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The Existence of Dominating Local Martingale Measures

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Abstract We prove that, for locally bounded processes, absence of arbitrage opportunities of the first kind is equivalent to the existence of a dominating local martingale measure. This is related to and motivated by results from the theory of filtration enlargements.

Keywords dominating local martingale measure; arbitrage of the first kind; fundamental theorem of asset pricing; supermartingale densities; Föllmer measure; enlargement of filtration; Jacod's criterion

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1 Introduction

It may be argued that the foundation of financial mathematics consists in giving a mathematical characterization of market models satisfying certain financial axioms. This leads to so-called *fundamental theorems of asset pricing.*

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Harrison and Pliska [HP81] were the first to observe that, on finite probability spaces, the absence of arbitrage opportunities (condition *no arbitrage, (NA)*) is equivalent to the existence of an equivalent martingale measure. A definite version was shown by Delbaen and Schachermayer [DS94]. Their result, commonly referred to as *the* Fundamental Theorem of Asset Pricing, states that for locally bounded semimartingale models there exists an equivalent probability measure under which the price process is a local martingale, if and only if the market satisfies the condition *no free lunch with vanishing risk (NFLVR)*. Delbaen and Schachermayer also observed that (NFLVR) is satisfied if and only if there are no arbitrage opportunities ((NA) holds), and if further it is not possible to make an unbounded profit with bounded risk (we say there are *no arbitrage opportunities of the first kind*, condition *(NA1)* holds). Since in finite discrete time, (NA) is equivalent to the existence of an equivalent martingale measure, it was then a natural question how to characterize continuous time market models satisfying only (NA) and not necessarily (NA1). For continuous price processes, this was achieved by Delbaen and Schachermayer [DS95b], who show that (NA) implies the existence of an *absolutely continuous* local martingale measure.

Here we complement this program by proving that for locally bounded processes, (NA1) is equivalent to the existence of a *dominating* local martingale measure. Apart from its mathematical interest, we believe that this result may also be relevant in financial applications. It is known that dominating local martingale measures are the appropriate pricing operators when working with continuous price paths satisfying (NA1) [Ruf13]. Moreover, in the duality approach to utility maximization, the dual variables are often given by suitable measures. There is a functional analytic approach, working with finitely additive measures [CSW01,KŽ03], and a probabilistic approach, working with countably additive measures on an extended probability space [FG06]. However, to the best of our knowledge there is no intrinsic characterization of these measures, they are described in terms of associated processes which may be interpreted as generalized Radon-Nikodym derivatives. While we do not believe that in general the space of dual variables is given by the dominating local martingale measures (see Remark 4.17), their closure seems to be a natural candidate.

Let us give a more precise description of the notions of arbitrage considered in this work, and of the obtained results.

In the first four sections of the paper, we fix the following probability space: Let E be a Polish space, and let $\Delta \notin E$ be a cemetery state. For all $\omega \in (E \cup \{\Delta\})^{[0,\infty)}$ define

$$\zeta(\omega) = \inf\{t \geq 0 : \omega(t) = \Delta\}.$$

Let $\Omega \subset (E \cup \{\Delta\})^{[0,\infty)}$ be the space of paths $\omega : [0,\infty) \to E \cup \{\Delta\}$, for which ω is càdlàg on $[0, \zeta(\omega))$, and for which $\omega(t) = \Delta$ for all $t \geq \zeta(\omega)$. For all $t \geq 0$ define $\mathcal{F}^0_t$ as the σ–algebra generated by the coordinate projections up to time t and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}^0_s$. Moreover, set $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}^0_t = \bigvee_{t\geq 0} \mathcal{F}_t$. We also assume that P is a given probability measure on $(\Omega, \mathcal{F})$ such that $P(\zeta < \infty) = 0$.

Note that $(\mathcal{F}_t)$ is not complete. We hope to convince the reader with the arguments in Appendix A that this does not pose any problem. While many of our auxiliary results hold on general probability spaces, the main results Theorem 1.5 and Theorem 1.6 need some topological assumptions on $(\Omega, \mathcal{F})$: on a general probability space, say with a complete filtration, we cannot hope to construct a probability measure which is not absolutely continuous. This is also why in model free financial mathematics, where non-dominated families of probability measures are considered, one usually works on suitable path spaces. We choose $(\Omega, \mathcal{F})$ as above because it allows us to apply the construction of Föllmer's measure given in [PR14].

Throughout, $S = (S_t)_{t\geq 0}$ denotes a d–dimensional adapted process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ which is almost surely right-continuous. We think of S as the (discounted) price process of d financial assets. Of course, the case of a finite time horizon $T > 0$ can be embedded by setting $S_{T+t} = S_T$ for all $t \geq 0$. *Semimartingales* are defined as usually, except that they are only almost surely càdlàg. Since $(\mathcal{F}_t)$ is not complete, a semimartingale does not need to be càdlàg for all $\omega \in \Omega$.

A *strategy* is a predictable process $H = (H_t)_{t\geq 0}$ with values in $\mathbb{R}^d$. If S is a semimartingale and $\lambda > 0$, then a strategy H is called λ*–admissible* (for S) if the stochastic integral $H \cdot S := \int_0^\cdot H_s \mathrm{d}S_s$ exists and satisfies $P((H \cdot S)_t \geq -\lambda) = 1$ for all $t \geq 0$. Here we write $xy = \sum_{k=1}^d x_k y_k$ for the usual inner product on $\mathbb{R}^d$. We define $\mathcal{H}_\lambda$ as the set of all λ–admissible strategies. For details on vector stochastic integration we refer to Jacod and Shiryaev [JS03].

If S is not a semimartingale, we can still integrate simple strategies against S. A *simple strategy* is a process of the form $H_t = \sum_{j=0}^{m-1} F_k 1_{(\tau_k, \tau_{k+1}]}(t)$ for $m \in \mathbb{N}$ and stopping times $0 \leq \tau_0 < \tau_1 < \cdots < \tau_m < \infty$, where for every $0 \leq k < m$ the random variable F_k is bounded and $\mathcal{F}_{\tau_k}$–measurable and takes its values in $\mathbb{R}^d$. The integral $H \cdot S$ is then defined as

$$(H \cdot S)_t = \sum_{k=0}^{m-1} F_k (S_{\tau_{k+1}\wedge t} - S_{\tau_k \wedge t}),$$

and λ–admissible strategies are defined analogously to the semimartingale case. We denote the set of simple λ–admissible strategies by $\mathcal{H}_{\lambda,s}$.

The set $\mathcal{W}_1$ consists of all wealth processes obtained by using 1–admissible strategies with initial wealth 1, and such that the terminal wealth is well defined, that is

$$\mathcal{W}_1 = \{1 + H \cdot S : H \in \mathcal{H}_1 \text{ and } (H \cdot S)_t \text{ almost surely converges as } t \to \infty\}. \tag{1.1}$$

Similarly, $\mathcal{W}_{1,s}$ is defined as $\mathcal{W}_{1,s} = \{1 + H \cdot S : H \in \mathcal{H}_{1,s}\}$. Note that the convergence condition in (1.1) is trivially satisfied for simple strategies. We will also need the following sets of terminal wealths:

$$\mathcal{K}_1 = \{X_\infty : X \in \mathcal{W}_1\} \qquad \text{and} \qquad \mathcal{K}_{1,s} = \{X_\infty : X \in \mathcal{W}_{1,s}\}. \tag{1.2}$$

We write $L^0 = L^0(\Omega, \mathcal{F}, P)$ for the space of real-valued random variables on $(\Omega, \mathcal{F})$, where we identify random variables that are P–almost surely equal. Recall that a family of random variables $\mathcal{X}$ is *bounded in probability*, or *bounded in* L^0, if $\lim_{m\to\infty} \sup_{X\in\mathcal{X}} P[|X| \geq m] = 0$.

Definition 1.1 A semimartingale S satisfies *no arbitrage of the first kind (NA1)* if $\mathcal{K}_1$ is bounded in probability. It satisfies *no arbitrage (NA)* if there is no $X \in \mathcal{K}_1$ with $X \geq 1$ and $P(X > 1) > 0$. If both (NA1) and (NA) hold, we say that S satisfies *no free lunch with vanishing risk (NFLVR)*.

Similarly, an almost surely right-continuous adapted process S satisfies *no arbitrage of the first kind with simple strategies (NA1$_s$)*, *no arbitrage with simple strategies (NA$_s$)*, or *no free lunch with vanishing risk with simple strategies (NFLVR$_s$)*, if $\mathcal{K}_{1,s}$ satisfies the corresponding conditions.

Heuristically, (NA) says that it is not possible to make a profit without taking a risk. (NA1), which is also referred to as "no unbounded profit with bounded risk" (NUPBR) [KK07], says that is not possible to make an unbounded profit if the risk remains bounded.

We want to construct dominating local martingale measures for S. When constructing absolutely continuous probability measures, it suffices to work with random variables. In Section 2 below, we argue that dominating measures correspond to nonnegative supermartingales with strictly positive terminal values. We also show that a dominating local martingale measure corresponds to a supermartingale density in the following sense.

Definition 1.2 Let $\mathcal{Y}$ be a family of stochastic processes. A *supermartingale density* for $\mathcal{Y}$ is an almost surely càdlàg and nonnegative supermartingale Z with $Z_\infty = \lim_{t\to\infty} Z_t > 0$, so that YZ is a supermartingale for all $Y \in \mathcal{Y}$. If all processes in $\mathcal{Y}$ are of the form $1 + (H \cdot S)$ for suitable integrands H, and if Z is a supermartingale density for $\mathcal{Y}$, then we will sometimes call Z a supermartingale density for S.

In the literature, supermartingale densities are usually referred to as *supermartingale deflators*. We think of a supermartingale density as the "Radon-Nikodym derivative" $\mathrm{d}Q/\mathrm{d}P$ of a dominating measure $Q \gg P$. This is why we prefer the term supermartingale density.

First we sketch an alternative proof of Rokhlin's theorem:

Theorem 1.3 ([Rok10], Theorem 2, see also [KK07], Theorem 4.12) *Let S be an adapted process, almost surely right-continuous (respectively a semimartingale). Then (NA1$_s$) (respectively (NA1)) holds if and only if there exists a supermartingale density for $\mathcal{W}_{1,s}$ (respectively for $\mathcal{W}_1$).*

The following corollary has also been known for a while:

Corollary 1.4 ([Ank05], Theorem 7.4.3 or [KP11]) *Let $S = (S^1, \dots, S^d)$ be an adapted process, almost surely right-continuous. If every component S^i of S is locally bounded from below and if S satisfies (NA1$_s$), then S is a semimartingale that satisfies (NA1), and any supermartingale density for $\mathcal{W}_{1,s}$ is also a supermartingale density for $\mathcal{W}_1$.*

Given a supermartingale density Z for S, we then apply Yoeurp's [Yoe85] results on Föllmer's measure [Föl72] together with the construction of [PR14], to obtain a dominating measure $Q \gg P$ associated to Z. Let γ be a right-continuous version of the density process $\gamma_t = \mathrm{d}P/\mathrm{d}Q|_{\mathcal{F}_t}$, and let τ be the first time that γ hits zero, $\tau = \inf\{t \geq 0 : \gamma_t = 0\}$. We define

$$S_t^{\tau-} = S_t 1_{\{t<\tau\}} + S_{\tau-} 1_{\{t\geq\tau\}} = S_t 1_{\{t<\tau\}} + \lim_{s\to\tau-} S_s 1_{\{t\geq\tau\}}.$$

Note that S and $S^{\tau-}$ are P–indistinguishable. In the predictable case, our main result is then:

Theorem 1.5 *Let S be a predictable semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. If Z is a supermartingale density for $\mathcal{W}_1$, then Z determines a probability measure $Q \gg P$ such that $S^{\tau-}$ is a Q–local martingale. Conversely, if $Q \gg P$ is a dominating local martingale measure for $S^{\tau-}$, then $\mathcal{W}_1$ admits a supermartingale density.*

Theorem 1.5 is false if S is not predictable, as we will demonstrate on a simple counterexample. But we will be able to exhibit a subset of supermartingale densities that do give rise to dominating local martingale measures. Conversely, every dominating local martingale measure for $S^{\tau-}$ corresponds to a supermartingale density, even for processes that are not predictable. In this way we obtain the following theorem, the main result of this paper. In the non-predictable case we build on results of [TS14] which are formulated for processes on finite time intervals. So in the theorem we let $T_\infty = \infty$ if S is predictable, and $T_\infty \in (0, \infty)$ otherwise.

Theorem 1.6 *Let $(S_t)_{t\in[0,T_\infty]}$ be a locally bounded, adapted process on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ which is almost surely right-continuous. Then S satisfies (NA1$_s$) if and only if there exists a dominating $Q \gg P$ such that $S^{\tau-}$ is a Q–local martingale.*

This work is motivated by insights from filtrations enlargements. A filtration $(\mathcal{G}_t)$ is called *filtration enlargement* of $(\mathcal{F}_t)$ if $\mathcal{G}_t \supseteq \mathcal{F}_t$ for all $t \geq 0$. A basic question is then under which conditions all members of a given family of $(\mathcal{F}_t)$–semimartingales are $(\mathcal{G}_t)$–semimartingales. We say that *Hypothèse* (H') is satisfied if all $(\mathcal{F}_t)$–semimartingales are $(\mathcal{G}_t)$–semimartingales. Given a $(\mathcal{F}_t)$–semimartingale that satisfies (NFLVR), i.e. for which there exists an equivalent local martingale measure, one might also ask under which conditions it still satisfies (NFLVR) under $(\mathcal{G}_t)$. It is well known, and we illustrate this in an example below, that the (NFLVR) condition is usually violated after filtration enlargements.

However, (NA1) is relatively stable under filtration enlargements. If $(\mathcal{G}_t)$ is an initial enlargement of $(\mathcal{F}_t)$, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X)$ for some random variable X and for every $t \geq 0$, then Jacod's criterion [Jac85] is a celebrated condition on X and $(\mathcal{F}_t)$ under which Hypothèse (H') is satisfied. We show that in fact Jacod's criterion implies the existence of a universal supermartingale density. A strictly positive process Z is called *universal supermartingale density* if ZM is a $(\mathcal{G}_t)$–supermartingale for every nonnegative $(\mathcal{F}_t)$–supermartingale M, which is obviously much stronger than Hypothèse (H'). Moreover, we will see that under Jacod's criterion every predictable process satisfying (NA1) under $(\mathcal{F}_t)$ also satisfies (NA1) under $(\mathcal{G}_t)$. Conversely, if $(\mathcal{G}_t)$ is a general (not necessarily initial) filtration enlargement, and if there exists a universal supermartingale density for $(\mathcal{G}_t)$, then a generalized version of Jacod's criterion is satisfied.

Section 2 describes the link to filtration enlargements in more detail. In Section 2 we also argue that a dominating local martingale measure should correspond to a supermartingale density. In Section 3 we discuss Rokhlin's theorem, for which we sketch an alternative proof. In Section 4 we prove that if S is predictable, then Z is a supermartingale density for S if and only if $S^{\tau-}$ is a local martingale under the Föllmer measure of Z. We also prove our main result, Theorem 1.6. In Section 5 we discuss the relation with filtration enlargements.

Relevant literature

To the best of our knowledge, supermartingale densities were first considered by Karatzas et al. [KLSX91]; see also Kramkov and Schachermayer [KS99] and Becherer [Bec01].

The semimartingale case of Theorem 1.3 was shown by Karatzas and Kardaras [KK07]. The generalized version which we will encounter in Section 3 is due to Rokhlin [Rok10].

It was first observed by Ankirchner [Ank05] that a locally bounded process satisfying (NA1_s) must be a semimartingale; see also Kardaras and Platen [KP11]. A slightly weaker version of this result is due to Delbaen and Schachermayer [DS94]. This part of Corollary 1.4 is an immediate consequence of Theorem 1.3. We get from [KP11] that (NA1_s) implies (NA1) for locally bounded processes and that supermartingale densities for $\mathcal{W}_{1,s}$ are supermartingale densities for $\mathcal{W}_1$.

Recently there has been an increased interest in Föllmer's measure, motivated by problems from mathematical finance. Föllmer's measure appears naturally in the construction and study of *strict local martingales*, i.e. local martingales that are not martingales. These are used to model bubbles in financial markets, see Jarrow et al. [JPS10]. A pioneering work on the relation between Föllmer's measure and strict local martingales is Delbaen and Schachermayer [DS95a]. Other references are Pal and Protter [PP10], Kardaras et al. [KKN14], and Carr et al. [CFR14]. The work most related to ours is Ruf [Ruf13], where it is shown that, in a diffusion setting, (NA1) implies the existence of a dominating local martingale measure. All these works study Föllmer measures of nonnegative local martingales.

To the best of our knowledge, the current work is the first in which the Föllmer measure of a supermartingale that is not a local martingale is used as a local martingale measure. In Föllmer and Gundel [FG06], supermartingales Z are associated to "extended martingale measures" P^Z. But by definition,

P^Z is an extended martingale measure if and only if Z is a supermartingale density. This does not obviously imply that $S^{\tau-}$ or S is a local martingale under P^Z – and as we will see it is not true in general.

Another related work is Kardaras [Kar10], where it is shown that (NA1) is equivalent to the existence of a finitely additive equivalent local martingale measure. Here we construct countably additive measures and therefore we lose the equivalence and only obtain dominating measures.

Our main motivation comes from filtrations enlargements, see for example Amendinger et al. [AIS98], Ankirchner [Ank05], and Ankirchner et al. [ADI06]. In these works it is shown that if M is a continuous local martingale and if $(\mathcal{G}_t)$ is a filtration enlargement, then under suitable conditions M is of the form $M = \widetilde{M} + \int_0^\cdot \alpha_s d\langle \widetilde{M} \rangle_s$, where $\widetilde{M}$ is a $(\mathcal{G}_t)$–local martingale. It is then a natural question whether there exists an equivalent measure Q that "eliminates" the drift, i.e. under which M is a $(\mathcal{G}_t)$–local martingale. In general, the answer to this question is negative. However, Ankirchner [Ank05], Theorem 9.2.7, observed that if there exists a well-posed utility maximization problem in the large filtration, then the *information drift* α must be locally square integrable with respect to $\widetilde{M}$. Here we show that this condition is in fact sufficient and necessary and we also give the corresponding results for discontinuous processes.

2 Motivation

We start with some motivating discussions. First we show that the (NFLVR) property is not very robust under filtration enlargements. Then we recall that if Jacod's criterion is satisfied, there still is a dominating local martingale measure. Finally we argue that under Jacod's criterion, (NA1) is often satisfied in the large filtration. So (NA1) respectively (NA1_s) seem to be related to the existence of dominating local martingale measures. Assuming that such a measure exists, we show that its Kunita-Yoeurp decomposition under P is supermartingale density.

Equivalent local martingale measures and filtration enlargements

Assume that $P(A) \in \{0, 1\}$ for all $A \in \mathcal{F}_0$ and that S is a one dimensional semimartingale modelling a complete market (i.e. for every $X \in L^\infty(\mathcal{F})$ there exists a predictable process H, integrable with respect to S, so that $X = X_0 + \int_0^\infty H_s \mathrm{d}S_s$ for some constant $X_0 \in \mathbb{R}$). Let X be a random variable that is not P–almost surely constant, and define the initially enlarged filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X)$ for $t \geq 0$. This is a toy model for insider trading: at time 0, the insider has the additional knowledge of the value of X. Assume now that Q is an equivalent $(\mathcal{G}_t)$–local martingale measure for S. Since X is not constant, there exists $A \in \sigma(X)$ with $P[A] \in (0, 1)$. Consider the $(Q, (\mathcal{F}_t))$–martingale $N_t = E_Q[1_A|\mathcal{F}_t]$, for $t \geq 0$. By completeness of the market, there exists a $(\mathcal{F}_t)$–predictable strategy H such that $N = Q[A] + \int_0^\cdot H_s \mathrm{d}S_s$. But then $\int_0^\cdot H_s \mathrm{d}S_s$

is a bounded $(Q, (\mathcal{G}_t))$–local martingale and thus a martingale. Using $A^c \in \mathcal{G}_0$ and that Q is equivalent to P, we derive the contradiction

$$0 = E_Q[1_{A^c} 1_A] = E_Q \left[1_{A^c} \left[Q(A) + \int_0^\infty H_s \mathrm{d}S_s \right] \right] = Q[A^c]Q[A] > 0.$$

So already in the simplest models incorporating information asymmetry, there may not exist an equivalent local martingale measure. If S is locally bounded, then by the Fundamental Theorem of Asset Pricing at least one of the conditions (NA) or (NA1) has to be violated.

Jacod's criterion and dominating local martingale measures

Here we consider again an initial filtration enlargement $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X)$ for $t \geq 0$, where X is a random variable. Jacod's criterion [Jac85] is a condition which implies that all $(\mathcal{F}_t)$–semimartingales remain $(\mathcal{G}_t)$–semimartingales. The following formulation was first found in Föllmer and Imkeller [FI93] and later generalized and carefully studied by Ankirchner et al. [ADI07]. Define the product space

$$\overline{\Omega} = \Omega \times \Omega, \qquad \overline{\mathcal{G}} = \mathcal{F} \otimes \sigma(X), \qquad \overline{\mathcal{G}}_t = \mathcal{F}_t \otimes \sigma(X),\, t \geq 0.$$

We consider two measures on $\overline{\Omega}$: the decoupling measure $\overline{Q} = P \otimes P|_{\sigma(X)}$, and $\overline{P} = P \circ \psi^{-1}$, where $\psi \colon \Omega \to \overline{\Omega}$, $\psi(\omega) = (\omega, \omega)$. The following result is then a reformulation of Jacod's criterion.

Theorem (Theorem 1 in [ADI07]) *If $\overline{P} \ll \overline{Q}$, then Hypothèse (H') holds, i.e. any $(\mathcal{F}_t)$–semimartingale is a $(\mathcal{G}_t)$–semimartingale.*

In this formulation it is quite obvious why Jacod's criterion works. Under the measure $\overline{Q}$, the additional information from X is independent of $\mathcal{F}$. Therefore, any $(\mathcal{F}_t)$–martingale M will be a $(\overline{\mathcal{G}}_t)$–martingale under $\overline{Q}$ (if we embed M from Ω to $\overline{\Omega}$ by setting $\overline{M}_t(\omega, \omega') = M_t(\omega)$). By assumption, $\overline{Q} \gg \overline{P}$, and therefore an application of Girsanov's theorem implies that $\overline{M}$ is a $\overline{P}$–semimartingale. But it is possible to show that if $\overline{M}$ is a $(\overline{P}, (\overline{\mathcal{G}}_t))$–semimartingale, then M is a $(P, (\mathcal{G}_t))$–semimartingale, which completes the argument.

The main argument was that Jacod's criterion gives us a dominating measure (on an enlarged space), under which any $(\mathcal{F}_t)$–martingale is a $(\mathcal{G}_t)$–martingale. But Jacod's criterion is always satisfied if X takes its values in a countable set, regardless of P and S. So if we recall our example of filtration enlargements in complete markets from above, then we see that Jacod's criterion may be satisfied although there is no equivalent local martingale measure under $(\mathcal{G}_t)$.

Utility maximization and filtration enlargements

Assume again that S models a complete market and that $(\mathcal{G}_t)$ is an initial enlargement satisfying Jacod's criterion. We then define the set of attainable terminal wealths $\mathcal{K}_1(\mathcal{F}_t)$ and $\mathcal{K}_1(\mathcal{G}_t)$ as in (1.2), using $(\mathcal{F}_t)$–predictable and $(\mathcal{G}_t)$–predictable strategies respectively. It is shown in Ankirchner's Ph.D. thesis ([Ank05], Theorem 12.6.1, see also [ADI06]), that the maximal expected logarithmic utility under $(\mathcal{G}_t)$ is given by

$$\sup_{X\in\mathcal{K}_1(\mathcal{G}_t)} E[\log(X)] = \sup_{X\in\mathcal{K}_1(\mathcal{F}_t)} E[\log(X)] + I(X,\mathcal{F}),$$

where $I(X,\mathcal{F})$ denotes the mutual information between X and $\mathcal{F}$. The mutual information is often finite, and therefore the maximal expected utility under $(\mathcal{G}_t)$ is often finite. But finite utility and (NA1) are equivalent:

Lemma 2.1 *The process S satisfies (NA1) under $(\mathcal{G}_t)$ if and only if there exists an unbounded increasing function U for which the maximal expected utility is finite, i.e. such that*

$$\sup_{X\in\mathcal{K}_1(\mathcal{G}_t)} E[U(X)] < \infty.$$

Proof This follows from Proposition 2.2 below.

In conclusion, we showed that (NFLVR) and thus (NA) or (NA1) is not very robust under filtration enlargements. We also observed that the maximal expected logarithmic utility in an enlarged filtration may be finite, and that this is only possible under the (NA1) condition. So (NA) seems to be the part of (NFLVR) which is less robust with respect to filtration enlargements; see also Remark 5.8 below. Moreover, Jacod's criterion is satisfied in the examples where (NA1) holds. As we saw above, Jacod's criterion gives us a dominating local martingale measure. Hence, (NA1) seems to be related to the existence of dominating local martingale measures.

Supermartingale densities

Now let $Q \gg P$ be a dominating local martingale measure for S. Let γ be the right-continuous density process, $\gamma_t = \mathrm{d}P/\mathrm{d}Q|_{\mathcal{F}_t}$, and set $\tau = \inf\{t \geq 0 : \gamma_t = 0\}$. Define $Z_t = 1_{\{t<\tau\}}/\gamma_t$. Let H be 1–admissible for S under Q, i.e. so that $Q[\int_0^t H_s \mathrm{d}S_s \geq -1] = 1$ for all $t \geq 0$. Let $s, t \geq 0$ and let $A \in \mathcal{F}_t$. We have

$$\begin{aligned} E_P[1_A Z_{t+s}(1+(H\cdot S)_{t+s})] &= E_Q\left[\gamma_{t+s}1_A \frac{1_{\{t+s<\tau\}}}{\gamma_{t+s}}(1+(H\cdot S)_{t+s})\right] \quad (2.1)\\ &\leq E_Q\left[1_A 1_{\{t<\tau\}}(1+(H\cdot S)_{t+s})\right]\\ &\leq E_Q\left[1_A 1_{\{t<\tau\}}(1+(H\cdot S)_{t})\right]\\ &= E_P[1_A Z_t(1+(H\cdot S)_t)], \end{aligned}$$

using in the second line that $1_A(1+(H\cdot S)_{t+s})$ is nonnegative, and in the third line that $1+(H\cdot S)$ is a Q–supermartingale. This indicates that Z should be a supermartingale density. Of course, here we only considered strategies that are 1–admissible under Q, and there might be strategies that are 1–admissible under P but not under Q. We will solve this by problem by considering dominating local martingale measures for $S^{\tau-}$ rather than for S.

The pair (Z,τ) is the *Kunita-Yoeurp decomposition* of Q with respect to P, a progressive Lebesgue decomposition on filtered probability spaces. It was introduced by Kunita [Kun76] in a Markovian context, and generalized to arbitrary filtered probability spaces by Yoeurp [Yoe85]. Namely, we have for all $t\geq 0$

1. $P[\tau=\infty]=1$,
2. $Q[\cdot\cap\{\tau\leq t\}]$ and P are mutually singular on $\mathcal{F}_t$,
3. for $A\in\mathcal{F}_t$ we have $Q[A\cap\{\tau>t\}]=E_P[1_A Z_t]$.

Note that the second property is a consequence of the first property.

Hence, our program will be to find a supermartingale density Z and to construct a measure Q and a stopping time τ such that (Z,τ) is the Kunita-Yoeurp decomposition of Q with respect to P. But the second part was already solved by [Yoe85], and Q will be the Föllmer measure of Z. Studying the relation between S and Z, we will see that $S^{\tau-}$ is a local martingale under Q.

Before getting to the main part of the paper, let us prove Lemma 2.1, which is a consequence of the following de la Vallée-Poussin type result for L^0–bounded families of random variables.

Proposition 2.2 *A family of random variables $\mathcal{X}$ is bounded in probability if and only if there exists an increasing and unbounded function U on $[0,\infty)$ for which*

$$\sup_{X\in\mathcal{X}} E[U(|X|)]<\infty.$$

In that case U can be chosen strictly increasing, concave, and such that $U(0)=0$.

Proof If such a U exists, then obviously $\mathcal{X}$ is bounded in L^0.

Conversely, assume that $\mathcal{X}$ is bounded in probability. Our construction of U is inspired by the proof of de la Vallée-Poussin's theorem. That is, we construct a function U of the form

$$U(x)=\int_0^x g(y)dy, \qquad \text{where} \qquad g(y)=g_k \text{ for } y\in[k-1,k),\ k\in\mathbb{N},$$

for a decreasing sequence of strictly positive numbers (g_k). This U will be strictly increasing, concave, and $U(0)=0$. It will be unbounded if and only if $\sum_{k=1}^\infty g_k=\infty$.

For such U we apply montone convergence and Fubini's theorem to obtain

$$\begin{aligned} E[U(|X|)] &= \sum_{m=1}^{\infty} E[U(|X|)1_{\{|X|\in[m-1,m)\}}] \leq \sum_{m=1}^{\infty}\sum_{k=1}^{m} g_k P[|X|\in[m-1,m)] \\ &= \sum_{k=1}^{\infty} g_k P[|X|\geq k-1] \leq \sum_{k=1}^{\infty} g_k F_{\mathcal{X}}(k-1), \end{aligned}$$

where $F_{\mathcal{X}}(k-1)=\sup_{X\in\mathcal{X}} P[|X|\geq k-1]$.

So the proof is complete if we can find a decreasing sequence (g_k) of positive numbers with $\sum_{k=1}^{\infty} g_k = \infty$ but $\sum_{k=1}^{\infty} g_k F_{\mathcal{X}}(k-1) < \infty$. By assumption, $(F_{\mathcal{X}}(k))$ converges to zero as k tends to ∞, and therefore it also converges to zero in the Cesàro sense. So for every $m\in\mathbb{N}$ there exists $K_m\in\mathbb{N}$ such that

$$\frac{1}{K_m}\sum_{k=1}^{K_m} F_{\mathcal{X}}(k-1) \leq \frac{1}{m}. \tag{2.2}$$

We may also assume that $K_m \geq m$. Define

$$g_k^m = \begin{cases} \frac{1}{mK_m}, & k\leq K_m, \\ 0, & k > K_m, \end{cases}$$

and let m_k denote the smallest m for which $g_k^m \neq 0$, that is $m_k := \min\{m\in\mathbb{N} : K_m \geq k\}$. By definition, $m_k \leq m_{k+1}$ for all k, and therefore the sequence (g_k), where

$$g_k = \sum_{m=1}^{\infty} g_k^m = \sum_{m=m_k}^{\infty} \frac{1}{mK_m} \leq \sum_{m=m_k}^{\infty}\frac{1}{m^2} < \infty,$$

is decreasing in k. Moreover, Fubini's theorem implies that

$$\sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} g_k^m = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} g_k^m = \sum_{m=1}^{\infty}\sum_{k=1}^{K_m}\frac{1}{mK_m} = \sum_{m=1}^{\infty}\frac{1}{m} = \infty,$$

and at the same time we get from (2.2)

$$\sum_{k=1}^{\infty} g_k F_{\mathcal{X}}(k-1) = \sum_{m=1}^{\infty}\sum_{k=1}^{K_m}\frac{F_{\mathcal{X}}(k-1)}{mK_m} \leq \sum_{m=1}^{\infty}\frac{1}{m^2} < \infty,$$

which completes the proof.

Remark 2.3 In Loewenstein and Willard [LW00], Theorem 1, it is shown that the utility maximization problem for Itô processes is well posed if and only if there is absence of a certain notion of arbitrage. Karatzas and Kardaras [KK07], Section 4.7 show for semimartingale models that (NA1) is the minimal property which has to be satisfied to obtain a well-posed utility maximization problem. Proposition 2.2 is much simpler and more obvious, but therefore also more robust. It is applicable in virtually any context, say non-semimartingale models with transaction costs under trading constraints. The family of portfolios need not even be convex.

Remark 2.4 Supermartingale densities are the dual variables in the duality approach to utility maximization [KS99]. Taking Proposition 2.2 into account, Theorem 1.3 therefore states that there exists a non-degenerate utility maximization problem if and only if the space of dual minimizers is nonempty. This insight might also be useful in more complicated contexts. As a sort of meta-theorem holding for many utility maximization problems, we expect that the space of dual variables is nonempty if and only if the space of primal variables is bounded in probability.

A first consequence is that any locally bounded process satisfying ($\mathrm{NA1}_s$) is a semimartingale. This follows from [Ank05], Theorem 7.4.3, where it is shown that finite utility implies the semimartingale property.

3 Existence of supermartingale densities

Now let us sketch an alternative proof of Theorem 1.3. In fact, we will obtain a generalized version which is known as Rokhlin's theorem.

A family of nonnegative stochastic processes $\mathcal{Y}$ is called *fork-convex*, see [Ž02] or [Rok10], if $Y_s = 0$ implies $Y_t = 0$ for all $0 \leq s \leq t < \infty$, and if further for all $Y^1, Y^2, Y^3 \in \mathcal{Y}$, for all $s \geq 0$, and for all $\mathcal{F}_s$–measurable random variables λ_s with values in $[0,1]$, we have that

$$Y_\cdot = 1_{[0,s)}(\cdot)Y^1_s + 1_{[s,\infty)}(\cdot)Y^1_s \left(\lambda_s \frac{Y^2_\cdot}{Y^2_s} + (1-\lambda_s)\frac{Y^3_\cdot}{Y^3_s}\right) \in \mathcal{Y}. \tag{3.1}$$

Here and in all that follows we interpret $0/0 = 0$. A fork-convex family of processes with $Y_0 = 1$ for all $Y \in \mathcal{Y}$ is convex. If moreover $\mathcal{Y}$ contains the constant process 1, then $\mathcal{Y}$ is stable under stopping at deterministic times. Rokhlin's theorem is the following.

Theorem 3.1 ([Rok10], Theorem 1) *Let $\mathcal{Y}$ be a fork-convex family of right-continuous and nonnegative processes containing the constant process 1 and such that $Y_0 = 1$ for all $Y \in \mathcal{Y}$. Let*

$$\mathcal{K} = \left\{ Y_\infty : Y \in \mathcal{Y}, Y_\infty = \lim_{t\to\infty} Y_t \text{ exists} \right\}.$$

Then $\mathcal{K}$ is bounded in probability if and only if there exists a supermartingale density for $\mathcal{Y}$.

Let us show the result for two time steps. First, we show that we can switch from boundedness in L^0 to boundedness in L^1 by a change of measure.

Lemma 3.2 *Let $\mathcal{X}$ be a convex family of nonnegative random variables. Then $\mathcal{X}$ is bounded in probability if and only if there exists a strictly positive random variable Z with*

$$\sup_{X\in\mathcal{X}} E[XZ] < \infty.$$

Proof The necessity is Theorem 1 of [Yan80] in combination with Remark (c) of [DM82], VIII-84; see also Lemma 2.3 of [BS99]. The sufficiency is easy.

This lemma is all we need to prove the result in two time steps:

Lemma 3.3 *Let $\mathcal{Y}$ be a L^1–bounded family of nonnegative processes indexed by $\{0,1\}$, adapted to a filtration $(\mathcal{F}_0, \mathcal{F}_1)$. Assume that $\mathcal{Y}$ is fork-convex and that $\mathcal{Y}$ contains a process of the form $(1, Y_1^*)$ for a strictly positive Y_1^*. Then there exists a strictly positive $\mathcal{F}_0$–measurable random variable Z for which $(Y_0 Z, Y_1)$ is a supermartingale for every $Y \in \mathcal{Y}$. The random variable Z can be chosen such that*

$$\sup_{Y\in\mathcal{Y}} E[Y_0 Z] \le \sup_{Y\in\mathcal{Y}} \max_{i=0,1} E[Y_i]. \tag{3.2}$$

Proof We define a nonnegative set function μ on $\mathcal{F}_0$ by setting

$$\mu(A) := \sup_{Y\in\mathcal{Y}} E[1_A Y_1/Y_0].$$

Let us apply the fork-convexity of $\mathcal{Y}$ to show that for every $Y \in \mathcal{Y}$ there exists $\widetilde{Y} \in \mathcal{Y}$ with $Y_1/Y_0 = \widetilde{Y}_1$. We take $s = 0$, $Y^1 = (1, Y_1^*)$, $Y^2 = Y$, and $\lambda_s = 1$ in (3.1). Then $\widetilde{Y} \in \mathcal{Y}$, where $\widetilde{Y}_0 = 1_{\{Y_0>0\}}$ and $\widetilde{Y}_1 = Y_1/Y_0$. In particular, it follows from the L^1–boundedness of $\mathcal{Y}$ that

$$\mu(A) = \sup_{Y\in\mathcal{Y}} E\left[1_A \frac{Y_1}{Y_0}\right] \le \sup_{\widetilde{Y}\in\mathcal{Y}} E[1_A \widetilde{Y}_1] < \infty$$

for all $A \in \mathcal{F}_0$. In fact, μ is a finite measure: Let $A, B \in \mathcal{F}_0$ be two disjoint sets and let $Y^A, Y^B \in \mathcal{Y}$. We take $s = 0$, $Y^1 = (1, Y_1^*)$, $Y^2 = Y^A$, $Y^3 = Y^B$, and $\lambda_s = 1_A$ in (3.1), so that

$$\widetilde{Y}_t = 1_{\{0\}}(t)\left(1_A 1_{\{Y_0^A>0\}} + 1_B 1_{\{Y_0^B>0\}}\right) + 1_{\{1\}}(t)\left(1_A \frac{Y_1^A}{Y_0^A} + 1_B \frac{Y_1^B}{Y_0^B}\right) \in \mathcal{Y}.$$

Note that $\widetilde{Y}_1/\widetilde{Y}_0 = \widetilde{Y}_1$, since we set $0/0 = 0$. Because A and B are disjoint, we have

$$1_{A\cup B} \frac{\widetilde{Y}_1}{\widetilde{Y}_0} = 1_A \frac{Y_1^A}{Y_0^A} + 1_B \frac{Y_1^B}{Y_0^B}.$$

As a consequence we obtain

$$\mu(A) + \mu(B) \le \sup_{\widetilde{Y}\in\mathcal{Y}} E\left[1_{A\cup B} \frac{\widetilde{Y}_1}{\widetilde{Y}_0}\right] = \mu(A \cup B).$$

But $\mu(A \cup B) \le \mu(A) + \mu(B)$ is obvious, and therefore μ is finitely additive. It is also obvious that $\mu(\bigcup_n A_n) \le \sum_n \mu(A_n)$ for every sequence (A_n) of disjoint sets in $\mathcal{F}_0$. Since the opposite inequality holds for any finitely additive nonnegative set function, μ is a finite measure on $\mathcal{F}_0$, absolutely continuous with respect to P. Hence, there exists a nonnegative $Z \in L^1(\mathcal{F}_0, P)$ with

$$\mu(A) = E[1_A Z] = \sup_{Y \in \mathcal{Y}} E\left[1_A \frac{Y_1}{Y_0}\right] \tag{3.3}$$

for all $A \in \mathcal{F}_0$. As a consequence, we get for any $Y \in \mathcal{Y}$ and $A \in \mathcal{F}_0$

$$E[1_A Y_0 Z] = \sup_{\widetilde{Y} \in \mathcal{Y}} E\left[1_A Y_0 \frac{\widetilde{Y}_1}{\widetilde{Y}_0}\right] \geq E\left[1_A Y_0 \frac{Y_1}{Y_0}\right] = E[1_A Y_1],$$

proving that $(Y_0 Z, Y_1)$ is a supermartingale provided that $E[Y_0 Z] < \infty$. But the bound (3.2) follows immediately from the fork-convexity of $\mathcal{Y}$ and the definition of Z, because the process $\widetilde{Y} = (Y_0 1_{\{Y_0^1 > 0\}}, Y_0 Y_1^1 / Y_0^1)$ is in $\mathcal{Y}$ for all $Y^1 \in \mathcal{Y}$. It remains to show that Z is strictly positive. By assumption, there exists $(1, Y_1^*) \in \mathcal{Y}$ with strictly positive Y_1^*. Since (Z, Y_1^*) is a supermartingale, also Z must be strictly positive.

The case of finitely many time steps easily follows by induction. From here we can use a compactness principle for nonnegative supermartingales such as Lemma 5.2 of Föllmer and Kramkov [FK97] to pass to the general case. It is also possible to use a Tychonoff theorem for *convex compactness*, a weak notion of compactness introduced by Žitković [Ž10]. This approach is carried out in [Per14], where also many counterexamples are given to show that all conditions of Theorem 3.1 are reasonably sharp. Alternatively, we may at this point just follow the arguments in the proof of Theorem 2 in [Rok10].

Corollary 3.4 *If $\mathcal{Y}$ is as in Theorem 3.1, then every $Y \in \mathcal{Y}$ is a semimartingale for which Y_t almost surely converges as $t \to \infty$.*

Proof Convergence follows because YZ is a nonnegative supermartingale and because Z converges to a strictly positive limit. The semimartingale property is obtained using Itô's formula and the strict positivity of Z.

To conclude the proof of Theorem 1.3, it suffices to show that $\mathcal{W}_1$ and $\mathcal{W}_{1,s}$ satisfy the assumptions of Theorem 3.1. This is easy and done for example in [Rok10]. Rokhlin only treats the case of $\mathcal{W}_1$ and $\mathcal{K}_1$, but the same arguments also work for $\mathcal{W}_{1,s}$ and $\mathcal{K}_{1,s}$.

Proof (Proof of Corollary 1.4) It suffices to argue for each component separately. Let S be locally bounded from below and assume that S satisfies ($\mathrm{NA1}_s$). Then also $S - S_0$ is locally bounded from below so that we can choose an increasing sequence of stopping times (τ_m) with $\lim_{m\to\infty} \tau_m = \infty$, and a sequence of strictly positive numbers (a_m), so that $(1 + a_m(S_{t\wedge\tau_m} - S_0))_{t\geq 0} \in \mathcal{W}_{1,s}$. By Corollary 3.4, the stopped process $S_{\cdot\wedge\tau_m}$ is a semimartingale for every m. But local semimartingales are semimartingales, see Protter [Pro04], Theorem II.6. Protter works with complete filtrations, but it follows from Lemma

A.3 in Appendix A that for every semimartingale in the completed filtration there exists an indistinguishable $(\mathcal{F}_t)$–semimartingale.

It remains to show that every supermartingale density for $\mathcal{W}_{1,s}$ is a supermartingale density for $\mathcal{W}_1$. But this is the content of [KP11], Section 2.2.

In the unbounded case S is not necessarily a semimartingale. A simple counterexample is given by a one dimensional Lévy-process with jumps that are unbounded both from above and from below, to which we add a fractional Brownian motion with Hurst index $H < 1/2$. The resulting process has infinite quadratic variation and is therefore not a semimartingale. But there are no 1–admissible simple strategies other than 0, so that $\mathcal{K}_{1,s} = \{1\}$ is obviously bounded in probability.

4 Construction of dominating local martingale measures

4.1 The Kunita-Yoeurp problem and Föllmer's measure

Now let Z be a strictly positive supermartingale with $Z_\infty > 0$ and $E_P(Z_0) = 1$. Our aim is to construct a dominating measure Q and a stopping time τ such that (Z, τ) is the Kunita-Yoeurp decomposition of Q with respect to P. We call this the *Kunita-Yoeurp problem*. Recall that (Z, τ) is the Kunita-Yoeurp decomposition of Q with respect to P if

1. $P[\tau = \infty] = 1$,
2. for $A \in \mathcal{F}_t$ we have

$$Q[A \cap \{\tau > t\}] = E_P[1_A Z_t]. \tag{4.1}$$

In this case it follows for any stopping time ρ and any $A \in \mathcal{F}_\rho$ that

$$Q[A \cap \{\tau > \rho\}] = E_P\left[1_{A \cap \{\rho < \infty\}} Z_\rho\right], \tag{4.2}$$

see for example [Yoe85], Proposition 4.

In general it is impossible to construct Q and τ without making further assumptions on the underlying probability space. Under certain topological assumptions on the probability space, Yoeurp [Yoe85] showed that one can always find an enlarged probability space where the Kunita-Yoeurp problem admits a solution (Q, τ). Indeed, Q can be chosen as the Föllmer measure [Föl72] of Z. Here we use the construction of [PR14], where it is shown that if P is a probability measure on the path space $(\Omega, \mathcal{F})$ described in the introduction, such that the explosion time to the cemetery state Δ is P–almost surely infinite, the Kunita-Yoeurp problem always admits a solution (Q, τ) on $(\Omega, \mathcal{F})$ and it is not necessary to enlarge the probability space. There it is also examined very precisely under which conditions Q and τ are unique (almost never).

So let (Q, τ) solve the Kunita-Yoeurp problem for Z and let us show that Q dominates P.

Lemma 4.1 *Let P and Q be two probability measures on $(\Omega, \mathcal{F})$ and let (Z, τ) be the Kunita-Yoeurp decomposition of Q with respect to P. Assume that $P[Z_\infty > 0] = 1$. Then $P \ll Q$.*

Proof Let $A \in \bigcup_{t\geq 0} \mathcal{F}_t$. Equation (4.1) and Fatou's lemma yield

$$Q[A \cap \{\tau = \infty\}] = \lim_{t\to\infty} Q[A \cap \{\tau > t\}] = \lim_{t\to\infty} E_P[Z_t 1_A] \geq E_P[Z_\infty 1_A].$$

By the monotone class theorem and since $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$, this inequality extends to all $A \in \mathcal{F}$. Since $P(Z_\infty > 0) = 1$, we conclude that $P \ll Q$.

4.1.1 Calculating expectations under Q

Here we collect important results of [Yoe85] that allow to rewrite certain expectations under Q as expectations under P. More precisely, let Z be a nonnegative supermartingale with $E(Z_0) = 1$ and with Doob-Meyer decomposition $Z = Z_0 + N - B$, where N is a local martingale starting in zero, and B is an adapted process, almost surely increasing and càdlàg, and let (Q, τ) solve the Kunita-Yoeurp problem for Z.

Lemma 4.2 ([Yoe85], Proposition I.9) *Let $Z = Z_0 + N - B$, and let τ and Q be as described above. Let $(\rho_m)_{m\in\mathbb{N}}$ be a localizing sequence for N such that every ρ_m is finite. Then we have for every bounded predictable process Y and for every $m \in \mathbb{N}$ that*

$$E_Q[Y_\tau^{\rho_m}] = E_P\left[Y_{\rho_m} Z_{\rho_m} + \int_0^{\rho_m} Y_s \mathrm{d}B_s\right]. \tag{4.3}$$

Corollary 4.3 *Let Y be a bounded adapted process that is P–almost surely càdlàg. Define*

$$Y_t^{\tau-}(\omega) := Y_t(\omega) 1_{\{t<\tau(\omega)\}} + \limsup_{s\to\tau(\omega)-} Y_s(\omega) 1_{\{t\geq\tau(\omega)\}}.$$

Let Z and (ρ_m) be as in Lemma 4.2. Then

$$E_Q[Y_{\rho_m}^{\tau-}] = E_P\left[Y_{\rho_m} Z_{\rho_m} + \int_0^{\rho_m} Y_{s-} \mathrm{d}B_s\right].$$

Proof This is a small generalization of (2.4) in [Yoe85]. Define $Y_t^-(\omega) = Y_{t-}(\omega) = \limsup_{s\to t-} Y_s$ for $t > 0$, and $Y_0^- = Y_0$. Then Y^- is predictable process, and therefore we can apply Lemma 4.2. Observe that

$$Y_{\rho_m}^{\tau-} = Y_{\rho_m} 1_{\{\tau>\rho_m\}} + Y_{\tau-} 1_{\{\tau\leq\rho_m\}} = Y_{\rho_m} 1_{\{\tau>\rho_m\}} + (Y^-)_\tau 1_{\{\tau\leq\rho_m\}}.$$

Now (4.2) implies that $E_Q[Y_{\rho_m} 1_{\{\tau>\rho_m\}}] = E_P[Y_{\rho_m} Z_{\rho_m}]$, whereas (4.3) and then again (4.2) applied to the second term on the right hand side give

$$\begin{aligned} E_Q[(Y^-)_\tau 1_{\{\tau\leq\rho_m\}}] &= E_Q[(Y^-)_\tau^{\rho_m}] - E_Q[(Y^-)_{\rho_m} 1_{\{\tau>\rho_m\}}] \\ &= E_P\left[Y_{\rho_m-} Z_{\rho_m} + \int_0^{\rho_m} Y_{s-} \mathrm{d}B_s\right] - E_P\left[Y_{\rho_m-} Z_{\rho_m}\right] \\ &= E_P\left[\int_0^{\rho_m} Y_{s-} \mathrm{d}B_s\right]. \end{aligned}$$

4.2 The predictable case

Let now S be a d–dimensional predictable semimartingale and let Z be a supermartingale density for $\mathcal{W}_1$. Here we examine the structure of S and Z closer. This will allow us to apply Lemma 4.2 to deduce that $S^{\tau-}$ is a local martingale under the dominating measure associated to Z.

Remark 4.4 Observe that, thanks to predictability, $S - S_0$ is almost surely locally bounded. In view of Corollary 1.4 it would therefore suffice to assume that Z is a supermartingale density for $\mathcal{W}_{1,s}$. Then S is a semimartingale and Z is a supermartingale density for $\mathcal{W}_1$.

Since $S - S_0$ is locally bounded, it is a special semimartingale. That is, there exists a unique decomposition $S = S_0 + M + D$, where M is a local martingale with $M_0 = 0$, and D is a predictable process of finite variation with $D_0 = 0$. Thus, $M = S - S_0 - D$ is predictable and therefore continuous. But then also D must be continuous, because (NA1) implies $\mathrm{d}D^i \ll \mathrm{d}\langle M^i\rangle$ for $i = 1, \ldots, d$, where $M = (M^1, \ldots, M^d)$ and $D = (D^1, \ldots, D^d)$; see for example [Ank05], Lemma 9.1.2. Otherwise, one could find a predictable process H^i which satisfies $H^i \cdot M^i \equiv 0$, but for which $H^i \cdot D^i$ is increasing. In conclusion, S must be continuous.

In fact, S must satisfy the *structure condition* as defined in Schweizer [Sch95]. Recall that $L^2_{\mathrm{loc}}(M)$ is the space of progressively measurable processes $(\lambda_t)_{t\geq 0}$ that are locally square integrable with respect to M, i.e. so that

$$\int_0^t \sum_{i,j=1}^d \lambda^i_s \lambda^j_s \mathrm{d}\langle M^i, M^j\rangle_s < \infty, \qquad t \geq 0.$$

Definition 4.5 Let $S = S_0 + M + D$ be a d–dimensional special semimartingale with locally square-integrable M. Define

$$C_t = \sum_{i=1}^d \langle M^i\rangle_t \qquad \text{and for } 1 \leq i, j \leq d: \qquad \sigma^{ij}_t = \frac{\mathrm{d}\langle M^i, M^j\rangle_t}{\mathrm{d}C_t}.$$

Note that σ exists by the Kunita-Watanabe inequality. Then S satisfies the *structure condition* if $\mathrm{d}D^i \ll \mathrm{d}\langle M^i\rangle$ for all $1 \leq i \leq d$, with predictable derivative $\alpha^i_t = \mathrm{d}D^i_t/\mathrm{d}\langle M^i\rangle_t$, and if there exists a predictable process $\lambda_t = (\lambda^1_t, \ldots, \lambda^d_t) \in L^2_{\mathrm{loc}}(M)$ so that for $i = 1, \ldots, d$ we have $\mathrm{d}C(\omega) \otimes P(\mathrm{d}\omega)$–almost everywhere

$$(\sigma\lambda)^i = \alpha^i \sigma^{ii}. \tag{4.4}$$

Note that λ might not be uniquely determined, but the stochastic integral $\int \lambda \mathrm{d}M$ does not depend on the choice of λ, see [Sch95]. If

$$\int_0^\infty \sum_{i,j=1}^d \lambda^i_t \sigma^{ij}_t \lambda^j_t \mathrm{d}C_t < \infty, \tag{4.5}$$

then we say that S satisfies the *structure condition until* ∞.

Recall that two one dimensional local martingales L and N are called *strongly orthogonal* if LN is a local martingale. If L and N are multidimensional, then we call them strongly orthogonal if all their components are strongly orthogonal. Also recall that the *stochastic exponential* of a semimartingale X is defined by the SDE

$$\mathcal{E}(X)_t = 1 + \int_0^t \mathcal{E}(X)_{s-} \mathrm{d}X_s, \qquad t \geq 0.$$

Let us write $\mathrm{d}X_t \sim \mathrm{d}Y_t$ if $\mathrm{d}(X-Y)_t$ is the differential of a local martingale.

Lemma 4.6 ([LŽ07], Proposition 3.2) *Let $S = S_0 + M + D$ be a predictable semimartingale and suppose that Z is a supermartingale density for S. Then S satisfies the structure condition until ∞, and*

$$\mathrm{d}Z_t = Z_{t-}(-\lambda_t \mathrm{d}M_t + \mathrm{d}N_t - \mathrm{d}B_t), \tag{4.6}$$

where λ satisfies (4.4) *and* (4.5)*, N is a local martingale that is strongly orthogonal to M, B is increasing, and $\mathcal{E}(N-B)_\infty > 0$.*

Conversely, if a predictable process S satisfies the structure condition until ∞, and if Z is defined by (4.6) *with $Z_0 = 1$, then Z is a supermartingale density for S.*

In particular, for predictable S the structure condition until ∞ is equivalent to (NA1).

Proof This is essentially Proposition 3.2 of Larsen and Žitković [LŽ07] in infinite time. We provide a slightly simplified version of their proof, because later we will need some results obtained during the proof.

Let Z be a supermartingale density. Since Z is strictly positive, it is of the form $\mathrm{d}Z_t = Z_{t-}(\mathrm{d}L_t - \mathrm{d}B_t)$ for a local martingale L and a predictable increasing process B. Since M is continuous, there exists a predictable process $\lambda \in L^2_{\mathrm{loc}}(M)$ such that $\mathrm{d}L_t = -\lambda_t \mathrm{d}M_t + \mathrm{d}N_t$, where N is a local martingale that is strongly orthogonal to all components of M, see [JS03], Theorem III.4.11. Moreover,

$$0 < Z_\infty = Z_0 \mathcal{E}(-\lambda \cdot M + N - B)_\infty = Z_0 \mathcal{E}(-\lambda \cdot M)_\infty \mathcal{E}(N-B)_\infty,$$

which is only possible if λ satisfies (4.5) and if $\mathcal{E}(N-B)_\infty > 0$. It remains to show that λ also satisfies (4.4).

Let H be a 1–admissible strategy. Write $W^H := 1 + H \cdot S$ for the wealth process generated by H. Then $W^H Z$ is a nonnegative supermartingale. Since Z is strictly positive, we must have $W^H_t \equiv 0$ for $t \geq \tau^H := \inf\{s \geq 0 : W^H_{s-} = 0 \text{ or } W^H_s = 0\}$. Therefore, we may assume without loss of generality that $H_t = H_t 1_{\{t < \tau^H\}}$ for all $t \geq 0$. Define $\pi_t := H_t / W^H_{t-}$, so that $W^H_t = 1 + \int_0^t \pi_s W^H_{s-} \mathrm{d}S_s$. In other words, every wealth process is of the form $W^H = \mathcal{E}(\pi \cdot S)$ for a suitable integrand π. We slightly abuse notation and write W^π instead of W^H.

Integration by parts applied to ZW^π gives

$$\begin{aligned} \mathrm{d}(ZW^\pi)_t &= W^\pi_{t-}\mathrm{d}Z_t + Z_{t-}\pi_t W^\pi_{t-}\mathrm{d}S_t + \mathrm{d}[W^\pi, Z]_t \\ &= W^\pi_{t-}Z_{t-}(-\lambda_t \mathrm{d}M_t + \mathrm{d}N_t - \mathrm{d}B_t) + Z_{t-}\pi_t W^\pi_{t-}(\mathrm{d}M_t + \mathrm{d}D_t) \\ &\quad + W^\pi_{t-}Z_{t-}\mathrm{d}[\pi\cdot(M+D), -\lambda\cdot M + N - B]_t \\ &\sim -W^\pi_{t-}Z_{t-}\mathrm{d}B_t + Z_{t-}\pi_t W^\pi_{t-}\mathrm{d}D_t + W^\pi_{t-}Z_{t-}\mathrm{d}\langle \pi\cdot M, -\lambda\cdot M\rangle_t, \end{aligned} \tag{4.7}$$

where we used that M and D are continuous and that N is strongly orthogonal to M.

Let now C and σ be as described in Definition 4.5. Then Theorem III.4.5 of [JS03] implies that the bracket $\langle \pi\cdot M, -\lambda\cdot M\rangle$ can be rewritten as

$$\begin{aligned} \mathrm{d}(ZW^\pi)_t &\sim W^\pi_{t-}Z_{t-}\left(-\mathrm{d}B_t + \pi_t\mathrm{d}D_t + \mathrm{d}\langle \pi\cdot M, -\lambda\cdot M\rangle_t\right) \\ &= W^\pi_{t-}Z_{t-}\Bigg(-\mathrm{d}B_t + \sum_{i=1}^d \pi^i_t\Big(\mathrm{d}D^i_t - \sum_{j=1}^d \sigma^{ij}_t\lambda^j_t \mathrm{d}C_t\Big)\Bigg). \end{aligned} \tag{4.8}$$

Assume that there exists $i\in\{1,\dots,d\}$ for which the continuous process of finite variation $X^i_t = D^i_t - \sum_{j=1}^d \int_0^\cdot \sigma^{ij}_s\lambda^j_s \mathrm{d}C_s$ is not evanescent. We claim that then there exists π for which the finite variation part of (ZW^π) is increasing on a small time interval. By the predictable Radon-Nikodym theorem of Delbaen and Schachermayer [DS95b], Theorem 2.1 b), we can find a predictable γ^i with values in $\{-1,1\}$ such that $\int_0^\cdot \gamma^i_s\mathrm{d}D^i_s = V^i$, where V^i denotes the total variation process of X^i. Note that [DS95b] work with complete filtrations, but we can apply Lemma A.2 to get rid of that assumption. Let now $m\in\mathbb{N}$ and set $\pi^j_t := m\delta_{ij}\gamma^i_t$ for $j=1,\dots,d$. Then

$$\mathrm{d}(ZW^\pi)_t \sim W^\pi_{t-}Z_{t-}\left(-\mathrm{d}B_t + m\mathrm{d}V^i_t\right).$$

Since V^i is an increasing process that is not constant, there exists m so that $-\mathrm{d}B_t + m\mathrm{d}V^i_t$ is locally strictly increasing with positive probability. Since π is bounded, we obtain that $W^\pi_{t-} > 0$ for all $t\geq 0$. Therefore, the finite variation part of $W^\pi Z$ is locally strictly increasing with positive probability, a contradiction to ZW^π being a supermartingale.

Thus, X^i is evanescent. Recall that $\mathrm{d}D^i \ll \mathrm{d}\langle M^i\rangle = \sigma^{ii}\mathrm{d}C$, and therefore there exists a predictable process α^i for which

$$0 \equiv \left(\mathrm{d}D^i_t - \sum_{j=1}^d \sigma^{ij}_t\lambda^j_t\mathrm{d}C_t\right) = \left(\alpha^i_t\sigma^{ii}_t - (\sigma_t\lambda_t)^i\right)\mathrm{d}C_t,$$

so that $\mathrm{d}C(\omega)\otimes P(\mathrm{d}\omega)$–almost everywhere $\alpha^i\sigma^{ii} = (\sigma\lambda)^i$ and thus λ satisfies (4.4).

The converse direction is easy and follows directly from (4.7).

Remark 4.7 For later reference we remark that if S is discontinuous, then a priori we only know that $\mathrm{d}Z_t = Z_{t-}(\mathrm{d}N_t - \mathrm{d}B_t)$, for a local martingale N and a predictable process of finite variation B. Similarly as in(4.7) we can show that then for $W^\pi = \mathcal{E}(\pi \cdot S)$

$$\mathrm{d}(W^\pi Z)_t \sim W^\pi_{t-} Z_{t-}(-\mathrm{d}B_t + \pi_t \mathrm{d}D_t + \mathrm{d}[\pi \cdot M, N]_t - \mathrm{d}[\pi \cdot D, B]_t). \qquad (4.9)$$

Here we used that if L is a local martingale and if D is predictable process of finite variation, then $[L, D]$ is a local martingale, see Proposition I.4.49 of [JS03].

If Z is a supermartingale density, then SZ is not necessarily a local martingale:

Corollary 4.8 *Let Z and S be as in Lemma 4.6. Then $S^i Z$ is a local supermartingale if and only if $S^i \geq 0$ on the support of the measure $\mathrm{d}B$. If $S^i \geq 0$ identically, then $S^i Z$ is a supermartingale.*

The process $S^i Z$ is a local martingale if and only if $S^i = 0$ on the support of the measure $\mathrm{d}B$.

Proof Integration by parts and (4.6) imply that

$$\begin{aligned} \mathrm{d}(S^i Z)_t &= Z_{t-}(\mathrm{d}M^i_t + \alpha^i_t \sigma^{ii}_t \mathrm{d}C_t) + S^i_{t-} Z_{t-}(-\lambda \mathrm{d}M_t + \mathrm{d}N_t - \mathrm{d}B_t) \\ &\quad - Z_{t-}(\sigma\lambda)^i_t \mathrm{d}C_t \\ &\sim -S^i_{t-} Z_{t-} \mathrm{d}B_t, \end{aligned}$$

where we used (4.4) in the second step. The claim now follows easily since nonnegative local supermartingales are supermartingales by Fatou's lemma.

Another consequence of Lemma 4.6 is that in the predictable case, the maximal elements among the supermartingale densities are always local martingales. This is important in the duality approach to utility maximization. For details we refer to [LŽ07].

We are now ready to prove Theorem 1.5, which is a consequence of the following corollary.

Corollary 4.9 *Let S be a predictable semimartingale, and let Z be a supermartingale density for S. Let τ be a stopping time and Q be a probability measure so that $(Z/E_P(Z_0), \tau)$ is the Kunita-Yoeurp decomposition of Q with respect to P. Then $S^{\tau-}$ is a Q–local martingale.*

Conversely, if $Q \gg P$ has the Kunita-Yoeurp decomposition (Z, τ) with respect to P and if $S^{\tau-}$ is a local martingale under Q, then Z is a supermartingale density for S.

Proof We first show that $S^{\tau-}$ is Q–almost surely locally bounded. For $n \in \mathbb{N}$ let $\widetilde{\rho}_n := \inf\{t \geq 0 : |S^{\tau-}_t| \geq n\}$. Since $S^{\tau-}$ was only required to be right-continuous P–almost surely and not identically, $\widetilde{\rho}_n$ is not necessarily a stopping time. But it is a stopping time in the filtration $(\mathcal{F}^Q_t)$ which is completed with respect to Q. By Lemma A.1, there exist $(\mathcal{F}_t)$–stopping times $(\rho_n)_{n\in\mathbb{N}}$ with $Q[\rho_n = \widetilde{\rho}_n] = 1$ for all n. Then $\sup_n \rho_n$ is a stopping time, and we obtain from (4.2) that

$$Q\big[\sup_n \rho_n < \tau\big] = E_P\left[Z_{\sup_n \rho_n} 1_{\{\sup_n \rho_n < \infty\}}\right] = 0, \tag{4.10}$$

because $P[\sup_n \rho_n < \infty] = 0$. But $S_t^{\tau-}$ is constant after τ, and therefore $\{\sup_n \rho_n \geq \tau\}$ is Q–almost surely contained in $\{\sup_n \rho_n = \infty\}$, so that $S^{\tau-}$ is Q–almost surely locally bounded.

Let now $(\sigma_n)_{n\in\mathbb{N}}$ be a localizing sequence of finite stopping times for the local martingale N, where $Z = Z_0 + N - B$, and define $\tau_n := \rho_n \wedge \sigma_n$. Let H be a strategy that is 1–admissible for $(S^{\tau-})^{\tau_n}$ under Q. Since $(H \cdot S^{\tau-}) = (H \cdot S)^{\tau-}$, we can apply Corollary 4.3 (which of course extends to nonnegative Y), to obtain

$$E_Q[1 + (H \cdot S^{\tau-})_{\tau_n}] = E_P\Big[(1 + (H \cdot S)_{\tau_n}) Z_{\tau_n} + \int_0^{\tau_n} (1 + (H \cdot S)_{s-}) \mathrm{d}B_s\Big].$$

But now (4.8) and (4.4) imply that for π as defined in the proof of Lemma 4.6, the process

$$(1 + (H \cdot S))Z + \int_0^{\cdot} (1 + (H \cdot S)_{s-}) \mathrm{d}B_s = W^\pi Z + \int_0^{\cdot} W^\pi_{s-} \mathrm{d}B_s$$

is a nonnegative P–local martingale starting in 1, and therefore $E_Q[(H \cdot S^{\tau-})_{\tau_n}] \leq 0$. Since $(S^{\tau-})^{\tau_n}$ is bounded, we easily conclude that it is a martingale.

The only remaining problem is that so far we only know that $Q[\sup_n \tau_n \geq \tau] = 1$ and not that $Q[\sup_n \tau_n = \infty] = 1$. But the same arguments also show that $(S^{\tau-})^{\rho_n \wedge \tau_m}$ is a martingale for all $n, m \in \mathbb{N}$, so that by bounded convergence

$$E_Q[(S^{\tau-})^{\rho_n}_{t+s} | \mathcal{F}_t] = \lim_{m\to\infty} E_Q[(S^{\tau-})^{\rho_n \wedge \tau_m}_{t+s} | \mathcal{F}_t] = \lim_{m\to\infty} (S^{\tau-})^{\rho_n \wedge \tau_m}_t = (S^{\tau-})^{\rho_n}_t$$

for all $s, t \geq 0$. As we saw above, $Q[\sup_n \rho_n = \infty] = 1$, and therefore $S^{\tau-}$ is a Q–local martingale.

Conversely, let $S^{\tau-}$ be a Q–local martingale, and let H be a 1–admissible strategy for S under P. Define $\rho := \inf\{t \geq 0 : (H \cdot S^{\tau-})_t < -1\}$. Then $P[\rho < \infty] = 0$ and therefore $Q[\rho < \tau] = 0$. Hence, H is 1–admissible for $S^{\tau-}$ under Q. Now we can repeat the arguments in (2.1), to obtain that $Z_t = 1_{\{t<\tau\}}/\gamma_t$ is a supermartingale density for S, where we denoted $\gamma_t := (\mathrm{d}P/\mathrm{d}Q)|_{\mathcal{F}_t}$.

Remark 4.10 For later reference, note that we only used once that S is predictable: it was only needed to obtain

$$E_P\left[(1 + (H \cdot S)_{\tau_n}) Z_{\tau_n} + \int_0^{\tau_n} (1 + (H \cdot S)_{s-}) \mathrm{d}B_s\right] \leq 1,$$

for which we applied Lemma 4.6 (and formula (4.8) from the proof of that lemma).

Remark 4.11 We argued above that a predictable process satisfying (NA1) must be continuous. Therefore, Theorem 1.5 is not much more general than Ruf [Ruf13], where it is shown that a diffusion S that satisfies (NA1) admits a dominating local martingale measure. However, [Ruf13] only shows that supermartingale densities which are local martingales correspond to local martingale measures. Here we show that in the predictable case this is true for all supermartingale densities.

4.3 The general case

We start the treatment of the non-predictable case with two examples that illustrate why it is natural to consider dominating local martingale measures for $S^{\tau-}$ rather than for S.

Example 4.12 If Q is a dominating local martingale measure for S, then S does not need to satisfy (NA1): Let τ be standard exponentially distributed under Q. Define the uniformly integrable martingale $S_t = e^t 1_{\{t<\tau\}}$ for $t \in [0,1]$ and set $\mathrm{d}P = S_1 \mathrm{d}Q$. Under P we have $S_t = e^t$ for all $t \in [0,1]$, so that S does not satisfy (NA1) under P, despite the fact that Q is a dominating martingale measure for S. Note that $S^{\tau-}_t = e^t$, $t \in [0,1]$, is not a Q–local martingale.

Recall that a stopping time τ is called *foretellable* under a probability measure P if there exists an increasing sequence (τ_n) of stopping times such that $P(\tau_n < \tau) = 1$ for every n, and such that $P(\sup_n \tau_n = \tau) = 1$. In this case (τ_n) is called an *announcing sequence* for τ. Every predictable time is foretellable under any probability measure, see Theorem I.2.15 and Remark I.2.16 of [JS03].

Example 4.13 Let S be a semimartingale under P and let $Q \gg P$ be a dominating measure with Kunita-Yoeurp decomposition (Z,τ) with respect to P. Assume that τ is not foretellable under Q. Then there exists an adapted process $\widetilde{S}$ which is P–indistinguishable from S and so that $\widetilde{S}$ is not a Q–local martingale: Let $x \in \mathbb{R}^d$ and define $\widetilde{S}^x_t = S_t 1_{\{t<\tau\}} + x 1_{\{t\geq\tau\}}$, which is P–indistinguishable from S since $P(\tau = \infty) = 1$. If $\widetilde{S}^x$ is a Q–local martingale, then $\tau^x_n = \inf\{t \geq 0 : |\widetilde{S}^x_t| \geq n\}$, $n \in \mathbb{N}$, defines a localizing sequence. In particular, $Q[\lim_{n\to\infty} \tau^x_n \geq \tau] = 1$. Since τ is not foretellable under Q, there must exist $n \in \mathbb{N}$ for which $Q[\tau^x_n \geq \tau] > 0$. Moreover, we have

$$E_Q[S_0] = E_Q[\widetilde{S}^x_{\tau^x_n}] = E_Q[S_{\tau^x_n} 1_{\{\tau^x_n < \tau\}}] + xQ[\tau^x_n \geq \tau].$$

Since $\tau^x_n = \tau^y_n$ for all $|x| < n$, $|y| < n$, this holds for all $|x| < n$ – a contradiction.

The examples show that given $Q \gg P$ it is important to choose a good version of S if we want to obtain a Q–local martingale. All results so far indicate that this good version should be $S^{\tau-}$. Maybe somewhat surprisingly, this is not true in general, as we demonstrate in the following example.

Example 4.14 Let $L_t = N^1_t - N^2_t + bt$, $t \in [0,1]$, where N^1 and N^2 are independent Poisson processes and $b \neq 0$. Let $a > |b|$ and let ρ be an exponential random variable with parameter a, independent of L. Define $\tau = \rho$ if $\rho \leq 1$, and $\tau = \infty$ otherwise. Consider the probability measure $\mathrm{d}P = e^a 1_{\{1<\tau\}}\mathrm{d}Q$. By independence of L and τ, L has the same distribution under P as under Q. The Kunita-Yoeurp decomposition of Q with respect to P is given by $((e^{-at})_{t\in[0,1]}, \tau)$.

We claim that $Z = e^{-a\cdot}$ is a supermartingale density for L. Let $(\pi_t W^\pi_{t-})$ be a strategy for L, where W^π is the wealth process obtained by investing in this strategy. Such a strategy is 1–admissible if and only if $|\pi_t| \leq 1$ for all $t \in [0,1]$. Moreover, we get from (4.9) that

$$d(ZW^\pi)_t \sim -W^\pi_{t-}Z_{t-}adt + Z_{t-}\pi_t W^\pi_{t-}bdt = W^\pi_{t-}Z_{t-}(\pi_t b - a)dt.$$

Since $W^\pi Z \geq 0$ and $\pi_t b - a < 0$ (recall that $a > |b|$), the drift rate is negative. Therefore, ZW^π is a local supermartingale, and since it is nonnegative, it is a supermartingale.

Now τ is independent from L under Q, and L has no fixed jump times. Hence,

$$Q[\Delta L_\tau \neq 0, \tau \leq 1] = \int_{[0,1]} Q[\Delta L_t \neq 0](Q \circ \tau^{-1})[\mathrm{d}t] = 0,$$

which implies that $L^{\tau-} = L^\tau$, and this is clearly no Q–local martingale.

Remark 4.15 In the preceding example it is possible to show that the modified process

$$\widetilde{L}_t = L^{\tau-}_t - \frac{b}{a} 1_{\{t\geq\tau\}} \tag{4.11}$$

is a Q–martingale. More generally, we expect that given a semimartingale S, a supermartingale density Z for S, and a measure $Q \gg P$ with Kunita-Yoeurp decomposition (Z, τ) with respect to P, there should always exist a version $\widetilde{S}$ that is P–indistinguishable from S and so that $\widetilde{S}$ is a Q–local martingale. But as (4.11) shows, we will need to take different $\widetilde{S}$ for different supermartingale densities. This seems somewhat unnatural, and we will not pursue it further.

Note that all three examples had one thing in common: τ was not foretellable under Q. We will see that things are much simpler if τ is foretellable under Q. But if $(\tau_n)_{n\in\mathbb{N}}$ is an announcing sequence for τ, then we obtain from (4.2) that $1 = Q[\tau_n < \tau] = E_P[Z_{\tau_n} 1_{\{\tau_n<\infty\}}]$ for all $n \in \mathbb{N}$ and $0 = Q[\sup_n \tau_n < \tau] = E_P[Z_{\sup_n \tau_n} 1_{\{\sup_n \tau_n<\infty\}}]$. Since Z is strictly positive, we conclude that $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence for Z under P, i.e. Z is a P–local martingale.

Therefore, we should look for supermartingale densities that are local martingales, so called *local martingale densities.* If $(S_t)_{t\in[0,T]}$ is one dimensional with finite terminal time $T < \infty$, it is shown by Kardaras [Kar12] that local martingale densities exist if and only if (NA1) is satisfied. Takaoka and

Schweizer [TS14] prove the same result in the multidimensional case with finite terminal time. See also the recent preprint Song [Son13] for an alternative proof.

Of course [Kar12, TS14, Son13] all work with complete filtrations, but as usual we remove this assumption with the help of Lemma A.2.

Lemma 4.16 *Let S be a locally bounded semimartingale, and let Z be a local martingale density for S. Let τ be a stopping time and Q be a probability measure such that $(Z/E_P(Z_0), \tau)$ is the Kunita-Yoeurp decomposition of Q with respect to P. Then $S^{\tau-}$ is a Q–local martingale.*

Conversely, if $Q \gg P$ has Kunita-Yoeurp decomposition (Z, τ) with respect to P, and if $S^{\tau-}$ is a Q–local martingale, then Z is a supermartingale density for S.

Proof The proof is similar to the one of Corollary 4.9. Recall from Remark 4.10 that we only used the predictability of S once in the proof of Corollary 4.9, namely to obtain

$$E_Q[(H \cdot S^{\tau-})_{\sigma_n}] \leq 0 \tag{4.12}$$

for all strategies H that are 1–admissible for $(S^{\tau-})^{\sigma_n}$ under Q. Here (σ_n) was a localizing sequence of finite stopping times for the local martingale N under P, where $Z = Z_0 + N - B$. So it suffices to show that (4.12) always holds if Z has the decomposition $Z = Z_0 + N$, i.e. if $B = 0$.

Let (σ_n) be a localizing sequence of finite stopping times for the local martingale Z under P, and let H be a strategy that is 1–admissible for $(S^{\tau-})^{\sigma_n}$ under Q (and then also for S^{σ_n} under P). We apply Corollary 4.3 with $B = 0$ and that Z is a supermartingale density to obtain

$$E_Q[1 + (H \cdot S^{\tau-})_{\sigma_n}] = E_P[(1 + (H \cdot S)_{\sigma_n})Z_{\sigma_n}] \leq 1.$$

From here on we can just copy the proof of Corollary 4.9.

Theorem 1.6 now easily follows by combining Theorem 1.3, Corollary 1.4, and Lemma 4.16.

Remark 4.17 There is another subset of supermartingale densities of which one might expect that they correspond to local martingale measures for $S^{\tau-}$: the maximal elements among the supermartingale densities. A supermartingale density Z is called *maximal* if it is indistinguishable from any supermartingale density Y that satisfies $Y_t \geq Z_t$ for all $t \geq 0$. If S is discontinuous, then some maximal supermartingale densities are no local martingales, see Example 5.1' of [KS99].

But such Z will usually not correspond to local martingale measures for $S^{\tau-}$. Assume for example that we are in the situation of Theorem 2.2 in [KS99]. That is, we have a dual optimizer Z and a primal optimizer H for a certain utility maximization problem. Then point iii) of this Theorem 2.2 states that $(1 + (H \cdot S))Z$ is a uniformly integrable martingale. If we assume now that Z

is not a local martingale, and if (τ_n) is a localizing sequence of finite stopping times for the local martingale part N of $Z = Z_0 + N - B$, then we obtain from Corollary 4.3 that

$$E_Q[1 + (H \cdot S^{\tau-})_{\tau_n}] = E_P[(1 + (H \cdot S)_{\tau_n})Z_{\tau_n}] + E_P\Big[\int_0^{\tau_n} (1 + (H \cdot S)_{s-})\mathrm{d}B_s\Big] = 1 + E_P\Big[\int_0^{\tau_n} (1 + (H \cdot S)_{s-})\mathrm{d}B_s\Big], \tag{4.13}$$

Since H is optimal, the wealth process $(1+(H\cdot S)_{s-})_{s\geq 0}$ will be strictly positive with positive probability. Since also $\mathrm{d}B \neq 0$ with positive probability, the expectation in (4.13) is strictly positive for sufficiently large n, and therefore $S^{\tau-}$ cannot be a Q–local martingale.

5 Relation to filtration enlargements

Here we show that Jacod's criterion for initial filtration enlargements is in fact a criterion for the existence of a universal supermartingale density (to be defined below) and we show that it preserves (NA1) for all continuous processes. For general filtration enlargements we show that if there exists a universal supermartingale density, then a generalized version of Jacod's criterion is necessarily satisfied.

5.1 Jacod's criterion and universal supermartingale densities

In this subsection we work on a general probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and not necessarily on a path space. Let $\mathcal{G}_t^0 = \mathcal{F}_t \vee \sigma(X)$, $t \geq 0$ be an initial filtration enlargement of $(\mathcal{F}_t)$ with the random variable X. We define the right-continuous regularization of $(\mathcal{G}_t^0)$ by setting $\mathcal{G}_t := \bigcap_{s>t} \mathcal{G}_s^0$ for all $t \geq 0$. Recall that Hypothèse (H') is satisfied if all $(\mathcal{F}_t)$–semimartingales are $(\mathcal{G}_t)$-semimartingales.

We now give the classical formulation of Jacod's criterion [Jac85]. For this purpose, we assume that X takes its values in a standard Borel $(\mathbb{X}, \mathcal{B})$. For the definition of standard Borel spaces see Parthasarathy [Par67], Definition V.2.2. For a detailed discussion see also Dellacherie [Del69], where standard Borel spaces are referred to as Lusin spaces. Note that $(\mathbb{X}, \mathcal{B})$ is a standard Borel space provided that $\mathbb{X}$ is a Polish space and $\mathcal{B}$ its Borel σ–algebra.

If X takes its values in a standard Borel space, then the regular conditional distribution

$$P_t[\omega, \mathrm{d}x] := P[X \in \mathrm{d}x | \mathcal{F}_t](\omega)$$

exists for all $t \geq 0$. We write P_X for the distribution of X. Jacod's criterion states that Hypothèse (H') is satisfied provided that for every $t \geq 0$ almost surely

$$P_t[\omega, \mathrm{d}x] \ll P_X[\mathrm{d}x]. \tag{5.1}$$

Note that this statement only makes sense if the set $\{\omega : P_t[\omega, \mathrm{d}x] \ll P_X[\mathrm{d}x]\}$ is $\mathcal{F}$–measurable. But since the σ–algebra of a standard Borel space is countably generated (see also [PR14]), it is easily verified that this is indeed the case. Below we give an alternative proof of Jacod's result, and we relate it to the existence of a *universal supermartingale density.*

First observe that Hypothèse (H') is satisfied if and only if all nonnegative $(\mathcal{F}_t)$–martingales are $(\mathcal{G}_t)$–semimartingales: This follows by decomposing every $(\mathcal{F}_t)$–local martingale into a sum of a locally bounded local martingale and a local martingale of finite variation, by observing that every bounded process can be made nonnegative by adding a deterministic constant, and from the fact that local semimartingales are semimartingales (see Protter [Pro04], Theorem II.6).

Definition 5.1 Let $(\mathcal{G}_t)$ be a filtration enlargement of $(\mathcal{F}_t)$. Let Z be a $(\mathcal{G}_t)$–adapted process that is almost surely càdlàg, such that $P[Z_t > 0] = 1$ for all $t \geq 0$. Then Z is called *universal supermartingale density* for $(\mathcal{G}_t)$ if ZM is a $(\mathcal{G}_t)$–supermartingale for every nonnegative $(\mathcal{F}_t)$–supermartingale M.

Note that here we do not require Z_∞ to be strictly positive, unlike in the previous sections. This is because here we are primarily interested in the semimartingale property and not whether (NA1) holds. Local semimartingales are semimartingales, and therefore it suffices to verify the $(\mathcal{G}_t)$–semimartingale property of M on every compact interval $[0, t]$.

The first result of this section shows that Jacod's criterion is not so much a criterion for Hypothèse (H') to hold, but rather for the existence of universal supermartingale densities.

Proposition 5.2 *Assume that Jacod's criterion* (5.1) *is satisfied. Then there exists a universal supermartingale density for* $(\mathcal{G}_t)$*.*

Proof 1. Let $t \geq 0$. Without loss of generality we may assume that $\mathrm{d}P_t[\omega, \cdot] \ll \mathrm{d}P_X[\cdot]$ for *all* $\omega \in \Omega$. This can be achieved by setting $P_t[\omega, \cdot] := 0$ on the measurable set $\{\omega : P_t(\omega)$ does not satisfy $P_t[\omega, \cdot] \ll P_X[\cdot]\}$. Now we can apply a theorem of Doob, see [YM78], according to which there exists a $\mathcal{F}_t \otimes \mathcal{B}$–measurable random variable $Y_t : \Omega \times \mathbb{X} \to \mathbb{R}_+$ so that for every $\omega \in \Omega$ we have P_X–almost surely

$$Y_t(\omega, x) = \frac{\mathrm{d}P_t[\omega, \cdot]}{\mathrm{d}P_X}(x).$$

Note that Yor and Meyer [YM78] do not require complete σ–algebras. Let now $t, s \geq 0$. We first show that $P \otimes P_X$–almost surely

$$\{(\omega, x) : Y_t(\omega, x) = 0\} \subseteq \{(\omega, x) : Y_{t+s}(\omega, x) = 0\}. \tag{5.2}$$

Note that $Y_{t+s} \geq 0$, and therefore Fubini's theorem and the tower property of conditional expectations imply that

$$\begin{aligned}\int_{\Omega\times\mathbb{X}} &1_{\{Y_t(\omega,x)=0\}} Y_{t+s}(\omega,x) P\otimes P_X[\mathrm{d}\omega,\mathrm{d}x] \\ &= \int_\Omega \int_{\mathbb{X}} 1_{\{Y_t(\omega,x)=0\}} P_{t+s}[\omega,\mathrm{d}x]P[\mathrm{d}\omega] \\ &= \int_\Omega \int_{\mathbb{X}} 1_{\{Y_t(\omega,X(\omega))=0\}} P[\mathrm{d}\omega] \\ &= \int_\Omega \int_{\mathbb{X}} 1_{\{Y_t(\omega,x)=0\}} P_t[\omega,\mathrm{d}x]P[\mathrm{d}\omega] = 0.\end{aligned}$$

2. Define $\widetilde{Z}_t(\omega,x) := 1_{\{Y_t(\omega,x)>0\}}/Y_t(\omega,x)$ and $Z_t(\omega) := \widetilde{Z}_t(\omega,X(\omega))$. This Z is $(\mathcal{G}_t)$–adapted by construction. Let now M be a nonnegative $(\mathcal{F}_t)$–supermartingale, and let $s,t\geq 0$, $A\in\mathcal{F}_t$, and $B\in\mathcal{B}$. Then we can apply the tower property to obtain

$$\begin{aligned}E&\left[1_A 1_B(X) M_{t+s} Z_{t+s}\right] \\ &= \int_\Omega 1_A(\omega) M_{t+s}(\omega) \int_{\mathbb{X}} 1_B(x) \widetilde{Z}_{t+s}(\omega,x) P_{t+s}[\omega,\mathrm{d}x]P[\mathrm{d}\omega] \\ &= \int_\Omega 1_A(\omega) M_{t+s}(\omega) \int_{\mathbb{X}} 1_B(x) \frac{Y_{t+s}(\omega,x)}{Y_{t+s}(\omega,x)} 1_{\{Y_{t+s}(\omega,x)>0\}} P_X[\mathrm{d}x]P[\mathrm{d}\omega] \\ &\leq \int_\Omega 1_A(\omega) M_{t+s}(\omega) \int_{\mathbb{X}} 1_B(x) 1_{\{Y_t(\omega,x)>0\}} P_X[\mathrm{d}x]P[\mathrm{d}\omega],\end{aligned}$$

where in the last step we applied (5.2). Using the $(\mathcal{F}_t)$–supermartingale property of M in conjunction with Fubini's theorem, we obtain

$$\begin{aligned}\int_\Omega &1_A(\omega) M_{t+s}(\omega) \int_{\mathbb{X}} 1_B(x) 1_{\{Y_t(\omega,x)>0\}} P_X[\mathrm{d}x]P[\mathrm{d}\omega] \\ &\leq \int_{\mathbb{X}} 1_B(x) \int_\Omega 1_A(\omega) M_t(\omega) 1_{\{Y_t(\omega,x)>0\}} P[\mathrm{d}\omega]P_X[\mathrm{d}x] \\ &= \int_\Omega 1_A(\omega) \int_{\mathbb{X}} 1_B(x) M_t(\omega) \widetilde{Z}_t(\omega,x) P_t[\omega,\mathrm{d}x]P[\mathrm{d}\omega] = E\left[1_A 1_B(X) M_t Z_t\right].\end{aligned}$$

The monotone class theorem allows to pass from sets of the form $A\cap X^{-1}(B)$ to general sets in $(\mathcal{G}_t^0)$, and therefore MZ is a $(\mathcal{G}_t^0)$–supermartingale. Taking $M\equiv 1$, we see that also Z is a $(\mathcal{G}_t^0)$–supermartingale.

3. Let us show that Z_t is P–almost surely strictly positive for every $t\geq 0$. For this purpose it suffices to show that $P[\omega : Y_t(\omega,X(\omega))=0]=0$. By the tower property we have

$$E[1_{\{Y_t(\cdot,X(\cdot))=0\}}] = \int_\Omega \int_{\mathbb{X}} 1_{\{Y_t(\omega,x)=0\}} P_t[\omega,\mathrm{d}x]P[\mathrm{d}\omega] = 0.$$

4. Z is not necessarily almost surely càdlàg, and also ZM is only a $(\mathcal{G}_t^0)$–supermartingale and not necessarily a $(\mathcal{G}_t)$–supermartingale. But to conclude the proof, it suffices to pass to the right limit process of Z. This can be done using standard techniques as the ones on page 59 of [EK86]. A similar construction (with an emphasis on the incomplete nature of our filtration) is carried out in the proof of Theorem 1.3.1 in [Per14].

Remark 5.3 If we are only interested whether Hypothèse (H') holds and not whether there exists a universal supermartingale density, then we can also work with $(\mathcal{G}_t^0)$ instead of its right-continuous regularization $(\mathcal{G}_t)$. Since Hypothèse (H') holds for $(\mathcal{G}_t)$ and since $(\mathcal{G}_t^0)$ is a filtration shrinkage of $(\mathcal{G}_t)$, Stricker's theorem implies that Hypothèse (H') is also satisfied for $(\mathcal{G}_t^0)$.

Next, we are interested in the (NA1) property under filtration enlargements. The following result is due to Ankirchner who proved that for continuous processes the structure condition is preserved under filtration enlargements satisfying Jacod's criterion. Here we give an alternative proof working directly with the definition of (NA1).

Proposition 5.4 ([Ank05], Theorem 2.1.11) *Assume that Jacod's criterion holds, let $T > 0$, and let $(S_t)_{t\in[0,T]}$ be a d–dimensional continuous $(\mathcal{F}_t)$–adapted process satisfying (NA1) under $(\mathcal{F}_t)$. Then S satisfies (NA1) under $(\mathcal{G}_t)$.*

Proof As in Section 2 we work with the equivalent formulation of Jacod's criterion given in [FI93]. Consider the product space $\overline{\Omega} := \Omega \times \mathbb{X}$ equipped with the σ–algebra $\overline{\mathcal{F}} := \mathcal{F} \otimes \mathcal{B}$ and the filtration $\overline{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{B}$, $t \geq 0$. Define $\psi \colon \Omega \to \overline{\Omega}$, $\psi(\omega) := (\omega, X(\omega))$ and the two probability measures $\overline{Q} := P \otimes P_X$ and $\overline{P} := P \circ \psi^{-1}$. We embed random variables from $(\Omega, \mathcal{F})$ to $(\overline{\Omega}, \overline{\mathcal{F}})$ by setting $\overline{Y}(\omega, x) = Y(\omega)$. Jacod's criterion is then equivalent to $\overline{P}|_{\overline{\mathcal{F}}_t} \ll \overline{Q}|_{\overline{\mathcal{F}}_t}$ for all $t \geq 0$. We define $\overline{\tau} := \inf\{t \geq 0 : \overline{\gamma}_t = 0\}$, where $\overline{\gamma}$ is a right-continuous version of the density process.

If now $(S_t)_{t\in[0,T]}$ satisfies (NA1) under P, then the independence of the coordinate projections under $\overline{Q}$ shows that $\overline{S}$ satisfies (NA1) under $\overline{Q}$ and has the same distribution as S under P. In particular, $\overline{S}$ is $\overline{Q}$–almost surely continuous and therefore also $\overline{S}^{\overline{\tau}-} = \overline{S}^{\overline{\tau}}$ satisfies (NA1) under $\overline{Q}$. Since any 1–admissible strategy for $\overline{S}$ under $\overline{P}$ is 1–admissible for $\overline{S}^{\overline{\tau}-}$ under $\overline{Q}$, the process $\overline{S}$ satisfies (NA1) under $\overline{P}$. Let now $\overline{H}$ be a 1–admissible strategy for $\overline{S}$ under $\overline{P}$, and let $m \in \mathbb{N}$. Then $\overline{P}(1 + (\overline{H} \cdot \overline{S})_T \geq m) = P(1 + (H(\cdot, X) \cdot S)_T \geq m)$, so that S satisfies (NA1) in the filtration $(\mathcal{G}_t^0)$. By right-continuity of S, we easily deduce the (NA1) property in $(\mathcal{G}_t)$.

Remark 5.5 If $\overline{\gamma}$ and $\overline{\tau}$ are as defined in the proof, if Z^S is a supermartingale density for the continuous process $(S_t)_{t\in[0,T]}$ under P in the filtration $(\mathcal{F}_t)$, and if $\overline{Z}_t = 1_{t<\overline{\tau}}/\overline{\gamma}_t$, $t \geq 0$, then one can show that $Y_t(\omega) = Z_t^S(\omega)\overline{Z}_t(\omega, X(\omega))$ defines a supermartingale density for S under $(\mathcal{G}_t)$. The key point is again that $\overline{S}^{\overline{\tau}-} = \overline{S}^{\overline{\tau}}$ under $\overline{Q}$, so that $\overline{S}^{\overline{\tau}-}$ satisfies (NA1) under $\overline{Q}$.

Remark 5.6 Continuity is necessary: Consider a process S starting in 1 and at time 1 jumping to 1/2 or 2, both with strictly positive probability. Then S satisfies (NA1) in its natural filtration, but of course not in the enlarged filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(S_1)$, $t \geq 0$. But S_1 is a discrete random variable and therefore Jacod's criterion is satisfied for $(\mathcal{G}_t)$. For a more detailed study of the discontinuous case, we refer to the recent preprint [AFK14], see also [ACDJ13].

Remark 5.7 At this point it is clear that to preserve the (NA1) property in infinite time, we would have to require $P_\infty[\omega, \mathrm{d}x] \ll P_X[\mathrm{d}x]$, where $P_\infty[\omega, \mathrm{d}x] = P[X \in \mathrm{d}x | \bigvee_{t\geq 0} \mathcal{F}_t]$. If $\mathcal{F} = \bigvee \mathcal{F}_t$, then this condition is satisfied if and only if X is a discrete random variable.

Remark 5.8 We could replace assumption (5.1) by $P_t[\omega, \mathrm{d}x] \gg P_X[\mathrm{d}x]$ and use the same proof as for Proposition 5.2 to obtain the existence of a nonnegative martingale Z such that ZM is a $(\mathcal{G}_t)$–supermartingale for every nonnegative $(\mathcal{F}_t)$–supermartingale M. In particular, then there exists a locally absolutely continuous measure Q so that every locally bounded $(P, (\mathcal{F}_t))$–local martingale is a $(Q, (\mathcal{G}_t))$–local martingale. Since (NA) is related to the existence of absolutely continuous local martingale measures [DS95b], this indicates that the (NA) property may be stable under such a "reverse Jacod criterion".

If $P_t[\omega, \mathrm{d}x] \sim P_X[\mathrm{d}x]$, we obtain a locally equivalent measure Q under which every nonnegative $(P, (\mathcal{F}_t))$–supermartingale is a nonnegative $(Q, (\mathcal{G}_t))$–supermartingale, and in particular every locally bounded $(P, (\mathcal{F}_t))$–local martingale is a $(Q, (\mathcal{G}_t))$–local martingale. This condition has been studied by Amendinger et al. [AIS98, Ame00] and is of course harder to satisfy than Jacod's criterion or the reverse Jacod criterion. In financial applications one may however assume that the "insider's knowledge" is perturbed by a small Gaussian noise which is independent of $\bigvee \mathcal{F}_t$ (or more generally by an independent noise whose distribution is equivalent to the Lebesgue measure). Then $P_\infty[\omega, \mathrm{d}x] \sim P_X[\mathrm{d}x]$ is always satisfied.

5.2 Universal supermartingale densities and the generalized Jacod criterion

In the previous section, we saw that Jacod's criterion is a sufficient condition for the existence of universal supermartingale densities under initial filtration enlargements. Here we show that for general filtration enlargements, a generalized version of Jacod's criterion is necessary for the existence of universal supermartingale densities.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space such that $(\Omega, \mathcal{F})$ is a standard Borel space; for example $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ could be a path space or the explosive path space on which we worked in Sections 1 to 4. We assume that there exists a filtration $(\mathcal{F}^0_t)_{t\geq 0}$ with $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}^0_s$ for all $t \geq 0$. We also assume that $(\mathcal{G}^0_t)_{t\geq 0}$ is a filtration enlargement of $(\mathcal{F}^0_t)$ such that $\mathcal{G}^0_t \subseteq \mathcal{F}$ is countably generated for every $t \geq 0$, that is there exists a sequence of sets $(B^t_n)_{n\in\mathbb{N}}$ with $\mathcal{G}^0_t = \sigma(B^t_1, B^t_2, \dots)$. Then $(\mathcal{G}_t)_{t\geq 0}$, defined by $\mathcal{G}_t := \bigcap_{s>t} \mathcal{G}^0_t$, is a filtration enlargement of $(\mathcal{F}_t)$.

The reason for choosing such a complicated set-up is that $\mathcal{G}_t$ will in general not be countably generated, but in our argumentation below we will need a countably generated filtration. On the other side, if we would only work with the non right-continuous filtration $(\mathcal{G}_t^0)$, then there would be little hope of constructing a right-continuous universal supermartingale density in the first place.

Since $(\Omega, \mathcal{F})$ is a standard Borel space, the regular conditional probabilities

$$P_t[\omega, \cdot] := P[\cdot|\mathcal{F}_t](\omega), \qquad t \geq 0,$$

exist. We say that the *generalized Jacod criterion* is satisfied if for all $s, t \geq 0$ almost surely

$$P_{t+s}|_{\mathcal{G}_t^0}[\omega, \cdot] \ll P_t|_{\mathcal{G}_t^0}[\omega, \cdot].$$

It is known that neither Jacod's criterion nor the generalized Jacod criterion are necessary conditions for Hypothèse (H') to hold. But the generalized Jacod criterion is a necessary condition for the existence of a universal supermartingale density for $(\mathcal{G}_t)$:

Proposition 5.9 *Assume that there exists a universal supermartingale density Z for $(\mathcal{G}_t)$. Then the generalized Jacod criterion is satisfied.*

Proof For all $A \in \mathcal{F}$, the process $M_t^A := E_P[1_A|\mathcal{F}_t]$, $t \geq 0$, is a nonnegative $(\mathcal{F}_t)$–martingale. Therefore, $M^A Z$ is a $(\mathcal{G}_t)$–supermartingale. Fix $s, t \geq 0$, let $A \in \mathcal{F}_{t+s}$, and $B \in \mathcal{G}_t$. Then

$$E\Big[1_A 1_B \frac{Z_{t+s}}{Z_t}\Big] \leq E\Big[\frac{1_B}{Z_t} M_t^A Z_t\Big] = E[1_A E[1_B|\mathcal{F}_t]].$$

The same inequality holds if we replace Z_{t+s}/Z_t by a version $\widetilde{Z}_{t+s}/\widetilde{Z}_t$ that is strictly positive for *all* $\omega \in \Omega$. Since the inequality holds for all $A \in \mathcal{F}_{t+s}$, we conclude that

$$\int 1_B(\omega') \frac{\widetilde{Z}_{t+s}}{\widetilde{Z}_t}(\omega') P_{t+s}[\omega, \mathrm{d}\omega'] \leq P_t[\omega, B] \text{ for almost all } \omega \in \Omega. \tag{5.3}$$

This looks promising. The only problem is that the null set outside of which the inequality holds may depend on B.

Now we use the assumption that $\mathcal{G}_t^0$ is countably generated, which gives us a countable algebra $\mathcal{H}_t$ with $\mathcal{G}_t^0 = \sigma(\mathcal{H}_t)$. From (5.3) we get a null set $\mathcal{N}$ such that for all $\omega \in \Omega \setminus \mathcal{N}$ and all $B \in \mathcal{H}_t$

$$\int 1_B(\omega') \frac{\widetilde{Z}_{t+s}}{\widetilde{Z}_t}(\omega') P_{t+s}[\omega, \mathrm{d}\omega'] \leq P_t[\omega, B]. \tag{5.4}$$

By the monotone class theorem, this extends to $B \in \sigma(\mathcal{H}_t) = \mathcal{G}_t^0$. To complete the proof, it suffices to recall that $\widetilde{Z}_{t+s}(\omega')/\widetilde{Z}_t(\omega') > 0$ for all $\omega' \in \Omega$.

Corollary 5.10 *Suppose that there exists a one dimensional continuous local martingale M which has the predictable representation property under $(\mathcal{F}_t)$. If under $(\mathcal{G}_t)$, the semimartingale decomposition of M is of the form*

$$M_t = \widetilde{M}_t + \int_0^t \alpha_s \mathrm{d}\langle \widetilde{M} \rangle_s, \qquad t \geq 0,$$

for a $(\mathcal{G}_t)$–local martingale $\widetilde{M}$ and a predictable integrand $\alpha \in L^2_{loc}(\widetilde{M})$, then the generalized Jacod criterion holds.

Proof The stochastic exponential $Z := \mathcal{E}(-\alpha \cdot M)$ is a universal supermartingale density.

Corollary 5.10 was previously shown in [IPW01] for initial enlargements and under the stronger assumption $E[\int_0^\infty \alpha_s^2 \mathrm{d}\langle \widetilde{M} \rangle_s] < \infty$.

Of course, the same argument works in a multidimensional setting: if $M = (M^1, \ldots, M^d)$ has the predictable representation property under $(\mathcal{F}_t)$, and if M satisfies the structure condition under $(\mathcal{G}_t)$, then the generalized Jacod criterion is satisfied.

A Incomplete filtrations

Here we collect some classical observations which allow to transfer results of other authors that were obtained under complete filtrations to our setting. There are at least two important monographs which avoid the use of complete filtrations as far as possible, Jacod [Jac79] and Jacod and Shiryaev [JS03]. Here we follow [JS03].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space with a right-continuous filtration $(\mathcal{F}_t)$. Write $\mathcal{F}^P$ for the P–completion of $\mathcal{F}$, and $\mathcal{N}^P$ for the P–null sets of $\mathcal{F}^P$. Then $\mathcal{F}^P_t = \mathcal{F}_t \vee \mathcal{N}^P$, $t \geq 0$, satisfies the usual conditions. It is well known and easy to show that for every random variable X on $(\Omega, \mathcal{F}^P)$ there exists a random variable Y on $(\Omega, \mathcal{F})$ with $P(X = Y) = 1$.

Recall that the optional σ–algebra over $(\mathcal{F}_t)$ is the σ–algebra on $\Omega \times [0, \infty)$ that is generated by all processes of the form $X_t(\omega) = 1_A(\omega) 1_{[r,s)}(t)$ for some $0 \leq r < s < \infty$ and $A \in \mathcal{F}_r$. The predictable σ–algebra over $(\mathcal{F}_t)$ is the σ–algebra on $\Omega \times [0, \infty)$ that is generated by all processes of the form $X_t(\omega) = 1_A(\omega)1_{\{0\}}(t) + 1_B(\omega)1_{(r,s]}(t)$ for some $0 \leq r < s < \infty$, for $A \in \mathcal{F}_0$, and $B \in \mathcal{F}_r$. Similarly we define the predictable and optional σ–algebras over $(\mathcal{F}^P_t)$.

The first result relates stopping times under $(\mathcal{F}_t)$ and under $(\mathcal{F}^P_t)$.

Lemma A.1 (Lemma I.1.19 of [JS03]) *Any stopping time on the completion $(\Omega, (\mathcal{F}^P_t))$ is almost surely equal to a stopping time on $(\Omega, (\mathcal{F}_t))$.*

We also have a comparable result on the level of processes.

Lemma A.2 *Any predictable (respectively optional) process on the completion $(\Omega, (\mathcal{F}^P_t))$ is indistinguishable from a predictable (respectively optional) process on $(\Omega, (\mathcal{F}_t))$.*

Proof The predictable case is Lemma I.2.17 of [JS03]. The proof of the optional case works exactly in the same way: the claim is trivial for the generating processes described above, and we can use the monotone class theorem to pass to indicator functions of general optional sets. Then we use monotone convergence to pass to general optional processes.

This allows us to deduce a similar result for càdlàg processes.

Lemma A.3 *Let S be a $(\mathcal{F}_t^P)$–adapted process that it almost surely càdlàg. Then S is indistinguishable from a $(\mathcal{F}_t)$–adapted process (which is then of course almost surely càdlàg as well).*

Proof Since $(\mathcal{F}_t^P)$ is complete, S admits an indistinguishable version $\widetilde{S}$ that is $(\mathcal{F}_t^P)$–adapted and càdlàg for *every* $\omega \in \Omega$. This $\widetilde{S}$ is optional, so now the result follows from Lemma A.2.

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