The existence of a real pole-free solution of the fourth order analogue of the Painlevé I equation

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Abstract

We establish the existence of a real solution $y(x,T)$ with no poles on the real line of the following fourth order analogue of the Painlevé I equation,

$$x = Ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_x^2 + 2yy_{xx}) + \frac{1}{240}y_{xxxx}\right).$$

This proves the existence part of a conjecture posed by Dubrovin. We obtain our result by proving the solvability of an associated Riemann-Hilbert problem through the approach of a vanishing lemma. In addition, by applying the Deift/Zhou steepest-descent method to this Riemann-Hilbert problem, we obtain the asymptotics for $y(x,T)$ as $x \to \pm\infty$.

1 Introduction

1.1 The P_I^2 equation

The first Painlevé equation is the second order differential equation

$$y_{xx} = 6y^2 + x. \tag{1.1}$$

This equation has higher order analogues of even order $2m$ for $m \geq 1$, which are collected, together with the first Painlevé equation itself, in the Painlevé I hierarchy, see e.g. [25, 27]. The second member in the hierarchy is the fourth order differential equation

$$x = -\left(\frac{1}{6}y^3 + \frac{1}{24}(y_x^2 + 2yy_{xx}) + \frac{1}{240}y_{xxxx}\right), \tag{1.2}$$

and has solutions that are meromorphic in the complex plane. In 1990, Brézin, Marinari, and Parisi [4] argued numerically that there exists a solution y to (1.2) with no poles on the real line, and with asymptotic behavior

$$y(x) \sim \mp|6x|^{1/3}, \qquad \text{as } x \to \pm\infty. \tag{1.3}$$

Moore [31] proved the existence of a unique real solution to (1.2) with asymptotic behavior given by (1.3), and he gave a line of argument why this solution is probably pole-free on the real line.

A generalization of (1.2) can be obtained by introducing an additional variable T, as done by Dubrovin in [14], so that we get the following differential equation for $y = y(x,T)$, which we denote as the P_I^2 equation (cf. [23] for $T = 0$),

$$x = Ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_x^2 + 2yy_{xx}) + \frac{1}{240}y_{xxxx}\right). \tag{1.4}$$

In recent work [14], Dubrovin conjectured (see Section 1.2 below for more details) the existence of a unique real solution to (1.4) with no poles on the real line. We prove the existence part of this conjecture.

Our result is the following.

Theorem 1.1 *There exists a solution $y(x,T)$ to the P_I^2 equation* (1.4) *with the following properties:*

(i) $y(x,T)$ is real valued and pole-free for $x, T \in \mathbb{R}$.

(ii) For fixed $T \in \mathbb{R}$, $y(x,T)$ has the following asymptotic behavior,

$$y(x,T) = \frac{1}{2} z_0 |x|^{1/3} + \mathcal{O}(|x|^{-2}), \qquad \text{as } x \to \pm\infty, \tag{1.5}$$

where $z_0 = z_0(x,T)$ is the real solution of

$$z_0^3 = -48\mathrm{sgn}(x) + 24 z_0 |x|^{-2/3} T. \tag{1.6}$$

Remark 1.2 Observe that z_0 is negative (positive) for $x > 0$ ($x < 0$) with the following asymptotic behavior as $x \to \pm\infty$,

$$z_0 = \hat{z}_0 - \mathrm{sgn}(x)\, \frac{2}{3}\, 6^{2/3} T |x|^{-2/3} + \mathcal{O}(|x|^{-4/3}), \qquad \hat{z}_0 = -\mathrm{sgn}(x)\, 2 \cdot 6^{1/3}, \tag{1.7}$$

so that the asymptotics (1.5) for y can be rewritten as, cf. (1.3)

$$y(x,T) = \mp (6|x|)^{1/3} \mp \frac{1}{3} 6^{2/3} T |x|^{-1/3} + \mathcal{O}(|x|^{-1}), \qquad \text{as } x \to \pm\infty. \tag{1.8}$$

Power expansions for solutions of (1.2) were found in [26].

Remark 1.3 One expects, see [31, Appendix A] for $T = 0$, that the solution y considered in Theorem 1.1 is uniquely determined by realness and the asymptotics (1.5).

1.2 Motivation

Hamiltonian perturbations of hyperbolic equations

Hyperbolic equations of the form

$$u_t + a(u) u_x = 0 \tag{1.9}$$

can be perturbed to a Hamiltonian equation of the form

$$u_t + a(u) u_x + \epsilon \left[b_1(u) u_{xx} + b_2(u) u_x^2 \right] + \epsilon^2 \left[b_3(u) u_{xxx} + b_4(u) u_x u_{xx} + b_5(u) u_x^3 \right] + \cdots = 0, \tag{1.10}$$

where ϵ is small and $b_1, b_2, \ldots$ are smooth functions. These equations have been studied by Dubrovin in [14], see also [13], where he formulated the universality conjecture about the behavior of a generic solution to a general perturbed Hamiltonian equation (1.10) near the point (x_0, t_0) of *gradient catastrophe* of the unperturbed solution (1.9). He argued that this behavior is described by a special solution to the P_I^2 equation (1.4). To be more precise, his conjecture is the following.

Conjecture 1.4 (Dubrovin, [14])

(i) Let $u_0 = u_0(x,t)$ be a smooth solution to the unperturbed equation (1.9)*, defined for all $x \in \mathbb{R}$ and $0 \le t < t_0$, and monotone in x for any t. Then there exists a solution $u = u(x,t;\epsilon)$ to the perturbed equation* (1.10) *defined on the same domain in the (x,t)-plane with the asymptotics as $\epsilon \to 0$ of the form*

$$u(x,t;\epsilon) = u_0(x,t) + \epsilon^2 u_1(x,t) + \epsilon^4 u_2(x,t) + o(\epsilon^4), \tag{1.11}$$

where u_1 and u_2 can be written down explicitly.

(ii) The ODE (1.4) *has a unique solution $y = y(x,T)$ smooth for all real $x \in \mathbb{R}$ for all values of the parameter T.*

(iii) The generic solution u described in part (i) of the conjecture can be extended up to $t = t_0+\delta$ for sufficiently small positive $\delta = \delta(\epsilon)$; near the point (x_0, t_0) it behaves in the following way

$$u(x,t;\epsilon) = u_0(x,t) + a\epsilon^{2/7} y\left(b\epsilon^{-6/7}(x - c(t-t_0) - x_0), d\epsilon^{-4/7}(t-t_0)\right) + \mathcal{O}(\epsilon^{4/7}),$$

for some constants a, b, c, d which depend on the hyperbolic equation, the solution u, and on the choice of perturbation. Here y is the unique smooth solution described in part (ii) of the conjecture.

So Theorem 1.1 in fact proves the existence part of part (ii) of Dubrovin's conjecture.

In [20], numerical calculations were done for the particular example (of a perturbed Hamiltonian equation) of the small dispersion limit of the KdV equation, see also [28, 29, 30, 35],

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \qquad \text{with initial condition} \qquad u(x,0) = u_0(x).$$

Before the time of gradient catastrophe t_0, solutions turn out to behave nicely. When approaching the critical time t_0, the slope of the function blows up near x_0, and at the critical time, fast oscillations near x_0 set in. The transition between the monotone behavior and the oscillations should be described in terms of the real pole-free solution to (1.4) we consider in this paper.

Random matrix theory

The local eigenvalue correlations of unitary random matrix ensembles on the space of $n \times n$ Hermitian matrices have universal behavior (when the size n of the matrices is going to infinity) in different regimes of the spectrum. In the bulk of the spectrum it is known, see e.g. [1, 8, 9, 32], that the local correlations can be expressed in terms of the sine kernel, while at the soft edge of the spectrum they generically (i.e. when the limiting mean eigenvalue density vanishes like a square root) can be expressed in terms of the Airy kernel, see e.g. [1, 9, 18, 34].

In the presence of singular points, one observes different types of limiting kernels in double scaling limits, see e.g. [2, 5, 6]. Near singular edge points, where the limiting mean eigenvalue density vanishes at a higher order than a square root (the regular case) the local eigenvalue correlations are expected [3] to be described in terms of functions associated with real pole-free solutions of the even members of the Painlevé I hierarchy. The particular case where the limiting mean eigenvalue density vanishes like a power 5/2, which is the lowest non-regular order of vanishing, should correspond with the real pole-free solution of P_I^2 considered in Theorem 1.1. We come back to this in [7].

1.3 Riemann-Hilbert problem and Lax pair for P_I^2

Consider the following Riemann-Hilbert (RH) problem for given complex parameters x and T, on a contour $\Sigma = \left(\cup_{j=0}^{6}\Sigma_j\right) \cup \mathbb{R}^-$, with $\Sigma_j = e^{j\frac{2\pi i}{7}}\mathbb{R}^+$, where each of the eight rays are orientated from 0 to infinity.

RH problem for Ψ:

(a) Ψ is analytic in $\mathbb{C} \setminus \Sigma$.

(b) Ψ satisfies the following jump relations on Σ, for some complex numbers $s_0, \ldots, s_6$ which do not depend on ζ, x, and T,

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Sigma_j \text{ for even } j, \tag{1.12}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ s_j & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Sigma_j \text{ for odd } j, \tag{1.13}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \mathbb{R}^-. \tag{1.14}$$

(c) There exist complex numbers y and h, which depend on x and T but not on ζ, such that Ψ has the following asymptotic behavior as $\zeta \to \infty$,

$$\Psi(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} N \left(I - h\sigma_3 \zeta^{-1/2} + \frac{1}{2} \begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix} \zeta^{-1} + \mathcal{O}(\zeta^{-2}) \right) e^{-\theta(\zeta;x,T)\sigma_3}, \tag{1.15}$$

where

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{1}{4}\pi i \sigma_3}, \qquad \theta(\zeta; x, T) = \frac{1}{105}\zeta^{7/2} - \frac{1}{3}T\zeta^{3/2} + x\zeta^{1/2}. \tag{1.16}$$

Remark 1.5 In [23], Kapaev uses a slightly modified RH problem for the P_I^2 equation with parameter $T = 0$. However a transformation shows that both RH problems are equivalent.

Remark 1.6 The RH problem for P_I^2 is similar to the RH problem for the Painlevé I equation, see [24]. The only differences are that, for Painlevé I, there are only six rays in the jump contour, and that the highest exponent of ζ in θ is $5/2$. For the m-th member of the Painlevé I hierarchy, there are $4 + 2m$ rays in the jump contour, and the highest exponent of ζ in θ is $m + 3/2$.

The complex numbers $s_0, \ldots, s_6$ are the Stokes multipliers and do not depend on x and T, so that varying the parameters x and T leads to a monodromy preserving deformation [15, 19, 21, 22]. The RH problem can only be solvable if the Stokes multipliers satisfy the relation

$$\begin{pmatrix} 1 & 0 \\ s_4 & 1 \end{pmatrix} \begin{pmatrix} 1 & s_5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_6 & 1 \end{pmatrix} \begin{pmatrix} 1 & s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.17}$$

As we will show in Section 2.3 (in fact we only treat one particular choice of Stokes multipliers, but the proof holds in general), a solution Ψ of the RH problem for Ψ also satisfies the following system of differential equations, which is the Lax pair for the P_I^2 equation,

$$\frac{\partial \Psi}{\partial \zeta} = U\Psi, \qquad \frac{\partial \Psi}{\partial x} = W\Psi, \tag{1.18}$$

where

$$U = \frac{1}{240} \begin{pmatrix} -4y_x\zeta - (12yy_x + y_{xxx}) & 8\zeta^2 + 8y\zeta + (12y^2 + 2y_{xx} - 120T) \\ U_{21} & 4y_x\zeta + (12yy_x + y_{xxx}) \end{pmatrix}, \tag{1.19}$$

$$U_{21} = 8\zeta^3 - 8y\zeta^2 - (4y^2 + 2y_{xx} + 120T)\zeta + (16y^3 - 2y_x^2 + 4yy_{xx} + 240x), \tag{1.20}$$

and

$$W = \begin{pmatrix} 0 & 1 \\ \zeta - 2y & 0 \end{pmatrix}. \tag{1.21}$$

This Lax pair appeared first in work of Moore [31] for $T = 0$ and was derived in [25] for general T. The compatibility condition of the Lax pair (1.18)–(1.21) is exactly the P_I^2 equation (1.4), see e.g. [25]. Different choices of Stokes multipliers $s_0, \dots, s_6$ correspond to different solutions of the P_I^2 equation. The particular solution we are interested in, is the unique solution with Stokes multipliers $s_1 = s_2 = s_5 = s_6 = 0$. It then follows by (1.17) that $s_0 = 1$ and $s_3 = s_4 = -1$. This choice of Stokes multipliers was suggested by Kapaev in [23], where he proved that the solution of (1.2) with asymptotics given by (1.3), if it exists, is indeed the one corresponding to $s_1 = s_2 = s_5 = s_6 = 0$, $s_0 = 1$, and $s_3 = s_4 = -1$. This proves the uniqueness part of Dubrovin's conjecture for the case $T = 0$. One can expect that similar arguments, based on the asymptotic solution of the direct monodromy problem, hold for $T \neq 0$ as well.

1.4 Outline of the rest of the paper

In the next section, we prove the first part (the existence part) of Theorem 1.1. In order to do this, we introduce in Section 2.1 a RH problem for Φ, which is equivalent to the RH problem for Ψ (the RH problem for P_I^2) with Stokes multipliers $s_1 = s_2 = s_5 = s_6 = 0$, $s_0 = 1$, and $s_3 = s_4 = -1$. Afterwards, we prove in Section 2.2 the solvability of the RH problem for Φ for real x and T by proving that the associated homogeneous RH problem has only the trivial solution. This approach is often referred to in the literature as a vanishing lemma, see e.g. [6, 9, 16, 17, 36]. We are only able to prove the vanishing lemma for real x and T due to symmetries in the RH problem. In Section 2.3 we show that Ψ satisfies a Lax pair of the form (1.18)–(1.21), with y given in terms of Φ. By compatibility of the Lax pair, it follows that y solves the P_I^2 equation, and by the solvability of the RH problem, y has no real poles.

In Section 3 we prove the second part (the asymptotics part) of Theorem 1.1. We do this by applying the Deift/Zhou steepest-descent method [11, 12] to the RH problem for Φ. In this method, we perform a series of transformations to reduce the RH problem for Φ to a RH problem that we can solve approximately for large $|x|$. By unfolding the series of the transformations, we obtain the asymptotics for y.

2 The existence of a real pole-free solution to P_I^2

2.1 Statement of an associated RH problem to P_I^2

Let $\Gamma = \bigcup_{j=1}^4 \Gamma_j$ be the contour consisting of four straight rays,

$$\Gamma_1 : \arg\zeta = 0, \qquad \Gamma_2 : \arg\zeta = \frac{6\pi}{7}, \qquad \Gamma_3 : \arg\zeta = \pi, \qquad \Gamma_4 : \arg\zeta = -\frac{6\pi}{7},$$

oriented as shown in Figure 1. We seek (for $x, T \in \mathbb{C}$) a 2×2 matrix valued function $\Phi(\zeta; x, T) = \Phi(\zeta)$ (we suppress notation of x and T for brevity) satisfying the following RH problem.

RH problem for Φ:

(a) Φ is analytic in $\mathbb{C} \setminus \Gamma$.

(b) Φ satisfies the following constant jump relations on Γ,

$$\Phi_+(\zeta) = \Phi_-(\zeta)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{2.1}$$

$$\Phi_+(\zeta) = \Phi_-(\zeta)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4, \tag{2.2}$$

$$\Phi_+(\zeta) = \Phi_-(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3. \tag{2.3}$$

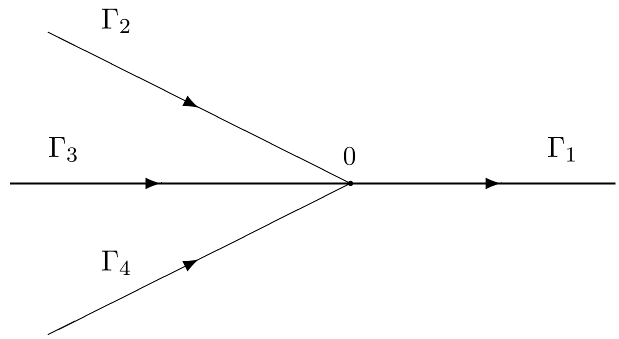

Figure 1: The oriented contour Γ consisting of the four straight rays Γ_1, Γ_2, Γ_3, and Γ_4.

(c) Φ has the following behavior at infinity,

$$\Phi(\zeta) = (I + \mathcal{O}(1/\zeta))\zeta^{-\frac{1}{4}\sigma_3} N e^{-\theta(\zeta;x,T)\sigma_3}, \qquad \text{as } \zeta \to \infty, \tag{2.4}$$

where N and θ are given by (1.16).

Remark 2.1 By multiplying Φ to the left with an appropriate matrix independent of ζ, see (2.32) below, we obtain by Proposition 2.5 the RH problem for Ψ, as stated in Section 1.3, for the particular choice of Stokes multipliers $s_1 = s_2 = s_5 = s_6 = 0$, $s_0 = 1$, and $s_3 = s_4 = -1$.

Remark 2.2 Let Φ be a solution of the RH problem. By using the jump relations (2.1)–(2.3) one has that $\det \Phi_+ = \det \Phi_-$ on Γ. This yields that $\det \Phi$ is entire. From (2.4) we have that $\det \Phi(\zeta) \to 1$ as $\zeta \to \infty$, and thus, by Liouville's theorem, we have that $\det \Phi \equiv 1$.

Now, suppose that $\tilde{\Phi}$ is a second solution of the RH problem. Then, since $\tilde{\Phi}$ and Φ satisfy the same jump relations on Γ, one has that $\tilde{\Phi}\,\Phi^{-1}$ is entire (observe that Φ^{-1} exists since $\det \Phi \equiv 1$). From (2.4) we have that $\tilde{\Phi}(\zeta)\Phi(\zeta)^{-1} \to I$ as $\zeta \to \infty$, and thus, by Liouville's theorem, we have that $\tilde{\Phi}\,\Phi^{-1} \equiv I$. We now have shown that if the RH problem for Φ has a solution, then this solution is unique.

2.2 Solvability of the RH problem for Φ

Here, our goal is to prove that the RH problem for Φ is solvable for $x, T \in \mathbb{R}$. Moreover, we will also strengthen the asymptotic condition (c) of the RH problem and prove analyticity properties in the variables x and T. In case $x = T = 0$, the solvability of the RH problem for Φ has been proven by Deift et al. in [9, Section 5.3]. The general case is analogous but for the convenience of the reader we will recall the different steps in the proof and indicate where we need the restriction to $x, T \in \mathbb{R}$. The result of this subsection is the following lemma.

Lemma 2.3 *For every $x_0, T_0 \in \mathbb{R}$, there exist complex neighborhoods $\mathcal{V}$ of x_0 and $\mathcal{W}$ of T_0 such that for all $x \in \mathcal{V}$ and $T \in \mathcal{W}$ the following holds.*

(i) The RH problem for Φ is solvable.

(ii) The solution Φ of the RH problem for Φ has a full asymptotic expansion in powers of ζ^{-1} as follows,

$$\Phi(\zeta; x, T) \sim \Big(I + \sum_{k=1}^{\infty} A_k \zeta^{-k}\Big)\zeta^{-\frac{1}{4}\sigma_3} N e^{-\theta(\zeta;x,T)\sigma_3}, \tag{2.5}$$

as $\zeta \to \infty$, uniformly in $\mathbb{C} \setminus \Gamma$. Here, the $A_k = A_k(x, T)$ are real-valued for $x, T \in \mathbb{R}$.

(iii) The solution Φ of the RH problem for Φ, as well as the A_k in (2.5), are analytic both as functions of x and T.

Remark 2.4 The important feature of this lemma is the following. In the next subsection we will show that $y = 2A_{1,11} - A_{1,12}^2$, where $A_{1,ij}$ is the (i,j)-th entry of A_1, is a solution to the P_I^2 equation. From the above lemma we then have that this y is real-valued and pole-free on the real axis, so that the first part of Theorem 1.1 is proven.

In order to prove Lemma 2.3, we transform, as in [9, Section 5.3], the RH problem for Φ into an equivalent RH problem for $\widehat{\Phi}$ such that the jump matrix for $\widehat{\Phi}$ is continuous on Γ and converges exponentially to the identity matrix as $\zeta \to \infty$ on Γ, and such that the RH problem for $\widehat{\Phi}$ is normalized at infinity. To do this, we introduce an auxiliary 2×2 matrix valued function M satisfying the following RH problem on a contour $\Gamma^\sigma = \bigcup_{j=1}^4 \Gamma_j^\sigma$ consisting of four straight rays

$$\Gamma_1^\sigma : \arg\zeta = 0, \qquad \Gamma_2^\sigma : \arg\zeta = \sigma, \qquad \Gamma_3^\sigma : \arg\zeta = \pi, \qquad \Gamma_4^\sigma : \arg\zeta = -\sigma, \tag{2.6}$$

where $\sigma \in (\frac{\pi}{3}, \pi)$. We orientate the straight rays from the left to the right, as shown in Figure 1 for the contour Γ. The dependence on the parameter σ is needed in Section 3. In this section, we take $\sigma = 6\pi/7$ fixed, so that $\Gamma^\sigma = \Gamma$.

RH problem for M:

(a) M is analytic in $\mathbb{C} \setminus \Gamma^\sigma$.

(b) M satisfies the following jump relations on Γ^σ,

$$M_+(\zeta) = M_-(\zeta) \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1^\sigma, \tag{2.7}$$

$$M_+(\zeta) = M_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2^\sigma \cup \Gamma_4^\sigma, \tag{2.8}$$

$$M_+(\zeta) = M_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3^\sigma. \tag{2.9}$$

(c) M has the following behavior at infinity,

$$M(\zeta) \sim \Big(I + \sum_{k=1}^{\infty} B_k \zeta^{-k}\Big)\zeta^{-\frac{1}{4}\sigma_3} N, \qquad \text{as } \zeta \to \infty, \tag{2.10}$$

uniformly for $\zeta \in \mathbb{C} \setminus \Gamma^\sigma$ and σ in compact subsets of $(\frac{\pi}{3}, \pi)$. Here, N is given by equation (1.16), and for $k \geq 1$,

$$B_{3k-2} = \begin{pmatrix} 0 & 0 \\ t_{2k-1} & 0 \end{pmatrix}, \qquad B_{3k-1} = \begin{pmatrix} 0 & \hat{t}_{2k-1} \\ 0 & 0 \end{pmatrix}, \qquad B_{3k} = \begin{pmatrix} \hat{t}_{2k} & 0 \\ 0 & t_{2k} \end{pmatrix}, \tag{2.11}$$

with

$$\hat{t}_k = \frac{\Gamma(3k + 1/2)}{36^k k! \Gamma(k + 1/2)}, \qquad t_k = -\frac{6k+1}{6k-1}\hat{t}_k. \tag{2.12}$$

It is well-known, see e.g. [8, 10], that there exists a unique solution M to the above RH problem given in terms of Airy functions Ai. The matrix valued function M is the so-called Airy parametrix and for the purpose of this paper we will not need its exact expression but refer the reader to [8, 10] for this.

We now define $\widehat{\Phi}(\zeta; x, T) = \widehat{\Phi}(\zeta)$ by

$$\widehat{\Phi}(\zeta) = \Phi(\zeta) e^{\theta(\zeta)\sigma_3} M(\zeta)^{-1}, \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma. \tag{2.13}$$

A straightforward calculation, using (2.1)–(2.4), (2.7)–(2.10), and $\theta_+(\zeta) + \theta_-(\zeta) = 0$ for $\zeta \in \mathbb{R}_-$, shows that $\widehat{\Phi}$ satisfies the following RH problem.

RH problem for $\widehat{\Phi}$:

(a) $\widehat{\Phi}$ is analytic in $\mathbb{C} \setminus \Gamma$.

(b) $\widehat{\Phi}_+(\zeta) = \widehat{\Phi}_-(\zeta)\hat{v}(\zeta)$ for $\zeta \in \Gamma$, where $v(\zeta) = v(\zeta; x, T)$ is given by

$$v(\zeta) = \begin{cases} M_-(\zeta) \begin{pmatrix} 1 & e^{-2\theta(\zeta)} - e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix} M_-(\zeta)^{-1}, & \text{for } \zeta \in \Gamma_1, \\ M_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2\theta(\zeta)} - e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} M_-(\zeta)^{-1}, & \text{for } \zeta \in \Gamma_2 \cup \Gamma_4, \\ I, & \text{for } \zeta \in \Gamma_3. \end{cases} \tag{2.14}$$

(c) $\widehat{\Phi}(\zeta) = I + \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty.$

Observe that the jump matrix v is indeed continuous on Γ and that it converges exponentially to the identity matrix as $\zeta \to \infty$ on Γ. This RH problem corresponds to the RH problem [9, (5.108)–(5.110)], and the only difference is that we now have a factor $e^{\pm 2\theta}$ (containing the x, T dependence) instead of $e^{\pm \zeta^{(4\nu+3)/2}}$ in the jump matrices.

Proof of Lemma 2.3 (i). From (2.13) it follows that proving the solvability of the RH problem for Φ is equivalent to proving the solvability of the RH problem for $\widehat{\Phi}$. By general theory of the construction of solutions of RH problems, this is reduced to the study of the singular integral operator,

$$C_v : L^2(\Gamma) \to L^2(\Gamma) : f \mapsto C_+ \left[f \left(I - v^{-1} \right) \right], \tag{2.15}$$

where v is the jump matrix (2.14) of the RH problem for $\widehat{\Phi}$, and where C_+ is the +boundary value of the Cauchy operator

$$Cf(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(s)}{s - z} ds, \qquad \text{for } z \in \mathbb{C} \setminus \Gamma.$$

Indeed, suppose that $I - C_v$ is invertible in $L^2(\Gamma)$. Then, there exists $\mu \in L^2(\Gamma)$ such that $(I - C_v)\mu = C_+(I - v^{-1})$, and it is immediate that

$$\widehat{\Phi}(\zeta) \equiv I + \frac{1}{2\pi i} \int_\Gamma \frac{(I + \mu(s))(I - v(s)^{-1})}{s - \zeta} ds, \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma, \tag{2.16}$$

is analytic in $\mathbb{C} \setminus \Gamma$ and satisfies (since $C_+ - C_- = I$) condition (b) of the RH problem for $\widehat{\Phi}$ in the so-called L^2-sense. However, as in [9, Step 3 of Sections 5.2 and 5.3], one can use the analyticity of v to show that $\widehat{\Phi}$ satisfies jump condition (b) in the sense of continuous boundary values, as well. Further, as in [9, Proposition 5.4], it follows from the exponential decaying of $I - v^{-1}$ as $\zeta \to \infty$ on Γ that the asymptotic condition (c) of the RH problem for $\widehat{\Phi}$ is also satisfied. We summarize that the RH problem for $\widehat{\Phi}$ is solvable, with solution given by (2.16), provided the singular integral operator $I - C_v$ is invertible in $L^2(\Gamma)$.

First, we consider the case $x, T \in \mathbb{R}$. For this case, we show that $I - C_v$ is invertible by showing that it is a Fredholm operator with zero index and kernel $\{0\}$. Exactly as in [9, Steps 1 and 2 of Section 5.3] one has that $I - C_v$ is a Fredholm operator with zero index. In this step, one does not need the restriction to real x and T. It remains to prove that the kernel of $I - C_v$ is $\{0\}$, and it is in this step that we will need the restriction that $x, T \in \mathbb{R}$. This is (again) as in [9, Section 5.3] but for the convenience of the reader we will indicate were we need x and T to be real.

Suppose there exists $\mu_0 \in L^2(\Gamma)$ such that $(I - C_v)\mu_0 = 0$. One can then show that the matrix valued function $\widehat{\Phi}_0$ defined by

$$\widehat{\Phi}_0(\zeta) \equiv \frac{1}{2\pi i} \int_\Gamma \frac{\mu_0(s)(I - v(s)^{-1})}{s - \zeta} ds, \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma, \tag{2.17}$$

is a solution to the RH problem for $\widehat{\Phi}$, but with the asymptotic condition (c) replaced by the *homogeneous* condition

$$\widehat{\Phi}_0(\zeta) = \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty, \text{ uniformly for } \zeta \in \mathbb{C} \setminus \Gamma. \tag{2.18}$$

Since $\mu_0 = \widehat{\Phi}_{0,+}$ (which follows from (2.17) together with $(I - C_v)\mu_0 = 0$), we need to show that $\widehat{\Phi}_0 \equiv 0$. Showing that a solution of the homogeneous RH problem is identically zero, is known in the literature as a vanishing lemma, see [9, 16, 17].

Now, let

$$\Phi_0(\zeta) = \widehat{\Phi}_0(\zeta) M(\zeta), \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma,$$

then it is straightforward to check, using (2.7)–(2.10), (2.14), and (2.18), that Φ_0 solves the following RH problem.

RH problem for Φ_0:

(a) Φ_0 is analytic in $\mathbb{C} \setminus \Gamma$.

(b) Φ_0 satisfies the following jump relations on Γ,

$$\Phi_{0,+}(\zeta) = \Phi_{0,-}(\zeta) \begin{pmatrix} 1 & e^{-2\theta(\zeta)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{2.19}$$

$$\Phi_{0,+}(\zeta) = \Phi_{0,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2\theta(\zeta)} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4, \tag{2.20}$$

$$\Phi_{0,+}(\zeta) = \Phi_{0,-}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3. \tag{2.21}$$

(c) $\Phi_0(\zeta) = \mathcal{O}(1/\zeta)\zeta^{-\frac{1}{4}\sigma_3} N, \qquad$ as $\zeta \to \infty$, uniformly for $\zeta \in \mathbb{C} \setminus \Gamma$.

Further, we introduce an auxiliary matrix valued function A with jumps only on $\mathbb{R}$, as follows, cf. [9, Equations (5.135)–(5.138)]

$$A(\zeta) = \begin{cases} \Phi_0(\zeta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{for } 0 < \arg\zeta < \frac{6\pi}{7}, \\ \Phi_0(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2\theta(\zeta)} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{for } \frac{6\pi}{7} < \arg\zeta < \pi, \\ \Phi_0(\zeta) \begin{pmatrix} 1 & 0 \\ -e^{2\theta(\zeta)} & 1 \end{pmatrix}, & \text{for } -\pi < \arg\zeta < -\frac{6\pi}{7}, \\ \Phi_0(\zeta), & \text{for } -\frac{6\pi}{7} < \arg\zeta < 0. \end{cases} \tag{2.22}$$

Using (2.19)–(2.21) and condition (c) of the RH problem for Φ_0 one can then check that A is a solution to the following RH problem.

RH problem for A:

(a) A is analytic in $\mathbb{C} \setminus \mathbb{R}$

(b) A satisfies the following jump relation on $\mathbb{R}$,

$$A_+(\zeta) = A_-(\zeta) \begin{pmatrix} 1 & -e^{2\theta_+(\zeta)} \\ e^{2\theta_-(\zeta)} & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \mathbb{R}_-, \tag{2.23}$$

$$A_+(\zeta) = A_-(\zeta) \begin{pmatrix} e^{-2\theta(\zeta)} & -1 \\ 1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \mathbb{R}_+, \tag{2.24}$$

(c) $A(\zeta) = \mathcal{O}(\zeta^{-3/4})$, as $\zeta \to \infty$, uniformly for $\zeta \in \mathbb{C} \setminus \mathbb{R}$.

Now, we define $Q(\zeta) = A(\zeta)A^*(\bar{\zeta})$, where A^* denotes the Hermitian conjugate of A. The matrix valued function Q is analytic in the upper half plane, continuous up to $\mathbb{R}$, and decays like $\zeta^{-3/2}$ as $\zeta \to \infty$. By Cauchy's theorem this implies, $\int_{\mathbb{R}} Q_+(s)ds = 0$. Using the jump relations (2.23) and (2.24) we then have,

$$\int_{\mathbb{R}_-} A_-(s) \begin{pmatrix} 1 & -e^{2\theta_+(s)} \\ e^{2\theta_-(s)} & 0 \end{pmatrix} A_-^*(s)ds + \int_{\mathbb{R}_+} A_-(s) \begin{pmatrix} e^{-2\theta(s)} & -1 \\ 1 & 0 \end{pmatrix} A_-^*(s)ds = 0.$$

Adding this to its Hermitian conjugate, and using the fact $\overline{\theta_+(s)} = \theta_-(s)$ for $s \in \mathbb{R}_-$ (which is true since $x, T \in \mathbb{R}$), we arrive at, cf. [9, Equation (5.146)]

$$\int_{\mathbb{R}_-} A_-(s) \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} A_-^*(s)ds + \int_{\mathbb{R}_+} A_-(s) \begin{pmatrix} 2e^{-2\theta(s)} & 0 \\ 0 & 0 \end{pmatrix} A_-^*(s)ds = 0. \tag{2.25}$$

This is the crucial step where we need x and T to be real. The latter relation implies that the first column of A_- is identically zero, and the jump relations (2.23) and (2.24) then imply that the second column of A_+ is identically zero, as well.

By writing out the RH conditions for each entry of A and using the vanishing of the first column of A_- and the second column of A_+, the matrix RH problem reduces to two scalar RH problems. The proof that the solutions of those scalar RH problems (and thus also the second column of A_- and the first column of A_+) are identically zero, is exactly as in [9, Step 3 of Section 5.3] using Carlson's theorem, see [33], and we will not go into detail about this. We then have shown that $A \equiv 0$, so that also $\widehat{\Phi}_0 \equiv 0$ and thus $\mu_0 \equiv 0$. We now have proven that $I - C_v$ is invertible for $x, T \in \mathbb{R}$, which implies that the RH problem for $\widehat{\Phi}$ (and thus also the RH problem for Φ) is solvable for $x, T \in \mathbb{R}$.

Next, fix $x_0, T_0 \in \mathbb{R}$. Above, we have shown that the singular integral operator $I - C_{v(\cdot\,;x_0,T_0)}$ is invertible. Since

$$I - C_{v(\cdot\,;x,T)} = \left(I - C_{v(\cdot\,;x_0,T_0)}\right)\left[I + \left(I - C_{v(\cdot\,;x_0,T_0)}\right)^{-1}\left(C_{v(\cdot\,;x_0,T_0)} - C_{v(\cdot\,;x,T)}\right)\right],$$

it then follows that $I - C_{v(\cdot\,;x,T)}$ is invertible provided

$$\left\|\left(I - C_{v(\cdot\,;x_0,T_0)}\right)^{-1}\left(C_{v(\cdot\,;x_0,T_0)} - C_{v(\cdot\,;x,T)}\right)\right\| < 1, .$$

where $\|\cdot\|$ denotes the operator norm. It is straightforward to check that there exist neighborhoods $\mathcal{V}$ of x_0 and $\mathcal{W}$ of T_0 such that for all $x \in \mathcal{V}$ and $T \in \mathcal{W}$,

$$\begin{aligned}\left\|C_{v(\cdot\,;x_0,T_0)} - C_{v(\cdot\,;x,T)}\right\| &\leq \left\|C_+\right\| \left\|v(\cdot\,;\,x_0,T_0) - v(\cdot\,;\,x,T)\right\|_{L^{(\infty)}(\Gamma)} \\ &< \left\|\left(I - C_{v(\cdot\,;x_0,T_0)}\right)^{-1}\right\|^{-1},\end{aligned}$$

which implies that the operator $I - C_{v(\cdot\,;x,T)}$ is invertible. Hence the RH problem for $\widehat{\Phi}$, and thus also the RH problem for Φ, is solvable for $x \in \mathcal{V}$ and $T \in \mathcal{W}$. This finishes the proof of the first part of the lemma. □

Proof of Lemma 2.3 (ii). It follows from the asymptotic expansion (2.10) of M together with $\Phi = \widehat{\Phi} M e^{-\theta\sigma_3}$, see (2.13), that we need to show that $\widehat{\Phi}$ has a full asymptotic expansion in powers of ζ^{-1}. Insert the relation

$$\frac{1}{s-\zeta} = -\sum_{k=1}^{n} s^{k-1}\zeta^{-k} + \frac{s^n}{\zeta^n(s-\zeta)}, \qquad \text{for } n \in \mathbb{N},$$

into the solution $\widehat{\Phi}$ of the RH problem for $\widehat{\Phi}$, which is given by (2.16). We then obtain for any $n \in \mathbb{N}$,

$$\widehat{\Phi} = I + \sum_{k=1}^{n} \widehat{B}_k \zeta^{-k} + \frac{1}{2\pi i}\int_\Gamma \frac{s^n(I+\mu(s))(I - v(\zeta)^{-1})}{\zeta^n(s-\zeta)} ds, \tag{2.26}$$

where

$$\widehat{B}_k = -\frac{1}{2\pi i}\int_\Gamma s^{k-1}(I+\mu(s))(I - v(s)^{-1}) ds. \tag{2.27}$$

As in [9, Proposition 5.4] one can check that the integral in (2.26) is of order $\mathcal{O}(\zeta^{-(n+1)})$ as $\zeta \to \infty$ uniformly for $\zeta \in \mathbb{C} \setminus \Gamma$. We then have shown that $\widehat{\Phi}$ has the following asymptotic expansion in powers of ζ^{-1},

$$\widehat{\Phi}(\zeta) \sim I + \sum_{k=1}^{\infty} \widehat{B}_k \zeta^{-k}, \qquad \text{as } \zeta \to \infty, \text{ uniformly for } \zeta \in \mathbb{C} \setminus \Gamma. \tag{2.28}$$

From (2.10), (2.28), and the fact that $\Phi = \widehat{\Phi} P e^{-\theta\sigma_3}$ it now follows that Φ has a full asymptotic expansion in the form (2.5), where (with $B_0 = \widehat{B}_0 = I$)

$$A_k = \sum_{j=0}^{k} B_j \widehat{B}_{k-j}. \tag{2.29}$$

It remains to show that the A_k are real-valued for $x, T \in \mathbb{R}$. It is straightforward to verify that for $x, T \in \mathbb{R}$ the matrix valued function $-i\overline{\Phi(\overline{\zeta}; x, T)}\sigma_3$ is a solution to the RH problem for Φ. By uniqueness we then have

$$\Phi(\zeta; x, T) = -i\overline{\Phi(\overline{\zeta}; x, T)}\sigma_3, \qquad \text{for } x, T \in \mathbb{R},$$

which yields

$$\Big(I + \sum_{k=1}^{\infty} A_k \zeta^{-k}\Big)\zeta^{-\frac{1}{4}\sigma_3} N e^{-\theta(\zeta;x,T)\sigma_3} = \Big(I + \sum_{k=1}^{\infty} \overline{A_k}\zeta^{-k}\Big)\zeta^{-\frac{1}{4}\sigma_3} N e^{-\theta(\zeta;x,T)\sigma_3},$$

and hence $A_k = \overline{A_k}$ for $x, T \in \mathbb{R}$. This proves the second part of the lemma. □

Proof of Lemma 2.3 (iii). We show that Φ and A_k are analytic in x, for $x \in \mathcal{V}$. The analyticity in T follows in a similar fashion. In order to show that $\widehat{\Phi}$ (and thus also Φ) is analytic for $x \in \mathcal{V}$ we need to show that, letting $h \to 0$ in the complex plane,

$$\lim_{h \to 0} \frac{1}{h}\big(\widehat{\Phi}(\zeta; x + h, T) - \widehat{\Phi}(\zeta; x, T)\big)$$

exists. Consider the 2×2 auxiliary matrix valued function $H(\zeta; x, T; h) = H(\zeta)$ defined as follows,

$$H(\zeta) = \widehat{\Phi}(\zeta; x + h, T)\widehat{\Phi}(\zeta; x, T)^{-1}, \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma. \tag{2.30}$$

Here we take h sufficiently small, so that $\Phi(\zeta; x + h, T)$ exists by part (i) of the lemma. It is straightforward to check that H satisfies the following RH problem.

RH problem for H:

(a) H is analytic in $\mathbb{C} \setminus \Gamma$.

(b) H satisfies the jump relation $H_+(\zeta) = H_-(\zeta)v_H(\zeta)$ for $\zeta \in \Gamma$, where

$$v_H(\zeta) = I + e^{-2\theta(\zeta;x,T)}(e^{-2h\zeta^{1/2}} - 1)\widehat{\Phi}_-(\zeta; x, T)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\widehat{\Phi}_-(\zeta; x, T)^{-1}, \qquad \zeta \in \Gamma_1,$$

$$v_H(\zeta) = I + e^{2\theta(\zeta;x,T)}(e^{2h\zeta^{1/2}} - 1)\widehat{\Phi}_-(\zeta; x, T)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\widehat{\Phi}_-(\zeta; x, T)^{-1}, \qquad \zeta \in \Gamma_2 \cup \Gamma_4,$$

$$v_H(\zeta) = I, \qquad \zeta \in \Gamma_3.$$

(c) $H(\zeta) = I + \mathcal{O}(1/\zeta)$, as $\zeta \to \infty$, uniformly for $\zeta \in \mathbb{C} \setminus \Gamma$.

Since $v_H(\zeta) = I + \mathcal{O}(h)$ as $h \to 0$ uniformly for $\zeta \in \Gamma$, where the $\mathcal{O}(h)$-term can be expanded into a full asymptotic expansion in powers of h, it follows as in [8, 10, 9] that

$$\widehat{\Phi}(\zeta; x + h, T)\widehat{\Phi}(\zeta; x, T)^{-1} = H(\zeta) = I + hH_1(\zeta; x, T) + \mathcal{O}(h^2), \qquad \text{as } h \to 0, \tag{2.31}$$

where H_1 is a 2×2 matrix valued function independent of h. This yields,

$$\lim_{h \to 0} \frac{1}{h}\big(\widehat{\Phi}(\zeta; x + h, T) - \widehat{\Phi}(\zeta; x, T)\big) = H_1(\zeta; x, T)\widehat{\Phi}(\zeta; x, T),$$

which implies that $\widehat{\Phi}$ (and thus also Φ) is analytic for $x \in \mathcal{V}$.

It remains to show that the matrix valued functions A_k are analytic for $x \in \mathcal{V}$. By (2.16), it is immediate that

$$\widehat{\Phi}_+(\zeta) = I + \mu(\zeta), \qquad \text{for } \zeta \in \Sigma,$$

so that μ is analytic for $x \in \mathcal{V}$. By (2.27) it then follows that $\widehat{B}_k$ is also analytic for $x \in \mathcal{V}$. This yields, by (2.29) and (2.11), the analyticity of A_k, and hence the last part of the lemma is proven. □

2.3 Proof of Theorem 1.1 (i)

In order to prove the existence part of Theorem 1.1 we proceed as follows. Introduce, for $x, T \in \mathbb{R}$, a 2×2 matrix valued function $\Psi(\zeta; x, T) = \Psi(\zeta)$ by multiplying the solution Φ of the RH problem for Φ to the left with an appropriate matrix independent of ζ,

$$\Psi(\zeta) = \begin{pmatrix} 1 & 0 \\ A_{1,12} & 1 \end{pmatrix} \Phi(\zeta), \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma. \tag{2.32}$$

Here $A_{1,12}$ is the $(1,2)$-entry of the 2×2 matrix $A_1 = A_1(x, T)$ appearing in the asymptotic expansion (2.5) of Φ at infinity. The important feature of this transformation is that Ψ satisfies the RH problem for P_I^2, see Section 1.3, as we will show in the following proposition.

Proposition 2.5 *The matrix valued function Ψ, defined by (2.32), is a solution to the RH problem for Ψ, see Section 1.3, with Stokes multipliers $s_1 = s_2 = s_5 = s_6 = 0$, $s_0 = 1$, and $s_3 = s_4 = -1$, and with the asymptotic condition (c) replaced by the stronger condition*

$$\Psi(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} N \widehat{\Psi}(\zeta) e^{-\theta(\zeta)\sigma_3}, \tag{2.33}$$

where $\widehat{\Psi}$ has a full asymptotic expansion in powers of $\zeta^{-1/2}$ as follows,

$$\widehat{\Psi}(\zeta; x, T) \sim I - h\sigma_3 \zeta^{-1/2} + \frac{1}{2}\begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix} \zeta^{-1}$$
$$+ \frac{1}{2}\sum_{k=1}^{\infty} \left[\begin{pmatrix} q_k & ir_k \\ ir_k & -q_k \end{pmatrix} \zeta^{-k-\frac{1}{2}} + \begin{pmatrix} v_k & iw_k \\ -iw_k & v_k \end{pmatrix} \zeta^{-k-1} \right], \tag{2.34}$$

as $\zeta \to \infty$ uniformly for $\zeta \in \mathbb{C} \setminus \Gamma$. Here, $y = y(x, T)$ is given by

$$y = 2A_{1,11} - A_{1,12}^2. \tag{2.35}$$

Further, $h = A_{1,12}$ and the q_k, r_k, v_k and w_k are some unimportant functions of x and T (independent of ζ).

Proof. The fact that Ψ satisfies conditions (a) and (b) of the RH problem for Ψ follows trivially from (2.32) together with conditions (a) and (b) of the RH problem for Φ. So, it remains to show that $\widehat{\Psi}$ given by

$$\widehat{\Psi} = N^{-1} \zeta^{\frac{\sigma_3}{4}} \Psi(\zeta) e^{\theta(\zeta)\sigma_3}, \tag{2.36}$$

satisfies an asymptotic expansion of the form (2.34) with y given by (2.35). It follows from (2.36), (2.32) and (2.5) that

$$\widehat{\Psi}(\zeta) \sim N^{-1} \zeta^{\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ A_{1,12} & 1 \end{pmatrix} \left[I + \sum_{k=1}^{\infty} A_k \zeta^{-k} \right] \zeta^{-\frac{\sigma_3}{4}} N$$
$$\sim N^{-1} \left(\sum_{k=0}^{\infty} \zeta^{\frac{\sigma_3}{4}} \tilde{A}_k \zeta^{-\frac{\sigma_3}{4}} \zeta^{-k} \right) N, \tag{2.37}$$

where

$$\tilde{A}_0 = \begin{pmatrix} 1 & 0 \\ A_{1,12} & 1 \end{pmatrix}, \qquad \text{and} \quad \tilde{A}_k = \begin{pmatrix} 1 & 0 \\ A_{1,12} & 1 \end{pmatrix} A_k, \qquad \text{for } k \geq 1.$$

Now, using the facts that $\tilde{A}_{0,11} = \tilde{A}_{0,22} = 1$, that $\tilde{A}_{0,12} = 0$ and that $\tilde{A}_{0,21} = \tilde{A}_{1,12} = A_{1,12}$ we find,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \zeta^{\frac{\sigma_3}{4}} \tilde{A}_k \zeta^{-\frac{\sigma_3}{4}} \zeta^{-k} \\
&\qquad = \sum_{k=0}^{\infty} \left[\begin{pmatrix} 0 & 0 \\ \tilde{A}_{k,21} & 0 \end{pmatrix} \zeta^{-k-\frac{1}{2}} + \begin{pmatrix} 0 & \tilde{A}_{k,12} \\ 0 & 0 \end{pmatrix} \zeta^{-k+\frac{1}{2}} + \begin{pmatrix} \tilde{A}_{k,11} & 0 \\ 0 & \tilde{A}_{k,22} \end{pmatrix} \zeta^{-k} \right] \\
&\qquad = I + A_{1,12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \zeta^{-1/2} + \begin{pmatrix} \tilde{A}_{1,11} & 0 \\ 0 & \tilde{A}_{1,22} \end{pmatrix} \zeta^{-1} \\
&\qquad\qquad + \sum_{k=1}^{\infty} \left[\begin{pmatrix} 0 & \tilde{A}_{k+1,12} \\ \tilde{A}_{k,21} & 0 \end{pmatrix} \zeta^{-k-\frac{1}{2}} + \begin{pmatrix} \tilde{A}_{k+1,11} & 0 \\ 0 & \tilde{A}_{k+1,22} \end{pmatrix} \zeta^{-k-1} \right].
\end{aligned}$$

Inserting this into (2.37) and using (1.16) we arrive at,

$$\begin{aligned}
\widehat{\Psi}(\zeta; x, T) \sim I - h\sigma_3 \zeta^{-1/2} &+ \frac{1}{2} \begin{pmatrix} \tilde{A}_{1,11} + \tilde{A}_{1,22} & i(\tilde{A}_{1,11} - \tilde{A}_{1,22}) \\ -i(\tilde{A}_{1,11} - \tilde{A}_{1,22}) & \tilde{A}_{1,11} + \tilde{A}_{1,22} \end{pmatrix} \zeta^{-1} \\
&+ \frac{1}{2} \sum_{k=1}^{\infty} \left[\begin{pmatrix} q_k & ir_k \\ ir_k & -q_k \end{pmatrix} \zeta^{-k-\frac{1}{2}} + \begin{pmatrix} v_k & iw_k \\ -iw_k & v_k \end{pmatrix} \zeta^{-k-1} \right], \qquad (2.38)
\end{aligned}$$

where $h = A_{1,12}$ and where the q_k, r_k, v_k, and w_k can be written down explicitly in terms of $\tilde{A}_k$ and $\tilde{A}_{k+1}$. Now, note that since $\det \Phi \equiv 1$ (see Remark 2.2) and since, by (2.5),

$$\det \Phi = 1 + (A_{1,11} + A_{1,22})\zeta^{-1} + \mathcal{O}(\zeta^{-2}), \qquad \text{as } \zeta \to \infty,$$

we have that $A_{1,22} = -A_{1,11}$. This together with the facts that $\tilde{A}_{1,11} = A_{1,11}$ and $\tilde{A}_{1,22} = A_{1,12}^2 + A_{1,22}$ yields,

$$\tilde{A}_{1,11} + \tilde{A}_{1,22} = A_{1,12}^2 = h^2, \qquad \tilde{A}_{1,11} - \tilde{A}_{1,22} = 2A_{1,11} - A_{1,12}^2 = y.$$

Inserting this into (2.38) the proposition is proven. □

The idea is now to show that Ψ satisfies the linear system of differential equations (1.18)–(1.21) with y given by (2.35), so that by compatibility of the Lax pair this y is a solution to the P_I^2 equation (1.4). Since by Lemma 2.3 the functions $A_{1,11}$ and $A_{1,12}$ are real-valued and pole-free for $x, T \in \mathbb{R}$ we have that y itself is real-valued and pole-free for $x, T \in \mathbb{R}$, so that the first part of Theorem 1.1 is proven.

Proof of Theorem 1.1 (i). Recall from the above discussion that we need to show that the matrix valued functions (note that, by Lemma 2.3 (iii) and (2.32), Ψ is differentiable with respect to x)

$$U = \frac{\partial \Psi}{\partial \zeta} \Psi^{-1} \qquad \text{and} \qquad W = \frac{\partial \Psi}{\partial x} \Psi^{-1}, \qquad (2.39)$$

are of the form (1.19) and (1.21), respectively, with y given by (2.35). Observe that, since Ψ has constant jump matrices, the derivatives $\frac{\partial\Psi}{\partial\zeta}$ and $\frac{\partial\Psi}{\partial x}$ have the same jumps as Ψ, and hence U and W are entire.

First, we focus on U. By (2.33),

$$U = -\frac{\partial\theta}{\partial\zeta}\zeta^{-\frac{\sigma_3}{4}}\left(N\widehat{\Psi}\sigma_3\widehat{\Psi}^{-1}N^{-1}\right)\zeta^{\frac{\sigma_3}{4}} + \begin{pmatrix}\mathcal{O}(\zeta^{-1}) & \mathcal{O}(\zeta^{-2})\\ \mathcal{O}(\zeta^{-1}) & \mathcal{O}(\zeta^{-1})\end{pmatrix}, \qquad \text{as } \zeta\to\infty. \tag{2.40}$$

Since $\det\Phi\equiv 1$ we obtain from (2.32) and (2.33) that $\det\widehat{\Psi}\equiv 1$, as well. Then, it is easy to verify that

$$\widehat{\Psi}\sigma_3\widehat{\Psi}^{-1} = \begin{pmatrix}1+2\widehat{\Psi}_{12}\widehat{\Psi}_{21} & -2\widehat{\Psi}_{11}\widehat{\Psi}_{12}\\ 2\widehat{\Psi}_{21}\widehat{\Psi}_{22} & -1-2\widehat{\Psi}_{12}\widehat{\Psi}_{21}\end{pmatrix} \equiv \begin{pmatrix}Q_{11} & -iQ_{12}\\ -iQ_{21} & -Q_{11}\end{pmatrix},$$

and hence, by (1.16),

$$N\widehat{\Psi}\sigma_3\widehat{\Psi}^{-1}N^{-1} = \begin{pmatrix}\frac{1}{2}(Q_{21}-Q_{12}) & -\frac{1}{2}(Q_{21}+Q_{12})-Q_{11}\\ \frac{1}{2}(Q_{21}+Q_{12})-Q_{11} & \frac{1}{2}(Q_{12}-Q_{21})\end{pmatrix}. \tag{2.41}$$

The asymptotics of the functions Q_{11}, Q_{12} and Q_{21} at infinity follow from the asymptotic behavior (2.34) of $\widehat{\Psi}$. We find, as $\zeta\to\infty$,

$$Q_{11} = 1+\frac{1}{2}y^2\zeta^{-2} + \big(yw_1 - \frac{1}{2}r_1^2\big)\zeta^{-3} + \mathcal{O}(\zeta^{-4}), \tag{2.42}$$

$$Q_{12} = y\zeta^{-1} + (r_1 - yh)\zeta^{-3/2} + \big(\frac{1}{2}yh^2 - hr_1 + w_1\big)\zeta^{-2} + \frac{1}{8}t\zeta^{-5/2} + u\zeta^{-3} + v\zeta^{-7/2} + \mathcal{O}(\zeta^{-4}), \tag{2.43}$$

$$Q_{21} = y\zeta^{-1} - (r_1 - yh)\zeta^{-3/2} + \big(\frac{1}{2}yh^2 - hr_1 + w_1\big)\zeta^{-2} - \frac{1}{8}t\zeta^{-5/2} + u\zeta^{-3} - v\zeta^{-7/2} + \mathcal{O}(\zeta^{-4}), \tag{2.44}$$

where t, u and v are some functions of x and T. Inserting (2.41)–(2.44) into (2.40) and using the fact that,

$$\frac{\partial\theta}{\partial\zeta} = \frac{1}{30}\zeta^{5/2} - \frac{1}{2}T\zeta^{1/2} + \frac{1}{2}x\zeta^{-1/2},$$

it is straightforward to check that,

$$U = \frac{1}{240}\begin{pmatrix}a\zeta + t & 8\zeta^2 + 8y\zeta + b + e\zeta^{-1}\\ 8\zeta^3 - 8y\zeta^2 + c\zeta + d & -a\zeta - t\end{pmatrix} + \begin{pmatrix}\mathcal{O}(\zeta^{-1}) & \mathcal{O}(\zeta^{-2})\\ \mathcal{O}(\zeta^{-1}) & \mathcal{O}(\zeta^{-1})\end{pmatrix},$$

with

$$a = 8r_1 - 8yh, \tag{2.45}$$

$$b = 4y^2 - 120T + 4yh^2 - 8hr_1 + 8w_1, \qquad c = 4y^2 - 120T - 4yh^2 + 8hr_1 - 8w_1, \tag{2.46}$$

$$d = 8yw_1 - 4r_1^2 + 120x + 120yT - 8u, \qquad e = 8yw_1 - 4r_1^2 + 120x - 120yT + 8u. \tag{2.47}$$

Since U is entire, it contains no negative powers of ζ. In particular $e = 0$, so that

$$d = d + e = 16yw_1 - 8r_1^2 + 240x. \tag{2.48}$$

We now have shown that,

$$U = \frac{1}{240}\begin{pmatrix} a\zeta + t & 8\zeta^2 + 8y\zeta + b \\ 8\zeta^3 - 8y\zeta^2 + c\zeta + d & -a\zeta - t \end{pmatrix}, \tag{2.49}$$

where a, b and c are given by (2.45) and (2.46), and where d is given by (2.48).

Next, we consider W. Observe that by (2.33),

$$W = \zeta^{-\frac{\sigma_3}{4}} N \frac{\partial\widehat{\Psi}}{\partial x}\widehat{\Psi}^{-1} N^{-1} \zeta^{\frac{\sigma_3}{4}} - \frac{\partial\theta}{\partial x}\zeta^{-\frac{\sigma_3}{4}}\left(N\widehat{\Psi}\sigma_3\widehat{\Psi}^{-1}N^{-1}\right)\zeta^{\frac{\sigma_3}{4}}. \tag{2.50}$$

From (2.34) we obtain

$$\begin{aligned} \zeta^{-\frac{\sigma_3}{4}} N \frac{\partial\widehat{\Psi}}{\partial x}\widehat{\Psi}^{-1} N^{-1} \zeta^{\frac{\sigma_3}{4}} &= \zeta^{-\frac{\sigma_3}{4}} N\left(-h_x\sigma_3\zeta^{-1/2} + \mathcal{O}(\zeta^{-1})\right)N^{-1}\zeta^{\frac{\sigma_3}{4}} \\ &= \begin{pmatrix} 0 & 0 \\ h_x & 0 \end{pmatrix} + \mathcal{O}(\zeta^{-1/2}), \end{aligned} \tag{2.51}$$

where h_x denotes the derivative of h with respect to x. Further, using (2.41)–(2.44) together with the fact that $\frac{\partial\theta}{\partial x} = \zeta^{1/2}$, we have

$$-\frac{\partial\theta}{\partial x}\zeta^{-\frac{\sigma_3}{4}}\left(N\widehat{\Psi}\sigma_3\widehat{\Psi}^{-1}N^{-1}\right)\zeta^{\frac{\sigma_3}{4}} = \begin{pmatrix} 0 & 1 \\ \zeta - y & 0 \end{pmatrix} + \mathcal{O}(\zeta^{-1}). \tag{2.52}$$

Inserting (2.51) and (2.52) into (2.50), and using the fact that W is entire (so that W contains no negative powers of ζ) we arrive at,

$$W = \begin{pmatrix} 0 & 1 \\ \zeta + (h_x - y) & 0 \end{pmatrix}. \tag{2.53}$$

We will now complete the proof by determining the functions a, b, c, d, t and h_x exclusively in terms of y, y_x, y_{xx}, and y_{xxx}, using the compatibility condition

$$\frac{\partial^2\Psi}{\partial\zeta\partial x} = \frac{\partial^2\Psi}{\partial x\partial\zeta}.$$

This condition is equivalent to $\dfrac{\partial U}{\partial x} - \dfrac{\partial W}{\partial\zeta} + UW - WU = 0$ and leads, after a straightforward calculation, to

$$C_0\zeta^2 + C_1\zeta + C_2 = 0,$$

where

$$C_0 = \begin{pmatrix} 8(h_x + y) & 0 \\ -8y_x - 2a & 8(h_x + y) \end{pmatrix}, \tag{2.54}$$

$$C_1 = \begin{pmatrix} a_x + 8y(h_x - y) + b - c & 8y_x + 2a \\ c_x - 2a(h_x - y) - 2t & -a_x - 8y(h_x - y) - b + c \end{pmatrix}, \tag{2.55}$$

$$C_2 = \begin{pmatrix} t_x + b(h_x - y) - d & b_x + 2t \\ d_x - 2t(h_x - y) - 240 & -t_x - b(h_x - y) + d \end{pmatrix}. \tag{2.56}$$

Since $C_0 = 0$ we deduce that $h_x = -y$, and hence by (2.53) W is of the form (1.21), and that $a = -4y_x$. By (2.45) we then have,

$$r_1 = -\frac{1}{2}y_x + yh. \tag{2.57}$$

Further, since $C_{1,11} = 0$ we then obtain from (2.46) that

$$w_1 = \frac{1}{4}y_{xx} + y^2 + \frac{1}{2}yh^2 - \frac{1}{2}y_x h. \tag{2.58}$$

Inserting the expressions (2.57) and (2.58) for r_1 and w_1 into the expressions (2.45), (2.46) and (2.48) for a, b, c and d, and using the fact that $t = -\frac{1}{2}b_x$ (since $C_{2,12} = 0$) we arrive at

$$a = -4y_x, \qquad b = 12y^2 + 2y_{xx} - 120T, \tag{2.59}$$

$$c = -4y^2 - 2y_{xx} - 120T, \qquad d = 16y^3 - 2y_x^2 + 4yy_{xx} + 240x, \tag{2.60}$$

$$t = -12yy_x - y_{xxx}. \tag{2.61}$$

Inserting the latter equations into (2.49) we have that U is of the form (1.19). Note that the fact that y satisfies the P_I^2 equation now follows from $C_{2,11} = 0$. This proves the first part of the theorem. □

Remark 2.6 Note that, by Lemma 2.3 (iii), we can safely differentiate y and h with respect to x, as we did in the above proof.

3 Asymptotic behavior of $y(x, T)$ as $x \to \pm\infty$

In this section we will determine for fixed $T \in \mathbb{R}$ the asymptotics (as $x \to \pm\infty$) of the particular solution $y(x, T)$ of the P_I^2 equation with no poles on the real line as constructed in the previous section and given by, cf. (2.35),

$$y = 2A_{1,11} - A_{1,12}^2. \tag{3.1}$$

Here, A_1 is the matrix valued function appearing in the asymptotic expansion (2.5) for Φ. So, it suffices to determine the asymptotics (as $x \to \pm\infty$) of the first row of A_1 which we will do by applying the Deift/Zhou steepest-descent method [8, 9, 10, 11, 12] to the RH problem for Φ.

3.1 Rescaling of the RH problem and deformation of the jump contour

Let $z_0 = z_0(x, T) \in \mathbb{R}$ (to be determined in Section 3.2) and let $\hat{\Gamma} = \bigcup_{j=1}^4 \hat{\Gamma}_j$ be the oriented contour through z_0 as shown in Figure 2. Here, the dotted lines are in fact Γ_2 and Γ_4, see Figure 1, and are not part of the contour. The precise form of the contour $\hat{\Gamma}$ (in particular of $\hat{\Gamma}_2$ and $\hat{\Gamma}_4$) will be determined below. Now, introduce the 2×2 matrix valued function $Y(\zeta; x, T) = Y(\zeta)$ as follows,

$$Y(\zeta) \equiv \begin{cases} \Phi(|x|^{1/3}\zeta), & \text{for } \zeta \in \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}, \\ \Phi(|x|^{1/3}\zeta)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } \zeta \in \mathrm{V}, \\ \Phi(|x|^{1/3}\zeta)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } \zeta \in \mathrm{VI}, \end{cases} \tag{3.2}$$

where Φ is the solution of the RH problem for Φ, see Section 2.1, and where the sets I,II,. . .,VI are defined by Figure 2. Then, it is straightforward to check, using (2.1)–(2.3), (2.5) and (1.16), that Y satisfies the following conditions.

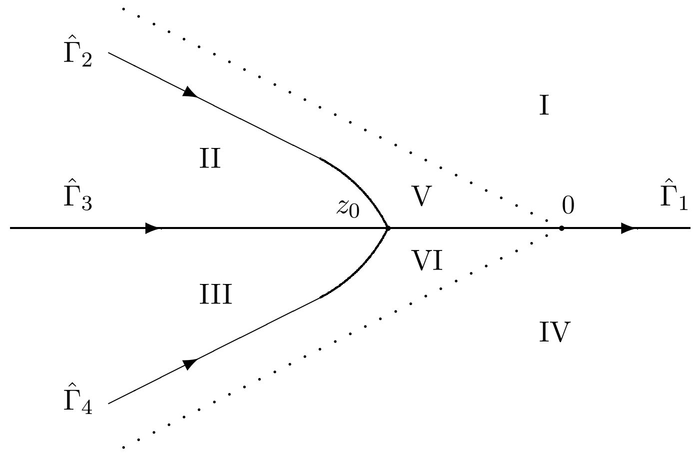

Figure 2: The contour $\hat{\Gamma} = \bigcup_{j=1}^{4} \hat{\Gamma}_j$. Note that the dotted lines are not part of the contour.

RH problem for Y:

(a) Y is analytic in $\mathbb{C} \setminus \hat{\Gamma}$.

(b) Y satisfies the same jump relations on $\hat{\Gamma}$ as Φ does on Γ. Namely,

$$Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \hat{\Gamma}_1, \tag{3.3}$$

$$Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \hat{\Gamma}_2 \cup \hat{\Gamma}_4, \tag{3.4}$$

$$Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \hat{\Gamma}_3. \tag{3.5}$$

(c) Y has the following behavior as $\zeta \to \infty$,

$$Y(\zeta) \sim \left(I + \sum_{k=1}^{\infty} A_k |x|^{-k/3} \zeta^{-k} \right) \zeta^{-\frac{\sigma_3}{4}} |x|^{-\frac{\sigma_3}{12}} N e^{-|x|^{7/6} \hat{\theta}(\zeta;x,T)\sigma_3}, \tag{3.6}$$

where

$$\hat{\theta}(\zeta; x, T) = \frac{1}{105}\zeta^{7/2} - \frac{1}{3}|x|^{-2/3} T \zeta^{3/2} + \operatorname{sgn}(x)\zeta^{1/2}. \tag{3.7}$$

3.2 Normalization of the RH problem for Y

In order to normalize the RH problem for Y at infinity we proceed as Kapaev in [24]. Introduce a function $g(\zeta; x, T) = g(\zeta)$ of the following form,

$$g(\zeta) = c_1(\zeta - z_0)^{7/2} + c_2(\zeta - z_0)^{5/2} + c_3(\zeta - z_0)^{3/2}. \tag{3.8}$$

where z_0 and the coefficients c_1, c_2, and c_3 are to be chosen independent of ζ (but possibly depending on x and T) in such a way that

$$g(\zeta) = \hat{\theta}(\zeta) + \mathcal{O}(\zeta^{-1/2}), \qquad \text{as } \zeta \to \infty. \tag{3.9}$$

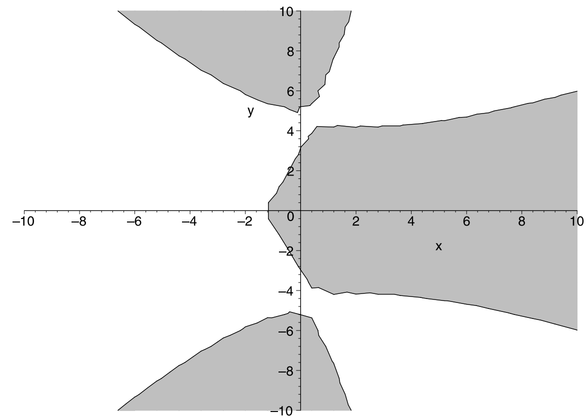

Figure 3: Contour plot of $\mathrm{Re}\, g$ for $T = 0$ and $x > 0$. The shaded areas indicate where $\mathrm{Re}\, g > 0$.

If we let $z_0 = z_0(x, T)$ be the real solution of the following third degree equation (which has one real and two complex conjugate solutions),

$$z_0^3 = -\mathrm{sgn}(x)48 + 24 z_0 |x|^{-2/3} T, \qquad \text{for } x \neq 0, \tag{3.10}$$

and if we set

$$c_1 = \frac{1}{105}, \qquad c_2 = \frac{1}{30} z_0, \qquad c_3 = \frac{1}{36} z_0^2 - \mathrm{sgn}(x)\frac{2}{3z_0}, \tag{3.11}$$

then it is straightforward to verify, using (3.7) and (3.8), that for ζ sufficiently large,

$$g(\zeta) = \hat{\theta}(\zeta) + \sum_{k=0}^{\infty} b_k \zeta^{-k-\frac{1}{2}}, \tag{3.12}$$

for some unimportant b_k's which depend only on x and T and which can be calculated explicitly. The latter equation yields that for ζ large enough,

$$e^{|x|^{7/6}(g(\zeta)-\hat{\theta}(\zeta))\sigma_3} = I + \sum_{k=1}^{\infty} d_k \sigma_3^k \zeta^{-k/2}, \tag{3.13}$$

where the coefficients d_k can also be calculated explicitly. Further, observe that by (3.13) we have $\det(I + \sum_{k=1}^{\infty} d_k \sigma_3^k \zeta^{-k/2}) = 1$, which yields

$$d_2 = \frac{1}{2} d_1^2. \tag{3.14}$$

Another crucial feature of the g-function is stated in the following proposition, which is important for the choice of the contour $\hat{\Gamma}$, and which is illustrated by Figure 3.

Proposition 3.1 *There exist constants $c > 0$, $\varepsilon_0 > 0$ and $x_0 > 0$ such that for $x \geq x_0$,*

$$\mathrm{Re}\, g(\zeta) > c|\zeta - z_0|^{7/2} > 0, \qquad \textit{as } \mathrm{Arg}(\zeta - z_0) = 0, \tag{3.15}$$

$$\mathrm{Re}\, g(\zeta) < -c|\zeta - z_0|^{7/2} < 0, \qquad \textit{as } \tfrac{6\pi}{7} - \varepsilon_0 \leq |\mathrm{Arg}(\zeta - z_0)| \leq \tfrac{6\pi}{7} + \varepsilon_0. \tag{3.16}$$

Proof. With $\zeta = z_0 + re^{i\phi}$ we have

$$r^{-7/2}\mathrm{Re}\, g(\zeta) = c_1 \cos(7\phi/2) + c_2 \cos(5\phi/2) r^{-1} + c_3 \cos(3\phi/2) r^{-2}, \tag{3.17}$$

where by using (3.10) and (3.11)

$$c_1 = \frac{1}{105}, \qquad c_2 = -\frac{1}{15}\mathrm{sgn}(x)6^{1/3} + \mathcal{O}(x^{-2/3}), \qquad c_3 = 6^{-1/3} + \mathcal{O}(x^{-2/3}), \tag{3.18}$$

as $|x| \to \infty$. Observe that the right hand side of (3.17) is a second degree equation in r^{-1}, so that it is straightforward to check that,

$$\min(r^{-7/6}\mathrm{Re}\, g(\zeta)) = c_1 - \frac{c_2^2}{4c_3} = \frac{1}{350} + \mathcal{O}(x^{-2/3}), \qquad \text{as } \phi = 0, \tag{3.19}$$

which yields already (3.15), and that

$$\max(r^{-7/6}\mathrm{Re}\, g(\zeta)) = c_1 \cos(7\phi/2) - \frac{c_2^2}{4c_3}\frac{\cos^2(5\phi/2)}{\cos(3\phi/2)}, \qquad \text{as } \pi/3 < |\phi| < \pi. \tag{3.20}$$

Further, since

$$\cos(7\phi/2) = -1, \qquad -\frac{\cos^2(5\phi/2)}{\cos(3\phi/2)} < 1.31, \qquad \text{as } \phi = \tfrac{6\pi}{7},$$

there exists, by continuity in ϕ, a constant $\varepsilon_0 > 0$ sufficiently small such that the following estimates hold,

$$\cos(7\phi/2) < -0.99, \qquad -\frac{\cos^2(5\phi/2)}{\cos(3\phi/2)} < 1.31, \qquad \text{as } \tfrac{6\pi}{7} - \varepsilon_0 \leq |\phi| \leq \tfrac{6\pi}{7} + \varepsilon_0.$$

This implies by (3.20) and (3.18) that

$$\max(r^{-7/6}\mathrm{Re}\, g(\zeta)) < -0.99c_1 + 1.31\frac{c_2^2}{4c_3} < -0.00069 + \mathcal{O}(x^{-2/3}), \qquad \text{as } \tfrac{6\pi}{7} - \varepsilon_0 \leq |\phi| \leq \tfrac{6\pi}{7} + \varepsilon_0, \tag{3.21}$$

which proves (3.16). □

Remark 3.2 Recall that the contour $\hat{\Gamma}$ (in particular $\hat{\Gamma}_2$ and $\hat{\Gamma}_4$) is not yet explicitly defined. For now, we choose $\hat{\Gamma}_2$ and $\hat{\Gamma}_4$ to lie in the sectors where (3.16) holds.

We are now ready to normalize the RH problem for Y at infinity. Let $S(\zeta; x, T) = S(\zeta)$ be the following 2×2 matrix valued function,

$$S(\zeta) = \begin{pmatrix} 1 & 0 \\ d_1|x|^{1/6} & 1 \end{pmatrix} Y(\zeta) e^{|x|^{7/6} g(\zeta)\sigma_3}, \qquad \text{for } \zeta \in \mathbb{C} \setminus \hat{\Gamma}, \tag{3.22}$$

where Y, g and d_1 are given by (3.2), (3.8) and (3.13), respectively. It is then straightforward to check, using (3.3)–(3.5), using the fact that $g_+(\zeta) + g_-(\zeta) = 0$ for $\zeta \in (-\infty, z_0)$, and using (3.6), (3.13), (1.16) and (3.14), that S satisfies the following conditions.

RH problem for S:

(a) S is analytic in $\mathbb{C} \setminus \hat{\Gamma}$.

(b) $S_+(\zeta) = S_-(\zeta)v_S(\zeta)$ for $\zeta \in \hat{\Gamma}$, where v_S is given by,

$$v_S(\zeta) = \begin{cases} \begin{pmatrix} 1 & e^{-2|x|^{7/6}g(\zeta)} \\ 0 & 1 \end{pmatrix}, & \text{for } \zeta \in \hat{\Gamma}_1, \\ \begin{pmatrix} 1 & 0 \\ e^{2|x|^{7/6}g(\zeta)} & 1 \end{pmatrix}, & \text{for } \zeta \in \hat{\Gamma}_2 \cup \hat{\Gamma}_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } \zeta \in \hat{\Gamma}_3. \end{cases} \tag{3.23}$$

(c) S has the following behavior as $\zeta \to \infty$,

$$S(\zeta) = \Bigg[I + \begin{pmatrix} 1 & 0 \\ d_1|x|^{1/6} & 1 \end{pmatrix} A_1 \begin{pmatrix} 1 & 0 \\ -d_1|x|^{1/6} & 1 \end{pmatrix} |x|^{-1/3}\zeta^{-1} \\ + \begin{pmatrix} \frac{1}{2}d_1^2 & -d_1|x|^{-1/6} \\ * & * \end{pmatrix} \zeta^{-1} + \mathcal{O}(\zeta^{-2})\Bigg] \zeta^{-\frac{\sigma_3}{4}}|x|^{-\frac{\sigma_3}{12}}N, \tag{3.24}$$

where the *'s denote unimportant functions depending only on x and T.

Remark 3.3 Note that by Proposition 3.1 the jump matrix v_S on $\hat{\Gamma}_1, \hat{\Gamma}_2$ and $\hat{\Gamma}_4$ converges exponentially fast (as $x \to \pm\infty$) to the identity matrix.

3.3 Parametrix for the outside region

From Remark 3.3 we expect that the leading order asymptotics of Φ will be determined by a matrix valued function $P^{(\infty)}$ (which will be referred to as the parametrix for the outside region) with jumps only on $(-\infty, z_0)$ satisfying there the same jump relation as S does. Let

$$P^{(\infty)}(\zeta) = |x|^{-\frac{\sigma_3}{12}}(\zeta - z_0)^{-\frac{\sigma_3}{4}}N, \qquad \text{for } \zeta \in \mathbb{C} \setminus (-\infty, z_0]. \tag{3.25}$$

Then, using (1.16) and the fact that $(\zeta - z_0)_-^{\frac{\sigma_3}{4}}(\zeta - z_0)_+^{-\frac{\sigma_3}{4}} = e^{-\frac{\pi i}{2}\sigma_3}$ for $\zeta \in (-\infty, z_0)$, we obtain that

$$\begin{aligned} P_+^{(\infty)}(\zeta) &= P_-^{(\infty)}(\zeta)N^{-1}(\zeta - z_0)_-^{\frac{\sigma_3}{4}}(\zeta - z_0)_+^{-\frac{\sigma_3}{4}}N \\ &= P_-^{(\infty)}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in (-\infty, z_0). \end{aligned} \tag{3.26}$$

Before we can do the final transformation $S \mapsto R$ we need to do a local analysis near z_0 since the jump matrices for S and $P^{(\infty)}$ are not uniformly close to each other in the neighborhood of z_0.

3.4 Parametrix near z_0

In this subsection, we construct the parametrix near z_0. We surround the fixed point $\hat{z}_0$, see (1.2), by a disk $U_\delta = \{z \in \mathbb{C} : |z - \hat{z}_0| < \delta\}$ with radius $\delta > 0$ (sufficiently small and which will be determined in Proposition 3.4 below as part of the problem) and we seek a 2×2 matrix valued function $P(\zeta; x, T) = P(\zeta)$ satisfying the following conditions.

RH problem for P:

(a) P is analytic in $U_\delta \setminus \hat{\Gamma}$.

(b) $P_+(\zeta) = P_-(\zeta) v_S(\zeta)$ for $\zeta \in \hat{\Gamma} \cap U_\delta$, where v_S is the jump matrix for S given by (3.23).

(c) $P(\zeta) P^{(\infty)}(\zeta)^{-1} = I + \mathcal{O}(x^{-1}), \qquad$ as $x \to \pm\infty$, uniformly for $\zeta \in \partial U_\delta$.

We start with constructing a matrix valued function satisfying conditions (a) and (b) of the RH problem. This is based upon the auxiliary RH problem for M with jumps on the contour Γ^σ, see Section 2.2. The idea is that, by (2.7)–(2.9), the matrix valued function $M(|x|^{7/9} f(z))$ will satisfy conditions (a) and (b) of the RH problem for P if we have appropriate biholomorphic maps f on U_δ which satisfy the following proposition.

Proposition 3.4 *There exists $x_1 \geq x_0 > 0$ and $\delta > 0$ such that for all $|x| \geq x_1$ there are biholomorphic maps $f = f(\cdot\,; x, T)$ on U_δ satisfying the following conditions.*

1. *There exists a constant c_0 such that for all $\zeta \in U_\delta$ and $|x| \geq x_1$ the derivative of f can be estimated by: $c_0 < |f'(\zeta)| < 1/c_0$ and $|\arg f'(\zeta)| < \varepsilon_0$ with ε_0 defined in Proposition* 3.1.
2. $f(U_\delta \cap \mathbb{R}) = f(U_\delta) \cap \mathbb{R}$ and $f(U_\delta \cap \mathbb{C}_\pm) = f(U_\delta) \cap \mathbb{C}_\pm$.
3. $\frac{2}{3} f(\zeta)^{3/2} = g(\zeta)$ for $\zeta \in U_\delta \setminus (-\infty, z_0]$.

Proof. One can verify, using (3.18), that there exists $x_1 \geq x_0 > 0$ sufficiently large and $\delta > 0$ sufficiently small, such that for all $|x| \geq x_1$ the function $f(\zeta; x, T) = f(\zeta)$ defined by

$$\begin{aligned} f(\zeta) &= \left(\frac{3}{2} c_3 + \frac{3}{2} c_1 (\zeta - z_0)^2 + \frac{3}{2} c_2 (\zeta - z_0) \right)^{2/3} (\zeta - z_0) \\ &= \left(\frac{3}{2} \frac{g(\zeta)}{(\zeta - z_0)^{3/2}} \right)^{2/3} (\zeta - z_0), \end{aligned} \tag{3.27}$$

is analytic for $\zeta \in U_\delta$, and that f is uniformly (in x and ζ) bounded in U_δ. By Cauchy's theorem for derivatives we then also have that f'' is uniformly (in x and ζ) bounded in U_δ for a smaller δ. Then, there exists a constant $C > 0$ such that

$$|f'(\zeta) - f'(z_0)| = \left| \int_{z_0}^{\zeta} f''(s) ds \right| \leq C |\zeta - z_0|, \qquad \text{for all } |x| \geq x_1 \text{ and } \zeta \in U_\delta.$$

Since, by (3.18), $f'(z_0) = (\frac{3}{2} c_3)^{2/3} \geq const > 0$ for $|x|$ large enough, this yields that for all $|x| \geq x_1$ (for a possible larger x_1) the functions f are injective and hence biholomorphic in U_δ (for a possible smaller δ) and that they satisy part 1 of the proposition.

The second part follows from the first part (for a possible smaller δ). The last part follows from the second part and from (3.27). $\square$

Now, let $|x| \geq x_1$ and $\sigma \in (\frac{\pi}{3}, \pi)$ (we will specify our choice of σ below), and recall that the contour $\hat{\Gamma}$ is not yet explicitly defined. We suppose that $\hat{\Gamma}$ is defined in U_δ as the pre-image of $\Gamma^\sigma \cap f(U_\delta)$ under the map f (so $\hat{\Gamma}$ depends on the parameters x and σ), where $\Gamma^\sigma = \cup_{j=1}^4 \Gamma_j^\sigma$ is the jump contour for M, as defined by (2.6). Then, we immediately have, by (2.7)-(2.9) and part 3 of Proposition 3.4, that $M(|x|^{7/9} f(\zeta))$ satisfies conditions (a) and (b) of the RH problem for P. Moreover, for any invertible analytic matrix valued function E in U_δ, one has that

$$P(\zeta) = E(\zeta) M(|x|^{7/9} f(\zeta)), \qquad \text{for } \zeta \in U_\delta \setminus \hat{\Gamma}, \tag{3.28}$$

satisfies also conditions (a) and (b) of the RH problem for P. We need E to be such that the matching condition (c) is satisfied as well. Let

$$E(\zeta) = |x|^{-\frac{\sigma_3}{12}}(\zeta - z_0)^{-\frac{\sigma_3}{4}}(|x|^{7/9}f(\zeta))^{\frac{\sigma_3}{4}}, \tag{3.29}$$

which of course is an invertible analytic matrix valued function in U_δ. Then, using (2.10), (2.11) and (3.25) we have,

$$P(\zeta)P^{(\infty)}(\zeta)^{-1} = I + \Delta_1|x|^{-1} + \Delta_2|x|^{-4/3} + \mathcal{O}\left(|x|^{-7/3}\right), \tag{3.30}$$

as $x \to \pm\infty$ uniformly for $\zeta \in \partial U_\delta$ and σ in compact subsets of $(\frac{\pi}{3}, \pi)$, where Δ_1 and Δ_2 are given by

$$\Delta_1 = \frac{1}{f(\zeta)}\left(\frac{\zeta - z_0}{f(\zeta)}\right)^{1/2}\begin{pmatrix} 0 & 0 \\ t_1 & 0\end{pmatrix}, \qquad \Delta_2 = \frac{1}{f(\zeta)^2}\left(\frac{\zeta - z_0}{f(\zeta)}\right)^{-1/2}\begin{pmatrix} 0 & \hat{t}_1 \\ 0 & 0\end{pmatrix}, \tag{3.31}$$

and where t_1 and $\hat{t}_1$ are unimportant constants given by (2.12). We then have shown that P defined by (3.28) satisfies the conditions of the RH problem for P. This ends the construction of the parametrix near z_0.

3.5 Final transformation

We will now perform the final transformation. Recall that the contour $\hat{\Gamma}$ is still not yet explicitly defined. We will now define it in terms of the (sufficiently large) parameter x.

Consider the fixed point $\hat{z}_0 + \delta e^{\frac{6\pi i}{7}}$ (which depends only on $\operatorname{sgn}(x)$) on ∂U_δ. Since $z_0 \to \hat{z}_0$ as $x \to \pm\infty$, see Remark 1.2, there exists $x_2 \geq x_1$ sufficiently large such that for all $|x| \geq x_2$,

$$\frac{6\pi}{7} - \varepsilon_0 < \arg(\hat{z}_0 + \delta e^{\frac{6\pi i}{7}} - z_0) < \frac{6\pi}{7} + \varepsilon_0,$$

where ε_0 is defined in Proposition 3.1. From Proposition 3.4 we then know that for $|x| \geq x_2$ there exists $\sigma = \sigma(x) \in (\frac{6\pi}{7} - 2\varepsilon_0, \frac{6\pi}{7} + 2\varepsilon_0)$ such that $f^{-1}(\Gamma_2^\sigma) \cap \partial U_\delta = \{\hat{z}_0 + \delta e^{\frac{6\pi i}{7}}\}$. By the symmetry $\overline{f(\zeta)} = f(\overline{\zeta})$ we then also have $f^{-1}(\Gamma_4^\sigma) \cap \partial U_\delta = \{\hat{z}_0 + \delta e^{-\frac{6\pi i}{7}}\}$. We now define $\hat{\Gamma}$ in U_δ (for $|x| \geq x_2$) as the inverse f-image of the contour Γ^σ. Outside U_δ, we take $\hat{\Gamma}_1 \cup \hat{\Gamma}_3 = \mathbb{R}$, $\hat{\Gamma}_2 = \{\hat{z}_0 + te^{6\pi i/7} : t \geq \delta\}$, and $\hat{\Gamma}_4 = \{\hat{z}_0 + te^{-6\pi i/7} : t \geq \delta\}$. Note that by Proposition 3.1,

$$\operatorname{Re} g(\zeta) > c|\zeta - z_0|^{7/2} \qquad \text{for } \zeta \in \hat{\Gamma}_1 \setminus U_\delta, \tag{3.32}$$

$$\operatorname{Re} g(\zeta) < -c|\zeta - z_0|^{7/2} \qquad \text{for } \zeta \in (\hat{\Gamma}_2 \cup \hat{\Gamma}_4) \setminus U_\delta. \tag{3.33}$$

Further define a contour Γ_R as $\Gamma_R = \hat{\Gamma} \cup \partial U_\delta$. This leads to Figure 4. Note that $\Gamma_R \cap U_\delta$ depends on x. However, the part of Γ_R outside U_δ is independent of x.

Now, we are ready to do the final transformation $S \mapsto R$. Define a 2×2 matrix valued function $R(\zeta; x, T) = R(\zeta)$ for $\zeta \in \mathbb{C} \setminus \Gamma_R$ as

$$R(\zeta) = \begin{cases} S(\zeta)P(\zeta)^{-1}, & \text{for } \zeta \in U_\delta \setminus \Gamma_R, \\ S(\zeta)P^{(\infty)}(\zeta)^{-1}, & \text{for } \zeta \text{ elsewhere}, \end{cases} \tag{3.34}$$

where P is the parametrix near z_0 given by (3.28), $P^{(\infty)}$ is the parametrix for the outside region given by (3.25), and S is the solution of the RH problem for S.

By definition, R has jumps on the contour Γ_R. However, S and P have the same jumps on $\Gamma_R \cap U_\delta$. Further, S and $P^{(\infty)}$ satisfy the same jump relation on $(-\infty, \hat{z}_0 - \delta)$. This yields that R has only jumps on the reduced system of contours $\hat{\Gamma}_R$ (which is independent of x), shown in Figure 5.

Using (3.34), (3.24) and (3.25) one can now show that R is a solution of the following RH problem on the contour $\hat{\Gamma}_R$.

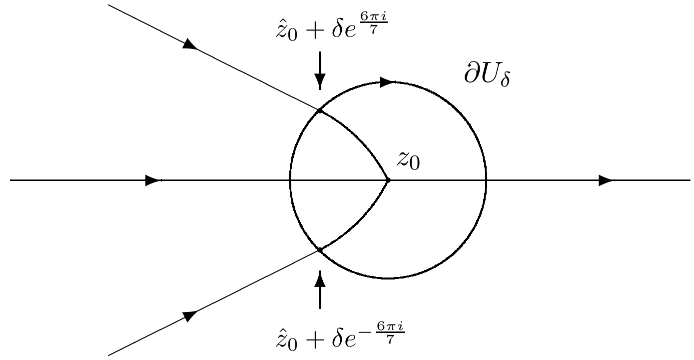

Figure 4: The contour $\Gamma_R = \hat{\Gamma}_R \cup \partial U_\delta$. The part of Γ_R inside U_δ depends on x. The rest of Γ_R is independent of x.

RH problem for R:

(a) R is analytic in $\mathbb{C} \setminus \hat{\Gamma}_R$.

(b) $R_+(\zeta) = R_-(\zeta) v_R(\zeta)$ for $\zeta \in \hat{\Gamma}_R$, with v_R given by

$$v_R(\zeta) = P^{(\infty)}(\zeta) v_S(\zeta) P^{(\infty)}(\zeta)^{-1}, \qquad \text{for } \zeta \in \hat{\Gamma}_R \setminus \partial U_\delta. \tag{3.35}$$

$$v_R(\zeta) = P(\zeta) P^{(\infty)}(\zeta)^{-1}, \qquad \text{for } \zeta \in \partial U_\delta. \tag{3.36}$$

(c) $R(\zeta) = I + \mathcal{O}(\zeta^{-1})$ as $\zeta \to \infty$.

Remark 3.5 Observe that by (3.30), (3.32) and (3.33) we have as $x \to \pm\infty$,

$$v_R(\zeta) = \begin{cases} I + \Delta_1 |x|^{-1} + \Delta_2 |x|^{-4/3} + \mathcal{O}(|x|^{-7/3}), & \text{uniformly for } \zeta \in \partial U_\delta, \\ I + \mathcal{O}(e^{-c|x|^{7/6}|\zeta - z_0|^{7/2}}) & \text{uniformly for } \zeta \in \hat{\Gamma}_R \setminus \partial U_\delta, \end{cases} \tag{3.37}$$

for some constant $\gamma > 0$, and where Δ_1 and Δ_2 are given by (3.31). As in [8, 9, 10], this yields that R itself is uniformly close to the identity matrix,

$$R(\zeta) = I + \mathcal{O}(x^{-1}), \qquad \text{as } x \to \pm\infty, \text{ uniformly for } \zeta \in \mathbb{C} \setminus \hat{\Gamma}_R.$$

Remark 3.6 Since $R(\zeta) = S(\zeta) P^{(\infty)}(\zeta)^{-1}$ for ζ large one can use (3.24), (3.25), and the fact that $(\zeta - z_0)^{\frac{\sigma_3}{4}} = \zeta^{\frac{\sigma_3}{4}} \left[I - \frac{1}{4} z_0 \sigma_3 \zeta^{-1} + \mathcal{O}(\zeta^{-2}) \right]$ as $\zeta \to \infty$, to strengthen condition (c) of the RH problem for R to

$$R(\zeta) = I + \frac{R_1}{\zeta} + \mathcal{O}(\zeta^{-2}), \qquad \text{as } \zeta \to \infty, \tag{3.38}$$

where R_1 is a 2×2 matrix valued function depending on x and T with $(1,1)$ and $(1,2)$ entries given by,

$$R_{1,11} = -\frac{z_0}{4} + \frac{1}{2} d_1^2 + |x|^{-1/3} A_{1,11} - d_1 |x|^{-1/6} A_{1,12}, \tag{3.39}$$

$$R_{1,12} = -d_1 |x|^{-1/6} + |x|^{-1/3} A_{1,12}. \tag{3.40}$$

From (3.37) it follows, as in [9], that

$$R_1 = -\operatorname{Res}(\Delta_1, z_0) |x|^{-1} - \operatorname{Res}(\Delta_2, z_0) |x|^{-4/3} + \mathcal{O}(|x|^{-7/3}), \qquad \text{as } x \to \pm\infty,$$

so that by (3.31),

$$R_{1,11} = \mathcal{O}(|x|^{-7/3}), \qquad R_{1,12} = \mathcal{O}(|x|^{-4/3}), \qquad \text{as } x \to \pm\infty. \tag{3.41}$$

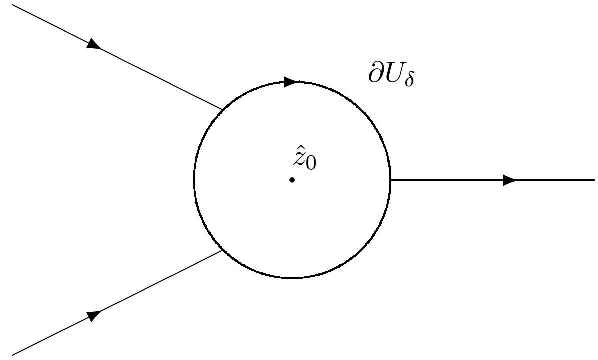

Figure 5: The reduced system of contours $\hat{\Gamma}_R$ independent of x.

3.6 Proof of Theorem 1.1 (ii)

We now have all the necessary ingredients to prove the second part of the main theorem.

Proof of Theorem 1.1 (ii). Recall that $y = 2A_{1,11} - A_{1,12}^2$. Using (3.39) and (3.40) one can then write y in terms of the $(1,1)$ and $(1,2)$ entries of R_1,

$$2A_{1,11} = \frac{1}{2}z_0|x|^{1/3} + 2|x|^{1/3}R_{1,11} - d_1^2|x|^{1/3} + 2d_1|x|^{1/6}A_{1,12},$$

$$A_{1,12}^2 = |x|^{2/3}R_{1,12}^2 - d_1^2|x|^{1/3} + 2d_1|x|^{1/6}A_{1,12},$$

so that

$$y = \frac{1}{2}z_0|x|^{1/3} + 2|x|^{1/3}R_{1,11} - |x|^{2/3}R_{1,12}^2. \tag{3.42}$$

Inserting (3.41) into the latter equation we obtain precisely (1.5). This finishes the proof of Theorem 1.1. □

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