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The Existence and Dimension of the Attractor for a 3D Flow of a Non-Newtonian Fluid subject to Dynamic Boundary Conditions *

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Abstract

We consider non-Newtonian incompressible 3D fluid of Ladyzhenskaya type, in the setting of the dynamic boundary condition. Assuming sufficient growth rate of the stress tensor with respect to the velocity gradient, we establish explicit dimension estimate of the global attractor in terms of the physical parameters of the problem.

1 Introduction.

The existence of global attractor, its finite-dimensionality, and possibly even the construction of a finite-dimensional exponential attractor belong to prototypical results of the dynamical theory of nonlinear evolutionary PDEs. These goals are often attained, as long as the system is well-posed and dissipative. The literature being too extensive to quote, let us mention the basic monographs [7], [8], [9], [22], [23]. On the other hand, an explicit dimension estimate of the attractor is a different matter, requiring additional tools from functional analysis, and considerably more demanding in view of the regularity of the underlying solution semigroup.

Focusing to the incompressible Navier-Stokes equations as a model problem, one can say that in 2D, the problem of the attractor dimension is rather well understood. Reasonable upper estimates are available for various domains, even unbounded ones, and the results are known to be sharp for the torus, see recent paper [12] and the references therein. For the 3D case, weak solutions exist globally, but the uniqueness remains a famous open problem even for the torus. One can still define (sort of) an attractor, but nothing can be said about its dimension. Consequently, various regularizations of the problem, more or less well-motivated physically, have been proposed, for which these problems were then successfully addressed, cf. for example [11] for the so-called Euler-Bardina regularization.

In the present paper, we consider one such classical modification, going back to Ladyzhenskaya [14], where additional gradient integrability is induced by a non-linear modification of the viscous stress tensor via the r-Laplacian type term $|\boldsymbol{Du}|^{r-2}\,\boldsymbol{Du}$. Thus, one the one hand, the problem becomes well-posed in 3D for values only slightly above the NSE-critical value $r = 2$. On the other hand, such a highest order nonlinearity brings additional complications to the analysis, as in particular higher regularity of weak solutions is difficult to obtain in dimensions other than two. Note that this so-called Ladyzhenskaya model is well-motivated physically [17].

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The problem of the attractor dimension, and more generally, the structural complexity of the dynamics, is presumable highly sensitive to the adopted boundary condition. Motivated by this, we further generalize our setting to allow for a non-linear evolution on $\partial\Omega$, which is driven by the normal stress force of the fluid, exerted across the boundary. Our result is new in particular by providing an explicit (asymptotic) dimension estimate for 3D fully non-linear problem, while remaining in the setting of weak solutions only.

Let us finally mention some related publications and results concerning our model, i.e. the Ladyzhenskaya r-fluid. For basic existence and uniqueness theory of weak solutions under dynamic boundary conditions, see recent paper [1], cf. also [18]. Existence of finite-dimensional exponential attractors was recently established in a rather general setting, but without explicit dimension estimates [21]. Concerning the Dirichlet boundary conditions, explicit dimension estimates in 3D setting were previously obtained in [4], to which the current paper is a direct generalization. Improved dimension estimates, based on the volume contraction method, were also obtained in the 2D setting by [13], and for suitably regularized problem again in 3D setting [19].

2 Formulation of the Problem and the Main Result.

We consider generalized Navier-Stokes equations with dynamic boundary condition on a bounded domain $\Omega \subset \mathbb{R}^3$, $\Omega \in \mathcal{C}^{0,1}$ and bounded time interval $(0,T)$. We denote space-time domain by $Q := (0,T) \times \Omega$, and by $\Gamma := (0,T) \times \partial\Omega$ the space-time boundary. We further denote unknown velocity by $v : Q \to \mathbb{R}^3$ and unknown pressure of the fluid by $\pi : Q \to \mathbb{R}$. The quantity $\boldsymbol{S}$ is called the extra stress tensor and here it is assumed to be a function of the symmetric velocity gradient $2\boldsymbol{D}\boldsymbol{v} = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T$. The external body force $\boldsymbol{f} : Q \to \mathbb{R}^3$ is independent of time.

An essential feature of our model is that we incorporate the so-called dynamic boundary condition, so that the tangential velocity component is subject to a certain non-linear response $\boldsymbol{s} = \boldsymbol{s}(\boldsymbol{v})$ on Γ. Our system thus reads

$$\begin{aligned}
\partial_t \boldsymbol{v} - \operatorname{div} \boldsymbol{S} + \operatorname{div} (\boldsymbol{v} \otimes \boldsymbol{v}) + \nabla \pi &= \boldsymbol{f} && \text{in } Q, && (2.1a)\\
\operatorname{div} \boldsymbol{v} &= 0 && \text{in } Q, && (2.1b)\\
\boldsymbol{v} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma, && (2.1c)\\
-(\boldsymbol{S}\boldsymbol{n})_\tau &= \alpha \boldsymbol{s} + \beta \partial_t \boldsymbol{v} && \text{on } \Gamma, && (2.1d)\\
\boldsymbol{v}(0) &= \boldsymbol{v}_0 && \text{in } \Omega \cup \partial\Omega. && (2.1e)
\end{aligned}$$

Concerning the constitutive functions $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{D}\boldsymbol{v})$ and $\boldsymbol{s} = \boldsymbol{s}(\boldsymbol{v})$, we assume polynomial growth in terms of certain r and $q \geq 2$. More precisely: for all $\boldsymbol{D}_1, \boldsymbol{D}_2 \in \mathbb{R}^{3\times 3}_{sym}$

$$\begin{aligned}
\boldsymbol{S}(\boldsymbol{0}) &= \boldsymbol{0},\\
|\boldsymbol{S}(\boldsymbol{D}_1) - \boldsymbol{S}(\boldsymbol{D}_2)| &\leq c_1 \left(\nu_1 + \nu_2 \left(|\boldsymbol{D}_1| + |\boldsymbol{D}_2|\right)^{r-2}\right) |\boldsymbol{D}_1 - \boldsymbol{D}_2|, \\
(\boldsymbol{S}(\boldsymbol{D}_1) - \boldsymbol{S}(\boldsymbol{D}_2)) : (\boldsymbol{D}_1 - \boldsymbol{D}_2) &\geq c_2 \left(\nu_1 + \nu_2 \left(|\boldsymbol{D}_1| + |\boldsymbol{D}_2|\right)^{r-2}\right) |\boldsymbol{D}_1 - \boldsymbol{D}_2|^2 .
\end{aligned} \tag{2.2}$$

Furthermore, it is assumed that $\boldsymbol{S}$ has a potential,

$$\begin{aligned}
\boldsymbol{S}(\boldsymbol{D}) &= \partial_{\boldsymbol{D}} \Phi \left(|\boldsymbol{D}|^2\right),\\
c_3 \left(\nu_1 + \nu_2 |\boldsymbol{D}|^{r-2}\right) |\boldsymbol{D}|^2 &\leq \Phi(\boldsymbol{D}) \leq c_4 \left(\nu_1 + \nu_2 |\boldsymbol{D}|^{r-2}\right) |\boldsymbol{D}|^2 .
\end{aligned} \tag{2.3}$$

Typical example is the so-called Ladyzhenskaya fluid

$$\boldsymbol{S}(\boldsymbol{D}) = \nu_1 \boldsymbol{D}\boldsymbol{v} + \nu_2 \left|\boldsymbol{D}\boldsymbol{v}\right|^{r-2} \boldsymbol{D}\boldsymbol{v} \tag{2.4}$$

Regarding the boundary nonlinearity $\boldsymbol{s}$, we require that for all $\boldsymbol{v}_1,\ \boldsymbol{v}_2 \in \mathbb{R}^3$

$$\begin{aligned} \boldsymbol{s}(\boldsymbol{0}) &= \boldsymbol{0}, \\ \left|\boldsymbol{s}(\boldsymbol{v}_1) - \boldsymbol{s}(\boldsymbol{v}_2)\right| &\leq c_5 \left|\boldsymbol{v}_1 - \boldsymbol{v}_2\right|, \\ \left(\boldsymbol{s}(\boldsymbol{v}_1) - \boldsymbol{s}(\boldsymbol{v}_2)\right) \cdot \left(\boldsymbol{v}_1 - \boldsymbol{v}_2\right) &\geq c_6 \left|\boldsymbol{v}_1 - \boldsymbol{v}_2\right|^2, \\ \boldsymbol{s}(\boldsymbol{v})\boldsymbol{v} &\geq c_7 \left(|\boldsymbol{s}|^{\bar{q}} + |\boldsymbol{v}|^q\right), \ \text{ where } {}^1\!/_q + {}^1\!/_{\bar{q}} = 1. \end{aligned} \tag{2.5}$$

Here, we also impose the existence of a potential, i.e.

$$\boldsymbol{s}(\boldsymbol{v}) = \partial_{\boldsymbol{v}} \mathcal{S}(\boldsymbol{v}) \tag{2.6}$$

Without loss of generality, let $\mathcal{S}(\boldsymbol{0}) = \boldsymbol{0}$. It is obvious that $\mathcal{S}$ obeys upper and lower q-growth bounds, in view of (2.5).

Our main result, stated somewhat informally, reads as follows.

Main Theorem. *Let $r > 12/5$ and $\boldsymbol{f} \in L^2(\Omega)$. Then the system (2.1a – 2.1e) has a global attractor in $L^2(\Omega) \times L^2(\partial\Omega)$. Moreover, its dimension can be explicitly estimated in terms of the data.*

See Theorem 4.1 below for a precise statement and proof. We note that the solutions are not uniquely determined by initial conditions in L^2 only. Yet they immediately become more regular (and hence unique), as follows from Theorems 3.2 and 3.3. This issue of initial nonuniqueness is easily avoided in our setting of short trajectories.

As a by-product of the time regularity, we obtain that the attractor is bounded in $W^{1,r}$, and the solutions on attractor are 1/2-Hölder continuous with values in L^2. One can expect that additional, i.e. spatial regularity is also available, so that the solutions would be in fact strong. We leave this problem to the forthcoming paper.

3 Well-posedness and Additional Time Regularity.

We carry out our analysis with dynamical boundary condition which includes the time derivative of the velocity $\boldsymbol{v}$ of the fluid weighted by the parameter β. This set up demands a specific type function spaces. First we introduce such function spaces and later we define the Gelfand triplet. We essentially follow the functional set up used in [1, Section 3].

For Ω a Lipschitz domain in $\mathbb{R}^d$, i.e., $\Omega \in \mathcal{C}^{0,1}$, $\beta \geq 0$ and $r \in (0, \infty)$, we define $\mathcal{V} \subset \mathcal{C}^{0,1}(\overline{\Omega}) \times \mathcal{C}^{0,1}(\partial\Omega)$ as

$$\mathcal{V} := \left\{(\boldsymbol{v}, \boldsymbol{g}) \in \mathcal{C}^{0,1}(\overline{\Omega}) \times \mathcal{C}^{0,1}(\partial\Omega) : \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, \text{ and } \boldsymbol{v} = \boldsymbol{g} \text{ on } \partial\Omega\right\}$$

With the help of $\mathcal{V}$, we define

$$V_r := \overline{\mathcal{V}}^{\|\cdot\|_{V_r}}, \text{ where } \|(\boldsymbol{v}, \boldsymbol{g})\|_{V_r} := \|\boldsymbol{v}\|_{W^{1,r}(\Omega)} + \|\boldsymbol{v}\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{L^2(\partial\Omega)}, \tag{3.1}$$

$$H := \overline{\mathcal{V}}^{\|\cdot\|_H}, \text{ where } \|(\boldsymbol{v},\boldsymbol{g})\|_H^2 := \|\boldsymbol{v}\|_{L^2(\Omega)}^2 + \beta\,\|\boldsymbol{g}\|_{L^2(\partial\Omega)}^2 \tag{3.2}$$

Note that H is a Hilbert space with respect to the above norm. We also remark that if $(\boldsymbol{v},\boldsymbol{g}) \in V_r$, then necessarily $\boldsymbol{g} = tr\ \boldsymbol{v}$. With some abuse of notation, V_r can thus be identified with its first component v.

Theorem 3.1. *Let $\boldsymbol{v}_0 \in H$, $\boldsymbol{f} \in L^{r'}(0,T;V_r')$, $T>0$ be given, and let $r \geq {}^{11}/_5$. Then there exists at least one weak solution $\boldsymbol{v}$ to* (2.1),

$$\begin{aligned} \boldsymbol{v} &\in L^\infty(0,T;H) \cap L^r(0,T;V_r), \\ \partial_t \boldsymbol{v} &\in L^{r'}(0,T;V_r'). \end{aligned} \tag{3.3}$$

The solution satisfies energy equality, and the initial condition $\boldsymbol{v}(0) = \boldsymbol{v}_0$ holds for the representative $\boldsymbol{v} \in C([0,T];H)$.

Proof. We only sketch the proof, referring to [1] for details. Take the scalar product of (2.1a) with an arbitrary $\boldsymbol{\varphi} \in V_r$, integrate the result over Ω, and use integration by parts to obtain

$$\int_\Omega [\partial_t \boldsymbol{v}\cdot\boldsymbol{\varphi} + (\boldsymbol{S} - \boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{\varphi} - \pi \operatorname{div}\boldsymbol{\varphi}]\ dx + \int_{\partial\Omega} [\pi\boldsymbol{I} + \boldsymbol{v}\otimes\boldsymbol{v} - \boldsymbol{S}]\,\boldsymbol{n}\cdot\boldsymbol{\varphi}\ dS = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\ dx \tag{3.4}$$

By utilizing the symmetry of $\boldsymbol{S}$, (2.1c), (2.1d), and the properties of $\boldsymbol{\varphi}$ ($\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω, $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$), we deduce the weak formulation

$$\int_\Omega \partial_t\boldsymbol{v}\cdot\boldsymbol{\varphi}\ dx + \beta\int_{\partial\Omega}\partial_t\boldsymbol{v}\cdot\boldsymbol{\varphi}\ dS + \int_\Omega [\boldsymbol{S}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{v}\otimes\boldsymbol{v}] : \nabla\boldsymbol{\varphi}\ dx + \alpha\int_{\partial\Omega}\boldsymbol{s}(\boldsymbol{v})\cdot\boldsymbol{\varphi}\ dS$$
$$= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\ dx \tag{3.5}$$

Formally, we set $\boldsymbol{\varphi} := \boldsymbol{v}$ in (3.5), and use

$$\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{v} dx = \int_\Omega \sum_{i,j=1}^d \boldsymbol{v}_i\boldsymbol{v}_j\partial_i\boldsymbol{v}_j dx = \frac{1}{2}\int_\Omega \sum_{i,j=1}^d \boldsymbol{v}_i\partial_i\,|\boldsymbol{v_j}|^2\,dx, \tag{3.6}$$

$$= \frac{1}{2}\left(-\int_\Omega \operatorname{div}\boldsymbol{v}\,|\boldsymbol{v}|^2\,dx + \int_{\partial\Omega}\boldsymbol{v}\cdot\boldsymbol{n}\,|\boldsymbol{v}|^2\,dS\right) = 0, \tag{3.7}$$

where we have used (2.1b), (2.1c). Thus we obtain,

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\boldsymbol{v}|^2\ dx + \beta\int_{\partial\Omega}|\boldsymbol{v}|^2\ dS\right) + \int_\Omega \boldsymbol{S}(\boldsymbol{D}\boldsymbol{v}) : \boldsymbol{D}\boldsymbol{v}\ dx + \alpha\int_{\partial\Omega}\boldsymbol{s}(\boldsymbol{v})\cdot\boldsymbol{v}\ dS = \langle\boldsymbol{f},\boldsymbol{v}\rangle_{V_r',V_r}. \tag{3.8}$$

For the right hand side of (3.8), we obtain by utilizing Korn's and Young's inequalities,

$$\begin{aligned} \langle\boldsymbol{f},\boldsymbol{v}\rangle_{V_r',V_r} \leq \|\boldsymbol{f}\|_{V_r'}\|\boldsymbol{v}\|_{V_r} \leq c_1\|\boldsymbol{f}\|_{V_r'}\|\boldsymbol{v}\|_{W^{1,r}} &\leq c_2(\varepsilon)\|\boldsymbol{f}\|_{V_r'}^{r'} + \frac{\varepsilon}{c_3}\|\boldsymbol{v}\|_{W^{1,r}}^r,\ (\varepsilon > 0) \\ &\leq c_2(\varepsilon)\|\boldsymbol{f}\|_{V_r'}^{r'} + \varepsilon\|\boldsymbol{D}\boldsymbol{v}\|_r^r + \varepsilon\|\boldsymbol{v}\|_H^r. \end{aligned}$$

Then by (2.2) and (2.5), we deduce,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}\|_H^2 + c_5\left[\nu_1\|\boldsymbol{D}\boldsymbol{v}\|_2^2 + \nu_2\|\boldsymbol{D}\boldsymbol{v}\|_r^r\right] + c_4\alpha\|\boldsymbol{v}\|_{L^q(\Gamma)}^q \leq c_2(\varepsilon)\|\boldsymbol{f}\|_{V_r'}^{r'} + \varepsilon\|\boldsymbol{v}\|_H^r. \tag{3.9}$$

We combine compactness and monotonicity arguments to obtain the existence of a solution as a limit of a suitable approximate problem, e.g. the Galerkin scheme. Remark that $r = {}^{11}/_{5}$ is the critical value which ensures that the convective term belongs to the proper dual space. Hence in particular, any weak solution is an admissible test function and the energy equality (3.8) holds. See [18] or [1]. □

Weak solutions are non-unique in general, unless additional regularity is assumed. In particular, analogously to [4, Theorem 3.2], one proves:

Theorem 3.2. *Let $\boldsymbol{u}, \boldsymbol{v}$ be weak solutions with $\boldsymbol{u}(0) = \boldsymbol{v}(0)$, and furthermore, let $\boldsymbol{v} \in L^{\frac{2r}{2r-3}}(0,T;V_r)$. Then $\boldsymbol{u} = \boldsymbol{v}$.*

Proof. Test the equation for $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}$ by $\boldsymbol{w}$. Using the identity

$$\int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{w} = \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{w} - \boldsymbol{w}\otimes\boldsymbol{v}) : \nabla\boldsymbol{w} = \int_\Omega (\boldsymbol{w}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{w} \tag{3.10}$$

(in view of $\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) = (\nabla\cdot\boldsymbol{u})\,\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}$) as well as (2.2), one obtains

$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{w}\|_2^2 + \beta\,\|\boldsymbol{w}\|_{L^2(\partial\Omega)}^2\right) + c_2\int_\Omega I^2(\boldsymbol{Du},\boldsymbol{Dv})\,dx + \alpha\int_{\partial\Omega}(\boldsymbol{s}(\boldsymbol{u}) - \boldsymbol{s}(\boldsymbol{v}))\,\boldsymbol{w}\,dS \le \int_\Omega |\boldsymbol{w}|^2\,|\nabla\boldsymbol{v}|\,dx, \tag{3.11}$$

where

$$I^2(\boldsymbol{Du},\boldsymbol{Dv}) := \left(\nu_1 + \nu_2\,(|\boldsymbol{Du}| + |\boldsymbol{Dv}|)^{r-2}\right)|\boldsymbol{Dw}|^2 \tag{3.12}$$

By monotonicity we have

$$\alpha\int_{\partial\Omega}(\boldsymbol{s}(\boldsymbol{u}) - \boldsymbol{s}(\boldsymbol{v}))\,\boldsymbol{w}\,dS \ge 0.$$

This yields

$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{w}\|_2^2 + \beta\,\|\boldsymbol{w}\|_{L^2(\partial\Omega)}^2\right) + c_2\int_\Omega I^2(\boldsymbol{Du},\boldsymbol{Dv})\,dx \le \int_\Omega |\boldsymbol{w}|^2\,|\nabla\boldsymbol{v}|\,dx \tag{3.13}$$

By Korn inequality (Lemma B.2 in the Appendix), we have

$$\int_\Omega I^2(\boldsymbol{Du},\boldsymbol{Dv})\,dx \ge \nu_1\int_\Omega |\boldsymbol{Dw}|^2\,dx + \nu_2\int_\Omega |\boldsymbol{Dw}|^r\,dx \ge c\nu_1\left(\|\boldsymbol{w}\|_{W^{1,2}(\Omega)}^2 - \|\boldsymbol{w}\|_{L^2(\partial\Omega)}^2\right). \tag{3.14}$$

We further estimate, using (B.2), cf. the Appendix,

$$\begin{aligned}\int_\Omega |\boldsymbol{w}|^2\,|\nabla\boldsymbol{v}|\,dx &\le \|\nabla\boldsymbol{v}\|_r\,\|\boldsymbol{w}\|_{\frac{2r}{r-1}}^2 \le c_3\,\|\nabla\boldsymbol{v}\|_r\,\|\boldsymbol{w}\|_2^{\frac{2r-3}{r}}\,\|\boldsymbol{w}\|_{W^{1,2}(\Omega)}^{\frac{3}{r}}, \\ &\le \frac{c_2}{4}\nu_1\,\|\boldsymbol{w}\|_{W^{1,2}(\Omega)}^2 + c_4\nu_1^{-\frac{3}{2r-3}}\,\|\nabla\boldsymbol{v}\|_r^{\frac{3}{2r-3}}\,\|\boldsymbol{w}\|_2^2.\end{aligned}$$

Then with (3.14) we obtain,

$$\begin{aligned}&\frac{d}{dt}\|\boldsymbol{w}\|_H^2 + c_5\nu_1\,\|\boldsymbol{w}\|_{W^{1,2}(\Omega)}^2 + c_5\int_\Omega I^2(\boldsymbol{Du},\boldsymbol{Dv})\,dx \\ &\qquad\le c_4\nu_1^{-\frac{3}{2r-3}}\,\|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^{\frac{3}{2r-3}}\,\|\boldsymbol{w}\|_2^2 + c_6\,\|\boldsymbol{w}\|_{L^2(\partial\Omega)}^2,\end{aligned}$$

$$\leq c_4 \nu_1^{-\frac{3}{2r-3}} \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^{\frac{3}{2r-3}} \|\boldsymbol{w}\|_2^2 + c_6 \|\boldsymbol{w}\|_{L^2(\partial\Omega)}^2 ,$$
$$\leq c_7 \left(\nu_1^{-\frac{3}{2r-3}} \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^{\frac{3}{2r-3}} + 1 \right) \|\boldsymbol{w}\|_H^2 . \tag{3.15}$$

Finally we apply Grönwall's lemma to deduce

$$\|\boldsymbol{w}(t)\|_2^2 \leq K \|\boldsymbol{w}(s)\|_2^2 , \quad 0 \leq s \leq t \leq T. \tag{3.16}$$

In particular, we have uniqueness. □

Now, we obtain additional time regularity of the solutions, together with an explicit estimate of the relevant norms, cf. [4, Theorem 3.3]. Symbol " $\lesssim$ " means an inequality up to some generic (i.e., independent of the data) constant $c_i > 0$.

Theorem 3.3. *Let $r > 12/5$, $\boldsymbol{f} \in L^2(\Omega)$. Then the weak solution has additional time regularity*

$$\boldsymbol{v} \in L^\infty(\tau, T; V_r),$$
$$\partial_t \boldsymbol{v} \in L^2(\tau, T; L^2(\Omega)).$$

Here $\tau \in (0, T)$ is arbitrary, and one can take $\tau = 0$ if $\boldsymbol{v}(0) \in V_r$.

Now let $\boldsymbol{\varphi} := \partial_t \boldsymbol{v}$

$$\int_\Omega |\partial_t \boldsymbol{v}|^2 \, dx + \beta \int_{\partial\Omega} |\partial_t \boldsymbol{v}|^2 \, dS + \int_\Omega \left[\nu_1 \boldsymbol{D}\boldsymbol{v} + \nu_2 |\boldsymbol{D}\boldsymbol{v}|^{r-2} \boldsymbol{D}\boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{v} \right] : \nabla \partial_t \boldsymbol{v} \, dx$$
$$+ \alpha \int_{\partial\Omega} \boldsymbol{s}(\boldsymbol{v}) \cdot \partial_t \boldsymbol{v} \, dS = \int_\Omega \boldsymbol{f} \cdot \partial_t \boldsymbol{v} \, dx$$
$$\|\partial_t \boldsymbol{v}\|_H^2 + \int_\Omega \left[\nu_1 \boldsymbol{D}\boldsymbol{v} + \nu_2 |\boldsymbol{D}\boldsymbol{v}|^{r-2} \boldsymbol{D}\boldsymbol{v} \right] : \partial_t \boldsymbol{D}\boldsymbol{v} \, dx + \int_\Omega (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) \cdot \partial_t \boldsymbol{v} \, dx + \alpha \int_{\partial\Omega} \boldsymbol{s}(\boldsymbol{v}) \cdot \partial_t \boldsymbol{v} \, dS$$
$$= \int_\Omega \boldsymbol{f} \cdot \partial_t \boldsymbol{v} \, dx$$

We estimate,

$$\int_\Omega (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) \cdot \partial_t \boldsymbol{v} \, dx \leq \|\boldsymbol{v}\|_{\frac{2r}{r-2}}^2 \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^2 + \frac{1}{4} \|\partial_t \boldsymbol{v}\|_2^2 . \tag{3.17}$$

Now by (2.6), we obtain,

$$\int_{\partial\Omega} \boldsymbol{s}(\boldsymbol{v}) \cdot \partial_t \boldsymbol{v} \, dS = \frac{d}{dt} \left(\int_{\partial\Omega} \boldsymbol{\mathcal{S}}(\boldsymbol{v}) \, dS \right) .$$

Then we obtain the following inequality,

$$\frac{1}{2} \|\partial_t \boldsymbol{v}\|_H^2 + \frac{d}{dt} \int_\Omega \Phi(\boldsymbol{D}\boldsymbol{v}) \, dx + \frac{d}{dt} \left(\int_{\partial\Omega} \boldsymbol{\mathcal{S}}(\boldsymbol{v}) \, dS \right) \leq c_8 \|\boldsymbol{v}\|_{\frac{2r}{r-2}}^2 \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^2 + \|\boldsymbol{f}\|_2^2 \tag{3.18}$$

This can be more compactly written as

$$\frac{1}{2} \|\partial_t \boldsymbol{v}\|_H^2 + \frac{d}{dt} U \leq c_8 \|\boldsymbol{v}\|_{\frac{2r}{r-2}}^2 \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^2 + \|\boldsymbol{f}\|_2^2 , \tag{3.19}$$

where

$$U = U(t) := 1 + \int_\Omega \Phi(\boldsymbol{D}\boldsymbol{v})\, dx + \int_{\partial\Omega} \boldsymbol{\mathcal{S}}(\boldsymbol{v})\, dS,$$
$$\text{and hence } U \sim 1 + \nu_1 \|\boldsymbol{D}\boldsymbol{v}\|_2^2 + \nu_2 \|\boldsymbol{D}\boldsymbol{v}\|_r^r + \|\boldsymbol{\mathcal{S}}(\boldsymbol{v})\|_{L^1(\partial\Omega)}, \tag{3.20}$$

by (2.3) and Korn's inequality (B.2). Now we distinguish two cases:

(i) case $r \in (12/5, 3]$. We claim

$$\|\boldsymbol{v}\|_{\frac{2r}{r-2}} \le c_9 \|\boldsymbol{v}\|_2^a \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^{1-a}, \quad a = \frac{5r-12}{5r-6}. \tag{3.21}$$

Note that $a > 0$ as $r > 12/5$. Then for $r \le 3$, the embedding $W^{1,r}(\Omega) \subset L^{\frac{3r}{3-r}}(\Omega)$ holds. We obtain

$$\|\boldsymbol{v}\|_{\frac{2r}{r-2}} \le \|\boldsymbol{v}\|_2^a \|\boldsymbol{v}\|_{\frac{3r}{3-r}}^{1-a} \le c_9 \|\boldsymbol{v}\|_2^a \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^{1-a}. \tag{3.22}$$

Then we estimate the first term on the right hand side of (3.19) and obtain

$$\begin{aligned}
\|\boldsymbol{v}\|_2^{\frac{2(5r-12)}{5r-6}} \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^{\frac{10r}{5r-6}} &\le \nu_2^{-\frac{10}{5r-6}} \|\boldsymbol{v}\|_2^{\frac{2(5r-12)}{5r-6}} \left[\nu_2 \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^r\right]^{\frac{10}{5r-6}}, \\
&\lesssim \nu_2^{-\frac{10}{5r-6}} \|\boldsymbol{v}\|_2^{\frac{2(5r-12)}{5r-6}} \left[\nu_2 \|\boldsymbol{D}\boldsymbol{v}\|_r^r + \nu_2 \|\boldsymbol{v}\|_2^r\right]^{\frac{10}{5r-6}}, \\
&\lesssim \nu_2^{-\frac{10}{5r-6}} \|\boldsymbol{v}\|_2^{\frac{2(5r-12)}{5r-6}} U^{\frac{10}{5r-6}} + \|\boldsymbol{v}\|_2^4.
\end{aligned}$$

This yields

$$\frac{d}{dt} U \le c_{10} \nu_2^{-\frac{10}{5r-6}} \|\boldsymbol{v}\|_2^{\frac{2(5r-12)}{5r-6}} U^{\frac{10}{5r-6}} + \|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2. \tag{3.23}$$

Dividing by $U^{1-\mu}$, where $\mu = \frac{2(5r-12)}{5r-6}$ yields,

$$\frac{d}{dt} U^\mu \le c_{10} \nu_2^{-\frac{10}{5r-6}} \|\boldsymbol{v}\|_2^{\frac{2(5r-12)}{5r-6}} U + \|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2. \tag{3.24}$$

Then we apply Grönwall's lemma to obtain the necessary bounds on U. It is worthwhile to note that

$$\left[\|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2\right] U^{\mu-1} = \left[\|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2\right] U^{\frac{5r-16}{5r-6}} \le \|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2.$$

The above property holds true because $U \ge 1$ and for $r \le 3$, we have $\frac{5r-16}{5r-6} < 0$.

(ii) case $r > 3$. Since $\dfrac{2r}{r-2} \in (2, 6)$, we use the interpolation Lemma B.1 to obtain,

$$\|\boldsymbol{v}\|_{\frac{2r}{r-2}} \lesssim \|\boldsymbol{v}\|_2^{\frac{r-3}{r}} \|\boldsymbol{v}\|_6^{\frac{3}{r}}. \tag{3.25}$$

Again by Lemma B.3 we obtain,

$$\|\boldsymbol{v}\|_{\frac{2r}{r-2}} \lesssim \|\boldsymbol{v}\|_2^{\frac{r-2}{r}} \|\boldsymbol{v}\|_{W^{1,3}(\Omega)}^{\frac{2}{r}} \le c_{11} \|\boldsymbol{v}\|_2^{\frac{r-2}{r}} \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^{\frac{2}{r}}. \tag{3.26}$$

Then right hand side of (3.19) can be estimated as

$$\|\boldsymbol{v}\|_2^{\frac{2(r-2)}{r}} \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^{\frac{2r+4}{r}} \le \|\boldsymbol{v}\|_2^{\frac{2(r-2)}{r}} \left[\|\boldsymbol{D}\boldsymbol{v}\|_r^r + \|\boldsymbol{v}\|_2^r\right]^{\frac{2r+4}{r^2}},$$
$$\le \nu_2^{-\frac{r^2}{2r+4}} \|\boldsymbol{v}\|_2^{\frac{2(r-2)}{r}} U^{\frac{2r+4}{r^2}} + \|\boldsymbol{v}\|_2^4. \tag{3.27}$$

This yields

$$\frac{d}{dt}U \le c_{12}\nu_2^{-\frac{r^2}{2r+4}} \|\boldsymbol{v}\|_2^{\frac{2(r-2)}{r}} U^{\frac{2r+4}{r^2}} + \|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2. \tag{3.28}$$

Take $\mu = \dfrac{2r+4}{r^2} - 1$. Then we consider two cases.

If $\dfrac{2r+4}{r^2} > 1$, i.e. $\mu > 0$, we divide (3.28) by U^μ. Thus we obtain

$$\frac{d}{dt}U^{1-\mu} \le c_{12}\nu_2^{-\frac{r^2}{2r+4}} \|\boldsymbol{v}\|_2^{\frac{2(r-2)}{r}} U + \|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2.$$

Similar to the previous case where $r \in (12/5, 3]$, we observe that

$$\left[\|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2\right] U^\mu \le \|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2.$$

If $\dfrac{2r+4}{r^2} \le 1$, i.e. $\mu \le 0$, we obtain by (3.28),

$$\frac{d}{dt}U \le c_{12}\nu_2^{-\frac{r^2}{2r+4}} \|\boldsymbol{v}\|_2^{\frac{2(r-2)}{r}} U + \|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2. \tag{3.29}$$

Then in both cases, we invoke Grönwall's lemma to obtain bounds on U.

4 Dimension of the Attractor

We follow the general scheme of method of trajectories presented in [16]. The main modification here is that we explicitly keep track of all a priori estimates.

Lemma 4.1. *There exists an absorbing, positively invariant set $\hat{\mathcal{B}} \subset H$ such that*

$$B_0 := \sup_{\boldsymbol{v}\in\hat{\mathcal{B}}} \|\boldsymbol{v}\|_H \le c_1 \min\left\{\kappa_1^{-1}\|\boldsymbol{f}\|_2, \left[\kappa_2^{-1}\|\boldsymbol{f}\|_2\right]^{\frac{1}{s-1}}\right\}, \tag{4.1}$$

where $s = \min\{r, q\}$, and $\kappa = \min\{\nu_2, \alpha\}$.

Proof. As in Theorem 3.1, we obtain

$$\frac{d}{dt}\|\boldsymbol{v}\|_H^2 + c_2\left[\nu_1\|\boldsymbol{D}\boldsymbol{v}\|_2^2 + \nu_2\|\boldsymbol{D}\boldsymbol{v}\|_r^r + \alpha\|\boldsymbol{v}\|_{L^q(\Gamma)}^q\right] \le c_3\|\boldsymbol{f}\|_2\|\boldsymbol{v}\|_H. \tag{4.2}$$

Then by dropping the term $\|\boldsymbol{D}\boldsymbol{v}\|_r^r$, we compute by Korn's inequality in Lemma B.2,

$$\frac{d}{dt}\|\boldsymbol{v}\|_H^2 + c_2\kappa_1\|\boldsymbol{v}\|_H^2 \le c_3\|\boldsymbol{f}\|_2\|\boldsymbol{v}\|_H, \text{ where } \kappa_1 = \min\{\nu_1, \alpha\} \tag{4.3}$$

Thus $\frac{d}{dt}\|\boldsymbol{v}\|_H^2 \le -\gamma\|\boldsymbol{v}\|_H^2$ if $\|\boldsymbol{v}\|_H > c_4\kappa_1^{-1}\|\boldsymbol{f}\|_2$ for some $\gamma > 0$. Now we drop the term $\|\boldsymbol{D}\boldsymbol{v}\|_2^2$ and obtain,

$$\frac{d}{dt}\|\boldsymbol{v}\|_H^2 + c_2\left[\nu_2\|\boldsymbol{D}\boldsymbol{v}\|_r^r + \alpha\|\boldsymbol{v}\|_{L^q(\Gamma)}^q\right] \le c_3\|\boldsymbol{f}\|_2\|\boldsymbol{v}\|_H. \tag{4.4}$$

Then we use the following estimate for $\|\boldsymbol{v}\|_H \ge 1$,

$$\nu_2\|\boldsymbol{D}\boldsymbol{v}\|_r^r + \alpha\|\boldsymbol{v}\|_{L^q(\Gamma)}^q \ge c_5\kappa\|\boldsymbol{v}\|_H^s, \text{ where } s = \min\{r,q\}, \text{ and } \kappa_2 = \min\{\nu_2,\alpha\}.$$

Thus we obtain $\frac{d}{dt}\|\boldsymbol{v}\|_H^2 \le -\gamma\|\boldsymbol{v}\|_H^2$ if $\|\boldsymbol{v}\|_H > c_6\left[\kappa^{-1}\|\boldsymbol{f}\|_2\right]^{\frac{1}{s-1}}$ for some $\gamma > 0$. Hence the conclusion follows. □

Lemma 4.2. *There exists an absorbing, positively invariant $\mathcal{B} \subset \hat{\mathcal{B}}$ such that $\mathcal{B}$ is closed in H, and*

$$B_r := \sup_{\boldsymbol{v}\in\hat{\mathcal{B}}}\|\boldsymbol{v}\|_{W^{1,r}} \le \begin{cases} c_{12}B_0^{\frac{5(5r-6)}{2(5r-11)}}, & r \in (12/5, 3], \\ c_{12}B_0^5, & r > 3. \end{cases} \tag{4.5}$$

Proof. Set

$$\mathcal{B} := \left\{\boldsymbol{v}(2T);\ \boldsymbol{v} \text{ is a weak solution on } [0,2T], \text{and } \boldsymbol{v}(0) \in \hat{\mathcal{B}}\right\},$$

and we take $T = B_0$. Recalling (4.3) and taking $U = \nu_1\|\boldsymbol{D}\boldsymbol{v}\|_2^2 + \nu_2\|\boldsymbol{D}\boldsymbol{v}\|_r^r$

$$\begin{aligned}\int_0^T U(t)\,dt &\lesssim \int_0^T \left[\|\boldsymbol{f}\|_2\|\boldsymbol{v}\|_H + \frac{d}{dt}\|\boldsymbol{v}\|_H^2\right]\,dt, \\ &\lesssim B_0^2 + TB_0 \le 2c_1B_0^2. \end{aligned}\tag{4.6}$$

By the mean value theorem of integrals, we obtain for $\tau \in (0,T)$ such that

$$U(\tau) \le c_2B_0. \tag{4.7}$$

Assume $r \le 3$. Integrating (3.24) over $(\tau, 2T)$ yields

$$\begin{aligned}U^\mu(2T) &\le c_3\nu_2^{-\frac{10}{5r-6}}B_0^{\frac{2(5r-12)}{5r-6}}\int_\tau^{2T}U(t)\,dt + \int_\tau^{2T}\left[\|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2\right]\,dt + U^\mu(\tau), \\ &\le c_4\nu_2^{-\frac{10}{5r-6}}B_0^{\frac{4(5r-9)}{5r-6}} + c_5B_0^\mu + c_6B_0^5 + c_7B_0\|\boldsymbol{f}\|_2^2.\end{aligned}$$

Here $\mu = \frac{2(5r-11)}{5r-6}$. It is reasonable to assume that $B_0 > 1$, and $\nu_1, \nu_2 < 1$, hence the largest term is $B_0^{5/\mu}$-term. The above estimate only gives an upper bound for $\|\boldsymbol{D}\boldsymbol{v}\|_r$. But by adding $\|\boldsymbol{v}\|_2$ to both sides we obtain an upper bound for $\|\boldsymbol{v}\|_{W^{1,r}}$. Then the desired estimate for B_r holds.

Then we compute for $r > 3$. Integrating

$$\begin{aligned}U(2T) &\le c_7\nu_2^{-\frac{r^2}{2r+4}}B_0^{\frac{2(r-2)}{r}}\int_\tau^{2T}U(t)\,dt + \int_\tau^{2T}\left[\|\boldsymbol{v}\|_2^4 + \|\boldsymbol{f}\|_2^2\right]dt + U(\tau), \\ &\le c_8\nu_2^{-\frac{r^2}{2r+4}}B_0^{\frac{4(r-1)}{r}} + c_9B_0^5 + c_{10}B_0\|\boldsymbol{f}\|_2^2 + c_{11}B_0.\end{aligned}$$

The largest term is the B_0^5-term. Hence the estimate follows. The closedness of $\mathcal{B}$ follows from the compactness of the set of weak solutions, which is part of the existence theory. See the reference for Theorem 3.1. □

4.1 Attractors and Method of Trajectories.

Observe that by Theorems 3.2, 3.3, the solution operator $S(t) : \boldsymbol{v}_0 \to \boldsymbol{v}(t)$ is well-defined for $\boldsymbol{v}_0 \in \mathcal{B}$. It follows that

$$\mathcal{A} = \omega(\mathcal{B}) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \overline{S(t)\mathcal{B}}^H \tag{4.8}$$

is the so-called *global attractor*. Our ultimate goal is to estimate its fractal dimension, defined as

$$d_f^H(\mathcal{A}) = \limsup_{\varepsilon \to 0+} \frac{\ln N_H(\mathcal{A}, \varepsilon)}{-\ln \varepsilon} \tag{4.9}$$

where $N_H(\mathcal{A}, \varepsilon)$ is the smallest number of ε-balls in the space H that cover $\mathcal{A}$. We employ the method of trajectories. Since the argument is very similar to [4], we only briefly sketch the main points. We refer to [16] for a more detailed description of the method; see also the introduction for other related references.

Let $\ell > 0$ be fixed; the exact value will specified in (4.12) below. The space of trajectories is defined as

$$\mathcal{B}_\ell = \{\chi \in H_\ell;\ \chi \text{ is a weak solution on } [0, \ell],\ \chi(0) \in \mathcal{B}\}\ , \tag{4.10}$$

with the underlying metric of $H_\ell = L^2(0, \ell; H)$. Note however that any trajectory χ has additional regularity, cf. Theorem 3.1. In particular, we always work with the representative $\chi \in C([0, \ell]; H)$, so that the value $\chi(t)$ is well-defined for any $t \in [0, \ell]$. The operators $\mathcal{L} : \mathcal{B}_\ell \to \mathcal{B}_\ell$, $b : \mathcal{B} \to \mathcal{B}_\ell$ and $e : \mathcal{B}_\ell \to \mathcal{B}$ are defined via the conditions

$$\begin{aligned} \mathcal{L}(\chi) = \psi &\iff \chi(\ell) = \psi(0)\,, \\ e(\chi) &= \chi(\ell)\,, \\ b(\boldsymbol{v}_0) = \chi &\iff \chi(0) = \boldsymbol{v}_0\,. \end{aligned}$$

Observe that $S(\ell) = e \circ b$ and $b \circ e = \mathcal{L}$, hence $\mathcal{L}$ is an equivalent (discrete) description of the dynamics of $S(t)$ on $\mathcal{B}_\ell = b(\mathcal{B})$. In particular, one has $\mathcal{A}_\ell = b(\mathcal{A})$, $\mathcal{A} = e(\mathcal{A}_\ell)$, where $\mathcal{A}_\ell$ is the global attractor for the dynamical system $(\mathcal{L}^n, \mathcal{B}_\ell)$.

In view of the Lipschitz continuity of operators e, b (see for example [16, Lemma 2.1], [16, Lemma 1.2])

$$d_f^H(\mathcal{A}) = d_f^{L^2(0,\ell;H)}(\mathcal{A}_\ell) \tag{4.11}$$

Thus, it suffices to estimate the last quantity. This will be done using the so-called smoothing property, see [16, Lemma 1.3]; see also [4, Theorem 4.1]. It remains to explicitly estimate the appropriate Lipschitz constants, which is done in the following lemma. Finally, the asymptotics of covering numbers is investigated in the Appendix.

Lemma 4.3. *Set*

$$\ell := \left[\nu_1^{-\frac{3}{2r-3}} B_r^{\frac{2r}{2r-3}} + 1\right]^{-1}. \tag{4.12}$$

Then for all $\chi, \psi \in \mathcal{A}_\ell$

$$\|\mathcal{L}\chi - \mathcal{L}\psi\|_{L^2(0,\ell;V_2)} \leq L_1 \|\chi - \psi\|_{H_\ell}\,, \tag{4.13}$$

$$\|\partial_t\mathcal{L}\chi - \partial_t\mathcal{L}\psi\|_{L^2(0,\ell;V_r')} \leq L_2 \|\chi - \psi\|_{H_\ell}\,, \tag{4.14}$$

where

$$L_1 = c_1 {\nu_1}^{-\frac{1}{2}} \ell^{-\frac{1}{2}}, \tag{4.15}$$
$$L_2 = U + W + Q, \tag{4.16}$$
$$U = c_2 \nu_1 L_1 (1 + M_r), \tag{4.17}$$
$$M_r = \nu_1^{-\frac{1}{2}} \nu_2^{\frac{1}{2}} B_r^{\frac{r-2}{2}}, \tag{4.18}$$
$$W = \begin{cases} c_4 B_0^{\frac{5r-12}{5r-6}} B_r^{\frac{6}{5r-6}}, \ r \in (12/5, 3], \\ c_4 B_0^{\frac{r-2}{r}} B_r^{\frac{2}{r}}, \ r > 3. \end{cases} \tag{4.19}$$

Proof. Let $\boldsymbol{u}, \boldsymbol{v}$ be two weak solutions on $[0, 2\ell]$ such that $\boldsymbol{u}|_{[0,\ell]} = \chi, \boldsymbol{u}|_{[0,\ell]} = \psi$, and set $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}$. In view of (4.12), (3.15) is rewritten as

$$\frac{d}{dt} \|\boldsymbol{w}\|_H^2 + c_5 \nu_1 \|\boldsymbol{w}\|_{W^{1,2}(\Omega)}^2 + c_5 \int_\Omega I^2(\boldsymbol{Du}, \boldsymbol{Dv}) \, dx \le c_7 \left[\nu_1^{-\frac{3}{2r-3}} B_r^{\frac{2r}{2r-3}} + 1 \right] \|\boldsymbol{w}\|_H^2 .$$

We replace norm of the second term of the left hand side with the equivalent norm $\| \ \|_{V_2}$.This yields,

$$\frac{d}{dt} \|\boldsymbol{w}\|_H^2 + c_8 \nu_1 \|\boldsymbol{w}\|_{V_2}^2 + c_8 \int_\Omega I^2(\boldsymbol{Du}, \boldsymbol{Dv}) \, dx \le c_7 \left[\nu_1^{-\frac{3}{2r-3}} B_r^{\frac{2r}{2r-3}} + 1 \right] \|\boldsymbol{w}\|_H^2 .$$

Then by (4.12) we obtain,

$$\frac{d}{dt} \|\boldsymbol{w}\|_H^2 + c_8 \nu_1 \|\boldsymbol{w}\|_{V_2}^2 + c_8 \int_\Omega I^2(\boldsymbol{Du}, \boldsymbol{Dv}) \, dx \le c_7 \ell^{-1} \|\boldsymbol{w}\|_H^2 . \tag{4.20}$$

Neglecting the positive terms of the left hand side, we obtain from Grönwall's Lemma

$$\|\boldsymbol{w}(t)\|_H^2 \le c_9 \|\boldsymbol{w}(s)\|_H^2 , \quad 0 < s < t < 2\ell, \tag{4.21}$$

where $c_9 = \exp\left((t-s)\ell^{-1}\right) \le \exp(2c_7)$. In other words, the smallness of ℓ eliminates the (exponential) dependence of the Lipschitz constant of $S(t)$ on the viscosities.

Integrating (4.20) over $(s, 2\ell)$, where $s \in (0, \ell)$ is fixed, one further derives

$$c_8 \nu_1 \int_s^{2\ell} \|\boldsymbol{w}(t)\|_{V_2}^2 \, dt + c_8 \int_s^{2\ell} \int_\Omega I^2(\boldsymbol{Du}, \boldsymbol{Dv}) \, dx dt \le \|\boldsymbol{w}(s)\|_H^2 + c_7 \ell^{-1} \int_s^{2\ell} \|\boldsymbol{w}(t)\|_H^2 \, dt.$$

By (4.21), we obtain $\displaystyle\int_s^{2\ell} \|\boldsymbol{w}(t)\|_H^2 \ dt \le 2c_9 \ell \|\boldsymbol{w}(s)\|_H^2$. By substituting this back in the above inequality, we obtain,

$$c_8 \nu_1 \int_s^{2\ell} \|\boldsymbol{w}(t)\|_{V_2}^2 \, dt + c_8 \int_s^{2\ell} \int_\Omega I^2(\boldsymbol{Du}, \boldsymbol{Dv}) \, dx dt \le c_{10} \|\boldsymbol{w}(s)\|_H^2 .$$

Integrating over $s \in (0, \ell)$ yields,

$$\ell \nu_1 \int_s^{2\ell} \|\boldsymbol{w}(t)\|_{V_2}^2 \, dt + \ell \int_s^{2\ell} \int_\Omega I^2(\boldsymbol{Du}, \boldsymbol{Dv}) \, dx dt \le c_{11} \int_0^{\ell} \|\boldsymbol{w}(s)\|_H^2 \, ds.$$

This proves (4.13). We also note here that

$$\left[\int_s^{2\ell}\int_\Omega I^2(\boldsymbol{Du},\boldsymbol{Dv})\ dxdt\right]^{1/2} \le c_{12}\nu_1^{\frac{1}{2}}L_1\left[\int_0^\ell \|\boldsymbol{w}(s)\|_H^2\,ds\right]^{1/2}. \tag{4.22}$$

To prove (4.14), (2.1) is used to get

$$\begin{aligned}
\|\partial_t\boldsymbol{w}\|_{L^2(\ell,2\ell;V_r')} &= \sup_\varphi \int_\ell^{2\ell}\langle\partial_t\boldsymbol{w},\varphi\rangle dt,\\
&= \sup_\varphi\Big[\underbrace{\int_\ell^{2\ell}\int_\Omega(\boldsymbol{S}(\boldsymbol{Du})-\boldsymbol{S}(\boldsymbol{Dv})):\boldsymbol{D}\varphi\,dxdt}_{I_1}+\underbrace{\int_\ell^{2\ell}\int_\Omega(\boldsymbol{u}\otimes\boldsymbol{u}-\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\varphi\,dxdt}_{I_2}\\
&\qquad +\alpha\underbrace{\int_\ell^{2\ell}\int_{\partial\Omega}(\boldsymbol{s}(\boldsymbol{u})-\boldsymbol{s}(\boldsymbol{v}))\,\boldsymbol{w}:\varphi\,dSdt}_{I_3}\Big]
\end{aligned}$$

where the supremum is taken over $\varphi\in L^2(\ell,2\ell;V_r)$ with $\|\varphi\|=1$. By Hölder inequality

$$I_1 \le \left[\int_\ell^{2\ell}\|\boldsymbol{S}(\boldsymbol{Du})-\boldsymbol{S}(\boldsymbol{Dv})\|_{r'}^2\,dt\right]^{1/2}.$$

Then by (2.2), (3.12) we obtain

$$|\boldsymbol{S}(\boldsymbol{Du})-\boldsymbol{S}(\boldsymbol{Dv})| \le I(\boldsymbol{Du},\boldsymbol{Dv})\left(\nu_1+\nu_2\left(|\boldsymbol{Du}|+|\boldsymbol{Du}|\right)^{r-2}\right)^{1/2},$$

hence

$$\begin{aligned}
\|\boldsymbol{S}(\boldsymbol{Du})-\boldsymbol{S}(\boldsymbol{Dv})\|_{r'} &= \left[\int_\Omega|\boldsymbol{S}(\boldsymbol{Du})-\boldsymbol{S}(\boldsymbol{Dv})|^{r'}\,dx\right]^{1/r'},\\
&\le \left[\int_\Omega I^{r'}(\boldsymbol{Du},\boldsymbol{Dv})\left(\nu_1+\nu_2\left(|\boldsymbol{Du}|+|\boldsymbol{Du}|\right)^{r-2}\right)^{r'/2}dx\right]^{1/r'},\\
&\le \left[\int_\Omega I^{sr'}(\boldsymbol{Du},\boldsymbol{Dv})dx\right]^{1/sr'}\left[\int_\Omega\left(\nu_1+\nu_2\left(|\boldsymbol{Du}|+|\boldsymbol{Du}|\right)^{r-2}\right)^{s'r'/2}dx\right]^{1/s'r'}.
\end{aligned}$$

We choose s, s' such that $1/s+1/s'=1, r's=2$ and $r's'=2r/r-2$. Then we obtain

$$\|\boldsymbol{S}(\boldsymbol{Du})-\boldsymbol{S}(\boldsymbol{Dv})\|_{r'} \le \left[\int_\Omega I^2(\boldsymbol{Du},\boldsymbol{Dv})dx\right]^{1/2}\underbrace{\left[\int_\Omega\left(\nu_1+\nu_2\left(|\boldsymbol{Du}|+|\boldsymbol{Du}|\right)^{r-2}\right)^{r/r-2}dx\right]^{r-2/2r}}_{M}.$$

Now we compute,

$$M=\nu_1^{1/2}\left[\int_\Omega\left(1+\nu_1^{-1}\nu_2\left(|\boldsymbol{Du}|+|\boldsymbol{Du}|\right)^{r-2}\right)^{r/r-2}dx\right]^{r-2/2r},$$

$$
\begin{aligned}
&\le c_{13}\nu_1^{1/2}\left[1+\nu_1^{-1}\nu_2\int_\Omega \left(|\boldsymbol{Du}|^r+|\boldsymbol{Du}|^r\right)dx\right]^{r-2/2r}, \\
&\le c_{14}\nu_1\left(1+M_r\right),
\end{aligned}
$$

cf (4.18). Note that the integral above cannot be bounded directly by B_r in (4.5). But U in Lemma 4.2 is bounded by B_r. Combining (4.15), (4.22)

$$
I_1 \le U\left\|\chi-\psi\right\|_{H_\ell}.
$$

Now we proceed to the estimate

$$
I_2 \le \int_\ell^{2\ell}\int_\Omega |\boldsymbol{w}|\left(|\boldsymbol{u}|+|\boldsymbol{v}|\right)|\nabla\varphi|\,dxdt \le \left[\int_\ell^{2\ell}\left\||\boldsymbol{w}|\left(|\boldsymbol{u}|+|\boldsymbol{v}|\right)\right\|dt\right]^{1/r'}.
$$

Then we compute

$$
\begin{aligned}
\left\||\boldsymbol{w}|\left(|\boldsymbol{u}|+|\boldsymbol{v}|\right)\right\|_{r'} &= \left[\int_\Omega |\boldsymbol{w}|^{r'}\left(|\boldsymbol{u}|+|\boldsymbol{v}|\right)^{r'}dx\right]^{1/r'}, \\
&\le c_{15}\left\|\boldsymbol{w}\right\|_2\left(\|\boldsymbol{u}\|_{\frac{2r}{r-2}}+\|\boldsymbol{v}\|_{\frac{2r}{r-2}}\right).
\end{aligned}
$$

We consider two cases

(i) case $r\in(12/5,3]$. Using (3.21), (4.5), we obtain

$$
\begin{aligned}
I_2 &\le c_{16}\left[\int_\ell^{2\ell}\|\boldsymbol{w}\|_2^2\,dt\right]^{1/2}\sup_{t\in(\ell,2\ell)}\left[\|\boldsymbol{u}\|_{\frac{2r}{r-2}}+\|\boldsymbol{v}\|_{\frac{2r}{r-2}}\right], \\
&\le c_{17}B_0^{\frac{5r-12}{5r-6}}B_r^{\frac{6}{5r-6}}\left[\int_\ell^{2\ell}\|\boldsymbol{w}\|_2^2\,dt\right]^{1/2}, \\
&\le W\left\|\chi-\psi\right\|_{H_\ell}.
\end{aligned}
$$

satisfying the first part of (4.19).

(ii) case $r>3$. Using (3.26), (4.5), we obtain

$$
\begin{aligned}
I_2 &\le c_{16}\left[\int_\ell^{2\ell}\|\boldsymbol{w}\|_2^2\,dt\right]^{1/2}\sup_{t\in(\ell,2\ell)}\left[\|\boldsymbol{u}\|_{\frac{2r}{r-2}}+\|\boldsymbol{v}\|_{\frac{2r}{r-2}}\right], \\
&\le c_{18}B_0^{\frac{r-2}{r}}B_r^{\frac{2}{r}}\left[\int_\ell^{2\ell}\|\boldsymbol{w}\|_2^2\,dt\right]^{1/2}, \\
&\le W\left\|\chi-\psi\right\|_{H_\ell}.
\end{aligned}
$$

satisfying the second part of (4.19). Finally, we estimate

$$
I_3 \le Q\left\|\chi-\psi\right\|_{H_\ell},
$$

with (2.5). This concludes the proof of the Lemma.

□

Now we formulate the main result.

Theorem 4.1. *Let the stress tensor satisfy* (2.2), (2.3) *with* $r > {}^{12}/_{5}$. *Then* (2.1) *has a global attractor* $\mathcal{A}$, *and its dimension can be estimated as*

$$d_f^H(\mathcal{A}) \leq c_{19} \left(L_1^4 + \ell L_1^{\frac{2(11r-6)}{3r}} L_2 \right) \ln L_1, \tag{4.23}$$

where L_1, L_2 *and* ℓ *are given in Lemma 4.3.*

Proof. Follows exactly along the arguments of [4, Theorem 4.1], using the estimates of Lemma 4.3 above and Lemma A.3 below. □

A Coverings and Fractal Dimension.

Now we present an elementary description of a class of Sobolev and Bochner spaces with fractional derivatives. These formulations will be used to obtain covering numbers for compact embeddings. We follow a similar technique used in [4, Section 7: Appendix], or [5, Secion 4]. Consider the following inhomogeneous Stokes problem,

$$\begin{cases} \partial_t \boldsymbol{v} - \operatorname{div} \boldsymbol{D}\boldsymbol{v} + \nabla\pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{v} = 0 & \text{in } Q, \quad \text{(A.1a)} \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad -(\boldsymbol{D}\boldsymbol{v})\boldsymbol{n} + \alpha \boldsymbol{v} = \beta \partial_t \boldsymbol{v} & \text{on } \Gamma, \quad \text{(A.1b)} \\ \boldsymbol{v}(0) = \boldsymbol{v}_0 & \text{in } \overline{\Omega}. \quad \text{(A.1c)} \end{cases}$$

Then the above dynamical system defines the operator $\mathcal{A}$ which generates a strongly continuous analytic semigroup on H with a compact resolvent with domain $\mathcal{D}(\mathcal{A}) \subset\subset V$, see [20, Theorem 1, p. 7]. We thus have linear (unbounded) operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to H$ satisfying,

$$(\boldsymbol{u}, \boldsymbol{\varphi})_V = (\mathcal{A}\boldsymbol{u}, \boldsymbol{\varphi})_H, \ \forall \boldsymbol{u} \in \mathcal{D}(\mathcal{A}), \ \forall \varphi \in V. \tag{A.2}$$

Moreover, from the same reference we have that $\mathcal{A}$ is surjective, and is also symmetric on its domain, i.e. for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{D}(\mathcal{A})$ we have

$$(\mathcal{A}\boldsymbol{u}, \boldsymbol{v})_H = (\boldsymbol{u}, \mathcal{A}\boldsymbol{v})_H. \tag{A.3}$$

Then by virtue of [2, Section 5, p. 168], we can define the domains fractional powers of the operator $\mathcal{A}$. Then by [24, Theorem 1.15.3, p. 114] for $0 \leq \theta \leq 1$,

$$\mathcal{D}(\mathcal{A}^\theta) \subset \left[L^2(\Omega) \times L^2(\partial\Omega), H^2(\Omega) \times L^2(\partial\Omega) \right]_\theta \hookrightarrow H^{2\theta}(\Omega) \times L^2(\partial\Omega), \tag{A.4}$$

where $H^s(\Omega) = W^{s,2}(\Omega)$, $s \in \mathbb{R}$. For more details on domains of fractional powers of matrix-valued operators, we refer to [15] and references therein. Let $\boldsymbol{w}_j, \lambda_j = 1, 2, \ldots$ be the eigenfunctions and eigenvalues of the operator $\mathcal{A}$ respectively.

$$-\operatorname{div} \boldsymbol{D}\boldsymbol{w}_j = \lambda_j \boldsymbol{w}_j, \quad \text{in } \Omega \tag{A.5a}$$

$$\operatorname{div} \boldsymbol{w}_j = 0, \quad \text{in } \Omega \tag{A.5b}$$

$$\boldsymbol{D}\boldsymbol{w}_j \boldsymbol{n} + \alpha \boldsymbol{w}_j = \lambda_j \beta \boldsymbol{w}_j \quad \text{in } \Gamma \tag{A.5c}$$

Note that we have taken $\partial_t \boldsymbol{v} = -\lambda_i \boldsymbol{v}$ for λ_i to be nonnegative. One can show that $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}$ is a basis for V and H, it is orthogonal in V and orthonormal in H. Moreover, we have $\lim_{i\to\infty} \lambda_i = +\infty$. See [18, Lemma 3.1]. We also have

$$Cj^{1/2} \leq \lambda_j \leq \widetilde{C} j^{2/3}, \tag{A.6}$$

for dimension $d = 3$ by [20, Section 3.2], and Lemma B.4 for some positive constants C, $\widetilde{C}$. For $b \in \mathbb{R}$, one introduces the space $\mathbb{H}^b := H^b(\Omega) \times L^2(\partial\Omega)$ as

$$\mathbb{H}^b = H^b(\Omega) \times L^2(\partial\Omega). \tag{A.7}$$

Let us define, $\mathbb{H}^{-b} := \left(H^b(\Omega)\right)' \times L^2(\partial\Omega)$, with the duality given by the generalized scalar product in H. Further, we define $\mathbb{H}^b$ as a class of interpolation spaces in the sense that $\left[\mathbb{H}^{b_1}, \mathbb{H}^{b_2}\right]_\alpha = \mathbb{H}^b$, where $b = (1-\alpha)b_1 + \alpha b_2$. To relate $\mathbb{H}^b$ to classical Sobolev spaces (product), observe that

$$\begin{aligned}
\|(\boldsymbol{u}, \boldsymbol{g})\|^2_{\mathbb{H}^0} &= \|(\boldsymbol{u}, \boldsymbol{g})\|^2_H = \|\boldsymbol{u}\|_2^2 + \beta \|\boldsymbol{g}\|^2_{L^2(\partial\Omega)}, \\
(\mathcal{A}\boldsymbol{u}, \boldsymbol{u})_H &= \left\|\mathcal{A}^{1/2}\boldsymbol{u}\right\|^2_H = \sum_j a_j^2 \lambda_j, \\
&= \|\boldsymbol{v}\|^2_{H^1(\Omega)} + \beta \|tr\ \boldsymbol{u}\|^2_{L^2(\partial\Omega)} \sim \|\boldsymbol{u}\|^2_{\mathbb{H}^1} \sim \|\boldsymbol{u}\|^2_V, \text{ and} \\
\text{compute } \mathcal{A}\boldsymbol{u} &= \sum_j a_j \mathcal{A}\boldsymbol{w}_j = \sum_j a_j \lambda_j \boldsymbol{w}_j, \text{ hence } \|\mathcal{A}\boldsymbol{u}\|^2_H = \|\boldsymbol{u}\|^2_{\mathbb{H}^2}.
\end{aligned}$$

Similarly, an orthonormal basis for $L^2(0,\ell)$ will be defined as

$$\varphi_0(t) = \ell^{-1/2}, \quad \varphi_k(t) = 2^{1/2}\ell^{-1/2}\cos(k\pi t\ell^{-1}), \quad k \geq 1. \tag{A.8}$$

One sets $\mu_0 = \ell^{-2},\ \mu_k = k^2\pi^2\ell^{-2}$. The space $H^a(0,\ell)$ is defined as

$$\|\phi\|^2_{H^a(0,\ell)} = \sum_k a_k^2 \mu_k^a, \quad a_k = \int_0^\ell \phi(t)\varphi_k(t)dt.$$

The seminorm $\dot{H}^a(0,\ell)$ will also be used,

$$\|\phi\|^2_{\dot{H}^a(0,\ell)} = \sum_{k\neq 0} a_k^2 \mu_k^a,$$

and the space $H_0^a(0,\ell)$, in the definition of which $\varphi_k(t)$s are replaced by

$$\psi_k(t) = 2^{1/2}\ell^{-1/2}\sin(k\pi t\ell^{-1}), \quad k \geq 0.$$

Note that

$$\begin{aligned}
\mu_k &\sim k^2\ell^{-2}, \\
|\varphi_k(t)|, |\psi_k(t)| &\leq c_1 \ell^{-1/2}.
\end{aligned} \tag{A.9}$$

The dependence on ℓ has to be carefully traced down, since $\ell << 1$ in the applications.

Now we combine $\boldsymbol{w}_j, \varphi_k$ to describe certain norms of fractional Bochner spaces. For $\boldsymbol{u}(x,t) : \Omega \times (0,\ell) \longrightarrow \mathbb{R}^3$, one sets

$$\begin{aligned} \|\boldsymbol{u}\|^2_{H^a(0,\ell;\mathbb{H}^b)} &= \sum_{j,k} a_j^2 \lambda_j^b \mu_k^a, \\ \|\boldsymbol{u}\|^2_{\dot{H}^a(0,\ell;\mathbb{H}^b)} &= \sum_{j,k\neq 0} a_j^2 \lambda_j^b \mu_k^a, \\ \text{where } a_{jk} &= \int_{\Omega\times\partial\Omega\times(0,\ell)} \boldsymbol{v}(x,t)\cdot \boldsymbol{w}_j(x)\varphi_k(t)dxdt. \end{aligned}$$

As above, there is the introduction $H_0^a(0,\ell;\mathbb{H}^b)$ using ψ_k in place of φ_k. It is straightforward to verify that

$$\|\boldsymbol{u}\|_{L^2(0,\ell;\mathbb{H}^b)} = \|\boldsymbol{u}\|_{H^0(0,\ell;\mathbb{H}^b)} = \|\boldsymbol{u}\|_{H_0^0(0,\ell;\mathbb{H}^b)}\,. \tag{A.10}$$

In the following Lemma from [4, Lemma 7.1], it is proven that the seminorm $\dot{H}^1(0,\ell)$ can be estimated in terms of the time derivative. The value of b given in (A.11) is obtained by (B.3).

Lemma A.1. *Let $r \geq 2$ and let b be given by*

$$b = \frac{5r-6}{2r}, \ \ i.e. \ b \geq 1. \tag{A.11}$$

Then

$$\|\boldsymbol{u}\|_{\dot{H}^1(0,\ell;\mathbb{H}^{-b})} \leq c_1 \|\partial_t \boldsymbol{u}\|_{L^2(0,\ell;V_r')}$$

Here ∂_t stands for the distributional derivative in $\Omega \times (0,\ell)$.

Then we obtain the following two Lemmas from [4, Lemma 7.2, Lemma 7.3] by devising similar computations.

Lemma A.2. *Let $r \geq 2, \ \ell > 0$ and $C_1, C_2 >> 1$. Denote*

$$\mathcal{M} = \Big\{ \boldsymbol{u} : \|\boldsymbol{u}\|_{L^2(0,\ell;V_2)} \leq C_1, \ \|\partial_t \boldsymbol{u}\|_{L^2(0,\ell;V_r')} \leq C_2 \Big\}.$$

There exists orthonormal projection $\mathcal{P}$ in $L^2(0,\ell;H)$ such that

$$dist\,(\mathcal{M}, \mathcal{P}(\mathcal{M})) \leq \frac{1}{\sqrt{8}}, \tag{A.12}$$

and

$$rank\ \mathcal{P} \leq c_2 \bigg(C_1^4 + \ell C_1^{\frac{2(11r-6)}{3r}} C_2 \bigg) \tag{A.13}$$

Proof. The proof of this lemma follows similar argumentation as [4, Lemma 7.2]. By virtue of (A.10) and Lemma A.1, $\mathcal{M}$ can described by $H^0(0,\ell;\mathbb{H}^1)$ and $H^0(0,\ell;\mathbb{H}^{-b})$, where $b = {}^{(5r-6)}/_{2r}$. Then the Fourier coefficients of $\boldsymbol{u} \in \mathcal{M}$ satisfy

$$\sum_{j,k} a_{jk}^2 \lambda_j \leq c_3 C_1^2, \quad \sum_{j,k\neq 0} a_{jk}^2 \lambda_j^{-b} \mu_k \leq c_4 C_2^2. \tag{A.14}$$

Hence it is enough to take $\mathcal{P}$ as the projection to the span of

$$\left\{ \boldsymbol{w}_j \varphi_k : \ \lambda_j \le 8c_3 C_1^2 \text{ and } \mu_k \le 8c_4 \lambda_j^b C_2^2 \right\}.$$

First, we show that (A.12) holds. First observe that, for $\boldsymbol{u} \in \mathcal{M}$,

$$\boldsymbol{u} = \sum_{j,k} a_{jk} \boldsymbol{w}_j \varphi_k, \quad \text{and} \quad \mathcal{P}\boldsymbol{u} = \sum_{\left\{\lambda_j \le 8c_3 C_1^2, \ \mu_k \le 8c_4 \lambda_j^b C_2^2\right\}} a_{jk} \boldsymbol{w}_j \varphi_k.$$

Now we estimate

$$\|\boldsymbol{u} - \mathcal{P}\boldsymbol{u}\|_H^2 = \sum_{\left\{\lambda_j > 8c_3 C_1^2, \ \text{ or } \mu_k > 8c_4 \lambda_j^b C_2^2\right\}} a_{jk}^2.$$

We further estimate above two different cases separately,

$$\|\boldsymbol{u} - \mathcal{P}\boldsymbol{u}\|_H^2 = \sum_{\left\{\lambda_j > 8c_3 C_1^2\right\}} a_{jk}^2, \quad \text{or} \quad \|\boldsymbol{u} - \mathcal{P}\boldsymbol{u}\|_H^2 = \sum_{\left\{\mu_k > 8c_4 \lambda_j^b C_2^2\right\}} a_{jk}^2.$$

Furthermore,

$$\|\boldsymbol{u} - \mathcal{P}\boldsymbol{u}\|_H^2 = \sum_{\left\{\lambda_j > 8c_3 C_1^2\right\}} a_{jk}^2 \lambda_j \frac{1}{\lambda_j}, \quad \text{or} \quad \|\boldsymbol{u} - \mathcal{P}\boldsymbol{u}\|_H^2 = \sum_{\left\{\mu_k > 8c_4 \lambda_j^b C_2^2\right\}} a_{jk}^2 \lambda_j^{-b} \mu_k \frac{\lambda_j^b}{\mu_k},$$

combining both cases, we obtain

$$\|\boldsymbol{u} - \mathcal{P}\boldsymbol{u}\|_H^2 \le \frac{1}{8}. \tag{A.15}$$

Hence (A.12). Now we recall (A.6) and (A.9), we estimate

$$\begin{aligned} \operatorname{rank} \mathcal{P} &\le \sum_{\left\{j \ \le \ c_5 C_1^4\right\}} \left(1 + C_2 \ell j^{b/3}\right) \le c_6 \left(C_1^4 + \ell C_2 C_1^{\frac{4(b+3)}{3}}\right), \\ &= c_6 \left(C_1^4 + \ell C_1^{\frac{2(11r-6)}{3r}} C_2\right), \end{aligned} \tag{A.16}$$

□

Lemma A.3. *The set $\mathcal{M}$ from Lemma A.2 can be covered by K balls of radii $1/2$ in $L^2(0, \ell; H)$, where*

$$\ln K \le c_7 \left(C_1^4 + \ell C_1^{\frac{2(11r-6)}{3r}} C_2\right) \ln C_1. \tag{A.17}$$

Remark A.1. The difference between the estimate obtained in [4, Lemma 7.2] and (A.13) is due to the difference between the lower and upper bounds of the eigenvalues in two cases. In the former, the authors had $\lambda_j \sim c_1 j^{2/3}$, and in our case we have $C j^{1/2} \le \lambda_j \le \widetilde{C} j^{2/3}$. This difference is also evident in the estimate (A.17).

B Appendix

Lemma B.1. *[3, Lemma II.2.33, p. 66] Let Ω be any open set of $\mathbb{R}^d$ and let $u \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \le p, q \le \infty$. Then for all r such that*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},\ 0 < \theta < 1.$$

we have $u \in L^r(\Omega)$, and

$$\|u\|_r \le \|u\|_p^{\theta} \|u\|_q^{1-\theta}. \tag{B.1}$$

Theorem B.1. *[3, p. 173] Let Ω be a Lipschitz domain in $\mathbb{R}^d$ with compact boundary. Let $p \in [1, \infty]$ and $q \in \left[p, \dfrac{pd}{d-p}\right]$. There exists a $C > 0$ such that*

$$\|\varphi\|_{L^q(\Omega)} \le C \|\varphi\|_{L^p(\Omega)}^{1+d/q-d/p} \|\varphi\|_{W^{1,p}(\Omega)}^{d/p-d/q}, \quad \textit{for all } \varphi \in W^{1,p}(\Omega). \tag{B.2}$$

Lemma B.2. *[6, Lemma 1.11, p. 63] Let $\Omega \in \mathcal{C}^{0,1}$ and $q \in (0, +\infty)$. Then there exists a positive constant C, depending only on Ω and q, such that for all $\boldsymbol{v} \in W^{1,q}(\Omega)$ which has the trace $tr\ \boldsymbol{v} \in L^2(\partial\Omega)$, the following inequality hold,*

$$\|\boldsymbol{w}\|_{W^{1,q}(\Omega)} \le C \left(\|\boldsymbol{D}\boldsymbol{v}\|_{L^q(\Omega)} + \|tr\ \boldsymbol{v}\|_{L^2(\partial\Omega)} \right),$$
$$\|\boldsymbol{w}\|_{W^{1,q}(\Omega)} \le C \left(\|\boldsymbol{D}\boldsymbol{v}\|_{L^q(\Omega)} + \|\boldsymbol{v}\|_{L^2(\Omega)} \right).$$

Theorem B.2. *[24, p328] Let Ω be an arbitrary bounded domain, $\Omega \subset \mathbb{R}^d$. Let $0 \le t \le s < \infty$ and $\infty > q \ge \widetilde{q} > 1$. Then, the following embedding holds true:*

$$W^{s,\widetilde{q}}(\Omega) \subset W^{t,q}(\Omega),\ s - \frac{d}{\widetilde{q}} \ge t - \frac{d}{q} \quad \square \tag{B.3}$$

Lemma B.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Then*

$$\|u\|_6 \le c_0 \|u\|_2^{\frac{1}{3}} \|u\|_{W^{1,3}(\Omega)}^{\frac{2}{3}}, \tag{B.4}$$

for any function $u \in W^{1,3}(\Omega)$.

Proof. By interpolation result (B.2), we obtain

$$\|u\|_6 \le c_1 \|u\|_3^{\frac{1}{2}} \|u\|_{W^{1,3}(\Omega)}^{\frac{1}{2}}. \tag{B.5}$$

Then by (B.1), we obtain

$$\|u\|_3 \le \|u\|_2^{\frac{1}{2}} \|u\|_6^{\frac{1}{2}}. \tag{B.6}$$

By combining above two inequalities, we obtain the result. $\square$

Lemma B.4. *Let the dimension of Ω be d. Then the eigenvalues $\{\lambda_j\}$ of the problem (A.5) are bounded above by $cj^{2/d}$ where $c > 0$.*

Proof. The asymptotic behavior of the eigenvalues λ_j as $j \to \infty$ can be estimated using the Rayleigh quotient

$$\mathcal{R}(\boldsymbol{u}) = \frac{\displaystyle\int_\Omega |\boldsymbol{D}\boldsymbol{u}|^2 \, dx + \alpha \int_{\partial\Omega} |\boldsymbol{u}|^2 \, dS}{\displaystyle\int_\Omega |\boldsymbol{u}|^2 \, dx + \beta \int_{\partial\Omega} |\boldsymbol{u}|^2 \, dS}. \tag{B.7}$$

With this notion we have

$$\lambda_j = \inf_{M \in X_j(V)} \sup_{\boldsymbol{u} \in M \setminus \{0\}} \mathcal{R}(\boldsymbol{u}), \tag{B.8}$$

where $X_j(V)$ is the j-dimensional subspaces of the space V with divergence free condition and zero normal component. Then we estimate

$$\mathcal{R}(\boldsymbol{u}) \le c \frac{\|\boldsymbol{u}\|^2_{W^{1,2}(\Omega)}}{\|\boldsymbol{u}\|_2^2}. \tag{B.9}$$

Therefore

$$\lambda_j \le c \inf_{M \in X_j(W)} \sup_{\boldsymbol{u} \in M \setminus \{0\}} \frac{\|\boldsymbol{u}\|^2_{W^{1,2}(\Omega)}}{\|\boldsymbol{u}\|_2^2} = c\mu_k, \tag{B.10}$$

where space W with divergence free and zero boundary conditions, i.e. $W \subset V$. Now this upper-bound $c\mu_k$ is related to the following Stokes-eigenvalue problem

$$\begin{aligned} \Delta \boldsymbol{u} + \nabla \pi &= \mu_k \boldsymbol{u} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

It is shown in [10] that $\mu_k \sim k^{2/d}$. Hence we have $\lambda_j \le c j^{2/d}$. ☐

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