The Exciting Forces on a Moving Body in Waves

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ABSTRACT

This paper generalizes the "Haskind Relations" for the exciting forces in waves, to include the effects of constant forward speed. The analysis assumes the fluid to be ideal and incompressible, and the disturbance of the free surface to be small. The analytical relations are derived for the exciting forces in regular waves, in terms of the radiation potential associated with the forced harmonic oscillations of the same body in calm water. For this purpose it is sufficient to know the far-field asymptotic form of the radiation potential. The results are applied to the case of a submerged ellipsoid, to give the six exciting forces and moments as functions of the wave length, heading angle, and forward velocity.

NOMENCLATURE

A	Wave amplitude
(a_1, a_2, a_3)	Semi-axis of ellipsoid
B_{jj}	Damping coefficients
C_{xj}	Exciting force coefficients
c	Forward velocity
D_j	Virtual-mass coefficients of the ellipsoid
g	Gravitational acceleration constant
H_j	Kochine functions, defined by equation (27)
h	Depth of submergence
i	$= \sqrt{-1}$
j	Index referring to direction of force or motion
j_n	Spherical Bessel function
K	Wave number, $K = \sigma^2/g$
n	Unit normal into body
P_j	Functions defining the far-field radiation
R	Polar coordinate
S	Submerged surface of body
t	Time
v_j	Velocity components
X_j	Exciting force in j'th direction

(x, y, z)	Cartesian coordinates
$\boldsymbol{\alpha}_j$	Displacement vector of the body
α_j	Green's integrals for the ellipsoid
β	Angle of incident wave system
θ	Polar coordinate
λ	Wavelength
ρ	Fluid density
σ	Circular frequency of incident waves in fixed coordinate system
τ	$= \omega c/g$
φ	Velocity potential
ω	Circular frequency of encounter

I. INTRODUCTION

In analysing the motions of a ship or submerged body in regular waves, it is customary to decompose the overall problem into two parts, one being the prediction of the exciting forces experienced by the body due to the waves, and the other being the prediction of the restoring forces experienced by the body due to its own motion. In the linearized theory the exciting forces can be found with the unsteady motions of the body suppressed, while the restoring forces can be found for the oscillating body with the incident waves suppressed. Superposition of the two separate problems will ultimately lead to a complete linear theory for motions in waves.

The present investigation is concerned primarily with the exciting forces in waves. A significant contribution to this aspect of the overall problem was made by Haskind [1][2], who derived certain new relations for the exciting forces acting on a fixed body in waves. These relations require only the solution of the forced oscillation problem in calm water, and thus circumvent the necessity of solving the problem of wave diffraction past the body. Haskind's relations have been used to solve certain two- and three-dimensional problems and to derive relations between the exciting and damping forces [2]. The utility of Haskind's relations is limited however by the restriction of zero forward speed, and it is to the removal of this limitation that the present work is directed. In the analysis to

[2]Numbers in square brackets denote references at end of paper.

follow it will be shown that such an extension is possible, provided the body is such as to make in its forward motion only a small disturbance of the free surface (i.e. to be consistent with the linearized free surface theory). Thus the present work is applicable to thin or slender ships, and to deeply-submerged bodies.

The analysis and the limitations of the present theory closely parallel the work of Timman and Newman [3] on cross-coupling. In fact a sidelight of the present analysis is the derivation of the same symmetry or reciprocity relations which were established in [3], without assuming the existence of a particular Green's function.

There is however an additional complication in the context of exciting forces, namely the appropriate boundary condition on the free surface. The total velocity potential can be decomposed into a steady term, due to the steady state forward velocity of the ship, and an unsteady term including both the incident wave system and the diffraction effects of the ship. In some cases the diffraction potential will be of the same order of magnitude as the product of the steady and incident wave terms, and as a result the non-linear effects associated with this product can not be neglected. This is in fact the case for the thin ship in head waves, as was shown by Newman [4]. In such a situation the diffraction potential will satisfy not the conventional linearized free surface condition, but an inhomogeneous boundary condition.

The present paper is restricted to diffraction potentials which satisfy the homogeneous free surface condition. Nevertheless there are two significant reasons for considering this case. Some of the important mathematical models for ships do lead to a homogeneous free surface condition; these include slender ships, deeply submerged ships, and thin ships in oblique waves. Secondly, if one is faced with the inhomogeneous situation, as in [4], the generalized Haskind relations as derived here can be applied to the homogeneous solution for the diffraction potential, leaving only a particular solution of the inhomogeneous free surface condition to be found from direct methods.

The final expressions derived here differ very little from the zero-speed relations of Haskind. It is only necessary to replace the incident wave potential φ_0 by the corresponding potential in a moving coordinate system, and the forced-oscillation radiation potential φ_j by the corresponding solution for a moving body, but with the direction of forward motion reversed. Moreover, as in the case of zero-speed, the exciting forces can be found in terms of the far-field radiation potentials and can be related to the damping coefficients of the body. Detailed results are presented for an ellipsoid moving under a free surface with oblique waves, and these results are shown to be a generalization of Havelock's theory [5] for the exciting forces on a submerged spheroid.

II. THE REVERSE FLOW RELATIONS

We introduce two separate boundary value problems, in which the respective directions of forward motion are reversed. In the first problem, we consider radiation from an oscillating rigid body, moving with mean velocity c in the direction of the $+$ x-axis, parallel to the plane of the free surface. If (x, y, z) is a coordinate system, moving with the mean position of the body, the total velocity potential can be written in the form

$$\Phi^{+}(x,y,z;t) = \phi^{+}(x,y,z) + e^{i\omega t}\sum_{j=1}^{6} V_j \varphi_j^{+}(x,y,z) \tag{1}$$

Here ϕ^{+} is the potential of the steady flow, due to the forward velocity of the body, and

$$\mathbb{V}^{+} = \nabla\phi^{+}$$

is the steady velocity field. The potentials φ_j^{+} represent the unsteady disturbance due to oscillatory motion in each of the six degrees of freedom with velocity amplitude V_j and frequency ω, and the real part is understood in expressions involving $e^{i\omega t}$. If in each of the six modes of oscillation the displacement vector of point on the body is denoted by the vector $e^{i\omega t}\boldsymbol{\alpha}_j(x,y,z)$ and $\mathbf{n}$ is the unit normal vector into the body, then the potential φ_j^{+} satisfies the boundary condition [3]

$$V_j \frac{\partial\varphi_j^{+}}{\partial n} = \left[i\omega\boldsymbol{\alpha}_j + \nabla\times(\boldsymbol{\alpha}_j\times\mathbb{V}^{+})\right]\cdot\mathbf{n} \tag{2}$$

on the mean position of the body. The term $i\omega \alpha_j \cdot n$ denotes the usual oscillatory normal velocity, while the remaining contribution on the right-hand-side includes the difference in the normal component of the steady velocity field between the exact and mean positions of the body. In addition the potentials φ_j^+ satisfy Laplace's equation, a radiation condition at infinity, and the linearized free surface condition

$$\omega^2 \varphi^+ + 2i\omega c \frac{\partial \varphi^+}{\partial x} - c^2 \frac{\partial^2 \varphi^+}{\partial x^2} - g \frac{\partial \varphi^+}{\partial z} = 0 \qquad \text{on} \quad z = 0 \tag{3}$$

where $z = 0$ is the plane of the undisturbed free surface and g is the gravitational constant.

Now we consider a second problem with the direction of forward motion reversed, so that the body moves in the -x direction. The velocity potential is then of the form

$$\Phi^-(x,y,z;t) = \phi^-(x,y,z) + e^{i\omega t} \varphi^-(x,y,z) \tag{4}$$

where the potential φ^- represents an arbitrary oscillatory disturbance, of frequency ω, and satisfies the free surface condition

$$\omega^2 \varphi^- - 2i\omega c \frac{\partial \varphi^-}{\partial x} - c^2 \frac{\partial^2 \varphi^-}{\partial x^2} - g \frac{\partial \varphi^-}{\partial z} = 0 \qquad \text{on} \quad z = 0 \tag{5}$$

The steady velocity field is

$$\mathbb{V}^- = \nabla \phi^-$$

and the linearized unsteady fluid pressure is

$$p = -\rho e^{i\omega t}\left[i\omega \varphi^- + \mathbb{V}^- \cdot \nabla \varphi^-\right] \tag{6}$$

where second-order terms in the oscillatory disturbance are neglected.

If the body is fixed in space, the j'th oscillatory force or moment excited by the fluid pressure on the body, due to the disturbance φ_j^-, is

$$X_j^- = -\rho e^{i\omega t} \iint\limits_S \left[i\omega \varphi^- + \mathbb{V}^- \cdot \nabla \varphi^-\right] f_j \, dS \tag{7}$$

where the symbol S denotes that the integration is over the submerged surface of the body, and f_j is the appropriate direction cosine:

$f_1 = \cos(n,x)$ $\qquad$ $f_4 = y \cos(n,z) - z \cos(n,y)$

$f_2 = \cos(n,y)$ $\qquad$ $f_5 = z \cos(n,x) - x \cos(n,z)$

$f_3 = \cos(n,z)$ $\qquad$ $f_6 = x \cos(n,y) - y \cos(n,x)$

These can be expressed in terms of the displacement vectors $\boldsymbol{\alpha}_j$ since

$$i\omega \boldsymbol{\alpha}_j \cdot \mathbb{n} = v_j f_j$$

Thus, using the boundary condition (2),

$$f_j = \frac{\partial \varphi_j^+}{\partial n} - \frac{1}{v_j} \mathbb{n} \cdot \nabla \times (\boldsymbol{\alpha}_j \times \mathbb{V}^+)$$

Substituting back in (7), we obtain the expression

$$X_j^- = -i\omega\rho e^{i\omega t} \iint_S \varphi^- \frac{\partial \varphi_j^+}{\partial n} dS$$
$$- \rho e^{i\omega t} \iint_S \{(\mathbb{V}^- \cdot \nabla \varphi^-)[\frac{\partial \varphi_j^+}{\partial n} - \frac{1}{V_j} \mathbb{n} \cdot \nabla \times (\boldsymbol{\alpha}_j \times \mathbb{V}^+)] \quad (8)$$
$$- \frac{i\omega}{V_j} \varphi^- \mathbb{n} \cdot \nabla \times (\boldsymbol{\alpha}_j \times \mathbb{V}^+)\} dS$$

Using Stokes' theorem and well-known vector identities, together with the boundary condition (2), the second surface integral in equation (8) can be reduced as follows

$$\iint_S \{(\mathbb{V}^- \cdot \nabla \varphi^-)[\frac{\partial \varphi_j^+}{\partial n} - \frac{1}{V_j} \mathbb{n} \cdot \nabla \times (\boldsymbol{\alpha}_j \times \mathbb{V}^+)] - \frac{i\omega}{V_j} \varphi^- \mathbb{n} \cdot \nabla \times (\boldsymbol{\alpha}_j \times \mathbb{V}^+)\} dS$$

$$= \frac{i\omega}{V_j} \iint_S \{(\mathbb{V}^- \cdot \nabla \varphi^-) \boldsymbol{\alpha}_j - \nabla \times [\varphi^- (\boldsymbol{\alpha}_j \times \mathbb{V}^+)]$$
$$+ \nabla \varphi^- \times (\boldsymbol{\alpha}_j \times \mathbb{V}^+)\} \cdot \mathbb{n}\, dS$$

$$= \frac{i\omega}{V_j} \iint_S \{(\mathbb{V}^- \cdot \nabla \varphi^-) \boldsymbol{\alpha}_j + (\nabla \varphi^-) \times (\boldsymbol{\alpha}_j \times \mathbb{V}^+)\} \cdot \mathbb{n}\, dS$$
$$- \frac{i\omega}{V_j} \oint \varphi^- (\boldsymbol{\alpha}_j \times \mathbb{V}^+) \cdot d\boldsymbol{\ell}$$

$$= \frac{i\omega}{V_j} \iint_S [(\mathbb{V}^- + \mathbb{V}^+) \cdot \nabla \varphi^-](\boldsymbol{\alpha}_j \cdot \mathbb{n})\, dS$$
$$- \frac{i\omega}{V_j} \oint \varphi^- (\boldsymbol{\alpha}_j \times \mathbb{V}^+) \cdot d\boldsymbol{\ell} \quad (9)$$

where the line integral is over the intersection, if any, of the body with the free surface. In the last equality we have used the boundary condition that $\mathbb{V}^{+} \cdot \mathbb{n} = 0$ on the body surface. Following the hypothesis of [3] we assume that to leading order on the body,

$$\mathbb{V}^{+} \cong \mathbb{V}^{-}$$

and on the intersection of the body with the free surface,

$$(\boldsymbol{\alpha}_j \times \mathbb{V}^{+}) \cdot d\boldsymbol{\ell} \cong 0$$

Although these hypotheses have not been proven, they appear to be consistent with the linearized free surface condition, and can be verified in the special cases of a thin, slender, or deeply submerged body. It follows that equation (9) is identically zero, and thus from equation (8),

$$X_j^{-} = -i\omega\rho e^{i\omega t} \iint_S \varphi^{-} \frac{\partial \varphi_j^{+}}{\partial n} \, dS \tag{10}$$

Equation (10) forms the basis for various reverse-flow relations. First we note that if φ^{-} is a radiation potential for forced oscillations in the i'th mode, say φ_i^{-}, then the cross-coupling force in the j'th direction is

$$F_{ji}^{-} = -i\omega\rho e^{i\omega t} \iint_S \varphi_i^{-} \frac{\partial \varphi_j^{+}}{\partial n} \, dS \tag{11}$$

Now from Green's theorem, and the fact that the potentials φ_i^- and φ_j^+ satisfy a radiation condition at infinity and the adjoint free surface conditions (3) and (5), it follows that

$$\iint_S \left(\varphi_i^- \frac{\partial \varphi_j^+}{\partial n} - \varphi_j^+ \frac{\partial \varphi_i^-}{\partial n} \right) dS$$

$$= - \iint_{FS} \left(\varphi_i^- \frac{\partial \varphi_j^+}{\partial z} - \varphi_j^+ \frac{\partial \varphi_i^-}{\partial z} \right)_{z=0} dx\, dy$$

$$= -\frac{1}{g} \iint_{FS} \Big[\varphi_i^- \left(\omega^2 \varphi_j^+ + 2i\omega c \frac{\partial \varphi_j^+}{\partial x} - c^2 \frac{\partial^2 \varphi_j^+}{\partial x^2} \right)$$
$$- \varphi_j^+ \left(\omega^2 \varphi_i^- - 2i\omega c \frac{\partial \varphi_i^-}{\partial x} - c^2 \frac{\partial^2 \varphi_i^-}{\partial x^2} \right) \Big] dx\, dy$$

$$= -\frac{1}{g} \iint_{FS} \frac{\partial}{\partial x} \Big[\varphi_i^- \left(i\omega c\, \varphi_j^+ - c^2 \frac{\partial \varphi_j^+}{\partial x} \right)$$
$$+ \varphi_j^+ \left(i\omega c\, \varphi_i^- + c^2 \frac{\partial \varphi_i^-}{\partial x} \right) \Big] dx\, dy$$

$$= \frac{1}{g} \oint \Big[\varphi_i^- \left(i\omega c\, \varphi_j^+ - c^2 \frac{\partial \varphi_j^+}{\partial x} \right)$$
$$+ \varphi_j^+ \left(i\omega c\, \varphi_i^- + c^2 \frac{\partial \varphi_i^-}{\partial x} \right) \Big] dy \qquad (12)$$

where FS denotes integration over the plane of the undisturbed free surface, exterior to the body surface S , and the contour integral is over the intersection of these two surfaces. For submerged bodies the line integral vanishes identically, and for slender or thin bodies it is small of the same order as the beam. Thus it follows that

$$\iint_S \varphi_i^- \frac{\partial \varphi_j^+}{\partial n} dS = \iint_S \varphi_j^+ \frac{\partial \varphi_i^-}{\partial n} dS \tag{13}$$

or, from (11),

$$F_{ji}^- = F_{ij}^+ \tag{14}$$

Equation (14) was established by Timman and Newman [3] after assuming the existence of a certain Green's function. The present proof overcomes the need for such an assumption.

It should be noted that the forces $F_{ij}^{\pm}$ were obtained from integration of the pressure over the mean position of the body, and as such they are not the complete oscillatory hydrodynamic restoring forces acting on the body. In addition there are forces in phase with the displacement and otherwise independent of the frequency ω, due to the unsteady movement of the body in the steady velocity field. Thus the symmetry relations (12) apply to the damping coefficients (and in fact to the entire frequency-dependent forces), but not to the total restoring forces.

III. THE EXCITING FORCES IN WAVES

In order to apply the basic reverse flow relation (10) to the problem of determining the exciting forces in waves, we now consider the case where the potential φ^- corresponds to the diffraction problem of a plane progressive wave system incident upon the steadily moving body. Thus

$$\varphi^- = \varphi_0^- + \varphi_7^- \tag{15}$$

where φ_0^- is the potential of a plane progressive wave of given amplitude and arbitrary angle of incidence, and φ_7^- is the scattering potential, representing the disturbance of the incident wave by the body. Both φ_0^- and φ_7^- satisfy the free surface condition (5) and φ_7^- satisfies a radiation condition at infinity. On the body the total (unsteady) normal velocity must vanish, or

$$\frac{\partial}{\partial n}\left(\varphi_0^- + \varphi_7^-\right) = 0 \qquad \text{on } S. \tag{16}$$

From Green's theorem (13),

$$\iint_S \varphi_7^- \frac{\partial \varphi_j^+}{\partial n}\, dS = \iint_S \varphi_j^+ \frac{\partial \varphi_7^-}{\partial n}\, dS \qquad (j = 1, 2, \ldots 6) \tag{17}$$

Substituting the potential (15) in equation (10), the exciting force in the j'th mode is obtained in the form

$$X_j^- = -i\omega\rho e^{i\omega t} \iint_S (\varphi_0^- + \varphi_7^-) \frac{\partial \varphi_j^+}{\partial n} dS \tag{18}$$

Using Green's theorem (17) it follows that

$$X_j^- = -i\omega\rho e^{i\omega t} \iint_S \left(\varphi_0^- \frac{\partial \varphi_j^+}{\partial n} + \varphi_j^+ \frac{\partial \varphi_7^-}{\partial n} \right) dS \tag{19}$$

and making use of the boundary condition (16) we obtain the expression

$$X_j^- = -i\omega\rho e^{i\omega t} \iint_S \left(\varphi_0^- \frac{\partial \varphi_j^+}{\partial n} - \varphi_j^+ \frac{\partial \varphi_0^-}{\partial n} \right) dS \tag{20}$$

Equation (20) represents the desired generalization of the Haskind relations for bodies moving with constant forward velocity. The exciting forces in waves are expressed in terms of the incident wave potential φ_0^- and the "adjoint" radiation potential φ_j^+, thus permitting the evaluation of the exciting forces without solving the diffraction problem for φ_7^-. Clearly the direction of forward motion can be reversed, and the corresponding forces for a body moving in the +x direction are given by the analogous expression

$$X_j^+ = -i\omega\rho e^{i\omega t} \iint_S \left(\varphi_0^+ \frac{\partial \varphi_j^-}{\partial n} - \varphi_j^- \frac{\partial \varphi_0^+}{\partial n} \right) dS \tag{21}$$

At zero speed the superscripts are superfluous and these results reduce[3] to the original expression derived by Haskind [1].

As in the case of zero speed, Green's theorem may be used to replace the surface of integration S in equations (20) and (21) by any closed surface surrounding the body, but now there is an additional contribution from a line integral along the free surface. Let S_∞ be a suitable control surface at infinity (e.g. a vertical cylinder passing from the free surface down to an infinite depth). Then, following the analysis of equation (12),

$$\iint_S \left(\varphi_j^+ \frac{\partial \varphi_0^-}{\partial n} - \varphi_0^- \frac{\partial \varphi_j^+}{\partial n}\right) dS$$

$$= - \iint_{FS+S_\infty} \left(\varphi_j^+ \frac{\partial \varphi_0^-}{\partial n} - \varphi_0^- \frac{\partial \varphi_j^+}{\partial n}\right) dS$$

$$= - \iint_{S_\infty} \left(\varphi_j^+ \frac{\partial \varphi_0^-}{\partial n} - \varphi_0^- \frac{\partial \varphi_j^+}{\partial n}\right) dS$$

$$+ \frac{1}{g}\left(\oint_{C_\infty} - \oint_C\right)\left[\varphi_0^-\left(i\omega c\,\varphi_j^+ - c^2 \frac{\partial \varphi_j^+}{\partial x}\right)\right.$$

$$\left. + \varphi_j^+\left(i\omega c\,\varphi_0^- - c^2 \frac{\partial \varphi_0^-}{\partial x}\right)\right] dy$$

$$= - \iint_{S_\infty} \left(\varphi_j^+ \frac{\partial \varphi_0^-}{\partial n} - \varphi_0^- \frac{\partial \varphi_j^+}{\partial n}\right) dS$$

$$+ \frac{c}{g} \oint_{C_\infty} \left[\varphi_0^-\left(i\omega\,\varphi_j^+ - c\,\frac{\partial \varphi_j^+}{\partial x}\right)\right.$$

$$\left. + \varphi_j^+\left(i\omega\,\varphi_0^- + c\,\frac{\partial \varphi_0^-}{\partial x}\right)\right] dy \qquad (22)$$

[3] The difference in sign of the present equations with respect to references [1] and [2] results from the opposite convention in defining the direction of the unit normal vector.

Here the contours C_∞ and C denote the intersections of the plane $z = 0$ with the surfaces S_∞ and S, respectively. Once more we have assumed that the contour integral over C, or the intersection of the body with the undisturbed free surface, will vanish, but now there are contributions both from S_∞ and C_∞ since the incident wave potential φ_0^- does not satisfy a radiation condition. Thus it follows that the exciting forces can be expressed entirely in the terms of the far-field asymptotic behaviour of the radiation potential φ_j^+, together with the known incident wave potential:

$$X_j^- = i\omega\rho e^{i\omega t}\left\{\iint_{S_\infty}\left(\varphi_0^-\frac{\partial\varphi_j^+}{\partial n} - \varphi_j^+\frac{\partial\varphi_0^-}{\partial n}\right)dS + \frac{c}{g}\oint_{C_\infty}\left[2i\omega\varphi_0^-\varphi_j^+ + c\,\varphi_j^+\frac{\partial\varphi_0^-}{\partial x} - c\,\varphi_0^-\frac{\partial\varphi_j^+}{\partial x}\right]dy\right\} \tag{23}$$

Similarly,

$$X_j^+ = i\omega\rho e^{i\omega t}\left\{\iint_{S_\infty}\left(\varphi_0^+\frac{\partial\varphi_j^-}{\partial n} - \varphi_j^-\frac{\partial\varphi_0^+}{\partial n}\right)dS - \frac{c}{g}\oint_{C_\infty}\left[2i\omega\varphi_0^+\varphi_j^- - c\,\varphi_j^-\frac{\partial\varphi_0^+}{\partial x} + c\,\varphi_0^+\frac{\partial\varphi_j^-}{\partial x}\right]dy\right\} \tag{24}$$

In order to proceed further we must substitute the appropriate functions for the potentials φ_0 and φ_j. The necessary integrals can be evaluated by the method of stationary phase, but the details of this analysis are lengthy and have been placed in the Appendix. The results are very simple:

$$X_j^{\pm} = -4i\rho g \frac{A}{\sigma} e^{i\omega t} P_j^{\pm}(\beta) \tag{25}$$

or

$$X_j^{\pm} = -i\omega\rho g \frac{A}{\sigma} e^{i\omega t} H_j^{\mp}\left(\frac{\sigma^2}{g}, \pi+\beta\right) \tag{26}$$

Here σ denotes the frequency of the incident wave system in a stationary reference frame, and β is its angle of incidence with respect to the x-axis. The functions $P_j^{\pm}$ and $H_j^{\pm}$ characterize the far-field behaviour of the potentials $\varphi_j^{\pm}$ and can be defined in various ways. The function P_j^{+} is defined by equation (A3) which gives the far-field representation of φ_j^{+}, and this function is known for submerged ellipsoids, thin ships, and slender ships. The functions $H_j^{\pm}$ are the "Kochin" functions, defined by the integrals

$$H_j^{\pm}(\lambda, \theta) = \iint_S \left(\frac{\partial \varphi_j^{\pm}}{\partial n} - \varphi_j^{\pm}\frac{\partial}{\partial n}\right) \exp\left[\lambda(z + ix\cos\theta + iy\sin\theta)\right] dS \tag{27}$$

We note that (26) and (27) are essentially identical to (20-21). Clearly, from (25-26), the functions P_j and H_j can be related to each other by the formula

$$P_j^{\pm}(\beta) = \frac{1}{4}\omega H_j^{\pm}\left(\frac{\sigma^2}{g}, \pi+\beta\right) \tag{28}$$

A relation between the exciting forces and the damping coefficients can be obtained with the aid of the equation

$$B_{jj}^{+} = -\frac{8\rho}{\pi\omega}\sum_{m=1}^{2}\int_{0}^{\pi-u_0}\frac{\lambda_m(u)}{(1+4\tau\cos u)^{1/2}}\left|P_j^{+}(u)\right|^2 s_m(u)\,du \tag{29}$$

which was derived in [6]. Here B_{jj} denotes the damping coefficient for forced excitation in the j'th mode, and

$$\lambda_m(u) = \frac{g}{2c^2\cos^2 u}\left[1+2\tau\cos u \pm (1+4\tau\cos u)^{1/2}\right] \qquad (m=1,2)$$

$$\tau = \omega c/g$$

$$s_1(u) = \frac{\cos u}{|\cos u|} \qquad u_0 = \begin{cases} 0 & \text{for } \tau \le 1/4 \\ \cos^{-1}\frac{1}{4\tau} & \text{for } \tau > 1/4 \end{cases}$$

$$s_2(u) = -1$$

It follows that by direct substitution of (34) that

$$B_{jj}^{+} = -\frac{1}{2\pi\rho g\omega A^2}\sum_{m=1}^{2}\int_{0}^{\pi-u_0}\frac{[\lambda_m(u)]^2}{(1+4\tau\cos u)^{1/2}}\left|X_j^{-}(u)\right|^2 s_m(u)\,du \tag{30}$$

The summation should be interpreted to include both possible wavelengths for a given angle of incidence and frequency of encounter.

IV. THE EXCITING FORCE ON A SUBMERGED ELLIPSOID

It is clear from the results of the preceding section that the exciting forces can be determined for any body, provided the far-field characteristics of the radiation potentials, are known for the body. Those classes of bodies for which this is the case include thin ships (longitudinal motions only), slender ships, and submerged ellipsoids. The last is a particularly interesting application for our results since in this case only the far-field potential is known, and thus the direct method for obtaining the exciting forces from integration of the near-field pressure is not practical. (However Havelock [5] has successfully dealt with this problem for the special case of an ellipsoid of revolution.) Thus we shall study the exciting forces on a submerged ellipsoid, using the functions P_j which were derived in [6].

We consider an ellipsoid which is defined by the equation

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{(z+h)^2}{a_3^2} = 1 \tag{31}$$

Thus (a_1, a_2, a_3) are the semi-lengths of the ellipsoid along its principal axes, which are taken to be parallel to the (x, y, z) axis, and the ellipsoid is a distance h below the undisturbed free surface. From reference [6] the functions P_j are then given by

$$P_1^+(\beta) = -2\pi i a_1 a_2 a_3 e^{-Kh} K \cos\beta \left(\omega \mathbb{D}_1 - cK \mathbb{D}_1 \cos\beta\right) j_1(q)/q$$

$$P_2^+(\beta) = -2\pi i a_1 a_2 a_3 e^{-Kh} K \sin\beta \left(\omega \mathbb{D}_2 - cK \mathbb{D}_1 \cos\beta\right) j_1(q)/q$$

$$P_3^+(\beta) = 2\pi a_1 a_2 a_3 e^{-Kh} K \left(\omega \mathbb{D}_3 - cK \mathbb{D}_1 \cos\beta\right) j_1(q)/q$$

$$P_4^+(\beta) = -2\pi i a_1 a_2 a_3 e^{-Kh} K^2 \sin\beta \left(a_2^2 - a_3^2\right) \left(\omega \mathbb{D}_4 - cK \mathbb{D}_1 \cos\beta\right) j_2(q)/q^2$$

$$P_5^+(\beta) = 2\pi i a_1 a_2 a_3 e^{-Kh} \Big\{ K^2 \cos\beta \left(a_1^2 - a_3^2\right) \left(\omega \mathbb{D}_5 - cK \mathbb{D}_1 \cos\beta\right) j_2(q)/q^2 - cK \left(\mathbb{D}_3 - \mathbb{D}_1\right) j_1(q)/q \Big\}$$

$$P_6^+(\beta) = -2\pi a_1 a_2 a_3 e^{-Kh} \Big\{ K^2 \cos\beta \sin\beta \left(a_1^2 - a_2^2\right) \left(\omega \mathbb{D}_6 - cK \mathbb{D}_1 \cos\beta\right) j_2(q)/q^2 - cK \sin\beta \left(\mathbb{D}_2 - \mathbb{D}_1\right) j_1(q)/q \Big\}$$

where $j_n(q)$ is the spherical Bessel function,

$$j_n(q) = \left(\frac{\pi}{2q}\right)^{1/2} J_{n+1/2}(q)$$

and

$$q = K\left[(a_1^2 - a_3^2)\cos^2\beta + (a_2^2 - a_3^2)\sin^2\beta\right]^{1/2}$$

The coefficients $\mathcal{D}_j$ are related to the virtual mass coefficients of the ellipsoid in an infinite fluid, and are defined by

$$\mathbb{D}_j = \frac{1}{2 - \alpha_j}$$

$$\mathbb{D}_{j+3} = \frac{a_{j+1}^2 - a_{j+2}^2}{2(a_{j+1}^2 - a_{j+2}^2) + (a_{j+1}^2 + a_{j+2}^2)(\alpha_{j+1} - \alpha_{j+2})}$$

(j = 1,2,3)

where

$$\alpha_j = a_1 a_2 a_3 \int_0^\infty \frac{d\lambda}{(a_j^2+\lambda)^{3/2}\left[(a_{j+1}^2+\lambda)(a_{j+2}^2+\lambda)\right]^{1/2}}$$

(j = 1,2,3)

Here we have utilized the cyclic convention, i.e.

$$a_4 = a_1,\ a_5 = a_2,\ a_6 = a_3,\ \alpha_4 = \alpha_1,\ \alpha_5 = \alpha_2,\ \alpha_6 = \alpha_3$$

Substituting the above results in (25) we can obtain analytic expressions for the six exciting forces, as functions of the speed, frequency, heading angle, depth of submergence, and the semi-lengths (a_1, a_2, a_3).

If $a_2 = a_3$, the ellipsoid reduces to an ellipsoid of revolution (spheroid) and our results reduce to those obtained by Havelock [5].

In order to present graphical results in a form similar to Havelock's, we define non-dimensional force coefficients equal to the amplitude of the exciting forces divided by the product of the ellipsoid's displacement, $\frac{4}{3}\pi\rho g\, a_1 a_2 a_3$, and the amplitude of the effective wave slope at the ellipsoid's axis, KAe^{-Kh}. The moment coefficients are non-dimensionalized with the product of the same factor and the length $2a_1$. Thus

$$\frac{X_j^-}{\frac{4}{3}\pi\rho g a_1 a_2 a_3 KAe^{-Kh}} = \frac{3}{\pi}\,\frac{P_j^+(\beta)e^{i\omega t}}{a_1 a_2 a_3 \sigma K e^{-Kh}} \equiv C_{xj}^- e^{i(\omega t+\varepsilon_j)} \qquad (j=1,2,3)$$

$$\frac{X_j^-}{\frac{8}{3}\pi\rho g a_1^2 a_2 a_3 KAe^{-Kh}} = \frac{3}{2\pi}\,\frac{P_j^+(\beta)e^{i\omega t}}{a_1^2 a_2 a_3 \sigma K e^{-Kh}} \equiv C_{xj}^- e^{i(\omega t+\varepsilon_j)} \qquad (j=4,5,6)$$

or, substituting for the functions P_j:

$$C_{x1}^- = 6\,\mathbb{D}_1 \cos\beta\; j_1(q)/q \tag{32}$$

$$C_{x2}^- = 6\sin\beta\left[\mathbb{D}_2 + \frac{c\sigma}{g}(\mathbb{D}_2 - \mathbb{D}_1)\cos\beta\right] j_1(q)/q \tag{33}$$

$$C_{x3}^- = 6\left[\mathbb{D}_3 + \frac{c\sigma}{g}(\mathbb{D}_3 - \mathbb{D}_1)\cos\beta\right] j_1(q)/q \tag{34}$$

$$C_{x4}^{-} = 3K\left(\frac{a_2^2 - a_3^2}{a_1}\right)\sin\beta\left[\mathbb{D}_4 + \frac{c\sigma}{g}\left(\mathbb{D}_4 - \mathbb{D}_1\right)\cos\beta\right] j_2(q)/q^2$$

(35)

$$C_{x5}^{-} = 3K\left(\frac{a_1^2 - a_3^2}{a_1}\right)\cos\beta\left[\mathbb{D}_5 + \frac{c\sigma}{g}\left(\mathbb{D}_5 - \mathbb{D}_1\right)\cos\beta\right] j_2(q)/q^2$$

$$- \frac{3c\sigma}{gKa_1}\left(\mathbb{D}_3 - \mathbb{D}_1\right) j_1(q)/q$$

(36)

$$C_{x6}^{-} = 3K\left(\frac{a_1^2 - a_2^2}{a_1}\right)\cos\beta\sin\beta\left[\mathbb{D}_6 + \frac{c\sigma}{g}\left(\mathbb{D}_6 - \mathbb{D}_1\right)\cos\beta\right] j_2(q)/q^2$$

$$- \frac{3c\sigma}{gKa_1}\sin\beta\left(\mathbb{D}_2 - \mathbb{D}_1\right) j_1(q)/q$$

(37)

It is important to note that q does not depend on the forward velocity c, and thus the excicing force coefficients depend <u>linearly</u> on the forward velocity; in fact the surge force coefficient C_{x1} is independent of the forward velocity. For beam waves ($\beta = 90^{\circ}$) the coefficient for surge is zero, as is suggested from symmetry considerations, and the heave, sway, and roll coefficients are independent of forward velocity, but the pitch and yaw coefficients are non-zero and depend linearly on the forward velocity:

$$C_{x5}^{-}\left(\frac{\pi}{2}\right) = -\frac{3c\sigma}{gKa_1}\left(\mathbb{D}_2 - \mathbb{D}_1\right)\frac{j_1\left(K(a_1^2-a_3^2)^{1/2}\right)}{K(a_1^2-a_3^2)^{1/2}} \tag{38}$$

$$C_{x6}^{-}\left(\frac{\pi}{2}\right) = -\frac{3c\sigma}{gKa_1}\left(\mathbb{D}_3 - \mathbb{D}_1\right)\frac{j_1\left(K(a_1^2-a_3^2)^{1/2}\right)}{K(a_1^2-a_3^2)^{1/2}} \tag{39}$$

These qualitative conclusions are in agreement with Havelock's results for a spheroid, but in one respect Havelock's conclusions do not carry over. This is with regard to the effect of the angle of incidence β. For the spheroid the functional dependence of the pitch and heave exciting forces on β can be inferred from the case $\beta = 0$ by replacing K and $c\sigma/g$ with $K \cos \beta$ and $c\sigma/g \cos \beta$. However for the ellipsoid, with $a_2 \neq a_3$, the function $\mathcal{I}$ will have a more complicated dependence on β and this simple "foreshortening" relationship breaks down.

Finally we note the obvious relations

$$C_{x1}(\beta) = -C_{x1}(\pi-\beta) \tag{40}$$

$$C_{x2}^{-}(\beta) = C_{x2}^{+}(\pi-\beta) \tag{41}$$

$$C_{x3}^{-}(\beta) = C_{x3}^{+}(\pi-\beta) \tag{42}$$

$$C_{x4}^{-}(\beta) = C_{x4}^{+}(\pi-\beta) \tag{43}$$

$$C_{x5}^{-}(\beta) = -C_{x5}^{+}(\pi-\beta) \tag{44}$$

$$C_{x6}^{-}(\beta) = -C_{x6}^{+}(\pi-\beta) \tag{45}$$

which are valid for any body with longitudinal symmetry.

Figures 1-11 illustrate the above equations. These show the six exciting force coefficients $C_{\bar{x}_j}$ for the two ellipsoids $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$ and $a_2/a_1 = 1/14$, $a_3/a_1 = 1/7$. (The same ellipsoids were used for the damping calculations of reference [5].) In these figures the abscissa is the wavelength ratio λ/L, and different curves represent the different heading angles 0°, 30°, 60°, 90°, 120°, 150°, and 180°. The solid curves represent zero forward speed and the dashed curves represent a Froude number of 1.0. In each figure the ellipse indicates the orientation of a typical transverse section. Since equations (32-37) are linear in the forward velocity c, Figures 1-11 can be used for any Froude number employing linear interpolation or extrapolation. (As a result the curves for 120°, 150°, and 180° are in fact redundant, since those could be found from equations (40-45) with extrapolation to a Froude number of -1.0.)

Figures 1-6 are for the "flat" ellipsoid, while Figures 7-11 are for the "thin" one. The surge force for the thin ellipsoid is deleted since, within the limits of graphical accuracy, the values are identical to those of Figure 1.

Figure 1 shows only the zero speed curves, since the surge exciting force is independent of the forward velocity. The same is true in sway, heave, and roll for the case of beam waves (90°).

Comparing the respective forces for the two ellipsoids, there is clearly a qualitative similarity in all cases, with quantitative differences roughly proportional to the difference in the projected areas for each component. In roll the only important difference between the two ellipsoids is in the sign of the exciting moment.

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APPENDIX

Evaluation of the Far-Field Integrals

Equation (23) is an expression for the exciting forces X_j^- in terms of surface and line integrals over a control surface at infinity. Here we shall evaluate these integrals. First we require the incident wave potential φ_o^- and the far-field form of the radiation potential φ_j^+. The incident wave potential is given by

$$\varphi_o^- = \frac{gA}{\sigma} \exp\left(Kz - iKx\cos\beta - iKy\sin\beta \right) \tag{A1}$$

where A is the wave amplitude, σ is the frequency in a stationary coordinate system, $K = \sigma^2/g$ is the wave number, and β is the angle of incidence with respect to the x-axis. The frequency of encounter in the moving (x, y, z) coordinate system is $\omega = \sigma + c\,K\cos\beta$. Solving this equation for σ gives

$$\sigma = \omega \left(\frac{-1 \pm (1 + 4\tau\cos\beta)^{1/2}}{2\tau\cos\beta} \right) \tag{A2}$$

where $\tau = \omega c/g$. We note that there are two possible values of σ, corresponding to the fact that there are always two possible wavelengths which give the same frequency of encounter in a moving coordinate system, for the same value of $\cos\beta$.

Turning to the radiation potential φ_j^+ , the desired far-field expansion has been derived for a submerged ellipsoid in reference [5], and the general form of this expression is valid for any body. Thus, for large values of the polar radius $R = \sqrt{x^2+y^2}$,

$$\varphi_j^+ = -\frac{1}{i\omega}\left(\frac{8}{\pi R}\right)^{1/2} \sum_{m,n} \left\{ \frac{\lambda_m(u_n)\sin^2\theta}{\sin^2 u_n \left|\frac{d\theta}{du_n}\cos(u_n-\theta)\right|} \right\}^{1/2} S_m(u_n)\,P_j^+(u_n)$$

$$\exp\left\{\lambda_m(u_n)\left[z+iR\cos(u_n-\theta)\right] \pm \frac{\pi i}{4}\right\} + O(1/R) \tag{A3}$$

where

$$x = R\cos\theta$$

$$y = R\sin\theta$$

$$\lambda_m(u) = \frac{g}{2c^2}\,\frac{1+2\tau\cos u \pm (1+4\tau\cos u)^{1/2}}{\cos^2 u}$$

$$S_1(u) = \frac{\cos u}{|\cos u|}$$

$$S_2(u) = -1$$

and the ($\pm$) sign in (27) is chosen to agree with the sign of

$$\frac{d\theta}{du_n}\cos(u_n-\theta)$$

The second summation is over the N-roots of the equation

$$\operatorname{ctn} \theta = - \frac{\sin^2 u_n \pm (1 + 4\tau \cos u_n)^{1/4}}{\sin u_n \cos u_n} \tag{A4}$$

satisfying the inequality

$$-\pi \leq u_n \leq |\theta| - \frac{\pi}{2}, \qquad |\theta| \leq \pi$$

The functions $P_j(u_n)$ in (A3) depend on the body grometry as well as the frequency and speed parameters. These functions, which are related to the "Kochin" functions, are known for certain bodies including submerged ellipsoids, slender bodies, and, for $j = 1, 3, 5$, for thin ships.

We now substitute the potentials (A1) and (A3) in equation (23), taking as the control surface S_∞ a circular cylinder of large radius R about the x-axis. Thus

$$\begin{aligned} X_j^- = i\omega\rho e^{i\omega t} \Big\{ & \int_{-\infty}^{0} dz \int_0^{2\pi} R\, d\theta \left(\varphi_0^- \frac{\partial \varphi_j^+}{\partial R} - \varphi_j^+ \frac{\partial \varphi_0^-}{\partial R} \right) \\ & + \frac{c}{g} \int_0^{2\pi} \cos\theta\, d\theta \Big[2i\omega\, \varphi_0^- \varphi_j^+ + c\, \varphi_j^+ \frac{\partial \varphi_0^-}{\partial x} \\ & - c\, \varphi_0^- \frac{\partial \varphi_j^+}{\partial x} \Big] \Big\} \end{aligned} \tag{A5}$$

or, after substituting for φ_0^- and φ_j^+ and performing the z-integration,

$$X_j^- = -\rho g \frac{A}{\sigma} e^{i\omega t}\left(\frac{8}{\pi}R\right)^{1/2}\int_0^{2\pi} d\theta \sum_{m,n}\left\{\frac{\lambda_m(u_n)\sin^2\theta}{\sin^2 u_n\left|\frac{d\theta}{du_n}\cos(u_n-\theta)\right|}\right\}^{1/2}$$

$$s_m(u_n)\,P_j^+(u_n)\,\exp\left\{i\lambda_m(u_n)R\cos(u_n-\theta) - iKR\cos(\beta-\theta) \pm \frac{\pi i}{4}\right\}$$

$$\left\{\frac{1}{\lambda_m+K}\left[i\lambda_m\cos(u_n-\theta)+iK\cos(\beta-\theta)\right] + \frac{c}{g}\cos\theta\left[2i\omega - icK\cos\beta - ic\lambda_m(u_n)\cos u_n\right]\right\}$$

(A6)

Since R is a large parameter, the integral can be evaluated by the method of stationary phase. Thus the only finite contribution as $R \to \infty$ is from values of θ such that

$$\frac{d}{d\theta}\left[\lambda_m(u_n)\cos(u_n-\theta) - K\cos(\beta-\theta)\right] = 0$$

or

$$0 = \left(\frac{\partial}{\partial\theta} + \frac{\partial u_n}{\partial\theta}\frac{\partial}{\partial u_n}\right)\left[\lambda_m(u_n)\cos(u_n-\theta) - K\cos(\beta-\theta)\right]$$

$$= \frac{\partial}{\partial\theta}\left[\lambda_m(u_n)\cos(u_n-\theta) - K\cos(\beta-\theta)\right]$$

$$= \lambda_m(u_n)\sin(u_n-\theta) - K\sin(\beta-\theta)$$

(A7)

since, in accordance with (A4), the roots, u_n are already determined from the condition $\frac{\partial}{\partial u}\lambda_m(u)\cos(u-\theta) = 0$. Clearly one point of stationary phase is obtained when θ, m, and n take the values such that $\beta = u_n$ and $\lambda_m(u_n) = K$. From the relation (A2) this is possible, since

$$K = \frac{\sigma^2}{g} = \frac{\omega^2}{g}\left(\frac{1 + 2\tau\cos\beta \pm (1 + 4\tau\cos\beta)^{1/2}}{2\tau^2\cos^2\beta}\right) \tag{A8}$$

Moreover this is the only root of (A7) which gives a contribution to the integral (A6), since for other possible roots of (A7) the integrand of (A6) will vanish. In order to evaluate (A6) we also need the relation

$$\left\{ \frac{d^2}{d\theta^2}\left[\lambda_m(u_n)\cos(u_n-\theta) - K\cos(\beta-\theta)\right]\right\}_{\substack{u_n=\beta \\ \lambda_m=K}}$$

$$= \left\{\left(\frac{d\theta}{du_n}\right)^{-2} \frac{\partial^2}{\partial u_n^2}\left[\lambda_m(u_n)\cos(u_n-\theta)\right]\right\}_{\substack{u_n=\beta \\ \lambda_m=K}}$$

$$= -\left[\frac{\lambda_m}{\frac{d\theta}{du_n}\cos(u_n-\theta)}\right]_{\substack{u_n=\beta \\ \lambda_m=K}} \tag{A9}$$

Substituting in (A6) we obtain, by the method of stationary phase,

$$X_j^- = -4i\rho g \frac{A}{\sigma} e^{i\omega t}\left|\frac{\sin\theta}{\sin\beta}\right| s_m(\beta)\, P_j^+(\beta)\left[\cos(\beta-\theta) + 2\cos\theta\left(\tau - \frac{c^2}{g}K\cos\beta\right)\right]$$

or, after substituting (A4) and performing some straightforward reduction,

$$X_j^- = -4 i \rho g \frac{A}{\sigma} e^{i\omega t} P_j^+(\beta) \tag{A10}$$

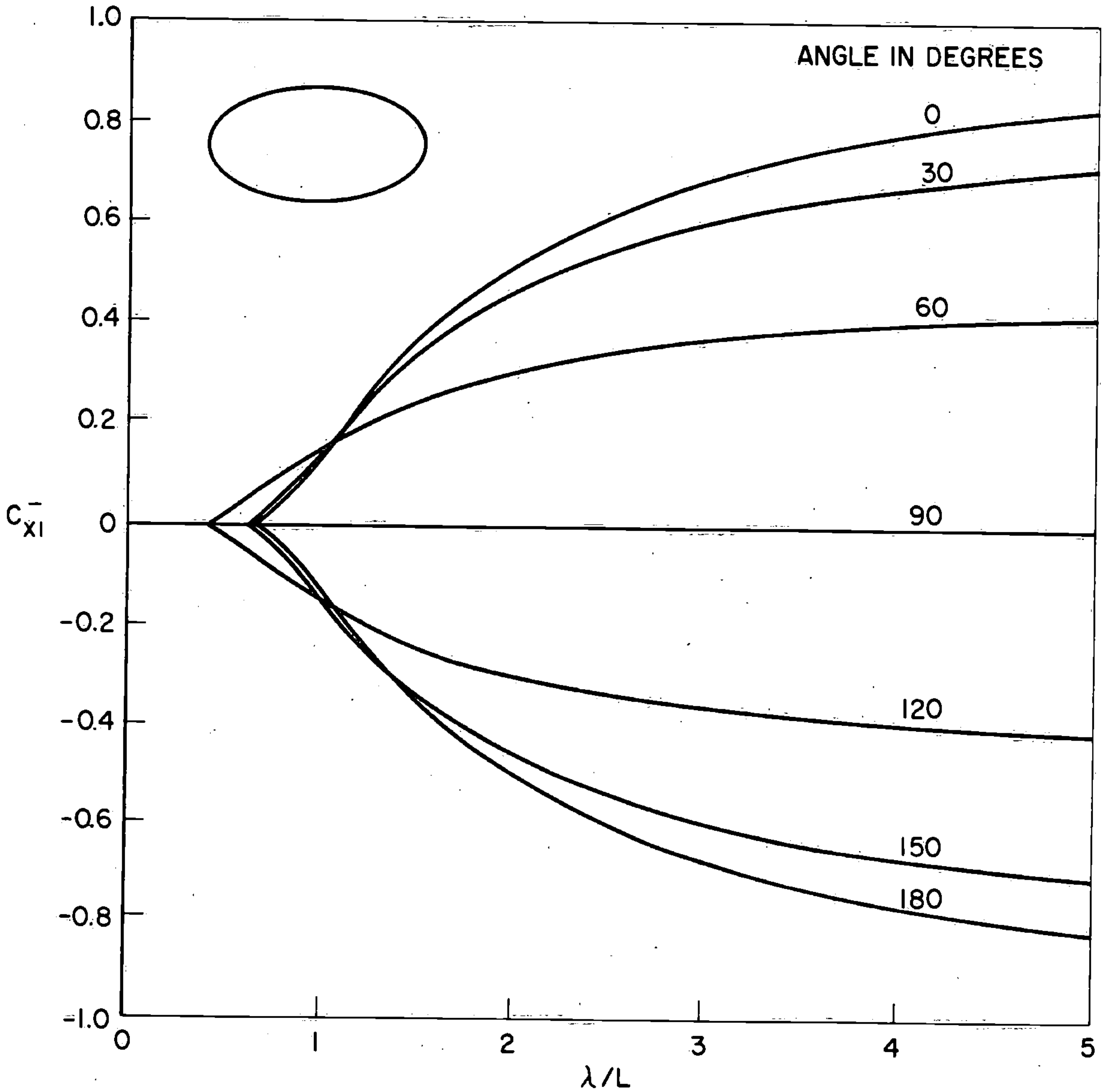

FIGURE 1 – SURGE EXCITING FORCE COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{7}$, $\frac{a_3}{a_1} = \frac{1}{14}$, FOR VARIOUS HEADING ANGLES

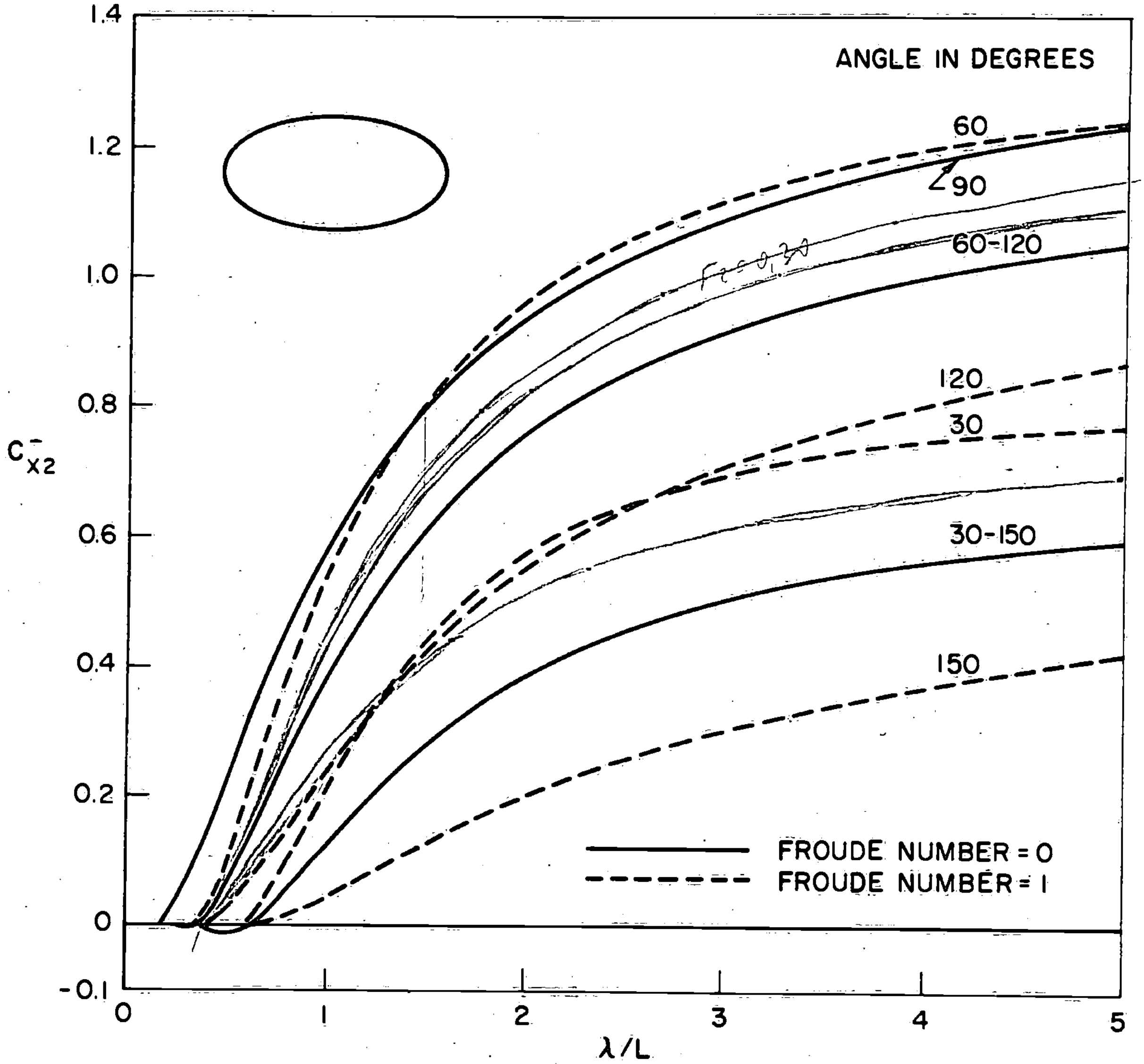

FIGURE 2 – SWAY EXCITING FORCE COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{7}$, $\frac{a_3}{a_1} = \frac{1}{14}$, FOR VARIOUS HEADING ANGLES

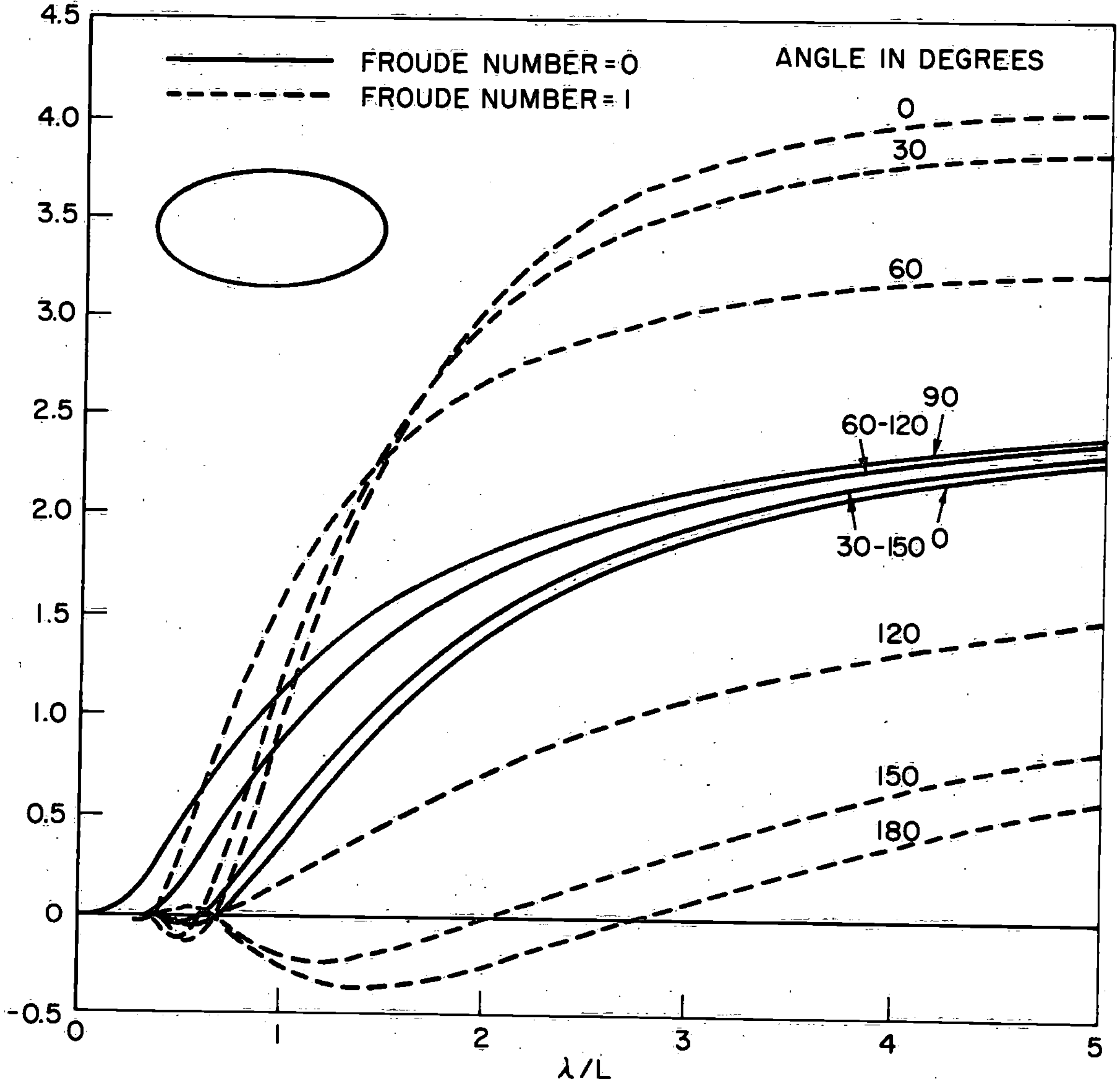

FIGURE 3 – HEAVE EXCITING FORCE COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{7}$, $\frac{a_3}{a_1} = \frac{1}{14}$, FOR VARIOUS HEADING ANGLES

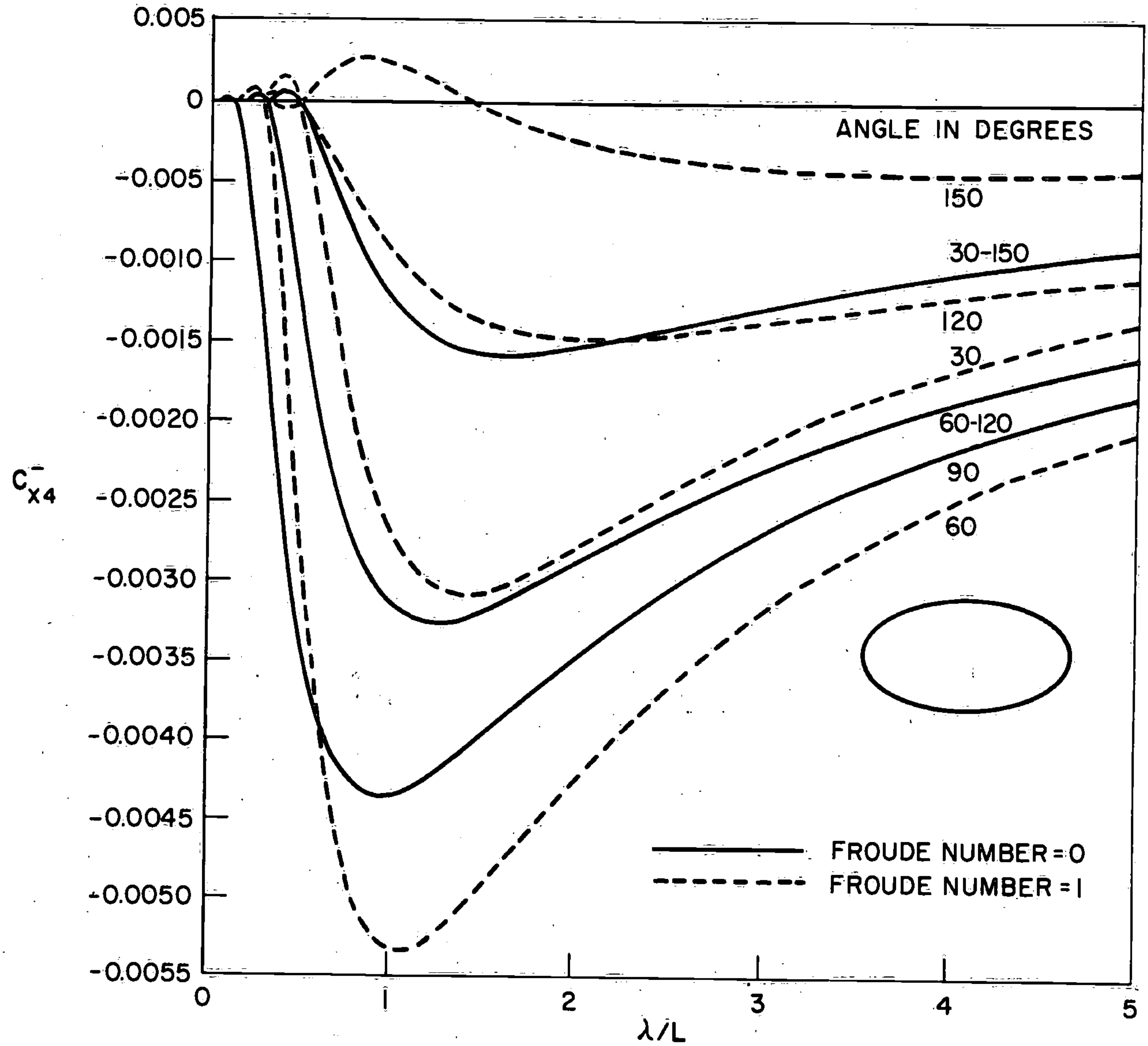

FIGURE 4 — ROLL EXCITING MOMENT COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{7}$, $\frac{a_3}{a_1} = \frac{1}{14}$, FOR VARIOUS HEADING ANGLES

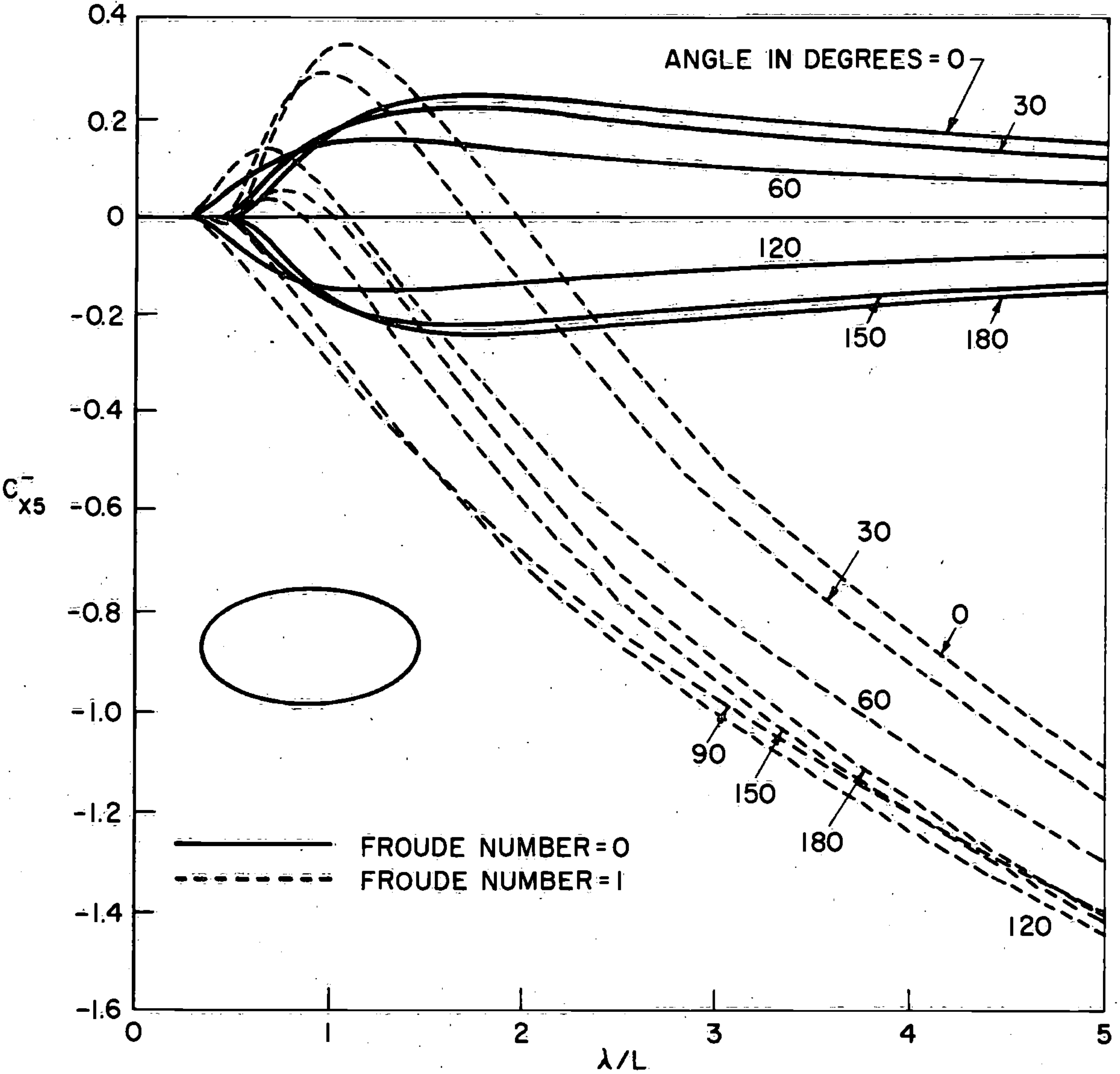

FIGURE 5 — PITCH EXCITING MOMENT COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{7}, \frac{a_3}{a_1} = \frac{1}{14}$, FOR VARIOUS HEADING ANGLES

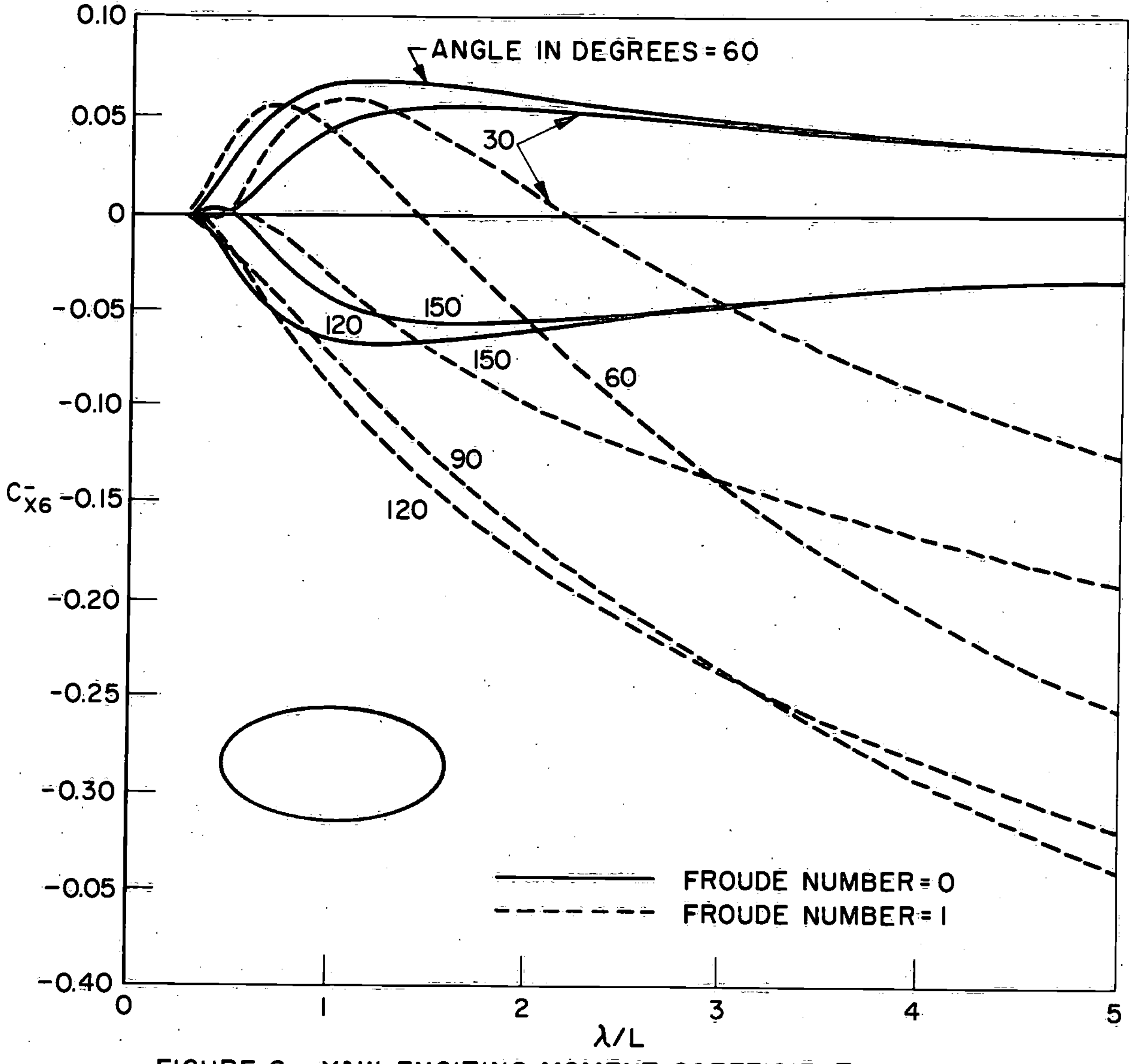

FIGURE 6 – YAW EXCITING MOMENT COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{7}$, $\frac{a_3}{a_1} = \frac{1}{14}$, FOR VARIOUS HEADING ANGLES

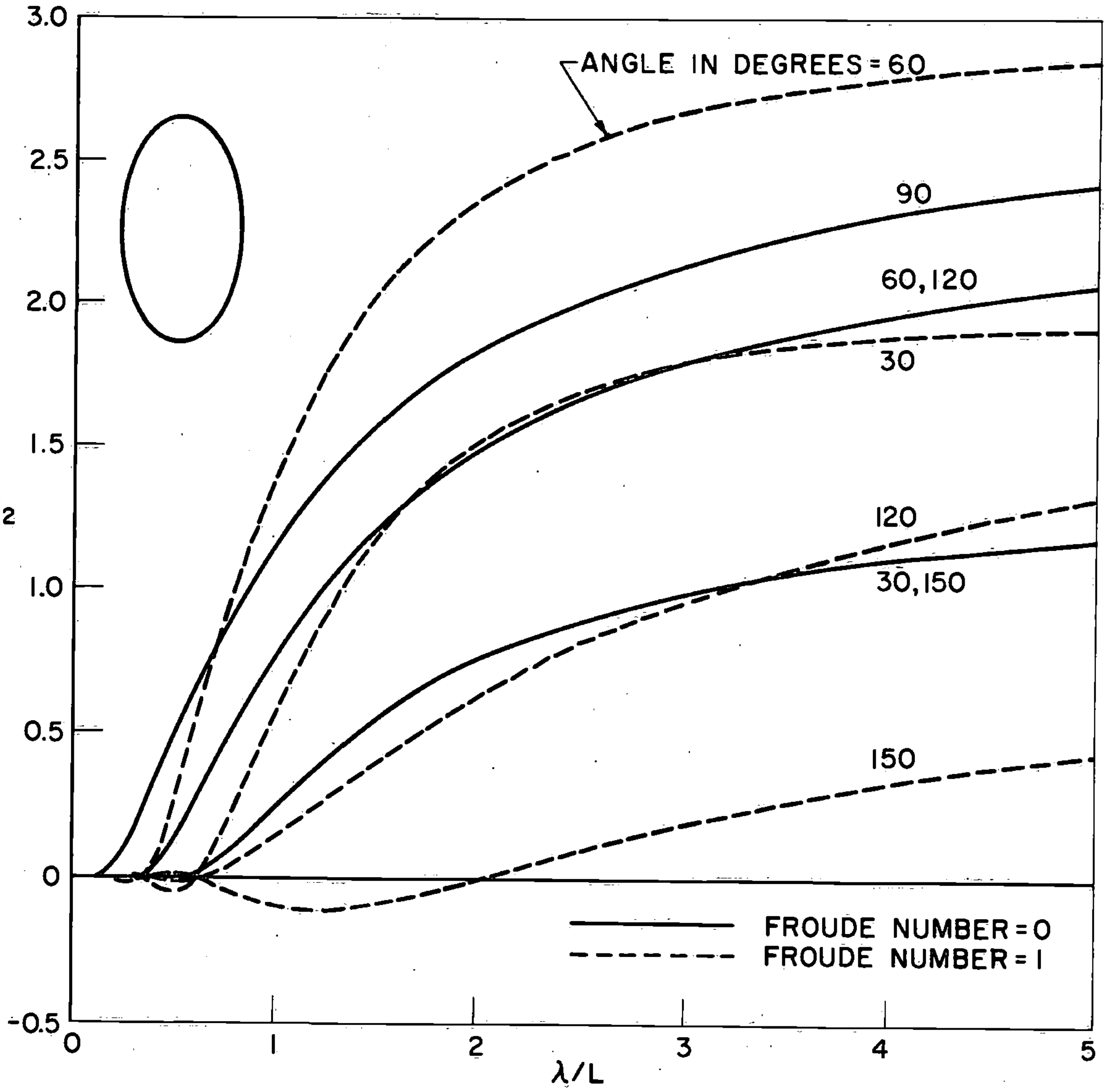

FIGURE 7 – SWAY EXCITING FORCE COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{14}$, $\frac{a_3}{a_1} = \frac{1}{7}$, FOR VARIOUS HEADING ANGLES

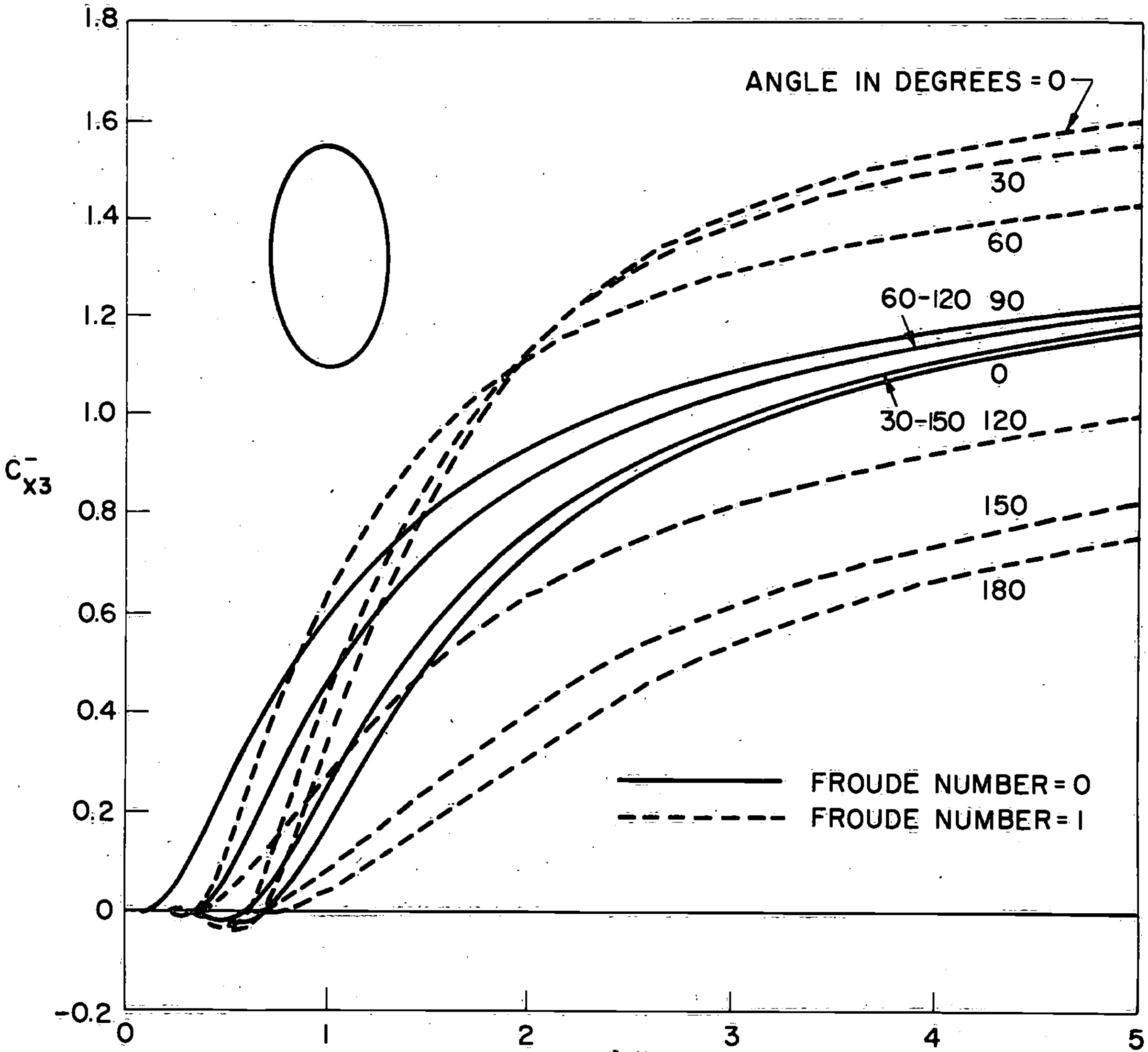

FIGURE 8 – HEAVE EXCITING FORCE COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{14}$, $\frac{a_3}{a_1} = \frac{1}{7}$, FOR VARIOUS HEADING ANGLES

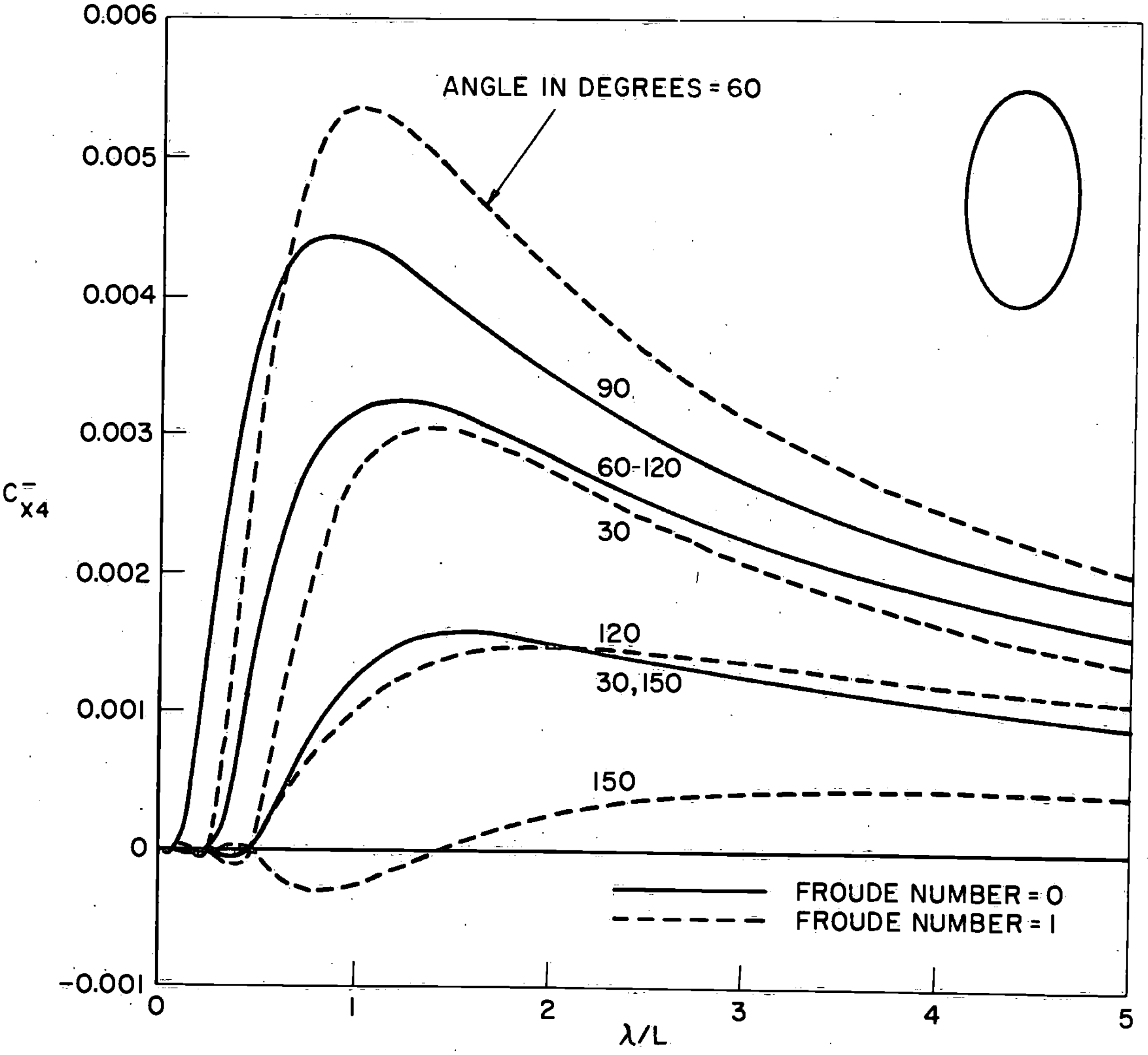

FIGURE 9 – ROLL EXCITING MOMENT COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{14}$, $\frac{a_3}{a_1} = \frac{1}{7}$, FOR VARIOUS HEADING ANGLES

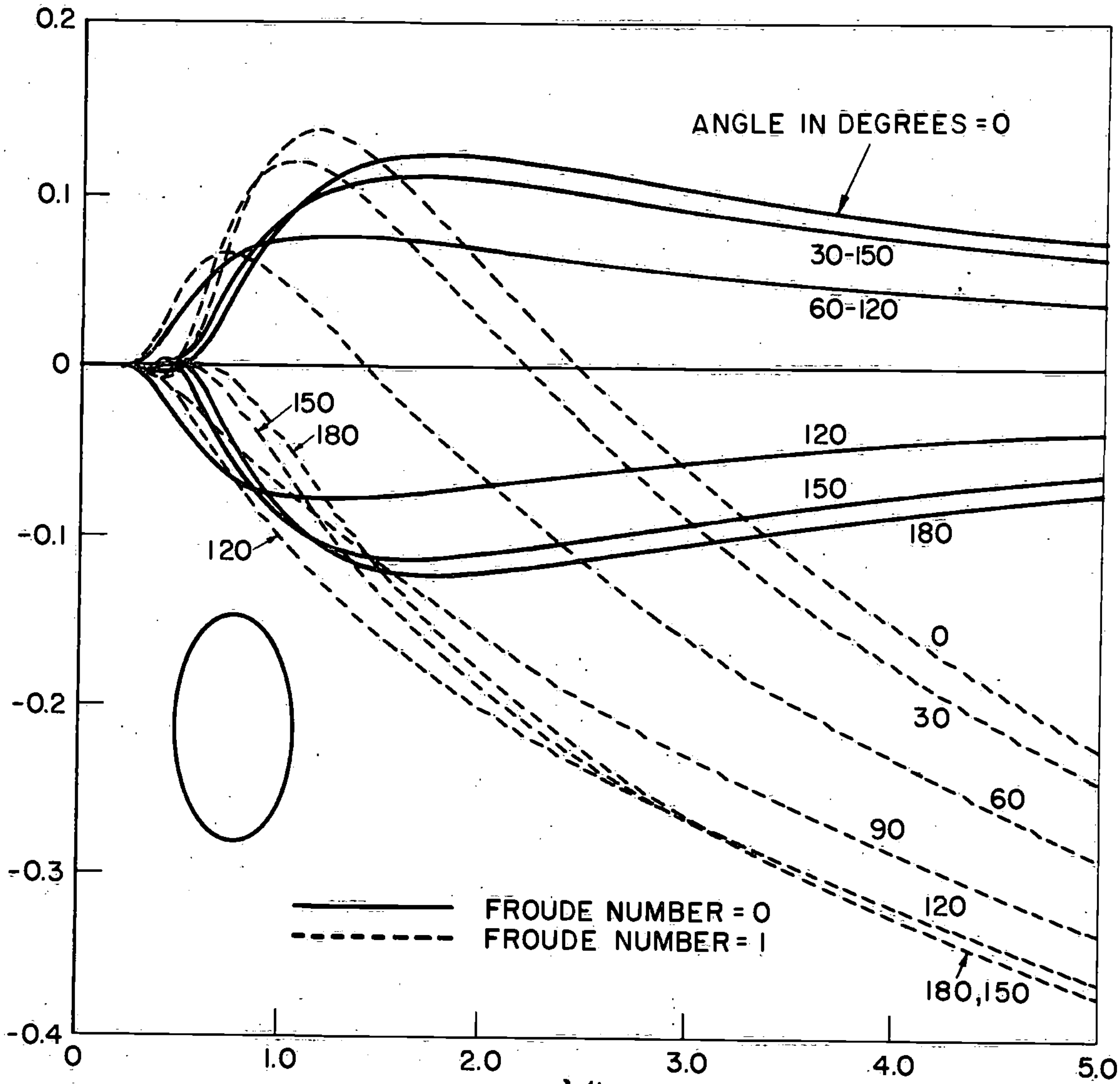

FIGURE 10 – PITCH EXCITING MOMENT COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{14}$, $\frac{a_3}{a_1} = \frac{1}{7}$, FOR VARIOUS HEADING ANGLES

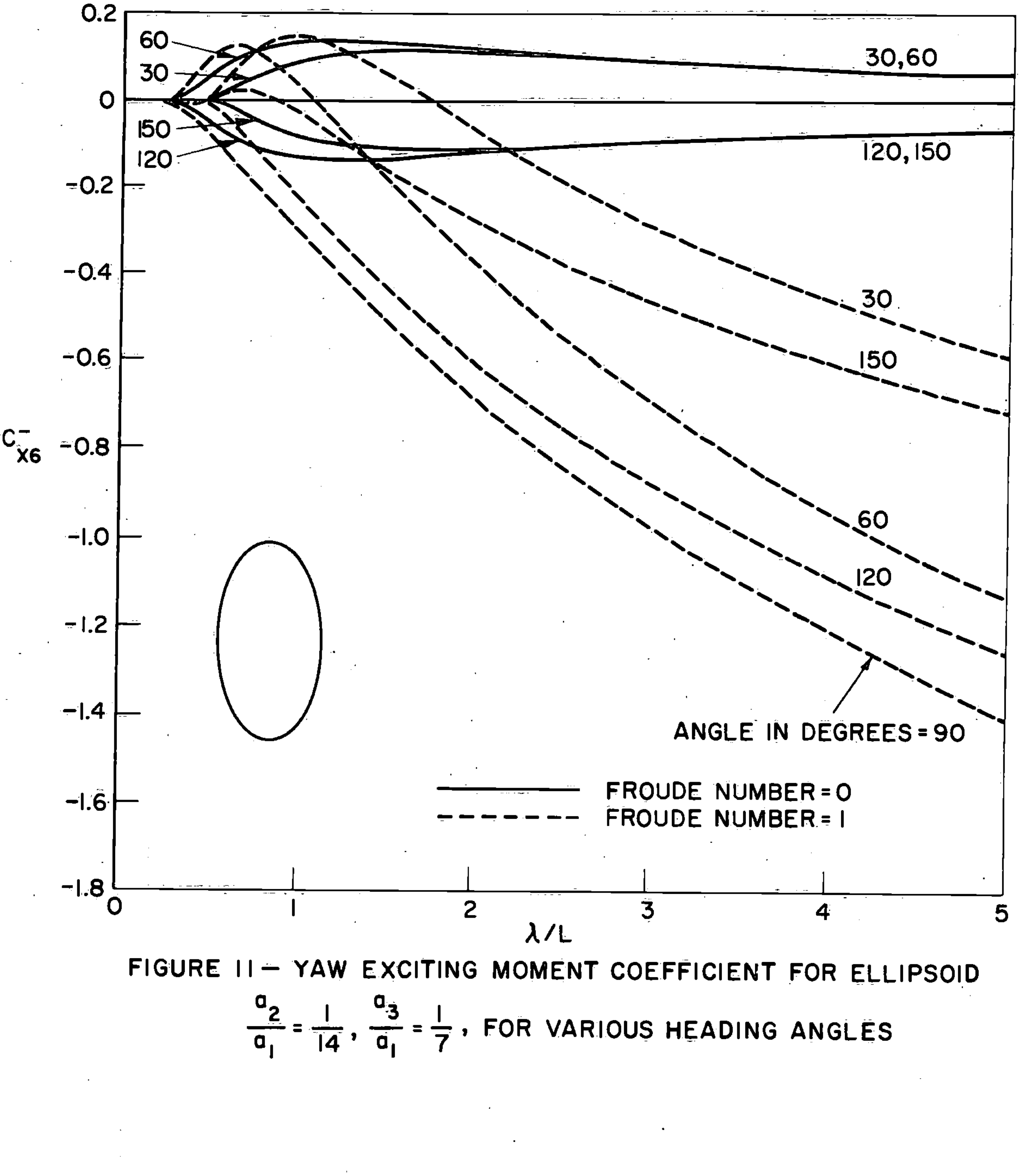

FIGURE 11 — YAW EXCITING MOMENT COEFFICIENT FOR ELLIPSOID $\frac{a_2}{a_1} = \frac{1}{14}$, $\frac{a_3}{a_1} = \frac{1}{7}$, FOR VARIOUS HEADING ANGLES