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THE EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract. This paper is concerned with periodic solutions of first-order nonlinear functional differential equations with deviating arguments. Some new sufficient conditions for the existence of periodic solutions are obtained. The paper extends and improves some well-known results.

Keywords: nonlinear functional differential equation, differential equation with deviating arguments, periodic solutions, coincidence degree theory

MSC 2000: 34B15, 34K13

1. Introduction

Recently, periodic solutions of functional differential equations have been extensively studied (see, e.g., [1]–[6]). In [1], the functional differential equation

$$\dot{x}(t) = b(t, x(t+\cdot)) + G(t, x(t+\cdot)) \tag{1}$$

is considered where $x(t) \in \mathbb{R}^n$, $x(t+\cdot) \in BC(\mathbb{R}, \mathbb{R}^n)$ is given by $x(t+\cdot)(s) = x(t+s)$, b and G are continuous and boundary operators from $\mathbb{R}\times BC(\mathbb{R}, \mathbb{R}^n)$ to $\mathbb{R}^n$ for any fixed $t \in \mathbb{R}$, $b(t, \varphi)$ is linear with respect to $\varphi \in BC(\mathbb{R}, \mathbb{R}^n)$, there exists a constant $T > 0$ such that $b(t+T, \varphi) = b(t, \varphi)$, $G(t+T, \varphi) = G(t, \varphi)$ for any $(t, \varphi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}^n)$. Moreover,

$$\lim_{\|\varphi\|\to\infty} \frac{|G(t,\varphi)|}{\|\varphi\|} = 0$$

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uniformly for $t \in \mathbb{R}$, where $|\cdot|$ and $\|\cdot\|$ denote the norms in $\mathbb{R}^n$ and $BC(\mathbb{R}, \mathbb{R}^n)$, respectively. In the theory of coincidence degree it is proved that if the linear equation $\dot{x}(t) = b(t, x(t+\cdot))$ has only the trivial T-periodic solution, then equation (1) has at least one T-periodic solution. Particularly, if $G(t,\varphi) = G(t)$, it is easy to show that equation (1) has a unique T-periodic solution. A specific example is given in [2] on how periodic solutions can be obtained for the functional differential equation

$$\dot{u}(t) = l(u(t)) + g(t) \tag{2}$$

where $l\colon C_\omega(\mathbb{R}) \to L(\mathbb{R})$ is a linear bounded operator and $g \in L_\omega(\mathbb{R})$, and an important particular case

$$\dot{u}(t) = \sum_{k=1}^{n} p_k(t) u(\tau_k(t)) + g(t) \tag{3}$$

is also studied.

In the present paper, we first consider the nonlinear equation with deviating argument

$$\dot{x}(t) = p(t) f\bigl(x(t - \tau(t))\bigr). \tag{4}$$

Some new optimal sufficient conditions are established for the existence of the trivial T-periodic solution of equation (4). Next, we consider the functional differential equation

$$\dot{x}(t) = p(t) f\bigl(x(t - \tau(t))\bigr) + g(t). \tag{5}$$

By the theory of coincidence degree, we obtain sufficient conditions that equation (5) has at least one T-periodic solution.

2. Main results and proofs

Theorem 1. *Assume that*

(a) $p, \tau \in C(\mathbb{R}, \mathbb{R})$, $p(t) \geqslant 0$, $p(t)$ *is not identically equal to zero for* $t \in \mathbb{R}$,

(b) *there exists a constant* $T > 0$ *such that* $p(t+T) = p(t)$, $\tau(t+T) = \tau(t)$ *for* $t \in \mathbb{R}$,

(c) f *is continuous and* $x(t) f(x(t)) > 0$, $(x(t) \neq 0)$. *If* $f(x(t))/x(t) \leqslant 1$ $(x(t) \neq 0)$ *and*

$$0 < \int_0^T p(t) < 4 \tag{6}$$

then equation (4) *has only the trivial* T*-periodic solution.*

P r o o f. Assume the contrary. Let there exist a nontrivial T-periodic solution x of equation (4); then $\max_{0\leqslant t\leqslant T}|x(t)|>0$. There are two cases:

Case 1. $M=\max_{0\leqslant t\leqslant T}x(t)\leqslant 0$, then $x(t)\leqslant 0$, $f(x)\leqslant 0$ since $p(t)\geqslant 0$, so $\dot{x}(t)\leqslant 0$, which is a contradiction with the assumption.

Case 2. $M=\max_{0\leqslant t\leqslant T}x(t)>0$, then there are two cases:

(i) $\min_{0\leqslant t\leqslant T}x(t)\geqslant 0$, then $x(t)\geqslant 0$, $f(x)\geqslant 0$ since $p(t)\geqslant 0$, so $\dot{x}(t)\geqslant 0$, which is a contradiction with the assumption.

(ii) $\min_{0\leqslant t\leqslant T}x(t)=-m<0$; choose $t_\star\in[0,T]$, $t^\star\in[t_\star,t_\star+T]$ such that $x(t_\star)=-m$, $x(t^\star)=M$, then $-m\leqslant f\bigl(x(t-\tau(t))\bigr)\leqslant M$.

Integrating equation (4) from $t_\star$ to $t^\star$ and from $t^\star$ to $t_\star+T$, respectively, we have

$$M+m=\int_{t_\star}^{t^\star}p(t)f\bigl(x(t-\tau(t))\bigr)\,\mathrm{d}t\leqslant M\int_{t_\star}^{t^\star}p(t)\,\mathrm{d}t \tag{7}$$

and

$$m+M=-\int_{t^\star}^{t_\star+T}p(t)f\bigl(x(t-\tau(t))\bigr)\,\mathrm{d}t\leqslant m\int_{t^\star}^{t_\star+T}p(t)\,\mathrm{d}t. \tag{8}$$

Therefore, summing the last two inequalities,

$$4\leqslant 2+\frac{M}{m}+\frac{m}{M}\leqslant\int_{t_\star}^{t_\star+T}p(t)\,\mathrm{d}t=\int_0^T p(t)\,\mathrm{d}t,$$

which is a contradiction with the condition (6). The proof is complete. □

R e m a r k. If equation (4) has only the trivial T-periodic solution, we cannot conclude that equation (5) has at least one T-periodic solution.

Theorem 2. *Assume that* $p,\tau,g\in C(\mathbb{R},\mathbb{R})$, $p(t)\geqslant 0$ *and* $p(t)$ *is not identically equal to zero for* $t\in\mathbb{R}$, *there exists a constant* $T>0$ *such that* $p(t+T)=p(t)$, $\tau(t+T)=\tau(t)$, $g(t+T)=g(t)$ *for* $t\in\mathbb{R}$, f *is continuous and* $x(t)f(x(t))>0$ $(x(t)\neq 0)$, $f(0)=0$. *Let the following conditions hold:*

(i) $|f(x(t))|\leqslant|x(t)|$ *for all* $x(t)\in\mathbb{R}$;

(ii) $0<\int_0^T p(t)\,\mathrm{d}t<4$;

(iii) $\int_0^T g(t)\,\mathrm{d}t=0$.

Then equation (5) *has at least one* T*-periodic solution.*

To prove the theorem, we first consider the auxiliary equation

$$\dot{x}(t)=\lambda p(t)f\bigl(x(t-\tau(t))\bigr)+\lambda g(t),\quad \lambda\in(0,1). \tag{9}$$

Lemma 1. *For each possible T-periodic solution x_λ of equation* (9), *if the conditions of Theorem* 2 *hold, then there exists a constant D which is independent of λ such that*

$$|x_\lambda(t)| \leqslant D, \quad t \in \mathbb{R}. \tag{10}$$

P r o o f. Let x denote x_λ. There are two possible cases:

Case 1. $x(t)$ is of a constant sign, i.e., either $x(t) \geqslant 0$ or $x(t) \leqslant 0$ for $t \in \mathbb{R}$. Integrating both sides of equation (9) from 0 to T, note that x is T-periodic and (iii) yields

$$\int_0^T p(t) f\bigl(x(t-\tau(t))\bigr) = \mathrm{d}t = 0. \tag{11}$$

From (11), in view of the fact that $p(t)$ is not identically equal to zero, we obtain that there exists $t_0 \in [0, T]$ such that $x(t_0) = 0$. Moreover, there exists $t_1 < t_0$ such that $|x(t_1)| = \|x\|_C$ with $\|x\|_C = \max\limits_{0 \leqslant t \leqslant T} x(t)$. Now integration of (9) on $[t_1, t_0]$ yields

$$\|x\|_C \leqslant \|g\|_L \tag{12}$$

where $\|g\|_L = \int_0^T |g(t)| \, \mathrm{d}t$.

Case 2. The function x assumes both positive and negative values. Let $I = [t_2, t_3]$, $J = [t_3, t_2 + T]$, where t_2 and t_3 are such that $t_2 < t_3 < t_2 + T$ and $x(t_2) = -\min\limits_{0 \leqslant t \leqslant T} x(t)$ and $x(t_3) = \max\limits_{0 \leqslant t \leqslant T} x(t)$. Then integration of (9) on I and J, in view of $-m \leqslant f\bigl(x(t-\tau(t))\bigr) \leqslant M$ and $\lambda \in (0,1)$, yields

$$m \leqslant M\Bigl(\int_I p(t)\,\mathrm{d}t - 1\Bigr) + \|g\|_L \tag{13}$$

and

$$M \leqslant m\Bigl(\int_J p(t)\,\mathrm{d}t - 1\Bigr) + \|g\|_L \tag{14}$$

where $M = \max\limits_{0 \leqslant t \leqslant T} x(t) > 0$, $m = -\min\limits_{0 \leqslant t \leqslant T} x(t) > 0$.

There are four cases:

Case a) $\int_I p(t)\,\mathrm{d}t \leqslant 1$ and $\int_J p(t)\,\mathrm{d}t \leqslant 1$. Then from (13) and (14) we get $\|x\|_C \leqslant \|g\|_L$.

Case b) $\int_I p(t)\,\mathrm{d}t \leqslant 1$ and $\int_J p(t)\,\mathrm{d}t > 1$. Then from (13) we have $m \leqslant \|g\|_L$, which together with (14) implies $M \leqslant \|p\|_L \, \|g\|_L$, i.e., $\|x\|_C \leqslant (\|p\|_L + 1)\|g\|_L$.

Case c) $\int_I p(t)\,\mathrm{d}t > 1$ and $\int_J p(t)\,\mathrm{d}t \leqslant 1$. Analogously to Case b, we obtain $\|x\|_C \leqslant (\|p\|_L + 1)\|g\|_L$.

Case d) $\int_I p(t)\,\mathrm{d}t > 1$ and $\int_J p(t)\,\mathrm{d}t > 1$. Then using (14) in (13) or (13) in (14) we have respectively

$$m \leqslant m\Big(\int_I p(t)\,\mathrm{d}t - 1\Big)\Big(\int_J p(t)\,\mathrm{d}t - 1\Big) + \Big(\int_I p(t)\,\mathrm{d}t - 1\Big)\|g\|_L + \|g\|_L \tag{15}$$

and

$$M \leqslant M\Big(\int_I p(t)\,\mathrm{d}t - 1\Big)\Big(\int_J p(t)\,\mathrm{d}t - 1\Big) + \Big(\int_J p(t)\,\mathrm{d}t - 1\Big)\|g\|_L + \|g\|_L. \tag{16}$$

Now, in view of the inequality $AB \leqslant (A+B)^2/4$ we have

$$\Big(\int_I p(t)\,\mathrm{d}t - 1\Big)\Big(\int_J p(t)\,\mathrm{d}t - 1\Big) \leqslant \frac{1}{4}\Big(\int_{I\cup J} p(t)\,\mathrm{d}t - 2\Big)^2. \tag{17}$$

Consequently, from (15) and (16) we obtain

$$\|x\|_C \leqslant \frac{1}{4}\Big(\int_0^T p(t)\,\mathrm{d}t - 2\Big)^2 \|x\|_C + \|p\|_L\|g\|_L. \tag{18}$$

Then, in view of condition (ii), we have

$$\|x\|_C \leqslant \Big(1 - \frac{1}{4}(\|p\|_L - 2)^2\Big)^{-1}\|p\|_L\|g\|_L. \tag{19}$$

When $m \geqslant M$, we can obtain the inequality (19) similarly according to (14).

Thus, in both cases, the estimate (10) holds with

$$D = \Big(1 + (\|p\|_L + 1)\Big(1 - \frac{1}{4}(\|p\|_L - 2)^2\Big)^{-1}\Big)\|g\|_L.$$

In order to prove Theorem 2, we also need the continuation theory of coincidence degree developed by Gains and Mawhin in [7]. □

Lemma 2 (Continuation theorem)**.** *Let X, Z be real Banach spaces, $L\colon \operatorname{dom} L \subset X \to Z$ a Fredholm operator with index zero and let $N\colon \bar\Omega \to Z$ be L-compact on $\bar\Omega$ where Ω is an open subset of X, let $Q\colon Z \to Z$ be a continuous projector with* $\operatorname{Im} L = \ker Q$ *and let $J\colon \operatorname{Im} Q \to \ker L$ be an isomorphism. Let*

(1) *$Lx \neq \lambda Nx$ for any $\lambda \in (0,1)$, $x \in \operatorname{dom} L \cap \partial\Omega$;*

(2) *$QNx \neq 0$ for $x \in \ker L \cap \partial\Omega$ and* $\deg_B(JQN, \ker L \cap Q, 0) \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar\Omega$.

P r o o f of Theorem 2. Let $X = Z = \{x \in C(\mathbb{R},\mathbb{R})\colon\ x(t+T) = x(t)\}$ with the norm $\|x\|_C = \max\limits_{0\leqslant t\leqslant T} |x(t)|$; $\operatorname{dom} L = X \cap C^1(\mathbb{R},\mathbb{R})$; $\Omega = \{x \in X\colon\ |x(t)| < \overline{D}\}$, where $\overline{D}$ is greater than D; let $L\colon\ \operatorname{dom} L \subset X \to X$ be the differential operator defined by $(Lx)(t) = \dot{x}(t)$, let $N\colon\ \bar{\Omega} \to Z$ be defined by $(Nx)(t) = p(t)f\bigl(x(t-\tau(t))\bigr) + g(t)$ and $J = \mathrm{id}$. Clearly, $\ker L = \mathbb{R}$. Defining the projectors $P = Q$ as follows:

$$Px(t) = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s. \tag{20}$$

Obviously, $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q$, L is a Fredholm operator with index zero and N is L-compact on $\bar{\Omega}$. According to the estimation of the periodic solution of equation (9), we have $Lx \neq \lambda Nx$, for all $x \in \operatorname{dom} L \cap \partial\Omega$, $\lambda \in (0,1)$. If $x \in \ker L \cap \partial\Omega$, then $x = \pm\overline{D}$, so

$$\begin{aligned} QNx &= \frac{1}{T}\int_0^T \bigl[p(t)f\bigl(x(t-\tau(t))\bigr) + g(t)\bigr]\,\mathrm{d}t \\ &= \frac{1}{T}\int_0^T p(t)f(\pm\overline{D})\,\mathrm{d}t = f(\pm\overline{D})\frac{1}{T}\int_0^T p(t)\,\mathrm{d}t \neq 0. \end{aligned}$$

Finally, consider the mapping

$$H(x,s) = sx + (1-s)f(x), \quad 0 \leqslant s \leqslant 1.$$

Since for every $s \in [0,1]$ and $x \in \ker L \cap \partial\Omega$, we have

$$xH(x,s) = sx^2 + (1-s)xf(x) > 0,$$

$H(x,s)$ is a homotopy. This shows that

$$\begin{aligned} \deg_B(JQN, \ker L \cap \Omega, 0) &= \deg_B(f, \ker L \cap \Omega, 0) \\ &= \deg_B(\mathrm{id}, \ker L \cap \Omega, 0) \neq 0. \end{aligned}$$

We have thus verified all the assumptions of the continuation theorem. Thus under the conditions of Theorem 2, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$. i.e., equation (5) has at least one T-periodic solution. The proof is complete. □

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