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RESEARCH ARTICLE

The eXogenous Kalman Filter (XKF)

Tor A. Johansen and Thor I. Fossen
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It is well known that the time-varying Kalman Filter (KF) is globally exponentially stable and optimal in the sense of minimum variance under some conditions. However, nonlinear approximations such as the extended KF linearizes the system about the estimated state trajectories, leading in general to loss of both global stability and optimality. Nonlinear observers tend to have strong, often global, stability properties. They are, however, often designed without optimality objectives considering the presence of unknown measurement errors and process disturbances. We study the cascade of a global nonlinear observer with the linearized KF, where the estimate from the nonlinear observer is an exogenous signal only used for generating a linearized model to the KF. It is shown that the two-stage nonlinear estimator inherits the global stability property of the nonlinear observer, and simulations indicate that local optimality properties similar to a perfectly linearized KF can be achieved. This two-stage estimator is called an eXogeneous KF (XKF).

1 Introduction

Estimation of states in nonlinear systems is a challenging problem. A wide range of methods have been proposed, where many of them fall into one of the following categories:

Methods based on local linearization (first-order Taylor series expansion): These methods are motivated by the global asymptotic stability and tunable performance of Luenberger observers for observable linear time-invariant (LTI) systems, Luenberger (1964), and the Kalman-Bucy filter (KF) for observable linear time-varying (LTV) systems, e.g. Kalman and Bucy (1961), Brown and Hwang (2012), Simon (2006). The KF gives optimal (minimum variance) filtering by selection of tuning parameters to match the variances of white measurement noise and process noise, regardless of the structure of the uniformly completely observable LTV system. By approximating the nonlinear system with a local linear approximation, usually referred to as a linearized model, these methods can in many cases be applied also to nonlinear systems.

Nonlinear observers usually take global asymptotic or exponential stability (or at least a large region of attraction) as the primary starting point for the design, and then employ tuning parameters to pursue desired performance. The design usually involves some kind of analysis based on Lyapunov stability theory that leads to conditions on the tunable parameters and system parameters, where particular structural properties of the nonlinear system are exploited in the design, e.g. Besancon (2007), Nijmeijer and Fossen (1999).

Both approaches have their strengths; the global (or large) region of attraction of nonlinear observers, and the relatively general applicability and good performance that is often achieved by nonlinear KF approximations. At the same time they both also have significant weaknesses. Since the state is unknown, the KF must usually rely on a linearization of the nonlinear model about a state estimate, leading to the Extended KF (EKF) or other variants such as the Unscented KF, Monto-Carlo filter, and particle filter,

Center for Autonomous Marine Operations and Systems (AMOS), Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim, Norway.

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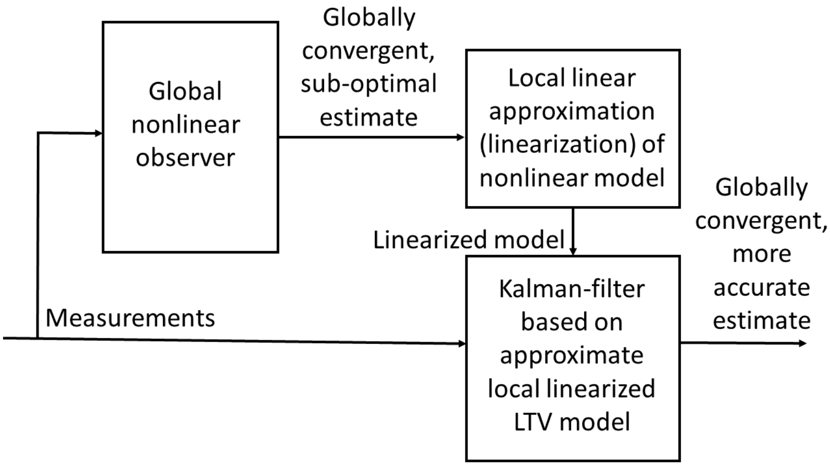
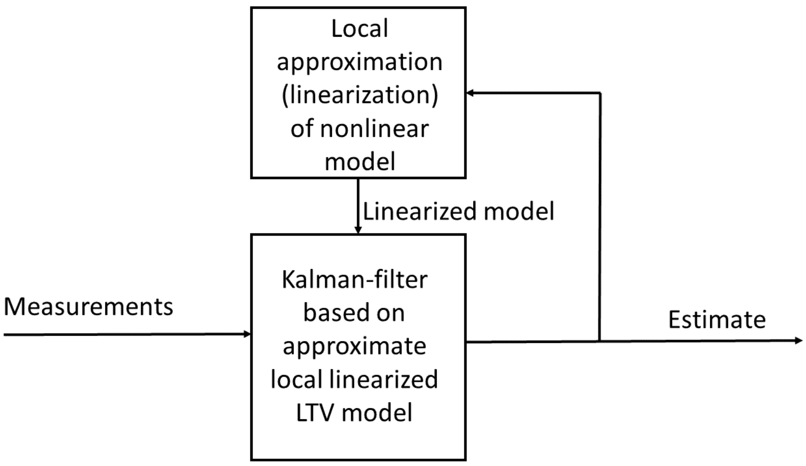


a) The eXogenous Kalman Filter (XKF) - cascaded nonlinear observer and linearized Kalman-filter.



(b) Extended Kalman Filter (EKF).

Figure 1. Proposed cascaded nonlinear observer and linearized Kalman-filter, and EKF for comparison. The difference between the two KFs is only the state estimate about which the linearization is made, where the proposed approach avoids the potentially de-stabilizing nonlinear feedback loop of the EKF.

that perform the linear approximation using different techniques, e.g. Gelb (1974), Julier and Uhlmann (2004), Brown and Hwang (2012), Gustafsson (2012). Regardless of the specific approach, global stability cannot be guaranteed in general, and existing stability analysis gives implicit conditions that cannot be verified a-priori as they depend on initial errors and system trajectories, e.g. Song and Grizzle (1995), Reif et al. (1998). Additionally, the errors and correlations due to linearization implies sub-optimality, although in some applications the degree of local sub-optimality is of little or no practical concern, Gelb (1974). The nonlinear observers, on the other hand, typically come without analysis of performance in terms of minimum variance, and may also lack a systematic approach to optimize their performance.

The problem addressed in this paper is then quite natural: Can a nonlinear observer and the Linearized KF (LKF) be combined to benefit from their complementary advantages, without inheriting their individual weaknesses?

One interesting approach is moving horizon estimation that combines a minimum variance cost function with constraints or special terms that ensure stability, e.g. Rao et al. (2003). While this is a general and powerful approach, it depends in general on solving non-convex numerical optimization problems in real time, which may not be feasible in all practical applications and typically leads to only local stability when gradient-based iterative optimization algorithms are employed.

Here we propose a practical solution to this problem. Although the approach is quite simple and general, it is to the best of our knowledge a novel method. Rather than linearizing the nonlinear system about the KF's own state estimate, one can linearize the nonlinear system about the estimated state trajectory coming from a global nonlinear observer. Since the linearization is made about an exogenous state trajectory, there is no nonlinear feedback loop that can destabilize the LKF, and it follows from nonlinear stability theory that the stability properties of the global nonlinear observer is inherited by the cascade, Panteley and Loria (1998), Loria and Panteley (2004), cf. Figure 1. For this reason, the approach is called the eXogenous Kalman Filter (XKF).

This paper is motivated by the benefits of final-stage LKF experienced in application studies on range-based navigation systems in Johansen et al. (2016), Johansen and Fossen (2016), where the EKF diverges due to the existence of local minimums, while a global observer in cascade with final-stage LKF provides both global stability and low variance of the estimates. These successful applications inspired this paper, where this simple and effective idea is presented in a more general framework, stability properties are formally analyzed, and some simple illustrative examples are provided.

2 Design and analysis

Consider the nonlinear system

$$\dot{x}(t) = f(x(t),t) + G(t)w(t) \tag{1}$$

$$y(t) = h(x(t),t) + e(t) \tag{2}$$

where f, G, h are smooth vector- or matrix-valued functions, x is the state vector, t is time, w is a vector of process disturbances, and e is a vector of measurement errors.

Let $\bar{x}$ be an estimate of x, and assume $\bar{x}$ is a bounded signal, given by a global nonlinear observer (NLO) with bounded error $\breve{x}(t) = x(t) - \bar{x}(t)$. Next, we consider the design of a 2nd-stage LKF, and regard $\bar{x}$ as an exogenous signal to this filter. A first-order Taylor series expansion of (1) about the trajectory $\bar{x}(t)$ gives

$$\dot{x}(t) = f(\bar{x}(t),t) + F(\bar{x}(t),t)\breve{x}(t) + G(t)w(t) + q(x(t),\bar{x}(t),t) \tag{3}$$

$$y(t) = h(\bar{x}(t),t) + H(\bar{x}(t),t)\breve{x}(t) + r(x(t),\bar{x}(t),t) + e(t) \tag{4}$$

where $q(\cdot)$ and $r(\cdot)$ are higher-order terms, and

$$F(\bar{x},t) = \frac{\partial f}{\partial x}(\bar{x},t), \;\; H(\bar{x},t) = \frac{\partial h}{\partial x}(\bar{x},t)$$

Since $\bar{x}(t)$ is bounded and f, h are smooth, there exist constants $k_q, k_r > 0$ such that the higher-order terms are bounded by

$$||q(t)|| \leq k_q ||\breve{x}(t)||^2, \;\; ||r(t)|| \leq k_r ||\breve{x}(t)||^2 \tag{5}$$

If the NLO converges, these terms vanish asymptotically, so in the 2nd-stage LKF we can neglect the higher-order terms and design a LKF based on the truncated LTV model. Using the first-order dynamics

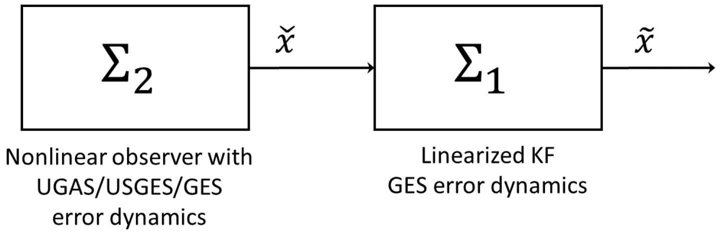


Figure 2. Cascaded error dynamics, where Σ_2 represents the NLO error dynamics and Σ_1 represents the LKF error dynamics.

of (3), i.e. neglecting the higher-order terms and the noise, and introducing a correction term using the first-order measurement model approximation (4), we define the estimator $\hat{x}$ for x by

$$\dot{\hat{x}}(t) = f(\bar{x}(t),t) + F(\bar{x},t)(\hat{x}(t) - \bar{x}(t)) + K(t)\left(y(t) - h(\bar{x}(t),t) - H(\bar{x}(t),t)(\hat{x} - \bar{x}(t))\right) \tag{6}$$

We remark that in contrast to typical nonlinear KF approximations, the estimator (6) uses the first-order model approximation rather than the full nonlinear model in the propagation and correction of the state estimate. This is crucial in order to avoid potential instabilities that might otherwise arise in particular if the system is not open loop stable.

Disregarding for the moment the linearization errors q and r, and the fact that $\bar{x}(t)$ depends on the measurements, we recall that the KF is designed to be optimal under the assumption that w is white noise with covariance matrix Q, e is white noise with covariance matrix R, and w and e are uncorrelated. Then the time-varying gain satisfies $K(t) = P(t)H^T(\bar{x}(t),t)R^{-1}$ where P is the time-varying symmetric positive definite solution to the Riccati equation

$$\dot{P}(t) = F(\bar{x}(t),t)P(t) + P(t)F^T(\bar{x}(t),t) + G(t)QG^T(t) - K(t)RK^T(t)$$

with $P(0)$ symmetric and positive definite, Kalman and Bucy (1961).

The state estimation error is $\tilde{x} := x - \hat{x} = \check{x} + \bar{x} - \hat{x}$. From (3)–(4) and (6) it follows that the error dynamics is LTV with a perturbation:

$$\Sigma_1 : \ \dot{\tilde{x}}(t) = A(\bar{x}(t),t)\tilde{x}(t) + d(t) \tag{7}$$

where

$$\begin{aligned} A(\bar{x}(t),t) &= F(\bar{x}(t),t) - K(t)H(\bar{x}(t),t) \\ d(t) &= q(x(t),\bar{x}(t),t) + K(t)r(x(t),\bar{x}(t),t) + K(t)e(t) + G(t)w(t) \end{aligned}$$

We let Σ_2 represent the dynamics of the NLO estimation error $\check{x}(t)$, cf. Figure 2 and analyze global stability conditions for the cascade $\Sigma_2 - \Sigma_1$ by employing results from Loria and Panteley (2004). Assumptions A1 and A2 are standard conditions that ensure boundedness and positive definiteness of the solution of the Riccati equation, and leads to nominal convergence of the KF, e.g. Kalman and Bucy (1961), Anderson (1971):

Assumption A1. The LTV system $(F(\bar{x}(t),t), G(t), H(\bar{x}(t),t))$ is uniformly completely observable and controllable.

Assumption A2. The LKF tuning parameters $P(0), Q, R$ are symmetric and positive definite.

A key assumption is that the NLO design leads to a global form of asymptotic stability of its error dynamics.

Assumption A3. The NLO error dynamics Σ_2 is Uniformly Globally Asymptotically Stable (UGAS), Uniformly Semi-Globally Exponentially Stable (USGES), or Globally Exponentially Stable (GES).

The theory in Loria and Panteley (2004) can be employed in any of these cases, where we remark that GES is stronger than USGES, which is stronger than UGAS. We can now present the main result for the stability of the unforced error dynamics:

Theorem 2.1 : *Suppose Assumptions A1-A3 hold. The origin $\check{x} = \tilde{x} = 0$ of the unforced error dynamics cascade $\Sigma_2 - \Sigma_1$ (with $w = 0$ and $e = 0$) inherits the stability properties of Σ_2.*

Proof. We note that A3 implies boundedness of $F(\bar{x}(t),t), G(t)$, and $H(\bar{x}(t),t)$ due to the smoothness of f, G, h. Assumptions A1 and A2 imply GES of the origin of the unforced error dynamics (7) with $d = 0$, Kalman and Bucy (1961), Anderson (1971). Note that $||d(t)|| \leq k_d ||\check{x}(t)||^2$ for some $k_d > 0$ due to (5). Since k_d is bounded and does not depend on $\tilde{x}$, the result follows from Theorem 2.1 and Proposition 2.3 in Loria and Panteley (2004). □

If w and e are bounded inputs, and the origin of the unforced NLO error dynamics is GES or USGES, it follows from Lemma 4.6 in Khalil (2002) that the origin of the error dynamics cascade $\Sigma_2 - \Sigma_1$ is input-to-state stable (ISS) with w and e as inputs, and that the solutions are uniformly ultimately bounded (UUB). This property may also hold in some cases when the origin of the unforced NLO error dynamics is UGAS, but the analysis would depend on the particular design of the NLO and its Lyapunov-function.

It should be emphasized that even with a Gaussian white noise assumption, the estimates of the state vector and covariance matrix may be biased and in general sub-optimal, like the EKF. The reason for this is the effect of the linearization error that is a random variable that may have a bias (due to errors resulting from the global nonlinear observer) and is correlated with the measurements. Like with nonlinear filtering in general, best practice may be to investigate the errors using simulation.

Formally, the stability properties are as strong as (but not stronger than) the NLO. The benefit of the approach compared to using the NLO is therefore related to the optimality of the KF, and that better performance can be achieved in many cases since a 'perfect LKF' can be approximated without the risk of divergence, in contrast to the EKF and similar nonlinear filter approximations. Other nonlinear filter approximations such as the moving horizon estimator, particle filter and Monte-Carlo filter are often more robust than the EKF and UKF, but their computational complexity are also significantly larger.

3 Illustrative examples

These examples are intended to illustrate the fundamental properties of the approach, and are therefore selected to be as simple as possible. We refer to Johansen et al. (2016), Johansen and Fossen (2016), Stovner et al. (2016) for more advanced examples and application case studies.

3.1 *Nonlinear measurement function*

Consider the nonlinear system with first-order dynamics and two measurements defined by the nominal model

$$\dot{x} = u \tag{8}$$

$$y_1 = |x - 1| \tag{9}$$

$$y_2 = |x + 2| \tag{10}$$

This can be viewed as a simple model of a mechanical system with linear motion in one degree of freedom, represented by the position x and driven by the velocity input u. There are two range measurements y_1 and y_2 relative to transponder or beacon positions $x = 1$ and $x = -2$, respectively.

A nonlinear observer can be designed for this system using the following nonlinear transform of the measurements

$$z = y_2^2 - y_1^2 \tag{11}$$

$$= x^2 + 4x + 4 - \left(x^2 - 2x + 1\right) \tag{12}$$

$$= 6x + 3 \tag{13}$$

which leads to the nonlinear observer

$$\dot{\bar{x}} = u + L\left(y_2^2 - y_1^2 - 6\bar{x} - 3\right) \tag{14}$$

with error dynamics $\breve{x} = x - \bar{x}$ given by the linear system

$$\Sigma_2 : \ \dot{\breve{x}} = -6L\breve{x} \tag{15}$$

Clearly, Σ_2 is GES for any constant $L > 0$ which should be chosen to achieve desired filtering bandwidth or pole locations. For completeness, the discrete-time implementation is given by the Euler discretization method

$$\bar{x}(k+1) = \bar{x}(k) + Tu(k) + TL\left(y_2^2(k) - y_1^2(k) - 6\bar{x}(k) - 3\right) \tag{16}$$

where T is the sampling period, and k is the discrete time index. The EKF estimate is

$$\hat{x}_E(k+1) = \hat{x}_E(k) + Tu(k) + K_E(k)\left(y(k) - C_E(k)\hat{x}_E(k)\right) \tag{17}$$

where the following linearization of the measurement matrix is used

$$C_E(k) = \begin{pmatrix} \text{sign}(\hat{x}_E(k) - 1) \\ \text{sign}(\hat{x}_E(k) + 2) \end{pmatrix} \tag{18}$$

and the gain matrix $K_E(k)$ is given by the discrete-time KF equations, Gelb (1974). In contrast, the 2nd-stage LKF of the XKF is defined by

$$\hat{x}(k+1) = \hat{x}(k) + Tu(k) + K(k)\left(y(k) - \bar{y}(k) - C(k)\left(\hat{x}(k) - \bar{x}(k)\right)\right) \tag{19}$$

where the measurement estimated based on the NLO estimate $\bar{y}(k) = (\bar{y}_1(k); \bar{y}_2(k))$ is defined by $\bar{y}_1(k) = |\bar{x}(k) - 1|$ and $\bar{y}_2(k) = |\bar{x}(k) + 2|$. The linearization of the measurement matrix is here based on the NLO estimate $\bar{x}(k)$:

$$C(k) = \begin{pmatrix} \text{sign}(\bar{x}(k) - 1) \\ \text{sign}(\bar{x}(k) + 2) \end{pmatrix} \tag{20}$$

and the gain matrix $K(k)$ is given by the discrete-time KF equations, Gelb (1974).

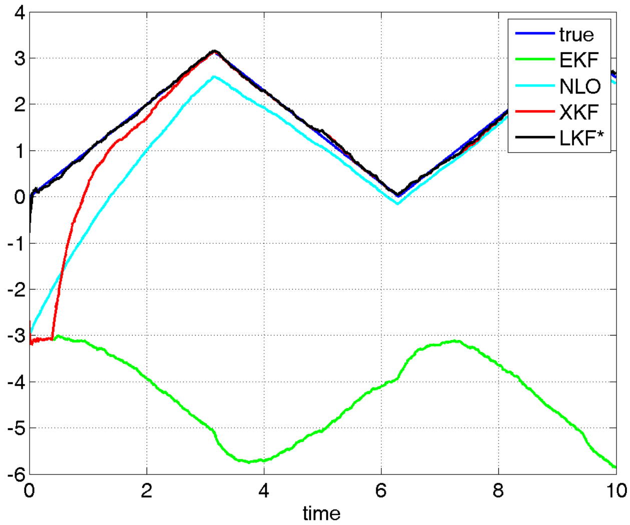


Figure 3. Simulation results, first example.

The simulation results shown in Figure 3 contain a comparison of transient performance of the NLO and XKF with an EKF as well as a perfect LKF linearized about the true (unknown) trajectory for reference (called LKF*). The control input u is a unit square wave signal with period 2π. It can be seen that with the initial state $x(0) = 0$ and initial estimates at $\hat{x}(0) = \bar{x}(0) = \hat{x}_E(0) = -3$, the EKF does not converge while all other estimators converge due to their GES property. In particular LKF* has a very accurate initial linearization that is exploited to make an accurate correction using the measurement already at time $k = 0$. In this example the measurement noise and process noise variances are $R = \text{diag}(1,\ 0.25)$, $Q = 1$, and the sampling interval is $T = 0.01$. The initial covariance estimate is $P(0) = 1$ in all cases, and the NLO gain is $L = 0.1$.

The steady state performance of the estimators is evaluated by choosing $\hat{x}(0) = \bar{x}(0) = \hat{x}_E(0) = x(0)$, i.e. perfect initialization. In this case also the EKF converges. The root-mean-squared error (RMSE) are computed by averaging of these simulation results as $\sigma_{LKF*} = 0.0461$, $\sigma_{EKF} = 0.0470$, $\sigma_{NLO} = 0.1330$ and $\sigma_{XKF} = 0.0471$. The results show that both the EKF and XKF come quite close to the perfect LKF*, despite the fact that the NLO has significantly higher errors. Hence, the errors in the NLO do not propagate very strongly into the XKF through the linearization process in this example.

3.2 *Nonlinear dynamics*

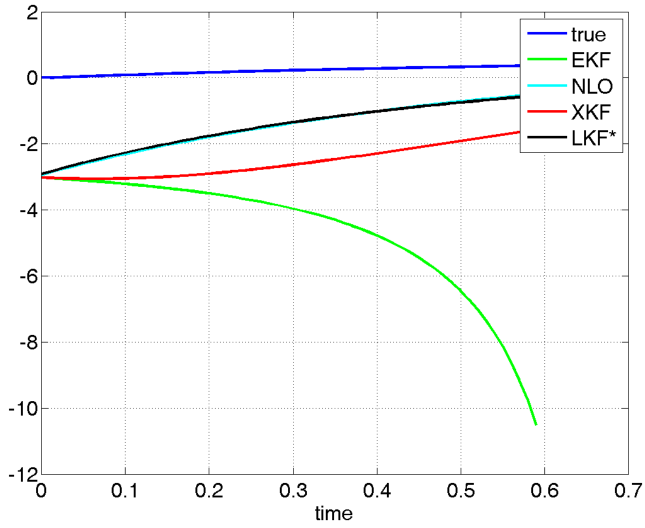
Consider the first-order nominal nonlinear dynamics

$$\dot{x} = -2x + x|x| + u \tag{21}$$

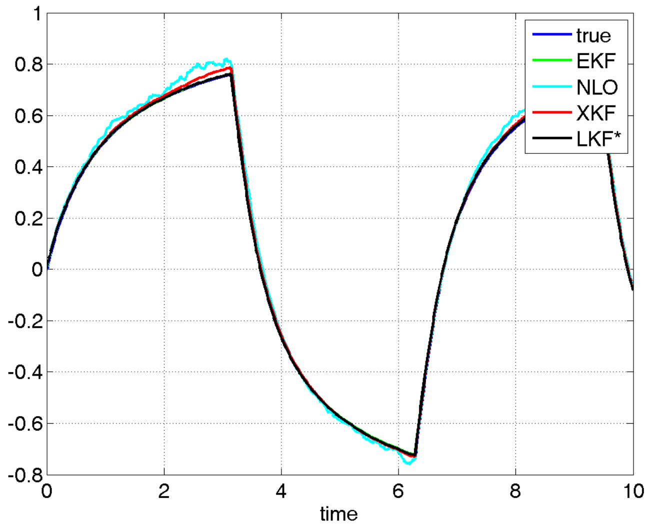
$$y = x \tag{22}$$

We note that this system is only locally stable. For this system an NLO can be desiged as

$$\dot{\bar{x}} = -2\bar{x} + \bar{x}|y| + u + L(y - \bar{x}) \tag{23}$$



(a) Initial condition $\hat{x}(0) = \eta(0) = -3$.



(b) Initial condition $\hat{x}(0) = \eta(0) = 0$.

Figure 4. Simulation results, second example. Note that the curves of the NLO and LKF* are almost indistinguishable in (a), while in (b) is curves of the EKF, XKF and LKF* are almost indistinguishable from the true state.

The error dynamics $\breve{x} = x - \bar{x}$ is given by the system

$$\Sigma_2 : \ \dot{\breve{x}} = (-2 + |y| - L)\,\breve{x} \tag{24}$$

Choosing $L = |y| + L_0$ leads to $\dot{\bar{x}} = -(2 + L_0)\bar{x}$ which is GES for any constant $L_0 > -2$ which should be chosen to achieve desired filtering performance. Discretization and design of EKF and XKF are similar to the previous example.

The simulation results shown in Figure 4a illustrate transient performance of the NLO and XKF in comparison with an EKF as well as a perfect LKF linearized about the true (unknown) trajectory for reference (LKF*). The control input u is a unit square wave signal with period 2π. It can be seen that with the initial state $x(0) = 0$ and initial estimates at $\hat{x}(0) = \bar{x}(0) = \hat{x}_E(0) = -3$, the EKF does not converge while all other estimators converge due to their GES property. In this example the measurement noise and process noise variances are $R = 0.25$ and $Q = 1$, and the sampling interval is $T = 0.01$. The initial covariance estimate is $P(0) = 1$ and the additional NLO gain is $L_0 = 0$.

The steady state performance of the estimators is evaluated by choosing $\hat{x}(0) = \bar{x}(0) = \hat{x}_E(0) = x(0)$, i.e. perfect initialization. In this case also the EKF converges, as shown in Figure 4b. The root-mean-squared errors (RMSE) are computed by averaging of these simulation results as $\sigma_{LKF*} = 0.0075$, $\sigma_{EKF} = 0.0071$, $\sigma_{NLO} = 0.0282$ and $\sigma_{XKF} = 0.0091$. The results show that both the EKF and XKF are not very far from the perfect LKF*, despite the fact that the NLO has significantly higher errors.

4 Conclusions

We study the cascade of a UGAS/USGES/GES nonlinear observer with the KF, where output of the nonlinear observer is only used for linearization. The cascade is called an eXogenous KF (XKF) since the linearization is based on an exogenous signal. It is shown that the XKF inherits the global stability properties of the nonlinear observer. Examples illustrate that the estimation error of the multi-stage filter can improve the performance of the NLO, and be close to the perfectly linearized KF. The examples also illustrate that the XKF can preserve stability when the EKF is not stable.

Acknowledgments

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