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THE EXISTENCE OF ALMOST TRANSLATION INVARIANT ULTRAFILTERS ON ANY SEMIGROUP

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ABSTRACT. We present a short ultrafilter proof of a result which has applications in combinatorial number theory and which has previously relied on the theory of compact semigroups.

The existence of non-fixed ultrafilters p on N such that whenever $A \in p$ one has $\{x \in N: A - x \in p\} \in p$, called almost translation invariant by Galvin, has always been of interest because it is closely related to the validity of Graham–Rothschild conjecture. In 1974, the conjecture was proved by Hindman [5] in ZFC. Combined with the proof in [4] which used the continuum hypothesis this yielded a CH proof of the existence of almost translation invariant ultrafilters. In 1975, Glazer proved their existence without using the continuum hypothesis [2]. His approach was to define an addition on ultrafilters on a semigroup S, so that the almost translation invariant ultrafilters become idempotents and then use Ellis' theorem (Lemma 2.9 of [3]) about the existence of idempotents in compact right topological semigroups. In this paper, we prove directly the existence of such ultrafilters on any cancellative semigroup using ultrafilter approach only.

1. **Definition.** Let $(S, +)$ be a semigroup, not necessarily commutative, and let $\mathscr{A}$, $\mathscr{B}$ be filters on S. For any $A \subseteq S$ we define

$$\Omega_{\mathscr{A}}(A) = \{x \in S: A - x \in \mathscr{A}\},$$
$$\mathscr{A} + \mathscr{B} = \{A \subseteq S: \Omega_{\mathscr{B}}(A) \in \mathscr{A}\},$$

where $A - x = \{y \in S: y + x \in A\}$.

The symbol $+$ is just a notion for the above described operation on filters. The fact that $\mathscr{A} + \mathscr{B}$ is a filter follows from steps (i) and (iii) of the following lemma. (In [1], Lemma 5.15 shows that $\mathscr{A} + \mathscr{B}$ is a filter on S and that $+$ is an associative operation on the set of filters. In the following discussion the associativity of $+$ is not needed.) All parts of the lemma are easy to prove.

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2. **Lemma.** *Let $\mathscr{A}$, $\mathscr{B}$ be arbitrary filters on S.*

(i) $\forall A, B \subseteq S\ \Omega_{\mathscr{A}}(A \cap B) = \Omega_{\mathscr{A}} \cap \Omega_{\mathscr{A}}(B)$.

(ii) $\forall A \subseteq S\ \Omega_{\mathscr{A}}(A^c) \subseteq (\Omega_{\mathscr{A}}(A))^c$; *equality holds if $\mathscr{A}$ is an ultrafilter.*

(iii) $\forall A, B \subseteq S, A \subseteq B \Rightarrow \Omega_{\mathscr{A}}(A) \subseteq \Omega_{\mathscr{A}}(B)$.

(iv) $\mathscr{A} \subseteq \mathscr{B} \Rightarrow \forall A \subseteq S\ \Omega_{\mathscr{A}}(A) \subseteq \Omega_{\mathscr{B}}(A)$.

(v) $\forall y \in S, \forall A \subseteq S\ \Omega_{\mathscr{A}}(A - y) = \Omega_{\mathscr{A}}(A) - y$.

(vi) $\forall A \subseteq S\ \Omega_{\mathscr{A}+\mathscr{B}}(A) = \Omega_{\mathscr{A}}(\Omega_{\mathscr{B}}(A))$.

3. **Theorem.** *Let $(S, +)$ be any semigroup, not necessarily commutative. Then there exists an ultrafilter p on S such that $\forall A \in p\ \Omega_p(A) \in p$. If S is cancellative then there also exists a nonfixed ultrafilter p having the same property.*

Proof. Zorn's Lemma gives the existence of a filter $\mathscr{F}_{\max}$ maximal with respect to the property $\forall A \in \mathscr{F}_{\max}\ \Omega_{\mathscr{F}_{\max}}(A) \in \mathscr{F}_{\max}$. (If S is cancellative, then the filter $\mathscr{F} = \{A \subseteq S\colon A^c \textit{ is finite}\}$ has this property, so we may choose $\mathscr{F}_{\max}$ to refine $\mathscr{F}$; then any ultrafilter refining $\mathscr{F}_{\max}$ is nonfixed.) Our aim now is to show that $\mathscr{F}_{\max}$ is an ultrafilter. Let p be an ultrafilter containing $\mathscr{F}_{\max}$. Consider $\mathscr{F} = \{A \subseteq S\colon \Omega_p(A) \in \mathscr{F}_{\max}\}$. Using Lemma 2 we can show that $\mathscr{F}$ is a filter containing $\mathscr{F}_{\max}$ and $\forall A \subseteq S$

$$\begin{aligned}\Omega_{\mathscr{F}}(A) &= \{y \in S\colon \Omega_p(A - y) \in \mathscr{F}_{\max}\}\\ &= \{y \in S\colon \Omega_p(A) - y \in \mathscr{F}_{\max}\} \text{ (Lemma 2(v))}\\ &= \Omega_{\mathscr{F}_{\max}}(\Omega_p(A)),\end{aligned}$$

so that $\forall A \in \mathscr{F}\ \Omega_{\mathscr{F}}(A) \in \mathscr{F}$ which implies that $\mathscr{F}_{\max} = \mathscr{F}$ from the maximality of $\mathscr{F}_{\max}$. Hence,

$$\forall A \subseteq S\ A \in \mathscr{F}_{\max} \textit{ iff } \Omega_p(A) \in \mathscr{F}_{\max}. \tag{1}$$

From Lemma 2 (ii) and (1) we can easily deduce that $\forall A \in p$, $\forall M \in \mathscr{F}_{\max}$, $M \cap \Omega_p(A) \neq \varnothing$ so that we can consider the filter $\mathscr{F}'$ generated by $\{\Omega_p(A) \cap M,\ A \in p,\ M \in \mathscr{F}_{\max}\}$. Clearly $\mathscr{F}'$ contains $\mathscr{F}_{\max}$. Now, $\forall A \in p$ we have $\Omega_p(A) \in \mathscr{F}'$, that is $A \in \mathscr{F}' + p$ and since p is an ultrafilter it follows that $\mathscr{F}' + p = p$. Next, using Lemma 2, we show that, for any $A \in p$ and any $M \in \mathscr{F}_{\max}$, $\Omega_{\mathscr{F}'}(\Omega_p(A) \cap M) \in \mathscr{F}'$: indeed,

$$\Omega_{\mathscr{F}'}(\Omega_p(A) \cap M) = \Omega_{\mathscr{F}'+p}(A) \cap \Omega_{\mathscr{F}'}(M) = \Omega_p(A) \cap \Omega_{\mathscr{F}'}(M)$$

and

$$\Omega_{\mathscr{F}'}(M) \supseteq \Omega_{\mathscr{F}_{\max}}(M).$$

Thus for $B \in \mathscr{F}'$, $\Omega_{\mathscr{F}'}(B) \in \mathscr{F}'$ and the maximality of $\mathscr{F}_{\max}$ gives $\mathscr{F}' = \mathscr{F}_{\max}$. So, for all $A \in p$, $\Omega_p(A) \in \mathscr{F}_{\max}$, which implies that $\mathscr{F}_{\max} = p$ by (1). Thus $\mathscr{F}_{\max}$ is an ultrafilter. □

In conclusion, we should point out that the structure of the above proof is basically the same as that of Ellis' algebraic proof though the details differ.

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