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The existence of irrational pencils on algebraic varieties

By

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We have a classical criterion whether a given algebraic variety has an irrational pencil or not.[1] We shall give here a purely algebro-geometric condition for a variety to have an irrational pencil. From this result we can derive somewhat interesting consequences concerning differential forms of the first kind on algebraic varieties. To our great regret, the method employed here is not applicable to the case of prime characteristic. We must restrict ourselves to the algebraic geometry over the field of all complex numbers.

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§ 1. Preliminary lemmas.

In this § we can omit the restriction on the characteristic of the universal domain.

LEMMA 1. *Let U be a variety[2] and V a complete variety without singular points such that there exists a rational map π from U onto V. Let f be a function on V, then $f\circ\pi$ is a function on U and we have*

$$(f\circ\pi)=\pi^{-1}((f)).$$

PROOF. Let k be a common field of definition for U, V, π and f, and P, Q corresponding generic points of U and V over k. We

1) Cf. Franchis (2). The numbers in bracket refer to the bibliography at the end of the paper.

2) Throughout this paper we shall follow the terminologies and notations used in Weil (7), we shall cite this book by (F).

shall put $F=f\circ\pi$, then it is clear that F is a function on $\boldsymbol{U}$ defined over k and $F(\boldsymbol{P})=f(\pi(\boldsymbol{P}))=f(\boldsymbol{Q})$. Let $\boldsymbol{\Gamma}_F$, $\boldsymbol{\Gamma}_f$ and $\boldsymbol{\Lambda}$ be the graphs of the functions $\boldsymbol{F}$, f and the map π respectively, and $\boldsymbol{T}$ the locus of $\boldsymbol{P}\times\boldsymbol{Q}\times f(\boldsymbol{Q})$ over k in $\boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{D}$, where $\boldsymbol{D}$ denotes as usual the projective straight line. Then we have, $\mathrm{pr}_{U\times V}\boldsymbol{T}=\boldsymbol{\Lambda}$, $\mathrm{pr}_{U\times D}\boldsymbol{T}=\boldsymbol{\Gamma}_F$. By definition

$$(f)=\mathrm{pr}_V[\boldsymbol{\Gamma}_f\cdot(\boldsymbol{V}\times\theta)]$$

where θ is the cycle of degree zero on the projective straight line $\boldsymbol{D}$ defined by $\theta=(0)-(\infty)$ and we have by (F)–VII, Th. 16.

$$\begin{aligned}\pi^{-1}((f))&=\mathrm{pr}_U[\boldsymbol{\Lambda}(\boldsymbol{U}\times(f))]\\ (1)\qquad &=\mathrm{pr}_U[\boldsymbol{\Lambda}\{\mathrm{pr}_{U\times V}(\boldsymbol{U}\times\boldsymbol{V}\times\theta)(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)\}]\\ &=\mathrm{pr}_U[(\boldsymbol{\Lambda}\times\boldsymbol{D})\{(\boldsymbol{U}\times\boldsymbol{V}\times\theta)(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)\}]\end{aligned}$$

if the right hand side is defined. Now we shall calculate the intersection product $(\boldsymbol{\Lambda}\times\boldsymbol{D})(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)$. Since $\boldsymbol{\Lambda}\times\boldsymbol{D}$ is not contained in $\boldsymbol{U}\times\boldsymbol{\Gamma}_f$ these two subvarieties intersect properly on $\boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{D}$. Now it is clear that $\boldsymbol{T}$ is a component of $(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)\cap(\boldsymbol{\Lambda}\times\boldsymbol{D})$, then we can put

$$(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)\cdot(\boldsymbol{\Lambda}\times\boldsymbol{D})=a\boldsymbol{T}+\sum_i a_i\boldsymbol{A}_i$$

Taking the algebraic projection on $\boldsymbol{U}\times\boldsymbol{V}$, we see that $a=1$ and $\mathrm{pr}_{U\times V}\boldsymbol{A}_i=0$. Hence $\boldsymbol{A}_i$ must be of the form $\boldsymbol{A}_i'\times\boldsymbol{D}$, where $\boldsymbol{A}_i'$ are subvarieties of $\boldsymbol{U}\times\boldsymbol{V}$ of dimension $m-1$. Writing $\boldsymbol{X}=\sum_i a_iA_i'$ we have

$$(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)\cdot(\boldsymbol{\Lambda}\times\boldsymbol{D})=\boldsymbol{T}+\boldsymbol{X}\times\boldsymbol{D}.$$

Thus we see that $\boldsymbol{U}\times\boldsymbol{V}\times\theta$ intersect properly with $(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)(\boldsymbol{\Lambda}\times\boldsymbol{D})$ and these three cycles intersect properly on $\boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{D}$ by (F)-VIII, Cor. of Th. 10. Moreover the cycle $(\boldsymbol{U}\times\boldsymbol{V}\times\theta)(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)$ is defined on $\boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{D}$. Hence the right hand side of (1) is defined by the same corollary. Now interchanging the order of intersection, we have

$$\begin{aligned}\pi^{-1}((f))&=\mathrm{pr}_U[\{(\boldsymbol{\Lambda}\times\boldsymbol{D})(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)\}(\boldsymbol{U}\times\boldsymbol{V}\times\theta)]\\ &=\mathrm{pr}_U[\{\mathrm{pr}_{U\times D}(\boldsymbol{\Lambda}\times\boldsymbol{D})(\boldsymbol{U}\times\boldsymbol{\Gamma}_f)\}(\boldsymbol{U}+\theta)]\\ &=\mathrm{pr}_U\{(\boldsymbol{\Gamma}_F+\mathrm{pr}_U\boldsymbol{X}\times\boldsymbol{D})(\boldsymbol{U}\times\theta)\}=(F)\end{aligned}$$

Thus the proof is completed. q. e. d.

COROLLARY. *Under the same hypothesis as in Lemma* 1, *if*

f is defined everywhere on V, then we have

$$(f\circ\pi)_0=\pi^{-1}((f)_0),\quad (f\circ\pi)_\infty=\pi^{-1}((f)_\infty).$$

PROOF. If is clear from the above proof, that it will be sufficient to show that the cycle X appeared in the proof is zero cycle. Suppose that A_i' is not zero and let B_i' be the projection of A_i' on V. Then we see that $B_i'\times D$ is contained in the graph Γ_f of f. Hence f is indeterminate for the point in let B_i', which proves our Corollary. q. e. d.

DEFINITION. *Let V be a variety, Γ a non singular curve of genus $g(\geqq 1)$ such that there exists a rational map π from V onto Γ. Then we can define the V-divisors $\pi^{-1}(M)$ for any point M of Γ. We get thus an algebraic system*[3] *$\{\pi^{-1}(M)\}$ which will be called an irractional pencil of genus g with the parameter curve Γ.*

Let U, V be two varieties, and π a rational map from U onto V. Let P and Q be corresponding generic points of U and V over a common field of definition k for U, V and π. Then we shall say that the rational map π is *separable* if $k(P)$ is separably generated over $k(Q)$. As we can easily see, this definition does not depend on the choice of k, P and Q. In this case, as is well known there exists a map $\delta\pi$ from the module of differential forms on V into the module of differential forms on U defined by

$$\delta\pi\Big(\sum_{i_1<\cdots<i_p}\varphi_{i_1\cdots i_p}\,d\xi_{i_1}\cdots d\xi_{i_p}\Big)=\sum_{i_1<\cdots<i_p}\varphi_{i_1\cdots i_p}\circ\pi\cdot(d(\xi_{i_1}\circ\pi))\cdots(d(\xi_{i_p}\circ\pi))$$

where φ's, ξ's are functions on V.

LEMMA 2. *Let U^m, V^n be complete varieties such that there exists a separable rational map π from U onto V and ω a differential form of the first kind on V. Then if V has a birationally equivalent non-singular complete model, the differential form $\delta\pi(\omega)$ on U is also of the first kind.*

PROOF. Suppose that $\delta\pi(\omega)$ is not of the first kind. Then there exists a birationally equivalent variety U' to U such that $\delta\pi(\omega)$ has a pole variety on U'[4]. Suppose that U has already this property and V has no singular point. Let A^{m-1} be a pole variety of $\delta\pi(\omega)$ and P a generic point of A over a common field of definition for U, V and A. Now by definition A is a simple subvariety[5] of U, hence the point $Q=\pi(P)$ is well defined. Let

3) Such algebraic system is often called an involutional system.

4) Cf. Koizumi (4).

5) The terminologies concerning differential forms we shall refer to the paper Nakai (5).

$\xi_1, \cdots, \xi_n$ be uniformizing parametiers on V at Q. Then expressing ω in the form

$$\omega = \sum_{i_1 < \cdots < i_p (\leq n)} \varphi_{i_1 \cdots i_p} d\xi_{i_1} \cdots d\xi_{i_p}$$

we see easily that the differential form

$$\delta\pi(\omega) = \sum_{i_1 < \cdots < i_p (\leq n)} \varphi_{i_1 \cdots i_p} \circ \pi \cdot d(\xi_{i_1} \circ \pi) \cdots d(\xi_{i_p} \circ \pi)$$

are also finite along A. This is a contradiction to our hypothesis. Thus the Lemma is proved. q. e. d.

Remark. We can take off the restriction that "*if V has a birationally equivalent non-singular complete model*", provided the following proposition is proved: *The differential form of the first kind has the property (F) defined in Koizumi* (4). But this is still an open question.

§ 2. The existence theorem.

THEOREM 1. *Let V^r be a complete variety without singular subvorieties of dimension $r-1$. Then V has an irrational pencil of genus ≥ 1, if and only if there exist two functions f and g such that the differential $\omega = fdg$ is of the first kind.*

PROOF. Suppose that V has an irrational pencil, then we see immediately, by Lemma 2, that V has the differential form of the first kind of the required form. Suppose that there exist two functions f and g on V such that $\omega = fdg$ is of the first kind. We shall first show that two functions f and g cannot be algebraically independent. Let P be a generic point of V over a common field of definition k for V, f and g, and put $f(P) = x$ and $g(P) = y$. Then if f and g are algebraically independent, x and y are independent variables over k. Let C be the locus of P over $K = \overline{k(x)} \cap k(P)$. Then $f_C =$ const. and g_C is not a constant function on C. Moreover since C has dimension $r-1$, it is a simple subvariety of V by our assumption. Hence if fdg is of the first kind on V, $f_C dg_C$ is also of the first kind on C.[6] But on any variety, the differential of a function cannot be of the first kind, and we have arrived at a contradiction. Hence under the assumption of the theorem, y is algebraic over $k(x)$. Let Γ be a non singular projective model of the function field K

6) Cf. Koizumi (4) or Kawahara (3).

over k, and $\boldsymbol{Q}$ a generic point of $\boldsymbol{\Gamma}$ over k such that $k(\boldsymbol{Q})=K$. Let f', g' be functions on $\boldsymbol{\Gamma}$ defined over k by $f'(\boldsymbol{Q})=x(=f(\boldsymbol{P}))$, $g'(\boldsymbol{Q})=y(=g(\boldsymbol{P}))$, and π the rational map from V onto $\boldsymbol{\Gamma}$ defined by $\pi(\boldsymbol{P})=\boldsymbol{Q}$. Then, by definition, $f=f'\circ\pi$, $g=g'\circ\pi$. We shall show that $\omega'=f'dg'$ is of the first kind on $\boldsymbol{\Gamma}$. Suppose that ω' is not of the first kind. Then since $\boldsymbol{\Gamma}$ has no singular point there exists a point $\boldsymbol{M}$ on $\boldsymbol{\Gamma}$ such $v_M((\omega'))<0$, i.e. $v_M((f'))+v_M(dg'))<0$. We shall put $v_M((f'))=m$, $v_M(dg'))=n$. Then for a suitable choice of a constant c we have $v_M((g'+c))=n+1$. Let $\pi^{-1}(\boldsymbol{M})=\boldsymbol{X}$ and $\boldsymbol{A}$ be a component of $\boldsymbol{X}$ with the multiplicity a. Then $v_A((f))=ma$ and $v_A((g+c))=(n+1)a$ by the Cor. of Lemma 1. Then $v_A((dg))=(n+1)a-1$ and $v_A((\omega))=a(m+n+1)-1$. But since $m+n<0$ we have $m+n+1\leq 0$. Then $v_A((\omega))<0$. This contradicts to the fact that ω is of the first kind. Hence the differential ω' is proved to be of the first kind. This proves that the genus of $\boldsymbol{\Gamma}$ is necessarily not less than 1. q. e. d.

§ 3. Applications.

THEOREM 2. *Let ω be a linear differential form of the first kind on a non-singular variety V, and* $\mathrm{L}(\omega)$ *the module of functions on V such that $(\varphi)+(\omega)\succ 0$. Let $l(\omega)$ be the dimension of* $\mathrm{L}(\omega)$ *over the field of constants, then if $l(\omega)>1$, V has an irrational pencil.*

PROOF. Suppose that $l(\omega)>1$. Then there exists a non constant function φ on V such that $(\varphi)+(\omega)\succ 0$. Then the differential $\varphi\omega$ is also of the first kind, since V has no singular point.[7] As is well known the differential forms of the first kind are harmonic, hence closed. Then we have

$$0=d(\varphi\cdot\omega)=d\varphi\wedge\omega+\varphi d\omega=d\varphi\wedge\omega.$$

From this we can conclude easily that there exists a rational function f on V such that $\omega=f\,d\varphi$. Now Th.2 is an immediate consequence of Th.1. q. e. d.

COROLLARY. *Let ω be a linear differential form of the first kind on a non-singular variety V. Then $l(\omega)=1$, if ω can not be written in the form fdg for any choice of rational functions f and g on V.*

THEOREM 3.[8] *Abelian varieties[9] cannot have irrational pencils of genus >1, or equivalently, the one dimensional subfield of the function field of an abelian variety cannot have a genus greater than* 1.

7) Cf. Koizumi (4) or Nakai (5).

PROOF. Suppose that an abelian variety A has an irrational pencil of genus >1, with the non-singular parameter curve Γ, and let us denote by π the rational map from A onto Γ. Let ω be a differential form of the first kind on Γ. Then $\delta\pi(\omega)$ is also of the first kind on A by Lemma 2. Now by hypothesis the genus of Γ is >1, hence the degree of the canonical divisor of Γ is >0. Then there exists a point Q on Γ such that $v_Q(\omega)>0$. The similar method used in the proof of Th. 1, combining the Cor. of Lemma 1, is applicable to show that the divisor of $\delta\pi(\omega)$ cannot be empty. On the other hand, on abelian varieties, the differential forms of the first kind are invariant differential forms. Hence we must have $(\delta\pi(\omega))=0$, and this is a contradiction. Thus we have the theorem. q. e. d.

From this theorem we see that when an abelian variety has an irrational pencil, it is necessarily of genus 1. In this case we can see easily that A is isogenous to the product of two abelian varieties (one of them is an elliptic curve).

COROLLARY. *Let A be a simple abelian variety of dimension $\geqq 2$, defined over a field k, and K the function field of A over k. Then any one dimensional subfield of K over k is a purely transcedental extension of k.*

PROOF. Let L be an intermediate field of K and k such that $\dim_k L=1$, and Γ a non-singular projective model of L over k. Then the genus of Γ is zero by the above theorem. Moreover we have a rational map from A onto Γ defined over k. Since A contains a rational point over k, Γ contains also a rational point with reference to k by Nishimura (6). From this we can conclude easily the assertion. q. e. d.

§ 4. A remark on the differential form of the first kind.

The device used to prove Th.2. can be applied to show the following

PROPOSION. *Let V^r $(r\geqq 2)$ be a non-singular projective model of an algebraic variety, and $\sum$ the linear system composed of all the hyperplane sections of V. Then the divieor of the linear differential*

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8) Prof. Igusa kindly communicated to me that this theorem is valid even in the case of prime characteristic. The corollary is also valid for a *perfect* field k in such a case.

9) The terminologies conserning abelian varieties are due to Weil (8).

forms of the first kind on V cannot contain any irreduble member of $\sum$ as a component..

This result is not new in the case when the characteristic of the universal domain of our algebraic geometry is 0.[10] But we'd like to point out here that this result comes from the closedness of the differential forms of the first kind on algebraic varieties. This fact may indicate some intrinsic properties of the differential forms of the first kind on algebraic varieties.

PROOF. Let ω be a linear differential forms of the first kind on V, and suppose that (ω) contains an irreducible member of $\sum$ which is not contained in the hyperplane defined by the equation $X_0=0$. Let $\sum_{i=0}^{N} c_i X_i=0$ be the defining equation for the hyperplane $\boldsymbol{H}'$ such that $\boldsymbol{H}\cdot\boldsymbol{V}=\boldsymbol{W}'$ is irreducible and contained in (ω). Let $\boldsymbol{k}$ be a field of definition for $\boldsymbol{V}$, ω,[11] containing $c_i (i=0, 1, \cdots, N)$ and $\boldsymbol{H}$ a generic hyperplane with reference to $\boldsymbol{k}$ defined by the equation

$$\sum_{i=0}^{N} u_i X_i=0$$

where (u) are $N+1$ independent variables over k. Let $\boldsymbol{P}=(1, x_1, \cdots, x_N)$ be a generic point of V over $k(u)$, and f_i the functions on V defined over k by $f_i(\boldsymbol{P})=x_i$. Let F be a function on V defined over $k(u)$ by

$$F=\left(\sum_{i=1}^{N} u_i f_i+u_0\right)\Big/\left(\sum_{i=1}^{N} c_i f_i+c_0\right)$$

Then $F\omega$ is also of the first kind, since $(F\omega)=(\omega)-W'+W>0$, where $W=V.H$. Then using the closedness property of the differential form of the first kind, we have $\omega=gdF$ by the similar reasoning as was used in the proof of Th. 2. But we know that $(dF)_\infty=-2W'$. Hence if we can show that $(dF)_0=0$, we have $\deg(\omega)=\deg((g))+\deg((dF))=-2(\deg V)<0$. It is impossible since ω is of the first kind. We shall show that $(dF)_0=0$ when $\dim V\geqq 2$. This is included in the following lemma *valid in the case of prime characteristic.*

LEMMA 3. *Let V^r $(r\geqq 2)$ be a variety in a projective space, k a field of definition for V, $\boldsymbol{P}=(1, x_1, \cdots, x_N)$ a generic point of V over*

10) Cf. e.g. Igusa (1)

11) Concerning the field of definition for a differential form see Nakai (5).

k and f_i the functions on V defined over k by $f_i(P)=x_i$. Let $u_1, \cdots, u_N$ be independent variables over $k(P)$ and φ a function on V defined over $k(u)$ by $\varphi=\sum_{i=1}^{N} u_i f_i$. Suppose that the intersection product of V with the hyperplane H_0 defined by the equation $X_0=0$ is defined and has no component whose multiplicity in $V \cdot H_0$ is congruent to zero mod p (the characteristic of the universal domain). Then we have $(d\varphi)_0=0$.

PROOF. By our assumption, the zero varieties of $d\varphi$ are not contained in the hyperplane H_0. Let A^{r-1} be any simple subvariety of V not contained in H_0. Then we can find uniformizing parameters on V along A among the functions f_i's. Let them be $f_1, \cdots, f_r$. Then we can write $d\varphi$ in the form $d\varphi=\sum_{i=1}^{r}\alpha_i df_i$, where $\alpha_j=u_j+\sum_{s=r+1}^{N} u_s(\partial f_s/\partial f_j)$, Suppose that A is contained in $(d\varphi)_0$, then A is a component of $(\alpha_j)_0$ for all $j=1, \cdots, r$. Let M be a generic point of A over the algebraic closure $\overline{k(u)}$ of $k(u)$. Then we have $k(M, u_{r+1}, \cdots, u_N) \ni u_1, \cdots, u_r$, since the functions $\partial f_s/\partial f_j$ are defined over k. Hence $\dim_{k(M)}(u) \leqq N-r$. On the other hand we have $\dim_{k(u)}(M)=r-1$, and then $r-1 \leqq \dim_k(M) \leqq r$. Combining these we get immediately $\dim_{k(M)}(u) \geqq N-1$. This contradicts to the preceeding result when $r \geqq 2$, thus A cannot be contained in $(d\varphi)_0$. Since this holds for any simple subvariety A^{r-1} of V we have $(d\varphi)_0=0$. q. e. d.

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Added in proof. Recently Prof. Matsusaka kindly communicated to me that the theorem 1 in this paper does not hold, in general, in the case of prime characteristic,