_p^*(r+1, \mathcal{F}, \chi, \phi)$$

is valid upon specialization at those P which satisfy (7.27). Since these P are Zariski dense in $\mathfrak{X}_{\mathrm{alg}}(\mathcal{I})$, they must coincide. Thus, we obtain the first part of the theorem.

Now assume, in addition, that θ is adapted to $\mathbf{f}_P$. To complete the proof of the theorem, it remains, by Lemma 7.10, to evaluate the following:

$$\Psi^{(\mathfrak{c}_\phi\mathfrak{c}_\theta p)}(k_P - 1, \mathbf{f}_P | V(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi), \mathbf{E}_{k_p-r-1}(\chi\phi\omega_F^{-r}, \epsilon_P\theta\omega_F^{1-k_P})|J_{\mathfrak{c}_\theta\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi p^{\alpha_P}}).$$

We calculate, using an identity similar to (7.12), and (7.18),

$$\begin{aligned}
&\Psi^{(\mathfrak{c}_\phi \mathfrak{c}_\theta p)}(k_P-1, \mathbf{f}_P | V(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi), \mathbf{E}_{k_p-r-1}(\chi\phi\omega_F^{-r}, \epsilon_P\theta\omega_F^{1-k_P})|J_{\mathfrak{c}_\theta \mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi p^{\alpha_P}})\\
&\quad = \mathcal{N}(\mathfrak{c}^{-1}_{\epsilon_P\theta\omega_F^{1-k_P}}\mathfrak{c}_\theta p^{\alpha_P})^{(k_P-r-1)/2} \times \Psi^{(\mathfrak{c}_\phi \mathfrak{c}_\theta p)}(k_P-1, \mathbf{f}_P|V(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi),\\
&\mathbf{E}_{k_p-r-1}(\chi\phi\omega_F^{-r}, \epsilon_P\theta\omega_F^{1-k_P})|J_{\mathfrak{c}_{\epsilon_P\theta\omega_F^{1-k_P}}\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi} V(\mathfrak{c}^{-1}_{\epsilon_P\theta\omega_F^{1-k_P}}\mathfrak{c}_\theta p^{\alpha_P}))\\
&\quad = \mathcal{N}(\mathfrak{c}^{-1}_{\epsilon_P\theta\omega_F^{1-k_P}}\mathfrak{c}_\theta p^{\alpha_P})^{(1-k_P-r)/2}\mathcal{N}(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi)^{-(k_P-1)}\\
&\qquad \times \Psi^{(\mathfrak{c}_\phi \mathfrak{c}_\theta p)}(k_P-1, \mathbf{f}_P|U(\mathfrak{c}^{-1}_{\epsilon_P\theta\omega_F^{1-k_P}}\mathfrak{c}_\theta p^{\alpha_P}),\\
&\mathbf{E}_{k_p-r-1}(\chi\phi\omega_F^{-r}, \epsilon_P\theta\omega_F^{1-k_P})|J_{\mathfrak{c}_\theta\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi p^{\alpha_P}} U(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi))\\
&\quad = \mathcal{N}(\mathfrak{c}^{-1}_{\epsilon_P\theta\omega_F^{1-k_P}}\mathfrak{c}_\theta p^{\alpha_P})^{(1-k_P-r)/2}\mathcal{N}(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi)^{-(k_P-1)}\alpha(\mathfrak{c}^{-1}_{\epsilon_P\theta\omega_F^{1-k_P}}\mathfrak{c}_\theta p^{\alpha_P}, \mathbf{f}_P)\\
&\qquad \times \Psi^{(\mathfrak{c}_\phi \mathfrak{c}_\theta p)}(k_P-1, \mathbf{f}_P, \mathbf{E}_{k_p-r-1}(\chi\phi\omega_F^{-r}, \epsilon_P\theta\omega_F^{1-k_P})|J_{\mathfrak{c}_\theta\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi} U(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi))
\end{aligned} \tag{7.34}$$

(note that $\mathfrak{c}^{-1}_{\epsilon_P\theta\omega_F^{1-k_P}}\mathfrak{c}_\theta p^{\alpha_P} = (\mathfrak{c}_{\epsilon_P\theta\omega_F^{1-k_P}})_p^{-1}\mathfrak{c}_{\theta,p}p^{\alpha_P}$, thus is divisible only by primes above p). By (3.17),

$$\begin{aligned}
&\mathbf{E}_{k_P-r-1}(\chi\phi\omega_F^{-r}, \epsilon_P\theta\omega_F^{1-k_P})|J_{\mathfrak{c}_{\epsilon_P\theta\omega_F^{1-k_P}}\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi}\\
&\quad = \frac{(-1)^{d(1-k_P)}}{\mathcal{N}\mathfrak{c}_{\epsilon_P\theta\omega_F^{1-k_P}}}\mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi\mathfrak{c}_{\epsilon_P\theta\omega_F^{1-k_P}})^{(k_P-r-1)/2}\mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi)^{1-(k_P-r-1)}\\
&\qquad \times \tau((\chi\phi\omega_F^{-r})^{-1})^{-1}\tau(\epsilon_P\theta\omega_F^{1-k_P})\mathbf{E}_{k_P-r-1}((\epsilon_P\theta\omega_F^{1-k_P})^{-1}, (\chi\phi\omega_F^{-r})^{-1}).
\end{aligned} \tag{7.35}$$

Now $\mathbf{E}_{k_P-r-1}((\epsilon_P\theta\omega_F^{1-k_P})^{-1}, (\chi\phi\omega_F^{-1})^{-r})$ is an eigenvector for $U(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi)$, with eigenvalue given by the normalized Fourier coefficient at the ideal $\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi$:

$$\begin{aligned}
&C(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi, \mathbf{E}_{k_P-r-1}((\epsilon_P\theta\omega_F^{1-k_P})^{-1}, (\chi\phi\omega_F^{-1})^{-r}))\\
&\quad = \sum_{\mathfrak{a}\mathfrak{b}=\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)}\mathfrak{c}_\phi} (\epsilon_P\theta\omega_F^{1-k_P})^{-1}(\mathfrak{a}) \times (\chi\phi\omega_F^{-1})^{-r}(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{k_P-r-2}\\
&\quad = (\epsilon_P\theta\omega_F^{1-k_P})^{-1}(\mathfrak{c}_\phi) \times (\chi\phi\omega_F^{-r})^{-1}(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)})\mathcal{N}(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)})^{k_P-r-2}.
\end{aligned} \tag{7.36}$$

Thus, it remains to evaluate $\Psi^{(\mathfrak{c}_\phi \mathfrak{c}_\theta p)}(k_p-1, \mathbf{f}_P, \mathbf{E}_{k_P-r-1}((\epsilon_P\theta\omega_F^{1-k_P})^{-1}, (\chi\phi\omega_F^{-r})^{-1})$. This is achieved by employing the following general identity, which is similar to (3.16).

Proposition 7.12 [Shi78]. *We have*

$$\Psi^{(\mathfrak{c}_{\zeta_1}\mathfrak{c}_{\zeta_2}p)}(\mathbf{f}_P, \mathbf{E}_l(\zeta_1, \zeta_2)) = (2\pi)^{-2ds}\Gamma(s)^d\Gamma(s+1-l)^d L(s+1-l, \mathbf{f}_P, \zeta_1)L(s, \mathbf{f}_P, \zeta_2). \tag{7.37}$$

Combining (7.20), (7.34)–(7.37), we obtain

$$\begin{aligned}
P(L_p^*(r+1, \mathcal{F}, \chi, \phi, \theta)) &= \frac{1}{\alpha(\mathfrak{c}_{\chi\omega_F^{-r}}, \mathbf{f}_P)}D_F^r\Gamma(r+1)^d\mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi)^{r+1}\\
&\quad \times \frac{L(r+1, \mathbf{f}_P, (\chi\phi\omega_F^{-r})^{-1})}{(-2\pi i)^{dr}\tau((\chi\phi\omega_F^{-r})^{-1})}\frac{T_P(\theta)}{\langle \mathbf{f}_P, \mathbf{f}_P\rangle'_{\mathfrak{n}p^{\alpha_P}}},
\end{aligned}$$

where $T_P(\theta) \in \mathbf{C}$ is independent of χ, r, and depends only on $P, \mathbf{f}_P$, and θ. Although the expression is complicated, all we really need is the fact that it is non-zero, by the assumption that θ is adapted to $\mathbf{f}_P$:

$$\begin{aligned} T_P(\theta) = {}& \pi^{d(k_P-1)}(2\pi)^{-2d(k_P-1)}(-2\pi i)^{-d}\tau(\epsilon_P\theta\omega_F^{1-k})\Gamma(k_P-1)^d \\ & \times \alpha((\mathfrak{c}_{\epsilon_P\theta\omega_F^{1-k_P}})_{,p}^{-1}p^{\alpha_P}, \mathbf{f}_P)\mathcal{N}(\mathfrak{n}^{-1}\mathfrak{c}_\theta^{(p)})^{k_P/2-1}\mathcal{N}(\mathfrak{c}_{\epsilon_P\theta\omega_F^{1-k_P}})^{k_P-2} \\ & \times \mathcal{N}(\mathfrak{c}_\theta p^{\alpha_P})^{1-k_P/2}\mathcal{N}(\mathfrak{c}_{\theta,p})^{k_P/2-1}L(k_P-1, \mathbf{f}_P, (\epsilon_P\theta\omega_F^{1-k_P})^{-1}). \end{aligned}$$

Thus, we obtain the form required for the value $P(L_p^*(r+1, \mathcal{F}, \chi, \phi, \theta))$, by setting

$$\Omega(\mathbf{f}_P, \theta) = \frac{\langle \mathbf{f}_P, \mathbf{f}_P\rangle'_{\mathfrak{n}p^{\alpha_P}}}{T_P(\theta)}. \qquad \square$$

8. Application to exceptional zero conjecture

8.1 In this section, we prove a special case of the exceptional zero conjecture for weight-two Hilbert modular forms, following the argument of Greenberg–Stevens [GS93, GS94]. In the following, for the existence of p-adic Galois representation associated to p-ordinary Hilbert eigenforms we refer to [Wil88, § 2] (the general case was treated by Taylor [Tay89] and Blasius–Rogawski [BR93]).

We first illustrate with the case of modular elliptic curves over totally real fields. Let E/F be an elliptic curve over the totally real field F, such that E has ordinary reduction (i.e. good ordinary or multiplicative) at all places $\mathfrak{p}$ above p.

Let $\rho_E : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbf{Q}_p)$ be the representation on the p-adic Tate module of E. Assume that E is modular, in the sense that there is a Hilbert newform $\mathbf{f}_E$ of weight two, tame level $\mathfrak{n}$, with trivial character, such that $\rho_E \cong \rho_{\mathbf{f}_E}$, where $\rho_{\mathbf{f}_E}$ is the p-adic Galois representation associated to $\mathbf{f}_E$ ($\mathbf{f}_E$ is necessarily p-ordinary). One has the equality of the L-function of E/F, and that of $\mathbf{f}_E$:

$$L(s, E/F) = L(s, \mathbf{f}_E). \tag{8.1}$$

Write $\alpha(\mathfrak{p}, E) = \alpha(\mathfrak{p}, \mathbf{f}_E)$, $\beta(\mathfrak{p}, E) = \beta(\mathfrak{p}, \mathbf{f}_E)$ (thus, $\beta(\mathfrak{p}, E) = \alpha(\mathfrak{p}, E)^{-1}\mathcal{N}\mathfrak{p}$ if E has good reduction at $\mathfrak{p}$, and zero otherwise).

Let $\mathbf{f}$ be the p-stabilization of $\mathbf{f}_E$, cf. (4.15):

$$\mathbf{f} = \mathbf{f}_E \Big| \prod_{\mathfrak{p}|p}(1 - \beta(\mathfrak{p}, E)V(\mathfrak{p})).$$

Then $\mathbf{f}$ is a p-ordinary newform of tame level $\mathfrak{n}$, with $U(\mathfrak{p})$-eigenvalue given by $\alpha(\mathfrak{p}, E)$ for all $\mathfrak{p}$ above p (thus $C(\mathfrak{p}, \mathbf{f}) = \alpha(\mathfrak{p}, E)$). Note that

$$L(s, \mathbf{f}) = \prod_{\mathfrak{p}|p}\left(1 - \frac{\beta(\mathfrak{p}, E)}{(\mathcal{N}\mathfrak{p})^s}\right)L(s, \mathbf{f}_E). \tag{8.2}$$

We can then apply our constructions from previous sections to $\mathbf{f}$. In particular, we fix a choice of θ adapted to $\mathbf{f}$, we have the p-adic L-function $L_p(s, \mathbf{f})$, which we define to be the p-adic L-function of E/F, also noted as $L_p(s, E/F)$. Similarly, we denote the transcendental factor $\Omega(\mathbf{f}, \theta)$ as $\Omega(E)$.

THEOREM 8.1. *Assume that for some place $\mathfrak{p}_0$ of F above p, E is split-multiplicative at the place $\mathfrak{p}_0$; equivalently, some place $\mathfrak{p}_0$ where the $U(\mathfrak{p}_0)$ eigenvalue of $\mathbf{f}$, $\alpha(\mathfrak{p}_0, \mathbf{f})$, is equal to one.*

Denote by $f_{\mathfrak{p}_0/p}$ *the residue field degree of* $F_{\mathfrak{p}_0}$ *over* $\mathbf{Q}_p$*, and by* $q_{E/F_{\mathfrak{p}_0}}$ *the Tate period associated to* $E/F_{\mathfrak{p}_0}$*. Then* $L_p(1, E/F) = 0$*, and we have the formula*

$$\frac{d}{ds} L_p(s, E/F)\Big|_{s=1} = f_{\mathfrak{p}_0/p} \frac{\log_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}_0}}}{\mathrm{ord}_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}_0}}} \times \prod_{\mathfrak{p} \neq \mathfrak{p}_0} \left(1 - \frac{1}{\alpha(\mathfrak{p}, E)}\right) \prod_{\mathfrak{p} | p} \left(1 - \frac{\beta(\mathfrak{p}, E)}{\mathcal{N}\mathfrak{p}}\right) \frac{L(1, E/F)}{\Omega(E)}. \tag{8.3}$$

Here $\log_p$ *is normalized so that* $\log_p(p) = 0$*.*

The quantity

$$f_{\mathfrak{p}_0/p} \frac{\log_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}_0}}}{\mathrm{ord}_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}_0}}}$$

will be denoted as $\mathcal{L}_{\mathfrak{p}_0}(E/F)$*, the* $\mathcal{L}$*-invariant at the place* $\mathfrak{p}_0$*.*

Theorem 8.1 is a special case of Theorem 8.2 below, in the following more general context. Let $\mathbf{f} \in S_2^{\mathrm{ord}}(\mathfrak{n}p, \psi)$ be a p-ordinary newform. For each $\mathfrak{p}$ above p, $\mathbf{f}$ is called multiplicative at $\mathfrak{p}$, if $\mathfrak{p}$ exactly divides the conductor of $\mathbf{f}$, but does not divide the conductor of its character ψ. In this case, we have $\alpha(\mathfrak{p}, \mathbf{f})^2 = 1$, and $\mathbf{f}$ is called split-multiplicative at $\mathfrak{p}$ if, furthermore, we have $\alpha(\mathfrak{p}, \mathbf{f}) = 1$.

Attached to $\mathbf{f}$ is the Hilbert modular p-adic Galois representation of $G_F = \mathrm{Gal}(\bar{F}/F)$:

$$\rho_{\mathbf{f}} : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

that is unramified outside $\mathfrak{n}p$, characterized by the condition

$$\mathrm{tr}\rho_{\mathbf{f}}(\mathrm{Frob}_{\mathfrak{q}}) = C(\mathfrak{q}, \mathbf{f})$$

for primes $\mathfrak{q}$ not dividing $\mathfrak{n}p$ (here $\mathrm{Frob}_{\mathfrak{q}}$ is a Frobenius element at the prime $\mathfrak{q}$). One also has $\det \rho_{\mathbf{f}} = \psi\chi_{\mathrm{cycl}}$, where $\chi_{\mathrm{cycl}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^{\times}$ is the p-adic cyclotomic character.

For $\mathfrak{p}$ above p, let $G_{F_{\mathfrak{p}}} = \mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ be the decomposition group at $\mathfrak{p}$, which by our chosen embedding of $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$, can be regarded as a subgroup of G_F. Then if $\mathbf{f}$ is split multiplicative at $\mathfrak{p}$, we have an exact sequence of $\mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ representations:

$$0 \to \overline{\mathbf{Q}}_p(\chi_{\mathrm{cycl}}) \to \rho_{\mathbf{f}}|_{G_{F_{\mathfrak{p}}}} \to \overline{\mathbf{Q}}_p \to 0. \tag{8.4}$$

This extension defines an element in $H^1(G_{F_{\mathfrak{p}}}, \overline{\mathbf{Q}}_p(\chi_{\mathrm{cycl}}))$, which by Kummer theory is isomorphic to

$$(\varprojlim_n F_{\mathfrak{p}}^{\times}/(F_{\mathfrak{p}}^{\times})^{p^n}) \otimes_{\mathbf{Z}_p} \overline{\mathbf{Q}}_p.$$

It is known that this extension element admits a *Tate period*, in the sense that there exists a (necessarily unique) $q_{\mathbf{f},\mathfrak{p}} \in F_{\mathfrak{p}}^{\times}$, not a p-adic unit, such that the extension element is given by the image of $q_{\mathbf{f},\mathfrak{p}} \otimes 1$.

The author is grateful to Hida for providing the following proof of the claim that the extension has a Tate period. If $[F : \mathbf{Q}]$ is odd, take a quaternion algebra B/F ramified exactly at all but one of the infinite places of F. On the other hand, if $[F : \mathbf{Q}]$ is even, we take a quaternion algebra B/F ramified exactly at the place $\mathfrak{p}$ and all but one of the infinite places. In both cases, the Jacquet–Langlands correspondence allows us to find a Hecke eigenform $\mathbf{f}_B$ on a Shimura curve S_B associated to B, with the same Hecke eigenvalues as our original Hilbert modular form $\mathbf{f}$, cf. [Car86b] (if $[F : \mathbf{Q}]$ is even, we need to use the assumption that $\mathbf{f}$ is multiplicative at $\mathfrak{p}$).

The p-adic Galois representation $\rho_{\mathbf{f}}$ is realized on the p-adic Tate module of the factor A of the Jacobian of S_B, corresponding to $\mathbf{f}_B$. More precisely, let $E = \mathrm{End}(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ be the endomorphism algebra of A. Then $\rho_{\mathbf{f}}$ is isomorphic to $T_pA \otimes_{E\otimes\mathbf{Q}_p} \overline{\mathbf{Q}}_p$, where the map from $E \otimes \mathbf{Q}_p$ to $\overline{\mathbf{Q}}_p$ is given by our fixed embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$.

By Blasius [Bla06], the p-adic Galois representation $\rho_{\mathbf{f}}$ fits into a strictly compatible system, which by uniqueness must coincide with the strictly compatible system $\{V_\lambda = T_lA \otimes_{E\otimes\mathbf{Q}_l} \overline{\mathbf{Q}}_l\}_\lambda$, here λ ranges over the places of E, while l is the place of $\mathbf{Q}$ lying under λ, so λ corresponds to an embedding of E into $\overline{\mathbf{Q}}_l$.

Thus by Blasius' results [Bla06], the assumption that $\mathbf{f}$ is split-multiplicative at $\mathfrak{p}$, implies that for λ not dividing p, $V_\lambda|_{G_{F_\mathfrak{p}}}$ is unipotent. By the semi-stable reduction theorem of Grothendieck [SGA7, Exposé IX, Proposition 3.5], A has to be split-multiplicative, and we have its Tate period.

If $[F : \mathbf{Q}]$ is odd, we can also appeal to the result of Carayol [Car86a], that the level group at $\mathfrak{p}$ is of $\Gamma_0(\mathfrak{p})$-type, so S_B and hence A have to be split-multiplicative reduction at $\mathfrak{p}$, and we have its Tate period.

THEOREM 8.2. *Assume that for some place $\mathfrak{p}_0$ of F above p, $\mathbf{f}$ is split-multiplicative at the place $\mathfrak{p}_0$. Let $q_{\mathbf{f},\mathfrak{p}_0}$ be its Tate period at the place $\mathfrak{p}_0$. Then $L_p(1,\mathbf{f}) = 0$, and we have the formula*

$$\frac{d}{ds}L_p(s,\mathbf{f})\bigg|_{s=1} = \mathcal{L}_{\mathfrak{p}_0}(\mathbf{f}) \prod_{\mathfrak{p}\neq\mathfrak{p}_0}\left(1 - \frac{1}{\alpha(\mathfrak{p},E)}\right)\frac{L(1,\mathbf{f})}{\Omega(\mathbf{f})}, \tag{8.5}$$

where

$$\mathcal{L}_{\mathfrak{p}_0}(\mathbf{f}) = f_{\mathfrak{p}_0/p}\frac{\log_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{\mathbf{f},\mathfrak{p}_0}}{\mathrm{ord}_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{\mathbf{f},\mathfrak{p}_0}}$$

is the $\mathcal{L}$-invariant of $\mathbf{f}$ at the place $\mathfrak{p}_0$.

The fact that Theorem 8.2 implies Theorem 8.1, follows by combining (8.1), (8.2), (8.5), and the well-known result that the Tate period of an elliptic curve, split multiplicative at the prime above p (which is defined using p-adic uniformization), coincides with the Tate period at that prime, defined by using the Galois representation on the p-adic Tate module.

Following Greenberg–Stevens, Theorem 8.2 follows from the properties of the two-variable p-adic L-function attached to the Hida family that lifts $\mathbf{f}$.

In the situation of Theorem 4.4, we take $\mathcal{F} \in \mathcal{S}(\mathfrak{n},\mathcal{I})$ to be a Hida family lifting $\mathbf{f}$, and $P \in \mathfrak{X}^{\mathrm{alg}}(\mathcal{I})$, with $P|_{\Lambda_F} = P_2$, such that $\mathbf{f} = P \circ \mathcal{F}$. We also have $\mathcal{I}_P$ being étale over Λ_{F,P_2} (recall that in this notation, $P_2 = P_{2,\epsilon}$, with $\epsilon = 1$). We have the two-variable p-adic L-function $L_p(s,\mathcal{F})$, which lies in $\mathcal{I}_P$ (see Definition 6.7 and (6.11)).

For this argument, we only need the deformation along the 'cyclotomic direction', i.e. we only need the use the weights $P_k \in \mathfrak{X}^{\mathrm{alg}}(\Lambda_F)$ with $k \geq 2$. Analytically, this can be interpreted as follows.

Let $\mathcal{R}$ be the subring of $\overline{\mathbf{Q}}_p[[\kappa-2]]$ consisting of formal power series in $\kappa-2$, with positive radius of convergence. We have an algebra homomorphism:

$$\widetilde{P_2} : \Lambda_F \to \mathcal{R}$$

by sending an element of the form $\langle[\mathfrak{l}]\rangle$ (with $\mathfrak{l}$ prime to $\mathfrak{n}p$) to the power series in $\mathcal{R}$ representing the analytic function $\kappa \mapsto \langle\mathfrak{l}\rangle_F^{\kappa-2}$ (recall that, in the notation of the paragraph following (4.5)), $\langle\mathfrak{l}\rangle_F = \langle\mathcal{N}(\mathfrak{l})\rangle_{\mathbf{Q}}$ is the composition of the norm map with the projection to the one-units).

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Now the ring $\mathcal{R}$ of convergent power series is Henselian ([Nag62, Theorem 45.5]). Hence, from the property that $\mathcal{I}_P$ is étale over Λ_{F,P_2}, the map $\widetilde{P_2}$ extends uniquely to $\mathcal{I}_P$ (which we still denote as $\widetilde{P_2}$):

$$\widetilde{P_2} : \mathcal{I}_P \to \mathcal{R}.$$

By applying this map to the Fourier coefficients of the Hida family $\mathcal{F}$, we obtain a formal Fourier expansion with coefficients in the ring $\mathcal{R}$, which we can specialize at integer weight $k \geq 2$, with k in a suitable p-adic neighbourhood of two, to obtain the Fourier expansion of a classical eigenform $\mathbf{f}_k$ (note that since the universal p-ordinary Hecke algebra $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})$ is finite over Λ_F, the Fourier coefficients of the eigenform $\mathcal{F}$ is generated over Λ_F by finitely many of them, hence there is a single radius of convergence which works for all of the coefficients).

Notation 8.3. We denote by $\alpha(\mathfrak{p}, \kappa)$ the image of the elements $\alpha(\mathfrak{p}, \mathcal{F})$ in $\mathcal{R}$. Similarly, for $s \in \mathbf{Z}_p$, denote by $L_p(s, \kappa)$ the image of $L_p(s, \mathcal{F})$ in $\mathcal{R}$. Thus, $L_p(s, \kappa)$ is a p-adic analytic function of the two variables s, κ, for $s \in \mathbf{Z}_p$, and κ in some p-adic disc around two. Note that $\alpha(\mathfrak{p}, 2) = \alpha(\mathfrak{p}, E)$, $L_p(s, 2) = L_p(s, \mathbf{f})$.

The theorem then follows from the following series of lemmas. For the first, we need a functional equation.

Proposition 8.4. *We have*

$$L_p(s, \kappa) = \epsilon_p \langle \mathfrak{n} \rangle_F^{\kappa/2 - s} L_p(\kappa - s, \kappa) \tag{8.6}$$

where $\epsilon_p = (-1)^e \epsilon_\infty$, with ϵ_∞ being the sign of the archimedean functional equation associated to the newform whose p-stabilization is $\mathbf{f}$, and e is the number of places above p where $\mathbf{f}$ is split multiplicative (i.e. those places $\mathfrak{p}$ with $\alpha(\mathfrak{p}, \mathbf{f}) = 1$).

Proof. This follows from the archimedean functional equation. For details, see Appendix Appendix B. □

In particular, specializing to $\kappa = 2$, we have

$$L_p(s, \mathbf{f}) = \epsilon_p \langle \mathfrak{n} \rangle_F^{1-s} L_p(2 - s, \mathbf{f}). \tag{8.7}$$

We begin to prove Theorem 8.2. It is clear, from (6.9), that $L_p(1, \mathbf{f}) = 0$. By considering partial derivatives of (8.6) at $(s, \kappa) = (1, 2)$, we obtain the following result.

Lemma 8.5. *We have*

$$\begin{aligned} \frac{\partial}{\partial s} L_p(s, \kappa) \bigg|_{(1,2)} &= 0 \quad \text{if } \epsilon_p = 1, \\ \frac{\partial}{\partial s} L_p(s, \kappa) \bigg|_{(1,2)} &= -2 \frac{\partial}{\partial \kappa} L_p(s, \kappa) \bigg|_{(1,2)} \quad \text{if } \epsilon_p = -1. \end{aligned} \tag{8.8}$$

Now, in the notation of Proposition 8.4, we have $\epsilon_p = (-1)^e \epsilon_\infty$, with $e \geq 1$. First suppose that $\epsilon_p = 1$. If $e = 1$, then $\epsilon_\infty = -1$, so $L(1, \mathbf{f}) = 0$. If $e \geq 2$, then $\prod_{\mathfrak{p} \neq \mathfrak{p}_0} (1 - (1/\alpha(\mathfrak{p}, \mathbf{f}))) = 0$. So, by (8.8), we see that Theorem 8.2 is trivially true if $\epsilon_p = 1$. Thus, we may assume that $\epsilon_p = -1$.

Lemma 8.6. *We have*

$$\frac{\partial}{\partial \kappa} L_p(s, \kappa) \bigg|_{(1,2)} = \frac{d}{d\kappa} \alpha(\mathfrak{p}_0, \kappa) \bigg|_{\kappa = 2} \prod_{\mathfrak{p} \neq \mathfrak{p}_0} \left(1 - \frac{1}{\alpha(\mathfrak{p}, \mathbf{f})} \right) \frac{L(1, \mathbf{f})}{\Omega(\mathbf{f})}. \tag{8.9}$$

Proof. As a consequence of Theorem 6.8, we have

$$L_p(1,\kappa)=\prod_{\mathfrak{p}|p}\left(1-\frac{1}{\alpha(\mathfrak{p},\kappa)}\right)L_p^*(\kappa), \tag{8.10}$$

where $L_p^*(\kappa)\in\mathcal{R}$ satisfies

$$L_p^*(2)=\frac{L(1,\mathbf{f})}{\Omega(\mathbf{f})}. \tag{8.11}$$

Thus, the result follows upon differentiation by using $\alpha(\mathfrak{p}_0,2)=\alpha(\mathfrak{p}_0,\mathbf{f})=1$. □

The following proposition is the final ingredient for the proof of Theorem 8.2.

PROPOSITION 8.7. *We have*

$$\frac{d}{d\kappa}\alpha(\mathfrak{p}_0,\kappa)\bigg|_{\kappa=2}=-\frac{1}{2}f_{\mathfrak{p}_0/p}\frac{\log_p\mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p}q_{\mathbf{f},\mathfrak{p}_0}}{\mathrm{ord}_p\mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p}q_{\mathbf{f},\mathfrak{p}_0}}. \tag{8.12}$$

This was proved by Greenberg–Stevens in the case where $F=\mathbf{Q}$. We follow their strategy.

First we introduce some terminology. Let G be a topological group. Fix a (continuous) representation:

$$\rho: G\to\mathrm{Aut}_{\overline{\mathbf{Q}}_p}(V)$$

with V a finite-dimensional vector space over $\overline{\mathbf{Q}}_p$.

Let $\widetilde{\overline{\mathbf{Q}}}_p$ be the ring of dual numbers over $\overline{\mathbf{Q}}_p$, i.e.

$$\widetilde{\overline{\mathbf{Q}}}_p=\overline{\mathbf{Q}}_p[t]/(t^2).$$

An infinitesimal deformation of ρ is a representation $\widetilde{\rho}$ on a free $\widetilde{\overline{\mathbf{Q}}}_p$-module $\widetilde{V}$:

$$\widetilde{\rho}: G\to\mathrm{Aut}_{\widetilde{\overline{\mathbf{Q}}}_p}(\widetilde{V})$$

such that the G-representations V and $\widetilde{V}/t\widetilde{V}$ are isomorphic.

We are going to apply Hida's deformation theory to construct deformations of Hilbert modular Galois representations.

As in our setting of Theorem 8.2, take a p-ordinary weight two Hilbert eigenform $\mathbf{f}\in S_2^{\mathrm{ord}}(\mathfrak{n}p^\alpha,\psi)$, and a Hida family $\mathcal{F}\in\mathcal{S}^{\mathrm{ord}}(\mathfrak{n},\mathcal{I})$ lifting $\mathbf{f}$. For each $P\in\mathfrak{X}^{\mathrm{alg}}(\mathcal{I})$, let $\mathbf{f}_P$ be the specialization of $\mathcal{F}$ at P. Let $\rho_{\mathbf{f}_P}$ be the p-adic Hilbert modular Galois representation associated to $\mathbf{f}_P$:

$$\rho_{\mathbf{f}_P}:\mathrm{Gal}(\bar{F}/F)\to\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

characterized by the condition that it is unramified outside $\mathfrak{n}p$, and satisfies

$$\mathrm{tr}\rho_{\mathbf{f}_k}(\mathrm{Frob}_{\mathfrak{q}})=C(\mathfrak{q},\mathbf{f}_k)$$

for primes $\mathfrak{q}$ not dividing $\mathfrak{n}p$. These Galois representations can be interpolated and form a deformation of $\rho_{\mathbf{f}}$. This is the content of the following theorem.

THEOREM 8.8 [Wil88, Theorem 2.2.1]. *With notation as above, there is a Galois representation*

$$\rho_{\mathcal{F}}:\mathrm{Gal}(\bar{F}/F)\to\mathrm{GL}_2(Q_{\mathcal{I}})$$

unramified outside $\mathfrak{n}p$*, satisfying*

$$\mathrm{tr}\rho_{\mathcal{F}}(\mathrm{Frob}_{\mathfrak{q}})=C(\mathfrak{q},\mathcal{F})$$

for primes $\mathfrak{q}$ *not dividing* $\mathfrak{n}p$.

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Furthermore, for $P \in \mathfrak{X}^{\mathrm{alg}}(\mathcal{I})$*, the representation* $\rho_{\mathcal{F}}$ *can be conjugated to take values in* $\mathrm{GL}_2(\mathcal{I}_P)$*, and whose specialization at* P*, obtained by composing with the reduction map* $\mathcal{I}_P \to \mathcal{I}_P/P\mathcal{I}_P$*, is isomorphic to* $\rho_{\mathbf{f}_P}$ *(up to extension of coefficient field to* $\overline{\mathbf{Q}}_p$*).*

For the proof of Proposition 8.7, we need to know the structure of $\rho_{\mathcal{F}}|_{G_{F_{\mathfrak{p}_0}}}$. This is given by Theorem 8.9 below (which is the analogue of (8.4)). To state this, we first have some notation.

Let $\langle[\cdot]\rangle : Z_F(\mathfrak{n}) \to \Lambda_F^{\times}$ be the (tautological) character sending $\mathfrak{l}$ to $\langle[\mathfrak{l}]\rangle$ for ideals $\mathfrak{l}$ prime to $\mathfrak{n}p$. By global class field theory, $Z_F(\mathfrak{n})$ is a quotient of $G_F = \mathrm{Gal}(\bar{F}/F)$. Hence, $\langle[\cdot]\rangle$ can be regarded as a character of G_F. From the property of the global Artin map, we see that, as a character of G_F, $P_k \circ \langle[\cdot]\rangle = \langle\chi_{\mathrm{cycl}}\rangle_{\mathbf{Q}}^{k-2}$.

On the local side, let $\sigma_{F_{\mathfrak{p}_0}} : F_{\mathfrak{p}_0}^{\times} \to G_{F_{\mathfrak{p}_0}}^{ab}$ be the local Artin map, normalized so that for π a uniformizer of $F_{\mathfrak{p}_0}$, $\sigma_{F_{\mathfrak{p}_0}}(\pi)$ gives the inverse of Frobenius on unramified extensions of $F_{\mathfrak{p}_0}$. With this normalization, we have, for any $q \in F_{\mathfrak{p}_0}^{\times}$,

$$\begin{aligned}\chi_{\mathrm{cycl}}(\sigma_{F_{\mathfrak{p}_0}}(q)) &= \chi_{\mathrm{cycl}}(\sigma_{\mathbf{Q}_p}(\mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p}(q))) \\ &= (\mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p}(q))p^{-\mathrm{ord}_p(\mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p}(q))}. \end{aligned} \tag{8.13}$$

Now going back to the situation of Theorem 8.8. Fix a $P \in \mathfrak{X}^{\mathrm{alg}}(\mathcal{I})$. Let M be a $\mathcal{I}_P$-lattice of $\rho_{\mathcal{F}}$, i.e. M is a free $\mathcal{I}_P$-module of rank two, with an action of G_F, such that $\rho_{\mathcal{F}} \cong M \otimes Q_{\mathcal{I}}$ as $Q_{\mathcal{I}}[G_F]$-modules. The next theorem gives the structure of M as a $\mathcal{I}_P[G_{F_{\mathfrak{p}_0}}]$-module. In the following, if $\phi : G_{F_{\mathfrak{p}_0}} \to \mathcal{I}_P$ is a character, then we denote by $\mathcal{I}_P(\phi)$ the free $\mathcal{I}_P$-module of rank one, with $G_{F_{\mathfrak{p}_0}}$-action given by ϕ.

Theorem 8.9 [Wil88, Theorem 2.2.2]. *We have an exact sequence of* $\mathcal{I}_P[G_{F_{\mathfrak{p}_0}}]$*-modules:*

$$0 \to \mathcal{I}_P(\chi_{\mathrm{cycl}}\langle[\cdot]\rangle\alpha^{-1}) \to M \to \mathcal{I}_P(\alpha) \to 0 \tag{8.14}$$

with $\alpha : G_{F_{\mathfrak{p}_0}} \to \mathcal{I}$ *is the unramified character sending the Frobenius to* $\alpha(\mathfrak{p}_0, \mathcal{F})$*.*

Proof of Proposition 8.7. We apply Theorem 8.9, taking $P \in \mathfrak{X}^{\mathrm{alg}}(\mathcal{I})$, $P|_{\Lambda_F} = P_2$, with $P \circ \mathcal{F} = \mathbf{f}$. By tensoring the exact sequence (8.14) with the map $\widetilde{P_2} : \mathcal{I}_P \to \mathcal{R}$, we obtain the exact sequence of $\mathcal{R}[G_{F_{\mathfrak{p}_0}}]$-modules:

$$0 \to \mathcal{R}(\chi_{\mathrm{cycl}}\langle[\cdot]\rangle_\kappa\alpha_\kappa^{-1}) \to M \otimes_{\mathcal{I}_P} \mathcal{R} \to \mathcal{R}(\alpha_\kappa) \to 0; \tag{8.15}$$

here $\langle[\cdot]\rangle_\kappa$ and α_κ denote the characters obtained by composing $\langle[\cdot]\rangle$, respectively α, with the map $\widetilde{P_2}$. With $t = \kappa - 2$, we have $\mathcal{R}/(t) = \overline{\mathbf{Q}}_p$, and $\mathcal{R}/(t^2) = \overline{\mathbf{Q}}_p[[t]]/(t^2) = \widetilde{\overline{\mathbf{Q}}}_p$. Hence,

$$\widetilde{\rho}_{\mathbf{f}} := M \otimes_{\mathcal{I}_P} \frac{\mathcal{R}}{(t^2)}$$

is an infinitesimal deformation of

$$\rho_{\mathbf{f}} \cong M \otimes_{\mathcal{I}_P} \frac{\mathcal{R}}{(t)}$$

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and we obtain the following commutative diagram of $\widetilde{\widetilde{\mathbf{Q}}}_p[G_{F_{\mathfrak{p}_0}}]$-modules (the vertical arrows being reduction modulo t).

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\widetilde{\mathbf{Q}}}_p(\chi_{\mathrm{cycl}}\langle[\cdot]\rangle_\kappa\alpha_\kappa^{-1}) \arrow[r] \arrow[d] & \widetilde{\rho}_{\mathbf{f}} \arrow[r] \arrow[d] & \widetilde{\widetilde{\mathbf{Q}}}_p(\alpha_\kappa) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \overline{\mathbf{Q}}_p(\chi_{\mathrm{cycl}}) \arrow[r] & \rho_{\mathbf{f}} \arrow[r] & \overline{\mathbf{Q}}_p \arrow[r] & 0 \end{tikzcd} \tag{8.16}$$

Now twist the upper row by $\langle[\cdot]\rangle_\kappa^{-1}\alpha_\kappa$. Since this character is congruent to one modulo t, we still have the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\widetilde{\mathbf{Q}}}_p(\chi_{\mathrm{cycl}}) \arrow[r] \arrow[d] & \widetilde{\rho}_{\mathbf{f}} \arrow[r] \arrow[d] & \widetilde{\widetilde{\mathbf{Q}}}_p(\langle[\cdot]\rangle_\kappa^{-1}\alpha_\kappa^2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \overline{\mathbf{Q}}_p(\chi_{\mathrm{cycl}}) \arrow[r] & \rho_{\mathbf{f}} \arrow[r] & \overline{\mathbf{Q}}_p \arrow[r] & 0 \end{tikzcd} \tag{8.17}$$

Recall that $q_{\mathbf{f},\mathfrak{p}_0}$ is the Tate period associated to the extension of the bottom row of (8.17). Now we invoke the lemma of Greenberg–Stevens ([GS94, Theorem 2.3.4], or [Hid, Theorem 4.7]): an argument involving local Tate duality [Hid, Theorem 4.7] shows that (8.17) implies

$$\frac{d}{d\kappa}(\langle[\cdot]\rangle_\kappa^{-1}\alpha_\kappa^2)(\sigma_{F_{\mathfrak{p}_0}}(q_{\mathbf{f},\mathfrak{p}_0}))\Big|_{\kappa=2} = 0. \tag{8.18}$$

Denote by $n = \mathrm{ord}_{\mathfrak{p}_0} q_{\mathbf{f},\mathfrak{p}_0} = f_{\mathfrak{p}_0/p}^{-1}\mathrm{ord}_p\mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{\mathbf{f},\mathfrak{p}_0}$. Then using (8.13), we compute

$$\begin{aligned} (\langle[\cdot]\rangle_\kappa^{-1}\alpha_\kappa^2)(\sigma_{F_{\mathfrak{p}_0}}(q_{\mathbf{f},\mathfrak{p}_0})) &= \langle\chi_{\mathrm{cycl}}(\sigma_{F_{\mathfrak{p}_0}}(q_{\mathbf{f},\mathfrak{p}_0}))\rangle_{\mathbf{Q}}^{2-\kappa}\alpha(\kappa,\mathfrak{p}_0)^{-2n} \\ &= \langle\mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p}(q_{\mathbf{f},\mathfrak{p}_0})\rangle_{\mathbf{Q}}^{2-\kappa}\alpha(\kappa,\mathfrak{p}_0)^{-2n}. \end{aligned} \tag{8.19}$$

Using the formula

$$\frac{d}{d\kappa}\langle\mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p}(q_{\mathbf{f},\mathfrak{p}_0})\rangle_{\mathbf{Q}}^{2-\kappa}\Big|_{\kappa=2} = -\log_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{\mathbf{f},\mathfrak{p}_0}$$

we see that (8.18) gives (upon differentiating (8.19))

$$-\log_p \mathcal{N}_{F_{\mathfrak{p}_0}/\mathbf{Q}_p} q_{\mathbf{f},\mathfrak{p}_0} - 2n\frac{d}{d\kappa}\alpha(\mathfrak{p}_0,\kappa)\Big|_{\kappa=2} = 0.$$

This finishes the proof of Proposition 8.7, and hence Theorem 8.2. □

Remark 8.10. In the definition of p-adic L-function for $\mathbf{f}$, there is an inherent choice for the transcendental factor $\Omega(\mathbf{f})$. It is clear that if $\Omega(\mathbf{f})'$ is a complex number such that $\Omega(\mathbf{f})'/\Omega(\mathbf{f}) \in \overline{\mathbf{Q}}^\times$, then it can also be used to define the p-adic L-function of $\mathbf{f}$, by an appropriate scaling. Theorem 8.1 is unaffected by this choice.

9. Exceptional zeros of higher order

The method of Greenberg–Stevens works only for exceptional zeros of order one. More generally, we have the following conjecture of Greenberg [Gre94] and Hida [Hid].

Let E/F be a modular elliptic curve over F. Assume that E is ordinary at all places above p, and let e be the number of places above p over which E is split-multiplicative. Equivalently, e is the number of places $\mathfrak{p}|p$ such that $\alpha(\mathfrak{p}, E) = 1$.

Let $\Omega(E)$ be the period which we used to define the p-adic L-function of E/F.

Conjecture 9.1. We have

$$L_p(s, E/F) = \mathcal{L}_p(E/F) \prod_{\substack{\mathfrak{p}|p \\ \alpha(\mathfrak{p},E)\neq 1}} \left(1 - \frac{1}{\alpha(\mathfrak{p}, E)}\right) \times \prod_{\mathfrak{p}|p}\left(1 - \frac{\beta(\mathfrak{p}, E)}{\mathcal{N}\mathfrak{p}}\right)\frac{L(1, E/F)}{\Omega(E)}(s-1)^e + \text{higher-order terms},$$

where $\mathcal{L}_p(E/F)$ is the $\mathcal{L}$-invariant of E, defined as follows:

$$\mathcal{L}_p(E/F) = \prod_{\substack{\mathfrak{p}|p \\ \alpha(\mathfrak{p},E)=1}} \mathcal{L}_{\mathfrak{p}}(E/F)$$

with

$$\mathcal{L}_{\mathfrak{p}}(E/F) = f_{\mathfrak{p}/p}\frac{\log_p \mathcal{N}_{F_{\mathfrak{p}}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}}}}{\mathrm{ord}_p \mathcal{N}_{F_{\mathfrak{p}}/\mathbf{Q}_p} q_{E/F_{\mathfrak{p}}}}$$

for prime $\mathfrak{p}$, where E becomes split-multiplicative, with Tate period $q_{E/F_{\mathfrak{p}}} \in F_{\mathfrak{p}}$.

Using the same technique as in § 8, one sees that the conjecture is equivalent to the following assertion: if $L_p(s, \kappa)$ is the associated two-variable p-adic L-function, then

$$\frac{\partial^i}{\partial s^i} L_p(s, \kappa)\bigg|_{(1,2)} = 0 \quad \text{for } i < e,$$

$$\frac{\partial^e}{\partial s^e} L_p(s, \kappa)\bigg|_{(1,2)} = (-2)^e \frac{\partial^e}{\partial \kappa^e} L_p(s, \kappa)\bigg|_{(1,2)}.$$

In the rest this section we examine the situation for forms base changed from $\mathbf{Q}$ to an abelian extension F.

Proposition 9.2. *Let $E/\mathbf{Q}$ be a (modular) elliptic curve over $\mathbf{Q}$, split-multiplicative at p. Let F be a totally real finite abelian extension of $\mathbf{Q}$, with p unramified in F. Then the conjecture is true for E/F.*

In fact, this proposition will follow as a corollary of the factorization formula of p-adic L-functions.

Proposition 9.3. *Let F be a totally real finite abelian extension of $\mathbf{Q}$, with p unramified in F. Let $H = \mathrm{Gal}(F/\mathbf{Q})$, $\widehat{H}$ its character group, identified as the group of Dirichlet characters associated to $F/\mathbf{Q}$. Given f a weight-two newform over $\mathbf{Q}$, let $\mathbf{f}$ be the base change of f to F. Suppose that $\Omega(f \otimes \phi)$ is the period used to define the p-adic L-function of $f \otimes \phi$. Then we can define the p-adic L-function of $\mathbf{f}$ by taking the period $\Omega(\mathbf{f})$ as $\prod_{\phi\in\widehat{H}} \Omega(f \otimes \phi)$. With this choice, we have the factorization*

$$L_p(s, \mathbf{f}) = \langle D_F\rangle_{\mathbf{Q}}^{s-1} \prod_{\phi\in\widehat{H}} L_p(s, f \otimes \phi).$$

Proof that Proposition 9.3 implies Proposition 9.2. Thus, let E be a elliptic curve over $\mathbf{Q}$, split-multiplicative at p, with Tate period $q_E \in \mathbf{Q}_p$. Let f be the weight-two newform associated to E, which is a p-ordinary newform. Let $\mathbf{f}$ its base change to F. Here $\mathbf{f}$ is again p-ordinary. In fact,

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we have the following relation

$$\alpha(\mathfrak{p}, \mathbf{f}) = \alpha(\mathcal{N}\mathfrak{p}, f) \quad \text{for } \mathfrak{p}|p. \tag{9.1}$$

By the original theorem of Greenberg–Stevens [GS93] (or Theorem 8.2 in the case $F = \mathbf{Q}$), the exceptional zero conjecture is true for those forms $f \otimes \phi$ which are split-multiplicative at p. Thus, for $\phi \in \widehat{H}$ with $\phi(p) = 1$, then

$$L_p(s, f \otimes \phi) = \mathcal{L}_p(f \otimes \phi) \frac{L(1, f \otimes \phi)}{\Omega(f \otimes \phi)}(s-1) + \text{higher-order terms} \tag{9.2}$$

with

$$\mathcal{L}_p(f \otimes \phi) = \mathcal{L}_p(f) = \mathcal{L}_p(E/\mathbf{Q}) = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}.$$

On the other hand, if $\phi(p) \neq 1$, then

$$L_p(s, f \otimes \phi) = \left(1 - \frac{1}{\phi(p)}\right) \frac{L(1, f \otimes \phi)}{\Omega(f \otimes \phi)} + \text{higher-order terms.} \tag{9.3}$$

In our case e is exactly the number of places of F above p, and an elementary argument (for example, [Was97, Theorem 3.7]) shows that it is equal to the number of $\phi \in \widehat{H}$ such that $\phi(p) = 1$. Furthermore, the residue field extension degree $f_{\mathfrak{p}/p} = d/e$ is the same for all of the primes $\mathfrak{p}$ above p, and similar considerations give

$$\prod_{\phi \in \widehat{H}, \phi(p) \neq 1} \left(1 - \frac{1}{\phi(p)}\right) = f_{\mathfrak{p}/p}^e. \tag{9.4}$$

Now, from Proposition 9.3,

$$L_p(s, E/F) = L_p(s, \mathbf{f}) = \prod_{\phi \in \widehat{H}, \phi(p) \neq 1} \left(1 - \frac{1}{\phi(p)}\right) \prod_{\phi \in \widehat{H}, \phi(p) = 1} \mathcal{L}_p(E/\mathbf{Q}) \prod_{\phi \in \widehat{H}} \frac{L(1, f \otimes \phi)}{\Omega(f \otimes \phi)}(s-1)^e + \text{higher-order terms.} \tag{9.5}$$

We have

$$\begin{aligned} \prod_{\phi \in \widehat{H}, \phi(p) \neq 1} \left(1 - \frac{1}{\phi(p)}\right) \prod_{\phi \in \widehat{H}, \phi(p) = 1} \mathcal{L}_p(E/\mathbf{Q}) &= (f_{\mathfrak{p}/p}\mathcal{L}_p(E/\mathbf{Q}))^e = \prod_{\mathfrak{p}|p} f_{\mathfrak{p}/p} \frac{\log_p q_E}{\mathrm{ord}_p q_E} \\ &= \prod_{\mathfrak{p}|p} f_{\mathfrak{p}/p} \frac{\log_p \mathcal{N}_{F_\mathfrak{p}/\mathbf{Q}_p} q_{E/F_\mathfrak{p}}}{\mathrm{ord}_p \mathcal{N}_{F_\mathfrak{p}/\mathbf{Q}_p} q_{E/F_\mathfrak{p}}} = \prod_{\mathfrak{p}|p} \mathcal{L}_\mathfrak{p}(E/F) \\ &= \mathcal{L}_p(E/F). \end{aligned} \tag{9.6}$$

By the classical factorization,

$$L(1, E/F) = L(1, \mathbf{f}) = \prod_{\phi \in \widehat{H}} L(1, f \otimes \phi) \tag{9.7}$$

the proposition follows, if we choose $\prod_{\phi \in \widehat{H}} \Omega(f \otimes \phi)$ as a period to define the p-adic L-function for E/F. □

For the proof of Proposition 9.3, we need a Gauss sum identity, analogous to that of Hasse–Davenport. In the following, for a Dirichlet character $\chi_\mathbf{Q}$, we denote by $c_{\chi_\mathbf{Q}}$ its conductor (as opposed to a German Gothic letter for Hecke characters over F).

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Lemma 9.4. *Let $\chi_{\mathbf{Q}}$ be a Dirichlet character, i.e. a finite-order Hecke character over $\mathbf{Q}$, whose conductor $c_{\chi_{\mathbf{Q}}}$ is prime to D_F. Let χ_F be the Hecke character of F obtained by composing with the norm map $\mathcal{N}$. Then we have the relation between Gauss sums:*

$$\tau(\chi_F) = \bigg(\prod_{\phi \in \widehat{H}} \phi(c_{\chi_{\mathbf{Q}}})\bigg)\chi(D_F)(\tau(\chi_{\mathbf{Q}}))^d. \tag{9.8}$$

Proof. This can be proved in the same way that Hasse deduced the conductor-discriminant relation from the functional equations for L-functions of Dedekind and that of Dirichlet (see e.g. [Was97, ch. 4]). By working with Hecke L-functions, we obtain the proof. For the detailed computations, the reader is referred to [Mok07]. □

Proof of Proposition 9.3. We follow the arguments of Gross [Gro80]. Let $\Omega(\mathbf{f})$ be a period used to define the p-adic L-function of $\mathbf{f}$ (say, given by Corollary 6.9). Let ν be the measure on $\overline{Z}_F(\mathfrak{r})$ corresponding to this p-adic L-function. Similarly, the p-adic L-functions of the forms $f \otimes \phi$ define measures ν_ϕ on the group $\overline{Z}_{\mathbf{Q}}(1)$. Finally, let ν_0 be the measure on $\overline{Z}_{\mathbf{Q}}(1)$ such that

$$\nu_0(\chi_{\mathbf{Q}}) = \chi_{\mathbf{Q}}(D_F)$$

for $\chi_{\mathbf{Q}}$ a character of $\overline{Z}_{\mathbf{Q}}(1)$.

The norm map extends to give $\mathcal{N} : \overline{Z}_F(\mathfrak{r}) \to \overline{Z}_{\mathbf{Q}}(1)$. By taking the push forward, we obtain a measure $\mathcal{N}_*(\nu)$ on $\overline{Z}_{\mathbf{Q}}(1)$. We would like to compare the following measures:

$$\mathcal{N}_*(\nu), \tag{9.9}$$

$$\nu_0 \prod_{\phi \in \widehat{H}} \nu_\phi, \tag{9.10}$$

here the product is the convolution product of measures on $\overline{Z}_{\mathbf{Q}}(1)$.

For this purpose, we evaluate (9.9) and (9.10) on all finite-order characters of $\overline{Z}_{\mathbf{Q}}(1)$. Let $\chi_{\mathbf{Q}}$ be such a character. Then by (6.9)

$$\begin{aligned}\mathcal{N}_*(\nu)(\chi_{\mathbf{Q}}) &= \nu(\chi_F) = L_p(1, \mathbf{f}, \chi_F)\\ &= \bigg(\prod_{\mathfrak{p}|p}\bigg(1 - \frac{\chi_F(\mathfrak{p})}{\alpha(\mathfrak{p}, \mathbf{f})}\bigg)\bigg)\frac{1}{\alpha(\mathfrak{c}_{\chi_F}, \mathbf{f})}\mathcal{N}(\mathfrak{c}_{\chi_F})\frac{L(1, \mathbf{f}, (\chi_F)^{-1})}{\tau((\chi_F)^{-1})\Omega(\mathbf{f})}.\end{aligned} \tag{9.11}$$

Similarly, noting that $\alpha(p, f \otimes \phi) = \phi(p)\alpha(p, f)$, we have

$$L_p(1, f \otimes \phi, \chi_{\mathbf{Q}}) = \bigg(1 - \frac{\chi_{\mathbf{Q}}(p)}{\phi(p)\alpha(p, f)}\bigg)\frac{1}{\phi(c_{\chi_{\mathbf{Q}}})\alpha(c_{\chi_{\mathbf{Q}}}, f)}c_{\chi_{\mathbf{Q}}}\frac{L(1, f \otimes \phi, (\chi_{\mathbf{Q}})^{-1})}{\tau((\chi_{\mathbf{Q}})^{-1})\Omega(f \otimes \phi)}. \tag{9.12}$$

We now compare the terms in the formulae. As before $f_{\mathfrak{p}/p}$ is the residue field extension degree of the primes $\mathfrak{p}$ above p. We have

$$\begin{aligned}\prod_{\mathfrak{p}|p}\bigg(1 - \frac{\chi_F(\mathfrak{p})}{\alpha(\mathfrak{p}, \mathbf{f})}\bigg) &= \prod_{\mathfrak{p}|p}\bigg(1 - \frac{\chi_{\mathbf{Q}}(\mathcal{N}(\mathfrak{p}))}{\alpha(\mathcal{N}(\mathfrak{p}), f)}\bigg) \quad \text{(by (9.1))}\\ &= \prod_{\mathfrak{p}|p}\bigg(1 - \frac{\chi_{\mathbf{Q}}(p)^{f_{\mathfrak{p}/p}}}{\alpha(p, f)^{f_{\mathfrak{p}/p}}}\bigg)\\ &= \prod_{\phi \in \widehat{H}}\bigg(1 - \frac{\chi_{\mathbf{Q}}(p)}{\phi(p)\alpha(p, f)}\bigg)\end{aligned} \tag{9.13}$$

where the last equality again follows from [Was97, Theorem 3.7].

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Since the conductor of χ_Q is relatively prime to D_F, we have $\mathfrak{c}_{\chi_F} = c_{\chi_{\mathbf{Q}}}\mathfrak{r}$, and hence

$$\mathcal{N}(\mathfrak{c}_{\chi_F}) = (c_{\chi_{\mathbf{Q}}})^d.$$

Hence

$$\alpha(\mathfrak{c}_{\chi_F}, \mathbf{f}) = \alpha(\mathcal{N}(\mathfrak{c}_{\chi_F}), f) = \alpha(c_{\chi_{\mathbf{Q}}}, f)^d. \tag{9.14}$$

On the other hand, Lemma 9.4 gives

$$\tau((\chi_F)^{-1}) = \left(\prod_{\phi\in\widehat{H}} \phi(c_{\chi_{\mathbf{Q}}})\right)\ \chi_{\mathbf{Q}}^{-1}(D_F)\tau((\chi_{\mathbf{Q}})^{-1})^d. \tag{9.15}$$

If we combine with the factorization

$$L(1, \mathbf{f}, \chi_F^{-1}) = \prod_{\phi\in\widehat{H}} L(1, f\otimes\phi, \chi_{\mathbf{Q}}^{-1}) \tag{9.16}$$

we see immediately that (9.10), when evaluated on $\chi_{\mathbf{Q}}$, gives

$$\begin{aligned}
&\chi_{\mathbf{Q}}(D_F)\prod_{\phi\in\widehat{H}} L_p(1, f\otimes\phi, \chi_{\mathbf{Q}})\\
&\quad= \chi_{\mathbf{Q}}(D_F)\prod_{\phi\in\widehat{H}}\left[\left(1 - \frac{\chi_{\mathbf{Q}}(p)}{\phi(p)\alpha(p, f)}\right)\frac{1}{\phi(c_{\chi_{\mathbf{Q}}})\alpha(c_{\chi_{\mathbf{Q}}}, f)}c_{\chi_{\mathbf{Q}}}\frac{L(1, f\otimes\phi, (\chi_{\mathbf{Q}})^{-1})}{\tau((\chi_{\mathbf{Q}})^{-1})\Omega(f\otimes\phi)}\right]\\
&\quad= \left(\prod_{\mathfrak{p}|p}\left(1 - \frac{\chi_F(\mathfrak{p})}{\alpha(\mathfrak{p}, \mathbf{f})}\right)\right)\frac{1}{\alpha(\mathfrak{c}_{\chi_F}, \mathbf{f})}\mathcal{N}(\mathfrak{c}_{\chi_F})\frac{L(1, \mathbf{f}, (\chi_F)^{-1})}{\tau((\chi_F)^{-1})\prod_{\phi\in\widehat{H}}\Omega(f\otimes\phi)}\\
&\quad= \frac{\Omega(\mathbf{f})}{\prod_{\phi\in\widehat{H}}\Omega(f\otimes\phi)}L_p(1, \mathbf{f}, \chi_F).
\end{aligned}$$

By Theorem 6.11, we can choose $\chi_{\mathbf{Q}}$, such that $L_p(1, f\otimes\phi, \chi_{\mathbf{Q}}) \neq 0$ for all $\phi\in\widehat{H}$. In particular we draw the following conclusion:

$$\frac{\Omega(\mathbf{f})}{\prod_{\phi\in\widehat{H}}\Omega(f\otimes\phi)} \in \overline{\mathbf{Q}}^{\times}.$$

Thus, we can use $\prod_{\phi\in\widehat{H}}\Omega(f\otimes\phi)$ as the period $\Omega(\mathbf{f})$ to define the p-adic L-function for $\mathbf{f}$. With this choice, (9.9) and (9.10) are equal when evaluated on any $\chi_{\mathbf{Q}}$. Hence, we conclude that

$$\mathcal{N}_*(\nu) = \nu_0\prod_{\phi\in\widehat{H}}\nu_\phi. \tag{9.17}$$

The proposition is proved by evaluating (9.17) at the character $\langle\cdot\rangle_{\mathbf{Q}}^{s-1}$.

Remark 9.5. Proposition 9.2 has an obvious generalization to Hilbert modular forms, which are obtained as base change from elliptic modular forms of weight two, split multiplicative at p. We leave this to the reader.

10. Concluding remarks

1. *Exceptional zero conjecture for higher weight forms.* Forms of weight greater than two that have exceptional zeros are no longer ordinary at p, and one needs to have a deformation theory for non-ordinary forms, and the construction of two-variable p-adic L-function associated to

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such families. In the case where $F = \mathbf{Q}$, this was constructed by Panchishkin [Pan03], using Rankin's method. Stevens also obtained a construction using the theory of over-convergent modular symbols. Recently, Emerton obtained a construction using the theory of locally analytic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. If one assumes, for example, that the prime p splits in the totally real field F, then his arguments may apply in the more general situation.

2. *Exceptional zero for the symmetric square of an elliptic curve.* Dabrowski–Delbourgo [DD97] constructed a p-adic L-function attached to the symmetric square of an elliptic modular form, using Rankin's method (by convoluting with a half-integral weight theta series). Their method should generalize to give a two-variable p-adic L-function of the symmetric square, Hilbert modular forms. This would have applications to the exceptional zero conjecture for symmetric square (the elliptic case is proved in an unpublished work of Greenberg–Tilouine).

3. The exceptional zero conjecture is consistent with the 'main conjecture'. In [Gre94], Greenberg studied the arithmetic p-adic L-function, constructed from Selmer group attached to the Galois representation. The main conjecture predicts that it agrees with the analytic p-adic L-function, up to an invertible function. He showed that the exceptional zero conjecture holds, with an arithmetic definition of $\mathcal{L}$-invariant conjecturally given by the formula as in (9.1).

4. For a connection between exceptional zeros, and extensions of p-adic automorphic representations, see Hida [Hid06].

5. The method of using Rankin integral representation to study algebraicity of zeta values has been generalized by Shimura [Shi00], to automorphic forms on unitary groups. One particular case, called UT (the associated symmetric space being a tube domain), has the appeal that the method is close to the Hilbert modular case. Recently, Hida [Hid] was able to extend his theory to these groups. Thus, one might be able to establish exceptional zero conjecture for these forms.

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Appendix A. Proofs of Propositions 5.5 and 5.6

In this appendix, we give the proofs of Propositions 5.5 and 5.6. The calculations can already be found in [Pan03] in the elliptic modular case, so it is included only for convenience of the reader.

Proof of Proposition 5.5. We analyse the Fourier coefficients of the form that appears in (5.14):

$$\mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|e. \tag{A1}$$

Following [Pan03], we use the trick that $U(p)$ is invertible on the image of e. So for fixed $\alpha \geq 1$, (A1) is equal to

$$\begin{aligned}&\mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|eU(p^\alpha)U(p^\alpha)^{-1}\\ &\quad= \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|U(p^\alpha)eU(p^\alpha)^{-1}.\end{aligned} \tag{A2}$$

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Thus, it suffices to interpolate

$$\mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|U(p^\alpha). \tag{A3}$$

We now calculate the Fourier coefficients of (A3). Thus, let $\xi \in F$, $\xi \gg 0$, $y \in \mathbf{A}_F^\times$, such that $\xi y\mathfrak{r} \subset \mathfrak{r}$, and without loss of generality we may assume that $y\mathfrak{r}$ is prime to p. From (2.16), and the definition of $U(p)$ in (2.23), we see that the Fourier coefficients of (A3) at the ideal $\xi y\mathfrak{r}$, is related to that of $\mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}$, and $\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)$, as follows:

$$\begin{aligned}
&C(\xi y\mathfrak{r}, \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|U(p^\alpha))\\
&\quad = C(p^\alpha\xi y\mathfrak{r}, \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p))\\
&\quad = \sum_{\substack{\xi_1+\xi_2=p^\alpha\xi\\ \xi_1,\xi_2\gg 0}} C(\xi_1 y\mathfrak{r}, \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00})C(\xi_2 y\mathfrak{r}, \mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p));\\
&C_0(\mathfrak{m}, \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|U(p)) = 0.
\end{aligned} \tag{A4}$$

In this calculation, we are implicitly using the fact, clear from (5.4), that

$$C_0(\mathfrak{m}, \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}) = 0$$
$$C(\xi_1 y\mathfrak{r}, \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}) = 0$$

if ξ_1 is not relatively prime to p. Thus, the constant term of $\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)$ (see (3.14)) does not enter (A4).

Substituting (3.14) and (5.4) into (A4), we obtain

$$\begin{aligned}
&C(\xi y\mathfrak{r}, \mathbf{E}_{k-r-1}(\chi\phi\omega_F^{-r}, \epsilon\theta\omega_F^{1-k})^{00}\mathbf{G}_{r+1}(\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}, \mathfrak{n}\mathfrak{c}_\phi\mathfrak{c}_\theta p)|U(p))\\
&= \sum_{\substack{\xi_1+\xi_2=p^\alpha\xi\\ \xi_1,\xi_2\gg 0}} \left(\sum_{\substack{\mathfrak{a}\mathfrak{b}=\xi_1 y\mathfrak{r}\\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r}, \mathfrak{a}\mathfrak{b}+p\mathfrak{r}=\mathfrak{r}}} \chi\phi\omega_F^{-r}(\mathfrak{a})\epsilon\theta\omega_F^{1-k}(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{k-r-2} \right.\\
&\qquad \left. \times \sum_{\substack{\xi_2=ed\\ e\in y^{-1}\mathfrak{r}, d\in\mathfrak{r}\\ d\mathfrak{r}+p\mathfrak{r}=\mathfrak{r}, d \bmod \mathfrak{r}^\times}} \mathcal{N}(ey\mathfrak{r})^r\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}(d\mathfrak{r}) \right).
\end{aligned} \tag{A5}$$

Thus, if we define

$$\mathcal{K}_\alpha(\chi, \phi, \theta, r)^{00} \in \mathcal{M}(\mathfrak{n}\mathfrak{c}_\phi^{(p)}\mathfrak{c}_\theta^{(p)}, \psi, \Lambda_F)\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\chi]$$

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by

$$C(\xi y\mathfrak{r}, \mathcal{K}_\alpha(\chi,\phi,\theta,r)^{00}) = \sum_{\substack{\xi_1+\xi_2=p^\alpha\xi \\ \xi_1,\xi_2\gg 0}} \left(\sum_{\substack{\mathfrak{ab}=\xi_1 y\mathfrak{r} \\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{ab}+p\mathfrak{r}=\mathfrak{r}}} \chi\phi\omega_F^{-r}(\mathfrak{a})\theta\omega_F^{-1}(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{-r}\langle[\mathfrak{b}]\rangle \right.$$

$$\left. \times \sum_{\substack{\xi_2=ed \\ e\in y^{-1}\mathfrak{r},d\in\mathfrak{r} \\ d\mathfrak{r}+p\mathfrak{r}=\mathfrak{r},d \bmod \mathfrak{r}^\times}} \mathcal{N}(ey\mathfrak{r})^r\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}(d\mathfrak{r}) \right)$$

$$C_0(\mathfrak{m}, \mathcal{K}_\alpha(\chi,\phi,\theta,r)^{00}) = 0 \tag{A6}$$

then $\mathcal{K}_\alpha(\chi,\phi,\theta,r)^{00}$ specializes to (A3), at $P_{k,\epsilon}$, $k\geq r+2$. It thus suffices to set

$$\mathcal{H}(\chi,\phi,\theta,r)^{00} := \mathcal{K}_\alpha(\chi,\phi,\theta,r)^{00}|eU(p^\alpha)^{-1} \tag{A7}$$

(which is clearly independent of the choice of α).

The assertions about $\mathcal{H}(\chi,\phi,\theta,r)^0$ is proved similarly. □

Proof of Proposition 5.6. Define the distributions $\mu_{\phi,\theta,r}$ as in (5.15). We need to check that it is a measure, i.e. bounded with respect to the norm (5.13). Thus, we consider a coset of the form $x+\overline{Z}_{F,\alpha}(\mathfrak{r})$. We need to calculate $\mu_{\phi,\theta,r}(x+\overline{Z}_{F,\alpha}(\mathfrak{r}))$, the value of $\mu_{\phi,\theta,r}$ on $x+\overline{Z}_{F,\alpha}(\mathfrak{r})$: denoting by h_α the cardinality of $\overline{Z}_F(\mathfrak{r})/\overline{Z}_{F,\alpha}(\mathfrak{r}) \cong \overline{\mathrm{Cl}}_{\mathrm{F}}(p^\alpha\mathfrak{r})$, we have

$$\begin{aligned} \mu_{\phi,\theta,r}(x+\overline{Z}_{F,\alpha}(\mathfrak{r})) &= \sum_{\chi \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})} \frac{\chi^{-1}(x)}{h_\alpha}\mathcal{H}(\chi,\phi,\theta,r)^{00} \\ &= \sum_{\chi \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})} \frac{\chi^{-1}(x)}{h_\alpha}\mathcal{K}_\alpha(\chi,\phi,\theta,r)^{00}|eU(p)^{-1} \end{aligned} \tag{A8}$$

(the notation $\chi \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})$ means that χ runs over the characters of $\overline{Z}_F(\mathfrak{r})$ factoring through $\overline{Z}_{F,\alpha}(\mathfrak{r})$, i.e. characters of $\overline{\mathrm{Cl}}_{\mathrm{F}}(p^\alpha\mathfrak{r})$). Thus, it suffices to show that

$$\sum_{\chi \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})} \frac{\chi^{-1}(x)}{h_\alpha}\mathcal{K}_\alpha(\chi,\phi,\theta,r)^{00} \tag{A9}$$

is uniformly bounded.

Using (A6) and the notation there, we have

$$C\left(\xi y\mathfrak{r}, \sum_{\chi \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})} \frac{\chi^{-1}(x)}{h_\alpha}\mathcal{K}_\alpha(\chi,\phi,\theta,r)^{00}\right)$$

$$= \sum_{\chi \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})} \frac{\chi^{-1}(x)}{h_\alpha} C(\xi y\mathfrak{r}, \mathcal{K}_\alpha(\chi,\phi,\theta,r)^{00})$$

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$$= \sum_{\chi \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})} \frac{\chi^{-1}(x)}{h_\alpha} \sum_{\substack{\xi_1+\xi_2=p^\alpha\xi \\ \xi_1,\xi_2\gg 0}} \left(\sum_{\substack{\mathfrak{a}\mathfrak{b}=\xi_1 y\mathfrak{r} \\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{a}\mathfrak{b}+p\mathfrak{r}=\mathfrak{r}}} \chi\phi\omega_F^{-r}(\mathfrak{a})\theta\omega_F^{-1}(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{-r}\langle[\mathfrak{b}]\rangle \right.$$

$$\left. \times \sum_{\substack{\xi_2=ed \\ e\in y^{-1}\mathfrak{r},d\in\mathfrak{r} \\ d\mathfrak{r}+p\mathfrak{r}=\mathfrak{r},d \bmod \mathfrak{r}^\times}} \mathcal{N}(ey\mathfrak{r})^r\psi(\chi\phi\theta)^{-1}\omega_F^{r+1}(d\mathfrak{r}) \right). \tag{A10}$$

Now interchange the order of summation: take the sum over χ mod $\overline{Z}_{F,\alpha}(\mathfrak{r})$ first. The finite sum

$$\sum_{\chi \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})} \chi(\mathfrak{a}d^{-1}x^{-1})$$

is zero, unless $\mathfrak{a} = dx\mathfrak{r} \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})$, in which case the sum is equal to h_α. Hence, the expression (A10) is equal to

$$\sum_{\substack{\xi_1+\xi_2=p^\alpha\xi \\ \xi_1,\xi_2\gg 0}} \left(\sum_{\substack{\mathfrak{a}\mathfrak{b}=\xi_1 y\mathfrak{r} \\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{a}\mathfrak{b}+p\mathfrak{r}=\mathfrak{r}}} \phi\omega_F^{-r}(\mathfrak{a})\theta\omega_F^{-1}(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{-r}\langle[\mathfrak{b}]\rangle \right.$$

$$\left. \times \sum_{\substack{\xi_2=ed \\ e\in y^{-1}\mathfrak{r},d\in\mathfrak{r} \\ d\mathfrak{r}+p\mathfrak{r}=\mathfrak{r},d \bmod \mathfrak{r}^\times \\ \mathfrak{a}=dx\mathfrak{r} \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})}} \mathcal{N}(ey\mathfrak{r})^r\psi(\phi\theta)^{-1}\omega_F^{r+1}(d\mathfrak{r}) \right) \tag{A11}$$

which is clearly uniformly bounded (independent of the coset $x + \overline{Z}_{F,\alpha}(\mathfrak{r})$).

Next we show the integration formula (5.16): let η be a finite-order character of $\overline{Z}_F(\mathfrak{r})$, then we would like to prove

$$\int_{\overline{Z}_F(\mathfrak{r})} \eta\langle\cdot\rangle_F^r \, d\mu_{\phi,\theta,0} = (-1)^{rd} \int_{\overline{Z}_F(\mathfrak{r})} \eta \, d\mu_{\phi,\theta,r}. \tag{A12}$$

This is equivalent to showing that for α large enough so that η factors through $\overline{Z}_\alpha(\mathfrak{r})$,

$$\sum_{x\in\overline{Z}/\overline{Z}_{F,\alpha}} \eta(x)\langle x\rangle_F^r \, \mu_{\phi,\theta,0}(x + \overline{Z}_{F,\alpha}(\mathfrak{r})) = (-1)^{rd} \int_{\overline{Z}(\mathfrak{r})} \eta \, d\mu_{\phi,\theta,r} + o(1). \tag{A13}$$

Here, by $o(1)$, we mean elements of $\mathcal{M}(\mathrm{lcm}(\mathfrak{n}, \mathfrak{c}_\phi^{(p)}\mathfrak{c}_\theta^{(p)}), \psi, \Lambda_F)$, whose norm goes to zero as $\alpha \to \infty$.

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With the notation of (A8), we show

$$\sum_{x\in\overline{Z}_F(\mathfrak{r})/\overline{Z}_{F,\alpha}(\mathfrak{r})}\eta(x)\langle x\rangle_F^r\sum_{\chi \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})}\frac{\chi^{-1}(x)}{h_\alpha}\mathcal{K}_\alpha(\chi,\phi,\theta,0)^{00}$$
$$=(-1)^{rd}\mathcal{K}_\alpha(\chi,\phi,\theta,r)^{00}+o(1) \tag{A14}$$

which clearly implies (A13).

By (A10), the Fourier coefficient of the left-hand side of (A14), at the ideal $\xi y\mathfrak{r}$, is given by

$$\sum_{x\in\overline{Z}_F(\mathfrak{r})/\overline{Z}_{F,\alpha}(\mathfrak{r})}\eta(x)\langle x\rangle_F^r\sum_{\substack{\xi_1+\xi_2=p^\alpha\xi\\ \xi_1,\xi_2\gg 0}}\left(\sum_{\substack{\mathfrak{a}\mathfrak{b}=\xi_1 y\mathfrak{r}\\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{a}\mathfrak{b}+p\mathfrak{r}=\mathfrak{r}}}\phi(\mathfrak{a})\theta\omega_F^{-1}(\mathfrak{b})\langle[\mathfrak{b}]\rangle\right.$$
$$\left.\times\sum_{\substack{\xi_2=ed\\ e\in y^{-1}\mathfrak{r},d\in\mathfrak{r}\\ d\mathfrak{r}+p\mathfrak{r}=\mathfrak{r},d \bmod \mathfrak{r}^\times\\ \mathfrak{a}=dx\mathfrak{r} \bmod \overline{Z}_{F,\alpha}(\mathfrak{r})}}\psi(\phi\theta)^{-1}\omega_F(d\mathfrak{r})\right)$$
$$=\sum_{\substack{\xi_1+\xi_2=p^\alpha\xi\\ \xi_1,\xi_2\gg 0}}\left(\sum_{\substack{\mathfrak{a}\mathfrak{b}=\xi_1 y\mathfrak{r}\\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{a}\mathfrak{b}+p\mathfrak{r}=\mathfrak{r}}}\sum_{\substack{\xi_2=ed\\ e\in y^{-1}\mathfrak{r},d\in\mathfrak{r}\\ d\mathfrak{r}+p\mathfrak{r}=\mathfrak{r},d \bmod \mathfrak{r}^\times}}\eta(d^{-1}\mathfrak{a})\phi(\mathfrak{a})\langle d^{-1}\mathfrak{a}\rangle_F^r\theta\omega_F^{-1}(\mathfrak{b})\langle[\mathfrak{b}]\rangle\right.$$
$$\left.\times\ \psi(\phi\theta)^{-1}\omega_F(d\mathfrak{r})\right)$$
$$=\sum_{\substack{\xi_1+\xi_2=p^\alpha\xi\\ \xi_1,\xi_2\gg 0}}\left(\sum_{\substack{\mathfrak{a}\mathfrak{b}=\xi_1 y\mathfrak{r}\\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{a}\mathfrak{b}+p\mathfrak{r}=\mathfrak{r}}}\sum_{\substack{\xi_2=ed\\ e\in y^{-1}\mathfrak{r},d\in\mathfrak{r}\\ d\mathfrak{r}+p\mathfrak{r}=\mathfrak{r},d \bmod \mathfrak{r}^\times}}\eta\phi\omega_F^{-r}(\mathfrak{a})\theta\omega_F^{-1}(\mathfrak{b})\mathcal{N}(\mathfrak{a})^r\mathcal{N}(d\mathfrak{r})^{-r}\langle[\mathfrak{b}]\rangle\right.$$
$$\left.\times\ \psi(\eta\phi\theta)^{-1}\omega_F^{r+1}(d\mathfrak{r})\right)$$

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$$= \sum_{\substack{\xi_1+\xi_2=p^\alpha\xi \\ \xi_1,\xi_2\gg 0}} \left(\sum_{\substack{\mathfrak{ab}=\xi_1 y\mathfrak{r} \\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{ab}+p\mathfrak{r}=\mathfrak{r}}} \sum_{\substack{\xi_2=ed \\ e\in y^{-1}\mathfrak{r},d\in\mathfrak{r} \\ d\mathfrak{r}+p\mathfrak{r}=\mathfrak{r},d \bmod \mathfrak{r}^\times}} \eta\phi\omega_F^{-r}(\mathfrak{a})\theta\omega_F^{-1}(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{-r}\mathcal{N}(d^{-1}\xi_1 y\mathfrak{r})^r\langle[\mathfrak{b}]\rangle \right.$$

$$\left. \times\, \psi(\eta\phi\theta)^{-1}\omega_F^{r+1}(d\mathfrak{r}) \right). \tag{A15}$$

Now $\xi_1+\xi_2=p^\alpha\xi$, and $\xi_1,\xi_2\gg 0$, so that

$$\mathcal{N}(\xi_1\mathfrak{r}) = (-1)^d\mathcal{N}(\xi_2\mathfrak{r}) + o(1)$$

and, hence, in the notation of (A15),

$$\begin{aligned}\mathcal{N}(d^{-1}\xi_1 y\mathfrak{r})^r &= (-1)^{dr}\mathcal{N}(d^{-1}\xi_2 y\mathfrak{r})^r + o(1)\\ &= (-1)^{dr}\mathcal{N}(ey\mathfrak{r})^r + o(1).\end{aligned}$$

It follows that up to terms of order $o(1)$, the expression (A15) is equal to

$$(-1)^{dr} \sum_{\substack{\xi_1+\xi_2=p^\alpha\xi \\ \xi_1,\xi_2\gg 0}} \left(\sum_{\substack{\mathfrak{ab}=\xi_1 y\mathfrak{r} \\ \mathfrak{a},\mathfrak{b}\subset\mathfrak{r},\mathfrak{ab}+p\mathfrak{r}=\mathfrak{r}}} \eta\phi\omega_F^{-r}(\mathfrak{a})\theta\omega_F^{-1}(\mathfrak{b})\mathcal{N}(\mathfrak{b})^{-r}\langle[\mathfrak{b}]\rangle \right.$$

$$\left. \times \sum_{\substack{\xi_2=ed \\ e\in y^{-1}\mathfrak{r},d\in\mathfrak{r} \\ d\mathfrak{r}+p\mathfrak{r}=\mathfrak{r},d \bmod \mathfrak{r}^\times}} \psi(\eta\phi\theta)^{-1}\omega_F^{r+1}(d\mathfrak{r})\mathcal{N}(ey\mathfrak{r})^r \right)$$

which is the Fourier coefficient of $(-1)^{rd}\mathcal{K}_\alpha(\eta,\phi,\theta,r)^{00}$ at the ideal $\xi y\mathfrak{r}$. $\square$

Appendix B. Proof of Theorem 8.2

Here we give the proof of Theorem 8.2. Thus, let $\mathbf{f}$ be a p-ordinary newform of tame level $\mathfrak{n}$, weight two, trivial character. Let $\mathcal{F}$ the Hida family lifting $\mathbf{f}$. Following the formalism of § 8, we only need to consider specializations of $\mathcal{F}$ along the cyclotomic direction. Thus, denote by $\mathbf{f}_k$ the specialization of $\mathcal{F}$ at weight k (thus, $\mathbf{f}_2=\mathbf{f}$).

We first consider the case where $k\equiv 2 \mod p-1$, in particular k is even. For these values of k, the form $\mathbf{f}_k$ also has trivial character.

Let $\mathbf{g}_k$ be the newform (in the usual sense) whose p-stabilization is $\mathbf{f}_k$. We use the archimedean functional equation for $\mathbf{g}_k$ to deduce the functional equation for the p-adic L-function for $\mathbf{f}_k$. As in § 8, we fix a character θ adapted to $\mathbf{f}_2$. Then θ is also adapted to $\mathbf{f}_k$ for $k\equiv 2 \mod p-1$.

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We denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_g$ the set of primes above p. Without loss of generality assume that the conductor of $\mathbf{g}_k$ is given by $\mathfrak{n}\mathfrak{p}_1 \cdots \mathfrak{p}_t$. Then $\mathbf{g}_k$ is Steinberg at $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, and by Atkin–Lehner theory, we have

$$\alpha(\mathfrak{p}_i, \mathbf{g}_k)^2 = (\mathcal{N}\mathfrak{p}_i)^{k-2}. \tag{B1}$$

One may thus write

$$\alpha(\mathfrak{p}_i, \mathbf{g}_k) = -c(k, \mathfrak{p}_i)(\mathcal{N}\mathfrak{p}_i)^{(k-2)/2}, \quad c(k, \mathfrak{p}_i) = \pm 1. \tag{B2}$$

(As a matter of fact, (B1) and (B2) imply, under the p-ordinary assumption, that $\mathbf{g}_k$ is Steinberg at some $\mathfrak{p}$ above p, only when $k = 2$.)

On the other hand, $\mathbf{g}_k$ is good ordinary at $\mathfrak{p}_{t+1}, \ldots, \mathfrak{p}_g$, and we have (in the notation of § 8):

$$\alpha(\mathfrak{p}, \mathbf{g}_k)\beta(\mathfrak{p}, \mathbf{g}_k) = (\mathcal{N}\mathfrak{p})^{k-1} \quad \text{for } \mathfrak{p} = \mathfrak{p}_{t+1}, \ldots, \mathfrak{p}_g. \tag{B3}$$

We have the functional equation for the archimedean L-function $L(s, \mathbf{g}_k)$ (see [Shi78]):

$$\frac{D_F^s}{(2\pi)^{ds}}\Gamma(s)^d L(s, \mathbf{g}_k) = (-1)^{dk/2} w(k)\mathcal{N}(\mathfrak{n}\mathfrak{p}_1 \cdots \mathfrak{p}_t)^{k/2-s} \frac{D_F^{k-s}}{(2\pi)^{d(k-s)}}\Gamma(k-s)^d L(k-s, \mathbf{g}_k) \tag{B4}$$

with $w(k) = \pm 1$ being the eigenvalue for the Atkin–Lehner operator acting on $\mathbf{g}_k$, i.e. $\mathbf{g}_k | J_{\mathfrak{n}\mathfrak{p}_1 \cdots \mathfrak{p}_t} = w(k)\mathbf{g}_k$. We write

$$\epsilon_\infty(k) = (-1)^{dk/2} w(k) \tag{B5}$$

the sign of the archimedean L-function.

To prove the functional equation for the p-adic L-function attached to $\mathbf{f}_k$, we need to consider the twisted form of (B4). Let $0 \le r \le k-2$ be an integer, χ a finite-order character of $\overline{Z}_F(\mathfrak{r})$. Renaming $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ if necessary, we suppose that $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ divide $\mathfrak{c}_{\chi\omega_F^{-r}}$, the conductor $\chi\omega_F^{-r}$, while $\mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_t$ do not. We also allow an auxiliary character ϕ, with conductor prime to $\mathfrak{n}\mathfrak{c}_\theta p$ as in § 6. We have the following functional equation:

$$\begin{aligned}
&\frac{1}{\tau((\chi\phi\omega_F^{-r})^{-1})}\frac{D_F^s}{(2\pi)^{ds}}\Gamma(s)^d L(s, \mathbf{g}_k, (\chi\phi\omega_F^{-r})^{-1}) \\
&\quad = (-1)^{dr}\epsilon_\infty(k)\left(\prod_{i=1}^{s} c(k, \mathfrak{p}_i)\right)\mathcal{N}(\mathfrak{n}\mathfrak{p}_{s+1}\cdots\mathfrak{p}_t)^{k/2-s}\mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi)^{k-2s}(\chi\phi\omega_F^{-r})^{-1}(\mathfrak{n}\mathfrak{p}_{s+1}\cdots\mathfrak{p}_t) \\
&\qquad \times \frac{1}{\tau(\chi\phi\omega_F^{-r})}\frac{D_F^{k-s}}{(2\pi)^{d(k-s)}}\Gamma(k-s)^d L(k-s, \mathbf{g}_k, \chi\phi\omega_F^{-r}).
\end{aligned} \tag{B6}$$

Now by Corollary 6.9,

$$\begin{aligned}
&L_p(r+1, \mathbf{f}_k, \chi, \phi) \\
&\quad = \prod_{\mathfrak{p}|p}\left(1 - \frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^r}{\alpha(\mathfrak{p}, \mathbf{f}_k)}\right) \\
&\qquad \times \frac{1}{\alpha(\mathfrak{c}_{\chi\omega_F^{-r}}, \mathbf{f}_k)} D_F^r\Gamma(r+1)^d\mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi)^{r+1}\frac{L(r+1, \mathbf{f}_k, (\chi\phi\omega_F^{-r})^{-1})}{(-2\pi i)^{dr}\tau((\chi\phi\omega_F^{-r})^{-1})\Omega(\mathbf{f}_k, \theta)}
\end{aligned}$$

$$= \prod_{\mathfrak{p}=\mathfrak{p}_1,\dots,\mathfrak{p}_t} \left(1 - \frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^r}{\alpha(\mathfrak{p}, \mathbf{g}_k)}\right)$$
$$\times \prod_{\mathfrak{p}=\mathfrak{p}_{t+1},\dots,\mathfrak{p}_g} \left(1 - \frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^r}{\alpha(\mathfrak{p}, \mathbf{g}_k)}\right)\left(1 - \frac{(\chi\phi\omega_F^{-r})^{-1}(\mathfrak{p})\beta(\mathfrak{p}, \mathbf{g}_k)}{(\mathcal{N}\mathfrak{p})^{r+1}}\right)$$
$$\times \frac{1}{\alpha(\mathfrak{c}_{\chi\omega_F^{-r}}, \mathbf{f}_k)} D_F^r \Gamma(r+1)^d \mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi)^{r+1} \frac{L(r+1, \mathbf{g}_k, (\chi\phi\omega_F^{-r})^{-1})}{(-2\pi i)^{dr}\tau((\chi\phi\omega_F^{-r})^{-1})\Omega(\mathbf{f}_k, \theta)}. \tag{B7}$$

Now we make the substitutions $r \to k-r-2$, $\chi \to \chi^{-1}$ and $\phi \to \phi^{-1}$. Using (B2), (B3), and the fact that $\chi^{-1}\omega_F^{-(k-r-2)} = (\chi\omega_F^{-r})^{-1}$ for $k \equiv 2 \mod p-1$, we obtain

$$L_p(k-r-1, \mathbf{f}_k, \chi^{-1}, \phi^{-1})$$
$$= \prod_{\mathfrak{p}=\mathfrak{p}_1,\dots,\mathfrak{p}_t} \left(1 - \frac{(\chi\phi\omega_F^{-r})^{-1}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-r-2}}{\alpha(\mathfrak{p}, \mathbf{g}_k)}\right)$$
$$\times \prod_{\mathfrak{p}=\mathfrak{p}_{t+1},\dots,\mathfrak{p}_g} \left(1 - \frac{(\chi\phi\omega_F^{-r})^{-1}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-r-2}}{\alpha(\mathfrak{p}, \mathbf{g}_k)}\right)\left(1 - \frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\beta(\mathfrak{p}, \mathbf{g}_k)}{(\mathcal{N}\mathfrak{p})^{k-r-1}}\right)$$
$$\times \frac{1}{\alpha(\mathfrak{c}_{\chi\omega_F^{-r}}, \mathbf{f}_k)} D_F^{k-r-2} \Gamma(k-r-1)^d \mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi)^{k-r-1} \frac{L(k-r-1, \mathbf{g}_k, \chi\phi\omega_F^{-r})}{(-2\pi i)^{d(k-r-2)}\tau(\chi\phi\omega_F^{-r})\Omega(\mathbf{f}_k, \theta)}$$
$$= \prod_{\mathfrak{p}=\mathfrak{p}_{s+1},\dots,\mathfrak{p}_t} \left(\frac{c(k,\mathfrak{p})(\mathcal{N}\mathfrak{p})^{k/2-r-1}}{\chi\phi\omega_F^{-r}(\mathfrak{p})}\right) \prod_{\mathfrak{p}=\mathfrak{p}_1,\dots,\mathfrak{p}_t} \left(1 - \frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^r}{\alpha(\mathfrak{p}, \mathbf{g}_k)}\right)$$
$$\times \prod_{\mathfrak{p}=\mathfrak{p}_{t+1},\dots,\mathfrak{p}_g} \left(1 - \frac{\chi\phi\omega_F^{-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^r}{\alpha(\mathfrak{p}, \mathbf{g}_k)}\right)\left(1 - \frac{(\chi\phi\omega_F^{-r})^{-1}(\mathfrak{p})\beta(\mathfrak{p}, \mathbf{g}_k)}{(\mathcal{N}\mathfrak{p})^{r+1}}\right)$$
$$\times \frac{1}{\alpha(\mathfrak{c}_{\chi\omega_F^{-r}}, \mathbf{f}_k)} D_F^{k-r-2} \Gamma(k-r-1)^d \mathcal{N}(\mathfrak{c}_{\chi\omega_F^{-r}}\mathfrak{c}_\phi)^{k-r-1}$$
$$\times \frac{L(k-r-1, \mathbf{g}_k, \chi\phi\omega_F^{-r})}{(-2\pi i)^{d(k-r-2)}\tau(\chi\phi\omega_F^{-r})\Omega(\mathbf{f}_k, \theta)}. \tag{B8}$$

Hence, by combining (B7) and (B8) with the archimedean functional equation (B6) evaluated at $s = r+1$, we obtain

$$L_p(r+1, \mathbf{f}_k, \chi, \phi) = \epsilon_\infty(k)\left(\prod_{i=1}^t c(k, \mathfrak{p}_i)\right)(-1)^{d(k/2-1)}(\chi\phi\omega_F^{-r})^{-1}(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{k/2-(r+1)}$$
$$\times L_p(k-r-1, \mathbf{f}_k, \chi^{-1}, \phi^{-1})$$
$$= \epsilon_\infty(k)\left(\prod_{i=1}^t c(k, \mathfrak{p}_i)\right)(-1)^{d(k/2-1)}\omega_F(\mathfrak{n})^{k/2-1}(\chi\phi)^{-1}(\mathfrak{n})\langle\mathfrak{n}\rangle_F^{k/2-(r+1)}$$
$$\times L_p(k-r-1, \mathbf{f}_k, \chi^{-1}, \phi^{-1}). \tag{B9}$$

Let us use the notation

$$\epsilon_p(k) = \epsilon_\infty(k)\left(\prod_{i=1}^t c(k, \mathfrak{p}_i)\right)(-1)^{d(k/2-1)}\omega_F(\mathfrak{n})^{k/2-1}; \tag{B10}$$

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note that $\omega_F(\mathfrak{n})^{k/2-1} = \pm 1$, thus $\epsilon_p(k) = \pm 1$. Also note $\epsilon_p(2) = \epsilon_\infty(2)(-1)^e$, where we recall that e is the number of $\{\mathfrak{p}|p\}$, with $\alpha(\mathfrak{p}, \mathbf{f}) = 1$. We then have

$$L_p(r+1, \mathbf{f}_k, \chi, \phi) = \epsilon_p(k)(\chi\phi)^{-1}(\mathfrak{n})\langle\mathfrak{n}\rangle_F^{k/2-(r+1)} L_p(k-r-1, \mathbf{f}_k, \chi^{-1}, \phi^{-1}).$$

So, by the Zariski density argument, we obtain the functional equation for the one-variable p-adic L-function:

$$\begin{aligned} L_p(s, \mathbf{f}_k, \chi, \phi) &= \epsilon_p(k)(\chi\phi)^{-1}(\mathfrak{n})\langle\mathfrak{n}\rangle_F^{k/2-s} L_p(k-s, \mathbf{f}_k, \chi^{-1}, \phi^{-1}) \\ &= \epsilon_p(k)(\chi\phi)^{-1}(\mathfrak{n})\langle\mathfrak{n}\rangle_F^{k/2-s} L_p(k-s, \mathbf{f}_k, \chi^{-1}, \phi^{-1}) \end{aligned} \tag{B11}$$

valid for $k \equiv 2 \mod p-1$, and k being sufficiently close to two p-adically. We claim that

$$\epsilon_p(k) = \epsilon_p(2) \tag{B12}$$

for these weights k. Indeed, by a theorem of Rohrlich (Theorem 6.4), we can choose χ, ϕ, so that $L_p(1, \mathbf{f}_2, \chi^{-1}, \phi^{-1}) \neq 0$. By continuity, $L_p(k-1, \mathbf{f}_k, \chi^{-1}, \phi^{-1}) \neq 0$ for $k \geq 2$,sufficiently close to two (p-adically). Take $s = 1$ in (B10). We see that there is a p-adic analytic function $F(k)$ in a p-adic disc around two, such that

$$\begin{gathered} \epsilon_p(2) = F(2), \\ \epsilon_p(k) = F(k), \quad k \equiv 2 \mod p-1, \quad k \text{ close to } 2. \end{gathered}$$

Since $\epsilon_p(k) = \pm 1$, it follows that $\epsilon_p(k) = \epsilon_p(2)$ for these values of k.

Now by the Zariski density of these weights, it follows finally that

$$L_p(s, \kappa, \chi, \phi) = \epsilon_p(2)(\chi\phi)^{-1}(\mathfrak{n})\langle\mathfrak{n}\rangle_F^{\kappa/2-s} L_p(\kappa-s, \kappa, \chi^{-1}, \phi^{-1}). \tag{B13}$$

Specializing (B12) to the case where χ, ϕ are trivial, we obtain Theorem 8.2. □

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