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Abstract

A complex Hadamard matrix, C , of order n has elements $1, -1, i, -i$ and satisfies $CC^* = nI_n$ where C^* denotes the conjugate transpose of C . Let $C = [c_{ij}]$ be a complex Hadamard matrix of order n . $S(C) = \sum c_{ij}$ is called the sum of C . $O(C) = |S(C)|$ is called the excess of C . We study the excess of complex Hadamard matrices. As an application many real Hadamard matrices of large and maximal excess are obtained.

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The Excess of Complex Hadamard Matrices

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Abstract. A complex Hadamard matrix, C , of order n has elements $1, -1, i, -i$ and satisfies $CC^* = nI_n$ where C^* denotes the conjugate transpose of C . Let $C = [c_{ij}]$ be a complex Hadamard matrix of order n . $S(C) = \sum_{ij} c_{ij}$ is called the sum of C . $\sigma(C) = |S(C)|$ is called the excess of C . We study the excess of complex Hadamard matrices. As an application many real Hadamard matrices of large and maximal excess are obtained.

1. Introduction

A complex Hadamard matrix, C , has elements $1, -1, i, -i$ where $i = \sqrt{-1}$, is necessarily of order 1 or $2c$ and satisfies $CC^* = 2cI_{2c}$ where C^* denotes the Hermitian conjugate (transpose, complex conjugate) of C . These were first introduced by Turyn [24], and further discussed by Seberry Wallis [23]. Their generalization to *complex orthogonal designs* is discussed by Geramita and Geramita [4].

Let $C = [a_{ij}]$ be a complex Hadamard matrix of order n and denote by $S(C)$ the sum of all entries of C , so $S(C) = \sum_{ij} a_{ij}$. $\sigma(C) = |S(C)|$ is called the *excess* of C . Let $\sigma_C(n) = \max\{\sigma(C) : C \text{ a complex Hadamard matrix of order } n\}$. The maximum excess problem has been extensively studied for real Hadamard matrices [1–3, 7–19, 25, 26]. In this paper we will study the excess of complex Hadamard matrices. As an application we get many real Hadamard matrices of large excess and new classes of Hadamard matrices of maximum excess.

2. Preliminaries

First we note some basic properties of complex Hadamard matrices of large excess:

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Lemma 1. Let $H = [h_{ij}]$ be a complex Hadamard matrix of order n . Let C_j denote the sum of the elements of the j th column of C , then

$$(i) \sum_j |C_j|^2 = n^2$$

$$(ii) \sum_j |C_j| \leq n\sqrt{n} \text{ with equality iff } |C_j| = \sqrt{n} \text{ for each } j$$

Proof.

(i) Let e be the $1 \times n$ matrix of ones, then

$$eHH^*e^t = [C_1 C_2 \cdots C_n][C_1 C_2 \cdots C_n]^* = \sum_j |C_j|^2$$

also, since $HH^* = nI$

$$eHH^*e^t = enIe^t = n^2$$

so

$$\sum_j |C_j|^2 = n^2$$

$$(ii) \sum_j (|C_j| - \sqrt{n})^2 = \sum_j |C_j|^2 + n^2 - 2\sqrt{n} \sum_j |C_j|$$

$$= 2n^2 - 2\sqrt{n} \sum_j |C_j| \geq 0 \text{ with equality iff } |C_j| = \sqrt{n} \text{ for each } j. \quad \square$$

A complex Hadamard matrix $H = [h_{ij}]$ of order n is called *regular* if $C_j = \sum_j h_{ij}$ remains the same for each j . In this case $|C_j| = \sqrt{n}$ for each j and the maximum excess of $n\sqrt{n}$ is attained. Since $C_j = a + bi$ for some integers a, b , it follows that for a regular complex Hadamard matrix of order n , n must be a sum of two squares. It is easy to see that an odd integer n is a sum of two squares iff $2^{n_0}n$ is a sum of two squares for some n_0 .

Conjecture 2. Suppose the odd integer m is a sum of two squares, then there is a regular complex Hadamard matrix of order $2m$.

If the odd integer m is not a sum of two squares, then there can be no regular complex Hadamard matrix of order $2^{n_0}m$ for any $n_0 \geq 1$. Consequently, if there exists a complex Hadamard matrix of order $n = 2^{n_0}m$, then the maximum excess of such a matrix is strictly less than $n\sqrt{n}$. The maximum excess problem in this case is hard and for many large values of n seems hopeless.

3. Basic Constructions

We begin with a lemma which implies that the construction of certain complex Hadamard matrices is equivalent to the Williamson construction of real Hadamard matrices.

Lemma 3. *There is an Hadamard matrix of order $4m$ of the form*

$$\begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}$$

if and only if there is a complex Hadamard matrix of order $2m$ of the form $\begin{bmatrix} S & T \\ -\bar{T} & \bar{S} \end{bmatrix}$ where \bar{T} denotes the complex conjugate of T .

Proof. To form the complex Hadamard matrix, let

$$X = \frac{1}{2}(A + B), \quad Y = \frac{1}{2}(A - B), \quad V = \frac{1}{2}(C - D), \quad W = \frac{1}{2}(C + D)$$

and the matrix is

$$\begin{bmatrix} X + iY & V + iW \\ -V + iW & X - iY \end{bmatrix}.$$

Conversely, if there is a complex Hadamard matrix of the form $\begin{bmatrix} S & T \\ -\bar{T} & \bar{S} \end{bmatrix}$, let

$$S = X + iY, \quad T = V + iW$$

Then

$$A = X + Y, \quad B = X - Y, \quad C = V + W, \quad D = W - V$$

can be used to form the required Hadamard matrix. \square

Starting with four Williamson matrices A, B, C, D of order m in Lemma 3, the derived complex Hadamard matrix will have a sum of $\sigma(A) + \sigma(B) + i(\sigma(C) + \sigma(D))$ and an excess of $\sqrt{[\sigma(A) + \sigma(B)]^2 + [\sigma(C) + \sigma(D)]^2}$. We use this method and known Williamson matrices [21, 22] to form Table 1.

Lemma 4. *Let $C = H + iK$ be a complex Hadamard matrix of order m . Then the matrix $H_{2m} = \begin{bmatrix} H + K & H - K \\ -H + K & H + K \end{bmatrix}$ is an Hadamard matrix of excess $2(\sigma(H) + \sigma(K))$ and the matrix*

$$H_{4m} = \begin{bmatrix} -(H + K) & H - K & H + K & H - K \\ H - K & H + K & H - K & -(H + K) \\ H + K & -H + K & H + K & H - K \\ H - K & H + K & -H + K & H + K \end{bmatrix}$$

is an Hadamard matrix of excess $8\sigma(H)$.

Table 1. * indicates the maximum excess

m	Complex order	Williamson decomposition	Sum	Excess
3	6	$3^2 + 1^2 + 1^2 + 1^2$	$6(2 + i)$	$6\sqrt{5}$
5	10	$3^2 + 3^2 + 1^2 + 1^2$	$10(3 + i)$	$10\sqrt{10}^a$
7	14	$3^2 + 3^2 + 3^2 + 1^2$	$14(3 + 2i)$	$14\sqrt{13}$
9	18	$3^2 + 3^2 + 3^2 + 3^2$	$18(3 + 3i)$	$18\sqrt{18}^a$
11	22	$5^2 + 3^2 + 3^2 + 1^2$	$22(4 + 2i)$	$22\sqrt{20}$
13	26	$5^2 + 3^2 + 3^2 + 3^2$	$26(4 + 3i)$	$26\sqrt{25}$
		$5^2 + 5^2 + 1^2 + 1^2$	$26(5 + i)$	$26\sqrt{26}^a$
15	30	$5^2 + 5^2 + 3^2 + 1^2$	$30(5 + 2i)$	$30\sqrt{29}$
17	34	$5^2 + 5^2 + 3^2 + 3^2$	$34(5 + 3i)$	$34\sqrt{34}^a$
19	38	$7^2 + 5^2 + 1^2 + 1^2$	$38(6 + i)$	$38\sqrt{37}$
21	42	$5^2 + 5^2 + 5^2 + 3^2$	$42(5 + 4i)$	$42\sqrt{41}$
23	46	$7^2 + 5^2 + 3^2 + 3^2$	$46(6 + 3i)$	$46\sqrt{45}$
25	50	$5^2 + 5^2 + 5^2 + 5^2$	$50(5 + 5i)$	$50\sqrt{50}^a$
		$7^2 + 7^2 + 1^2 + 1^2$	$50(7 + i)$	$50\sqrt{50}^a$

Table 2. * indicates maximum possible excess achieved; ^b indicates maximum real excess known

Complex m	Sum	Excess $\sigma(C_m)$	Excess $\sigma(H_{2m})$	Excess $\sigma(H_{4m})$
6	$6(2 + i)$	$6\sqrt{5}$	36^a	
10	$10(3 + i)$	$10\sqrt{10}^a$	80^a	240^b
14	$14(3 + 2i)$	$14\sqrt{13}$	140^a	
18	$18(3 + 3i)$	$18\sqrt{18}^a$	216^a	
22	$22(4 + 2i)$	$22\sqrt{20}$	264	
26	$26(4 + 3i)$	$26\sqrt{25}$	364^a	
	$26(5 + i)$	$26\sqrt{26}^a$		1040^b
30	$30(5 + 2i)$	$30\sqrt{29}$	420	
34	$34(5 + 3i)$	$34\sqrt{34}^a$	544	
38	$38(6 + i)$	$38\sqrt{37}$		1824^b
42	$42(5 + 4i)$	$42\sqrt{41}$	756^a	
46	$46(6 + 3i)$	$46\sqrt{45}$	828	2204^b
50	$50(5 + 5i)$	$50\sqrt{50}^a$	1000^a	
	$50(7 + i)$	$50\sqrt{50}^a$		2800^b

Proof. Note that C being a complex Hadamard matrix implies that $HH' + KK' = mI$ and $HK' = KH'$. Consequently, H_{2m} and H_{4m} are Hadamard. The excess part is trivial. \square

Suppose now that $\sigma(C) = a + bi$ and $|\sigma(c)| = \sqrt{a^2 + b^2}$ is close to $n\sqrt{n}$, then if a is close to b , H_{2m} will have a large excess, whereas, if $a > b$ and a is far from b , then H_{4m} will have a large excess. Table 2 shows how this property can be effectively used to find real Hadamard matrices of large excess.

Theorem 5. *Let $p \equiv 1 \pmod 4$ be a prime power, then there is a complex Hadamard matrix of order $p + 1$ with excess $(p + 1)\sqrt{p}$.*

Proof. It is known that if $p \equiv 1 \pmod 4$ is a prime power, then there are two circulant symmetric matrices, A and B , A with zero diagonal and all other elements ± 1 , B with elements ± 1 , of order $\frac{1}{2}(p + 1)$ which satisfy $AA^T + BB^T = pI$ [6, 23].

Let a, b be the row sums of A, B respectively, then $p = a^2 + b^2$, where b is odd.

Now, the matrix $C_{p+1} = \begin{bmatrix} I + iA & B \\ B & -I + iA \end{bmatrix}$ is a complex Hadamard matrix with sum $(p + 1)(b + ia)$ and consequently it has an excess of $(p + 1)\sqrt{a^2 + b^2} = (p + 1)\sqrt{p}$.

Corollary 6. *Let $p = k^2 + (k + 1)^2 \equiv 1 \pmod 4$ be a prime power. Furthermore, suppose p is a sum of two squares only as above. Then there is an Hadamard matrix of order $2(p + 1)$ with maximal excess of $2(p + 1)(2k + 1)$.*

Proof. Using the matrix C_{p+1} of the preceding theorem in Lemma 4, we get a Hadamard matrix of order $2(p + 1)$ with excess $2(p + 1)(2k + 1)$. It is shown in [17] that this is the maximum excess possible. Table 3 is drawn from this corollary. \square

Lemma 7. *Let $q^2 = p^{2\alpha} \equiv 1 \pmod 4$ be a prime power which is decomposable as sum of two squares in only one way, namely, $q^2 = p^{2\alpha} = (p^\alpha)^2 + 0 = q^2 + 0$. Then there is a regular complex Hadamard matrix of order $q^2 + 1$ and an Hadamard matrix of order $4(q + 1)$ with excess of $8q(q^2 + 1)$.*

Table 3. * indicates maximum possible excess

Prime $p \equiv (\pmod 4)$	Hadamard matrix order	Excess
$41 = 4^2 + 5^2$	84	756*
$61 = 5^2 + 6^2$	124	1364*
$113 = 7^2 + 8^2$	228	3420*
$181 = 9^2 + 10^2$	364	6916*
$313 = 12^2 + 13^2$	628	15700*
$421 = 14^2 + 15^2$	844	24476*

Table 4. * indicates maximum excess known

Prime $q^2 = p^{2\alpha} \equiv 1 \pmod 4$	Hadamard matrix order	Excess
7^2	200	2800*
3^4	328	5904*
11^2	488	10736*

Proof. Consider the matrices A, B in the proof of Theorem 5. Let $D_{q^2+1} = \begin{bmatrix} iI + A & B \\ B & iI - A \end{bmatrix}$, then D_{q^2+1} is regular. Using D_{q^2+1} in Lemma 4, we get an Hadamard matrix $H_{4(q^2+1)}$ with excess $8q(q^2 + 1)$. \square

Table 4 is drawn from this lemma.

4. The Main Theorem

Kounias and Farmakis [17] showed that the maximal excess of a Hadamard matrix of order $4m(m - 1)$ is given by

$$\sigma(4m(m - 1)) \leq 4(m - 1)^2(2m + 1).$$

Kharaghani [9] showed that this maximal excess can be attained if m is the order of a skew-Hadamard matrix. Koukouvinos and Seberry [14] subsequently proved that this maximal excess is attained if $m = 2 \pmod{4}$ is the order of a conference matrix.

In this section we will show that some special kinds of complex Hadamard matrices can be used to construct Hadamard matrices which attain the above upperbound. This covers both values of m which are the order of a skew-Hadamard matrix and $m \equiv 2 \pmod{4}$ which is the order of a conference matrix.

Theorem 8. *If there is a skew-complex Hadamard matrix of order m and a core of order $m - 1$, then there is a Hadamard matrix of order $4m(m - 1)$ with maximum excess of $4(m - 1)^2(2m + 1)$.*

Proof. Note that m , being the order of a complex Hadamard matrix, is an even integer. We will divide the proof in two parts:

- (i) $m - 1 \equiv 1 \pmod{4}$. We may assume that the skew-complex Hadamard matrix is of the form

$$\begin{bmatrix} + & i & i & \cdots & i \\ i & & & & \\ i & & B + I & & \\ \vdots & & & & \\ i & & & & \end{bmatrix},$$

where $B^* = -B$ and I is the identity matrix.

Insert the matrix J_{m-1} , the matrix of ones of order $m - 1$, on the main diagonal and elsewhere $I + i$ core. By Corollary 6.8 of [23] the resulting matrix is a complex Hadamard matrix of order $m(m - 1)$. The sum of each of the top $m - 1$ rows of this matrix is $m - 1 + (m - 1)i$ and the remaining rows have a sum of $m - 1 + i$ each. Thus, the total sum of this matrix is

$$(m - 1)[(m - 1) + (m - 1)i] + (m - 1)^2[m - 1 + i]$$

Using this Hadamard matrix and the two special Hadamard matrices, H and K .

$$H = \begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{bmatrix}, \quad K = \begin{bmatrix} - & + & + & + \\ + & - & + & + \\ - & - & + & - \\ - & - & - & + \end{bmatrix}$$

we can construct a Hadamard matrix H' of order $4m(m - 1)$ following Turyn's method [24].

The sum of the matrix H' is $8[(m - 1)^2 + (m - 1)^3]$. Furthermore, the matrix H' will have a block $J \times H$ on the upper left corner. Half of the rows and columns corresponding to this block have zero sum. Pick all such rows and columns corresponding to the $(3, 1)$ elements of H and multiply them by a minus. As a result a total of $4(m - 1)^2$ will be added to the previous sum [9]. The resulting Hadamard matrix will have the maximum excess of $8[(m - 1)^2 + (m - 1)^3] + 4(m - 1)^2 = 4(m - 1)^2[2m + 1]$.

- (ii) Let $m - 1 \equiv 3 \pmod{4}$. We may assume that the skew-complex Hadamard matrix is of the form

$$\begin{bmatrix} - & + & + & \cdots & + \\ + & & & & \\ + & & & & \\ \vdots & & & B + I & \\ + & & & & \end{bmatrix}$$

where $B^* = -B$.

Insert the matrix J_{m-1} on the main diagonal and elsewhere I + core. Follow a similar procedure as part (i) to construct the required Hadamard matrix.

Example. Consider the skew-complex Hadamard matrix

$$i \begin{bmatrix} -i & + & + & + & + & + \\ + & -i & + & - & - & + \\ + & + & -i & + & - & - \\ + & - & + & -i & + & - \\ + & - & - & + & -i & + \\ + & + & - & - & + & -i \end{bmatrix}$$

Let

$$C = \begin{bmatrix} 0 & + & - & - & + \\ + & 0 & - & + & - \\ - & - & 0 & + & + \\ - & + & + & 0 & - \\ + & - & + & - & 0 \end{bmatrix}$$

be a core of order 5. Then

$$H' = \begin{bmatrix} J \times H & I \times K - C \times H & I \times K - C \times H & I \times K - C \times H & I \times K - C \times H & I \times K - C \times H \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where

$$J \times H = \begin{bmatrix} H & H & H & H & H \\ H & H & H & H & H \\ H & H & H & H & H \\ H & H & H & H & H \\ H & H & H & H & H \end{bmatrix},$$

$$I \times K + C \times H = \begin{bmatrix} K & H & -H & -H & H \\ H & K & -H & H & -H \\ -H & -H & K & H & H \\ -H & H & H & K & -H \\ H & -H & H & -H & K \end{bmatrix}$$

$$H = \begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{bmatrix}, \quad K = \begin{bmatrix} - & + & + & + \\ + & - & + & + \\ - & - & + & - \\ - & - & - & + \end{bmatrix}$$

The total sum of H' is 1200. Following the procedure in the theorem, one should multiply the following rows and columns of H' by minus;

$$\begin{aligned} \text{row number:} & \quad 3, 7, 11, 15, 19 \\ \text{column number:} & \quad 1, 5, 9, 13, 17. \end{aligned}$$

As a result the new matrix will have a sum of 1300 which is the maximum possible sum for the order $n = 120$.

Example.

Start with the matrix $\begin{bmatrix} - & + & + & + \\ + & + & - & + \\ + & + & + & - \\ + & - & + & + \end{bmatrix}$. Let $C = \begin{bmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{bmatrix}$ be a core of

order 3.

Then

$$H' = \begin{bmatrix} -J \times H & (I + C) \times H & (I + C) \times H & (I + C) \times H \\ (I + C) \times H & \cdot & \cdot & \cdot \\ (I + C) \times H & \cdot & \cdot & \cdot \\ (I + C) \times H & \cdot & \cdot & \cdot \end{bmatrix}$$

where

$$J \times H = \begin{bmatrix} H & H & H \\ H & H & H \\ H & H & H \end{bmatrix}, \quad (I + C) \times H = \begin{bmatrix} H & H & -H \\ -H & H & H \\ H & -H & H \end{bmatrix},$$

$$H = \begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{bmatrix}.$$

H' has a sum of 288. Following the procedure in the theorem, one should multiply the following rows and columns of H' by minus;

$$\begin{aligned} \text{row number:} & \quad 4, 8, 12 \\ \text{column number:} & \quad 1, 5, 9 \end{aligned}$$

As a result, the new matrix will have a sum of 324 which is the maximum possible sum for the order $n = 48$.

Theorem 8 is more powerful than those of maximal excess found in [9] and [14] and induces and unifies those results.

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