THE EXPECTED ORDER OF A RANDOM PERMUTATION

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ABSTRACT

Let μ_n be the expected order of a random permutation, that is, the arithmetic mean of the orders of the elements in the symmetric group S_n. We prove that $\log \mu_n \sim c\sqrt{(n/\log n)}$ as $n \to \infty$, where

$$c = 2\sqrt{\left(2\int_0^\infty \log\log\left(\frac{e}{1-e^{-t}}\right)dt\right)}.$$

1. *Overview*

If σ is a permutation on n letters, let $N_n(\sigma)$ be the order of σ as a group element. For a typical permutation, N_n is about $n^{\log n/2}$. To make this precise, we quote a stronger result of Erdős and Turán [5].

THEOREM 1. *For any fixed x,*

$$\lim_{n\to\infty} \frac{\#\left\{\sigma \in S_n : \log(N_n(\sigma)) < \frac{1}{2}\log^2 n + \frac{x}{\sqrt{3}}\log^{3/2} n\right\}}{n!} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

Many authors have given their own proofs of this remarkable theorem. For a survey of these and related results, see [2].

Let

$$\mu_n \overset{\text{def}}{=} \frac{1}{n!}\sum_{\sigma \in S_n} (\text{order of } \sigma)$$

be the expected order of a random permutation. The problem of estimating μ_n was first raised by Erdős and Turán [6]. Note that x is fixed in Theorem 1. It will not help us estimate μ_n because we cannot ignore the tail of the distribution. There are some permutations for which N_n is quite large. In fact, Landau proved that

$$\max_{\sigma \in S_n} N_n(\sigma) = e^{\sqrt{n \log n}(1+o(1))}.$$

It turns out that a small set of exceptional permutations contributes significantly to μ_n. Erdős and Turán determined that $\log \mu_n = O(\sqrt{(n/\log n)})$. (A proof appears in [12, 13].) This paper contains sharper estimates. We prove the following asymptotic formula.

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THEOREM 2.

$$\log \mu_n \sim c\sqrt{\left(\frac{n}{\log n}\right)} \quad \textit{where } c = 2\sqrt{\left(2\int_0^\infty \log\log\left(\frac{e}{1-e^{-t}}\right)dt\right)}.$$

First we give a brief overview of the proof. Consider the generating function

$$F(t) = \sum_n B_n e^{-nt} = (1-e^{-t})^{-1} \prod_{\text{primes } p} (1-\log(1-e^{-pt})).$$

One can think of B_n as the sum of the weights of a certain set of weighted partitions. By classical methods, one can easily prove that $\log B_n \sim c\sqrt{(n/\log n)}$. Our goal is to prove that $\log \mu_n \sim \log B_n$.

The connection between permutations and partitions is that the cycle lengths of a permutation on n letters form a partition λ of the integer n (written $\lambda \vdash n$). By a well-known theorem of Cauchy, the number of permutations of n letters with c_i cycles of length i is

$$\frac{n!}{c_1!c_2!\dots c_n!1^{c_1}2^{c_2}\dots n^{c_n}}.$$

If $\lambda = \{\lambda_1, \lambda_2, \dots\} = \{1^{c_1}, 2^{c_2}, \dots, n^{c_n}\}$, define

$$W(\lambda) := \frac{\mathrm{LCM}(\lambda_1, \lambda_2, \dots)}{c_1!c_2!\dots c_n!1^{c_1}2^{c_2}\dots n^{c_n}}.$$

Since the order of a permutation is the least common multiple of its cycle lengths, we have

$$\mu_n = \sum_{\lambda \vdash n} W(\lambda).$$

For each fixed integer $s \geqslant 2$, we shall construct a set P_n^s of partitions of n. Obviously, a lower bound can be obtained by considering only the contribution from elements of P_n^s:

$$\mu_n \geqslant \sum_{\lambda \in P_n^s} W(\lambda). \tag{$*$}$$

The idea is to choose P_n^s in such a way that the right-hand side of ($*$) is both easy to estimate and large enough to give a good bound. Consider the generating function

$$F_s(t) = \sum_n B_n^{(s)} e^{-nt} := (1-e^{-t})^{-1} \prod_{\text{primes } p>s} \left(1+e^{-pt}+\frac{e^{-2pt}}{2}+\frac{e^{-3pt}}{3}+\frac{e^{-4pt}}{4}+\dots+\frac{e^{-spt}}{s}\right).$$

We shall choose P_n^s in such a way that

$$\sum_{\rho \in P_n^s} W(\rho) \geqslant \left(\frac{1}{2n^2}\right) B_n^{(s)}.$$

For large s, this is a good approximation; for any $\varepsilon > 0$, one can choose s so that, for $n > n_0(\varepsilon, s)$, one has

$$\log \mu_n \geqslant \log\left(\frac{B_n^{(s)}}{2n^2}\right) \geqslant (1-\varepsilon)\log B_n.$$

The upper bound is both more difficult and harder to outline. A sieve-like argument is used to prove that the lower bound is sharp, that is, that $\log \mu_n \leqslant (1+o(1)) \log B_n$.

2. *The lower bound*

Let G_m^s be the set of partitions of m into distinct parts of the form $(d \cdot p)$, where $d \leqslant s, p$ is a prime that is larger than s, and each prime larger than s divides at most one part. In other words, $\# G_m^s$ is the coefficient of x^m in

$$\prod_{\text{primes } p > s} (1 + x^p + x^{2p} + \ldots + x^{sp}).$$

For technical reasons, it is not convenient to estimate $\sum_{\lambda \in G_n^s} W(\lambda)$. (Roughly: the Tauberian side conditions are difficult to verify.) Instead, we consider a closely related set. Suppose that $\lambda \in G_m^s$ for some $m < n$. Then one can obtain a partition of n by adjoining a part of size $(n-m)$. Think of partitions as multisets. Adjoining a part corresponds to taking a multiset union (that is, with multiplicities). Let A_n^s be the set of partitions that can be obtained in this way; in other words, let

$$A_n^s := \{\lambda \cup \{(n-m)\} : m < n \text{ and } \lambda \in G_m^s\}.$$

Of course, it may happen that a partition $\rho \in A_n^s$ can be formed in more than one way; one can have $\rho = \lambda \cup \{(n-m)\} = \lambda' \cup \{(n-m')\}$. But we claim that ρ can be formed in at most n ways. To see this, note that λ is completely determined by ρ and m. In other words, if $\lambda \cup \{(n-m)\} = \lambda' \cup \{(n-m)\}$, then $\lambda = \lambda'$. Since there are at most n possible values of m, it follows that there are at most n decompositions of ρ of the form $\rho = \lambda \cup \{(n-m)\}$. Finally, let $P_n^s = G_n^s \cup A_n^s$. Then we have

$$\mu_n \geqslant \sum_{\rho \in P_n^s} W(\rho) \geqslant \frac{1}{n} \sum_{m<n} \sum_{\lambda \in G_m^s} W(\lambda \cup \{(n-m)\}) + \frac{1}{n} \sum_{\lambda \in G_n^s} W(\lambda).$$

For $\lambda = \{d_1 p_1, d_2 p_2, \ldots\} \in G_m^s$,

$$W(\lambda) \geqslant \frac{1}{d_1 d_2 \ldots}.$$

But then

$$W(\lambda \cup \{(n-m)\}) \geqslant \frac{\mathrm{LCM}(\lambda_1, \lambda_2, \ldots)}{2 \cdot \lambda_1 \lambda_2 \ldots \lambda_n \cdot n} \geqslant \frac{1}{2n \cdot d_1 d_2 \ldots}.$$

Hence

$$\begin{aligned} \mu_n &\geqslant \frac{1}{2n^2} \sum_{m \leqslant n} \sum_{\lambda \in G_m^s} W(\lambda) \\ &\geqslant \frac{1}{2n^2} \mathrm{Coefficient}_{e^{-nt}}(F_s(t)) = \frac{B_n^{(s)}}{2n^2}. \end{aligned}$$

To estimate B_n and $B_n^{(s)}$, we need the following lemma.

Lemma 1 (Hardy–Ramanujan [9]). *Let $f(t) = \sum_n a_n e^{-nt}$, and suppose that*
(1) $a_n \geqslant 0$, *and*

(2) $\log f(t) \sim \dfrac{A}{t\log\left(\frac{1}{t}\right)}$ *as* $t \to 0^+$, *where* A *is a fixed positive constant.*

Then

$$\log\left(\sum_{k=1}^{n} a_k\right) \sim 2\sqrt{(2A)}\sqrt{(n/\log n)} \quad as \ n \to \infty.$$

Now let

$$h_s(r) := \log\left(1 + e^{-r} + \frac{e^{-2r}}{2} + \frac{e^{-3r}}{3} + \ldots + \frac{e^{-sr}}{s}\right),$$

and let $h(r) := \log(1 - \log(1 - e^{-r}))$. We need the following.

LEMMA 2.

$$\log F_s(t) \sim \frac{1}{t\log\left(\frac{1}{t}\right)} \int_0^\infty h_s(u)\,du \quad as \ t \to 0^+.$$

Proof. First, we remark that this lemma is quite similar to Lemma 1 in [6]. At the suggestion of a referee, we are providing a fairly detailed proof. In this lemma, $\pi(x)$ denotes the number of primes less than or equal to x. We have

$$\log F_s(t) = -\log(1 - e^{-t}) + \sum_{\text{primes } p > s} h_s(pt).$$

Expressing this as a Stieltjes integral and integrating by parts, we obtain

$$\int_s^\infty h_s(rt)\,d\pi(r) - \log(1 - e^{-t}) = -\pi(s)\,h_s(st) - \int_s^\infty \pi(r)\,th_s'(rt)\,dr - \log(1 - e^{-t}).$$

Using the prime number theorem in the form

$$\pi(r) = \int_2^r \frac{dr}{\log r} + O(re^{-c\sqrt{(\log r)}}),$$

we obtain

$$-\int_s^\infty \int_2^r \frac{dx}{\log x}\,th_s'(rt)\,dr + E(t),$$

where

$$E(t) = -\log(1 - e^{-t}) + O\left(t\int_s^\infty h_s'(rt)\,re^{-c\sqrt{(\log r)}}\,dr\right).$$

Integrating by parts again, we have

$$\int_s^\infty \frac{h_s(rt)\,dr}{\log r} + E(t).$$

Later, $E(t)$ will be shown to be negligible. For now, we concentrate on the main term, splitting the interval of integration into three parts. Let

$$\xi_1 := \frac{1}{t\log^3\frac{1}{t}} \quad \text{and} \quad \xi_2 := \frac{1}{t}\log^3\frac{1}{t}.$$

Then

$$\int_s^\infty \frac{h_s(rt)\,dr}{\log r} = \int_s^{\xi_1} + \int_{\xi_1}^{\xi_2} + \int_{\xi_2}^\infty .$$

The idea is that $\log r$ is nearly constant on the middle interval, and this makes the integral easy to estimate. The other two intervals contribute negligibly. For $r \in [\xi_1, \xi_2]$ we have $\log r = \log(1/t) + O(\log\log(1/t))$. Hence

$$\int_{\xi_1}^{\xi_2} \frac{h_s(rt)}{\log r}\,dr = \frac{1}{t\log\frac{1}{t}} \int_{\xi_1 t}^{\xi_2 t} h_s(u)\,du + O\left(\frac{\log\log\frac{1}{t}}{t\log^2\frac{1}{t}}\right)$$

$$\sim \frac{1}{t\log\frac{1}{t}} \int_0^\infty h_s(u)\,du \quad \text{as } t \to 0^+.$$

This is the main term. The appendix contains a proof that both $E(t)$ and the remaining two integrals are negligible.

COROLLARY.

$$\log B_n^{(s)} \sim k_s \sqrt{(n/\log n)} \quad \textit{as } n \to \infty, \quad \textit{where } k_s = 2\sqrt{\left(2\int_0^\infty h_s(r)\,dr\right)}.$$

Proof. Apply Lemma 1 to $f_s(t) := (1-e^{-t})F_s(t)$.

By arguments similar to those in Lemma 2, we obtain the following.

LEMMA 3.

$$\log F(t) \sim \frac{1}{t\log\frac{1}{t}} \int_0^\infty h(r)\,dr \quad \textit{as } t \to 0^+.$$

COROLLARY.

$$\log(B_n) \sim c\sqrt{(n/\log n)} \quad \textit{as } n \to \infty, \quad \textit{where } c = 2\sqrt{\left(2\int_0^\infty h(r)\,dr\right)}.$$

Finally, we use the monotone convergence theorem to complete the proof of the lower bound:

$$\lim_{s\to\infty} \int_0^\infty h_s(r)\,dr = \int_0^\infty \lim_{s\to\infty} h_s(r)\,dr = \int_0^\infty h(r)\,dr.$$

Hence $\log \mu_n \geqslant (\log B_n)(1-o(1))$.

3. *The upper bound*

We had

$$\mu_n = \sum_{\lambda\vdash n} \frac{\mathrm{LCM}(\lambda_1, \lambda_2, \ldots)}{c_1!\ldots c_n!\,2^{c_2}\ldots n^{c_n}} \leqslant \sum_{\lambda\vdash n} \frac{\mathrm{LCM}(\lambda_1, \lambda_2, \ldots)}{2^{c_2}3^{c_3}\ldots n^{c_n}}.$$

Observe that $2^{c_2} 3^{c_3} \dots n^{c_n} = \lambda_1 \lambda_2 \lambda_3 \dots$. (Recall that $c_i = c_i(\lambda)$ = the number of parts of size i in λ.) We therefore have

$$\mu_n \leqslant \sum_{\lambda \vdash n} \frac{\mathrm{LCM}(\lambda_1, \lambda_2, \dots)}{\lambda_1 \lambda_2 \dots}.$$

Think of partitions as multisets. For each partition λ, we shall choose partitions π and ω such that $\lambda = \pi \cup \omega$. (To make this paper self-contained, we repeat a few arguments from **[12]**.) The decomposition will have the following two properties:

(1) the least common multiple of the parts of π is equal to the least common multiple of the parts of λ;

(2) if $\lambda = \pi \cup \omega$ and $\lambda' = \pi' \cup \omega'$, then $\lambda = \lambda'$ if and only if $\pi = \pi'$ and $\omega = \omega'$. One can think of π as a kind of minimal generating set. Suppose m is the least common multiple of the parts of a partition λ. Let $p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$ be the prime factorization of m ($p_i < p_j$ for $i < j$). We define $\pi = \{\pi_1, \pi_2, \dots, \pi_t\}$ as follows. Let π_1 be the smallest part of λ that is divisible by $p_1^{e_1}$. Now suppose that $\pi_1, \pi_2, \dots, \pi_\ell$ have been defined. If each $p_i^{e_i}$ divides some π_j with $1 \leqslant j \leqslant \ell$, then set $t = \ell$ and stop. Otherwise, let $k = \min\{i \mid p_i^{e_i}$ divides none of $\pi_1, \pi_2, \dots, \pi_\ell\}$, and let $\pi_{\ell+1}$ be the smallest part of λ that is divisible by $p_k^{e_k}$. Finally, we define $\pi := \{\pi_1, \pi_2, \dots, \pi_t\}$. The procedure must terminate (in fact, $t \leqslant s$), so π is well defined. Given λ, let ω be the remaining parts, that is, $\omega = \lambda - \pi$ and $\lambda = \pi \cup \omega$. This is a convenient place to define a certain function α, which will play an important role later. Suppose $\pi = \{\pi_1, \pi_2, \dots\}$, and suppose $m = p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$ is the least common multiple of the parts of π. Then define

$$\alpha(i,j) = \alpha(i,j,\pi) := \begin{cases} e_j & \text{if } i = \min\{k \mid p_j^{e_j} \text{ divides } \pi_k\} \\ 0 & \text{else.} \end{cases}$$

For future reference, we make the following simple observation: for each j, $\sum_i \alpha(i,j) = e_j$.

Continuing where we left off, we have

$$\begin{aligned} \sum_{\lambda \vdash n} \frac{\mathrm{LCM}(\lambda_1, \lambda_2, \dots)}{\lambda_1 \lambda_2 \dots} &= \sum_{\pi, \omega} \frac{\mathrm{LCM}(\pi_1, \pi_2, \dots)}{\pi_1 \pi_2 \dots \omega_1 \omega_2 \dots} \\ &= \sum_{\pi} \frac{\mathrm{LCM}(\pi_1, \pi_2, \dots)}{\pi_1 \pi_2 \dots} \sum_{\omega} \frac{1}{\omega_1 \omega_2 \dots}. \end{aligned}$$

The inner sum is uniformly $O(n)$ by a theorem of Lehmer **[10]**. Since a factor of n is negligible, we need only estimate

$$\sum_{\pi} \frac{\mathrm{LCM}(\pi_1, \pi_2, \dots)}{\pi_1 \pi_2 \dots}.$$

We exploit cancellation by factoring the parts of our partitions. For $i = 1, \dots, t$, let

$$d_i = \prod_{j=1}^{s} p_j^{\alpha(i,j)}.$$

The d_i have deliberately been defined in such a way that they are pairwise relatively prime and their product is $\mathrm{LCM}(\pi_1, \pi_2, \dots)$. Note that d_i divides π_i for each i. Let $d_i' = \pi_i / d_i$, and let $\vec{d}' = \langle d_i' \rangle_{i=1}^{t}$. Then, for any of our 'generating partitions' π, we have

$$\frac{\mathrm{LCM}(\pi_1, \pi_2, \dots)}{\pi_1 \pi_2 \dots} = \frac{d_1 d_2 \dots}{(d_1 d_1')(d_2 d_2') \dots}. \qquad (**)$$

REMARK. We can think of $D=\{d_1,d_2,\ldots\}$ as a multiset (partition), or alternatively as a vector (sequence, ordered partition). There is no inconsistency, because only one ordering of the parts of D can occur. This is discussed in our earlier paper [12], where we used the notation $\vec{d}$ instead of D. (The notation has been changed to avoid writing unions of vectors.)

From (**) we have

$$\sum_{\pi}\frac{\operatorname{LCM}(\pi_1,\pi_2,\ldots)}{\pi_1\pi_2\ldots}=\sum_{D}\sum_{\vec{d}'}\frac{d_1 d_2\ldots}{d_1 d_1' d_2 d_2'\ldots}=\sum_{D}\sum_{\vec{d}'}\frac{1}{d_1' d_2'\ldots}.$$

In order to estimate this expression, we first decompose the Ds. Let $z:=\sqrt{n}/(\log^2 n)$, and write $D=S\cup P\cup R$, where

S consists of those parts of D that are divisible by some prime less than z,

P consists of those parts of D that are primes greater than or equal to z, and

R consists of the remaining parts of D.

Then each D corresponds to a triple (S,P,R), and we have

$$\sum_{(S,P,R)}\sum_{\vec{d}'}\frac{1}{d_1' d_2'\ldots}\leqslant\underbrace{\Bigl(\sum_{S}\sum_{\vec{j}}\frac{1}{j_1 j_2\ldots}\Bigr)}_{T_1}\cdot\underbrace{\Bigl(\sum_{P}\sum_{\vec{j}}\frac{1}{j_1 j_2\ldots}\Bigr)}_{T_2}\cdot\underbrace{\Bigl(\sum_{R}\sum_{\vec{j}}\frac{1}{j_1 j_2\ldots}\Bigr)}_{T_3}.$$

In T_1, the inner sum ranges over all sequences $\langle j_i\rangle$ for which $s_1 j_1+s_2 j_2+\ldots\leqslant n$. Similarly for T_2 and T_3. This overestimate gives an upper bound that is surprisingly sharp. We shall prove that T_1 and T_3 are small.

Estimates for T_1: Recall that $S\subseteq D$, and the parts of D are pairwise relatively prime. Hence each prime less than z divides at most one part of S. This implies that S has less than z parts. Thus there are at most n^z different possible S, and the inner sum of T_1 is at most $(\sum_{j=1}^{n}1/j)^z$. We therefore have

$$T_1\leqslant n^z\Bigl(\sum_{j=1}^{n}\frac{1}{j}\Bigr)^z=e^{o(\sqrt{(n/\log n)})}.$$

Estimates for T_3: The parts of R have at least two prime divisors (counting multiplicities), each of which is greater than $z=\sqrt{n}/(\log^2 n)$. They therefore have exactly two prime divisors, since the product of more than two such primes would be larger than n. Therefore R can have at most $\log^4 n$ parts (otherwise R would sum to more than n). Thus there are at most $n^{\log^4 n}$ possible R, and for each R the inner sum is at most $(\sum_{j=1}^{n}1/j)^{\log^4 n}$. Thus $T_3=e^{o(\sqrt{(n/\log n)})}$.

Finally, recall that

$$F(t):=\sum_{n}B_n e^{-nt}=(1-e^{-t})^{-1}\prod_{\text{primes }p}\Bigl(1+e^{-pt}+\frac{e^{-2pt}}{2}+\ldots\Bigr).$$

Hence

$$\begin{aligned}T_2&=\operatorname{Coefficient}_{e^{-nt}}\Bigl\{\frac{1}{(1-e^{-t})}\prod_{p\geqslant z}\Bigl(1+e^{-pt}+\frac{e^{-2pt}}{2}+\frac{e^{-3pt}}{3}+\frac{e^{-4pt}}{4}+\ldots\Bigr)\Bigr\}\\&<\operatorname{Coefficient}_{e^{-nt}}\{F(t)\}=B_n.\end{aligned}$$

One could certainly obtain slightly sharper estimates by a more careful treatment of the generating function $F_s(t)$ for an optimal choice of s. But a new idea will be needed for a really significant improvement. It is probably too much to ask for an asymptotic formula for μ_n, but perhaps others can obtain upper and lower bounds that differ by a polynomial factor.

Appendix

Our estimates for the error terms depend on the following easily verified facts:

(1) $h_s(u) < h(u)$ for all $u > 0$;

(2) $h(u) \sim \log\log\left(\frac{1}{u}\right)$ as $u \to 0^+$;

(3) $h(u) = O(e^{-u})$ as $u \to \infty$;

(4) $h(u)$ is a non-increasing function of u.

The first of the three error terms is

$$\int_s^{\xi_1} \frac{h_s(rt)}{\log r}\,dr.$$

Using first (1) and then (2), we have

$$\int_s^{\xi_1} \frac{h_s(rt)}{\log r}\,dr < \int_2^{\xi_1} \frac{h(rt)}{\log r}\,dr < \frac{\xi_1}{\log 2} h(\xi_1 t) \ll \xi_1 \log\log\left(\log^3 \frac{1}{t}\right) = o\left(\frac{\log\log \frac{1}{t}}{t \log^2 \frac{1}{t}}\right).$$

The second error term is

$$\int_{\xi_2}^{\infty} \frac{h_s(rt)}{\log r}\,dr < \frac{1}{\log \xi_2} \int_{\xi_2}^{\infty} h(rt)\,dr.$$

By (3) this is

$$\ll \frac{1}{\log \xi_2} \int_{\xi_2}^{\infty} e^{-rt}\,dr = o\left(\frac{\log\log \frac{1}{t}}{t \log^2 \frac{1}{t}}\right).$$

Finally, we must show that $E(t)$ is negligible. Integrating by parts, we obtain

$$E(t) \ll h_s(rt)\, r e^{-c\sqrt{(\log r)}} \Big|_s^{\infty} - \int_s^{\infty} h_s(rt) \left\{ e^{-c\sqrt{(\log r)}} - \frac{c e^{-c\sqrt{(\log r)}}}{2\sqrt{(\log r)}} \right\} dr$$

$$\ll \int_s^{\infty} h(rt)\, e^{-c\sqrt{(\log r)}}\,dr = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_s^{1/\sqrt{t}} \qquad I_2 = \int_{1/\sqrt{t}}^{(1/t)\log^2(1/t)} \quad \text{and} \quad I_3 = \int_{(1/t)\log^2(1/t)}^{\infty}.$$

Each of these is easy to estimate:

$$I_1 < \frac{1}{\sqrt{t}} \max_{r \in [s,\, t^{-1/2}]} h(rt)\, e^{-c\sqrt{(\log r)}} \ll \frac{h(st)}{\sqrt{t}} \ll \frac{\log\log \frac{1}{t}}{\sqrt{t}} = o\left(\frac{\log\log \frac{1}{t}}{t \log^2 \frac{1}{t}}\right);$$

$$I_2 = \int_{t^{-1/2}}^{(1/t)\log^2(1/t)} h(rt)\, e^{-c\sqrt{(\log r)}}\, dr < \left(\frac{1}{t}\log^2\frac{1}{t}\right) h(\sqrt{t}) \exp\left[-c\sqrt{(\log(t^{-1/2}))}\right]$$

$$\ll \frac{\left(\log^2 \frac{1}{t}\right)\log\log\frac{1}{t}}{t} \exp\left[-c\sqrt{\log(t^{-1/2})}\right] = o\left(\frac{\log\log \frac{1}{t}}{t \log^2 \frac{1}{t}}\right);$$

$$I_3 \ll \int_{(1/t)\log^2(1/t)}^{\infty} e^{-rt}\, e^{-c\sqrt{(\log r)}}\, dr = o\left(\frac{\log\log \frac{1}{t}}{t \log^2 \frac{1}{t}}\right).$$

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