The Annals of Statistics
1974, Vol. 2, No. 3, 415-436

THE EXPECTED SAMPLE SIZE OF SOME TESTS OF POWER ONE

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To Jerzy Neyman
on his 80th birthday
Pro summo beneficio eius maximas patri nostrum
omnium gratias agimus

Asymptotic approximations to the expected sample size are given for a class of tests of power one introduced in [10]. Comparisons are made with the method of mixtures of likelihood ratios, and an application is given to Breiman's gambling theory for favorable games.

1. Introduction and summary. Let $f_\theta(x)$, $\theta \in \Omega$, be a one-parameter family of probability densities with respect to some σ-finite measure ν on the Borel sets of the line. Denote by P_θ the probability measure under which random variables $x_1, x_2, \cdots$ are independent with the common probability density function $f_\theta(x)$. Suppose $\theta_0 \in \Omega$ and $0 < \alpha < 1$. By a size α test of power one of the hypothesis $\theta \leqq \theta_0$ against the alternative $\theta > \theta_0$ we mean a stopping rule T for the sequence $x_1, x_2, \cdots$ such that

$$P_\theta\{T < \infty\} \leqq \alpha \qquad \text{for every} \quad \theta \leqq \theta_0 \tag{1}$$

and

$$P_\theta\{T < \infty\} = 1 \qquad \text{for every} \quad \theta > \theta_0 . \tag{2}$$

Among such rules we wish to find one which in some sense minimizes $E_\theta T$ for all $\theta > \theta_0$.

If $\Omega - \{\theta_0\}$ consists of a single point $\theta' > \theta_0$, our problem is solved by the following "one-sided" sequential probability ratio tests. For any $b > 1$ let $N = N(\theta', b) =$ first $n \geqq 1$ such that $\prod_1^n (f_{\theta'}(x_k)/f_{\theta_0}(x_k)) \geqq b$, $= \infty$ if no such n occurs. It may be shown that

$$P_{\theta_0}\{N < \infty\} \leqq b^{-1} , \tag{3}$$

so (1) is satisfied at least for all $b \geqq \alpha^{-1}$, and for any stopping rule T for the sequence $x_1, x_2, \cdots$ such that

$$P_{\theta_0}\{T < \infty\} \leqq P_{\theta_0}\{N < \infty\} , \qquad E_{\theta'}(T) \geqq E_{\theta'}(N) \qquad \text{(cf. [2], page 107 ff.)} .$$

Received October 1972; revised June 1973.

[1] Research supported by Public Health Service Grant No. 5-R01-GM-16895-03. Reproduction in whole or in part is permitted for any purpose of the United States Government. The research of the second author was partially supported by a NSF Postdoctoral Fellowship.

AMS 1970 *subject classifications.* Primary 62L10; Secondary 60G40.

Key words and phrases. Sequential tests, tests of power one, expected sample size.

These considerations motivate the following approach to the general problem. For each $n = 0, 1, 2, \cdots$ let $\theta_{n+1} = \theta_{n+1}(x_1, \cdots, x_n)$ be a Borel measurable function of the indicated variables such that

$$\theta_{n+1} \geqq \theta_0 .$$

(In particular, θ_1 is some constant $\geqq \theta_0$.)
Define

$$z_n = \prod_1^n f_{\theta_i}(x_i)/f_{\theta_0}(x_i) \qquad (n = 1, 2, \cdots), \tag{4}$$

and for any fixed $b > 0$ let

$$\begin{aligned} T &= \text{first} \quad n \geqq 1 \quad \text{such that} \quad z_n \geqq b \\ &= \infty \quad \text{if no such} \quad n \quad \text{occurs.} \end{aligned} \tag{5}$$

It was shown in [10] under a general assumption on the family $\{f_\theta : \theta \in \Omega\}$ (which holds for the general one-parameter Koopman–Darmois–Pitman exponential family) that $P_\theta\{T < \infty\} \leqq b^{-1}$ for all $\theta \leqq \theta_0$, and hence that (1) holds for all $b \geqq \alpha^{-1}$.

The primary purpose of this paper is to give asymptotic approximations for exponential families $f_\theta(x)$ to $E_\theta T$ as $\theta \downarrow \theta_0$ (b fixed) and as $b \to \infty$, with various sequences $\{\theta_n\}$. Since the present proofs of the main results are quite complicated even for normally distributed x's, we give details only in this case, and in Section 8 indicate the straightforward but tedious modifications necessary to handle non-normal exponential families.

Specifically, writing $\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)$ and $s_n = \sum_1^n x_i$, we prove as a first result

THEOREM 1. *If* $f_\theta(x) = \varphi(x - \theta)$ *and* $\theta_{n+1} = \theta_0 + (n^{-1}s_n - \theta_0)^+$ $(n = 1, 2, \cdots)$, *then for T defined by* (5)

$$E_\theta T \sim P_{\theta_0}\{T = \infty\}(\theta - \theta_0)^{-2} \log[(\theta - \theta_0)^{-1}] \qquad \textit{as} \quad \theta \downarrow \theta_0 . \tag{6}$$

According to a result of Farrell [4] (cf. Section 5), the smallest possible asymptotic expected value as $\theta \downarrow \theta_0$ for a stopping rule T such that $P_{\theta_0}\{T < \infty\} < 1$ is $2P_{\theta_0}\{T = \infty\}(\theta - \theta_0)^{-2} \log\log[(\theta - \theta_0)^{-1}]$, which is smaller than (6). This raises the question of whether for some other sequence $\{\theta_n\}$ we can achieve this "optimal" asymptotic expected sample size. Theorem 2 answers this question in the affirmative. We write $\log\log = \log_2$, etc., and put $a_n = (2\log_2^+ n + 3\log_3^+ n)^{\frac{1}{2}}$.

THEOREM 2. *If* $f_\theta(x) = \varphi(x - \theta)$ *and*

$$\theta_{n+1} = \theta_0 + (n^{-1}s_n - \theta_0) \cdot I\{s_n - n\theta_0 > n^{\frac{1}{2}}a_n\} \qquad (n = 1, 2, \cdots),$$

then for T defined by (5)

$$E_\theta T \sim 2P_{\theta_0}\{T = \infty\}(\theta - \theta_0)^{-2} \log_2[(\theta - \theta_0)^{-1}] \qquad \textit{as} \quad \theta \downarrow \theta_0 . \tag{7}$$

Theorem 1 is proved in Section 3, which also contains some related results. In Sections 4 and 5 we prove Theorem 2 and sketch a simplified proof of Farrell's theorem based on our techniques. Section 6 contains a comparison of these results with those obtained by the method of mixtures in [8] and [9]. In Section 7 we find an asymptotic approximation to $E_\theta T$ as $b \to \infty$, and in Section 9 we give an application of our results to Breiman's gambling theory [1].

2. Some fundamental lemmas. In Sections 2–7 we assume that $f_\theta(x) = \varphi(x-\theta)$ and take $\theta_0 = 0$, so that with $c = \log b$ (5) becomes

$$(8) \qquad T = T(c) = \text{first} \quad n \geqq 1 \quad \text{such that} \quad \textstyle\sum_1^n (\theta_k x_k - \frac{1}{2}\theta_k^2) \geqq c$$
$$= \infty \quad \text{if no such} \quad n \quad \text{occurs.}$$

By the results of [10], $P_\theta\{T < \infty\} \leqq b^{-1} = e^{-c}$ for all $\theta \leqq 0$. We shall also assume that for each $\theta > 0$ the sequence $\{\theta_n\}$ is such that

$$(9) \qquad E_\theta \theta_n^4 < \infty \qquad \text{and} \qquad E_\theta(\theta_n - \theta)^4 = O(n^{-2}) \qquad \text{as } n \to \infty\,.$$

The important results of this section are Lemma 1, which states that $E_\theta T < \infty$ for $\theta > 0$, Lemma 5, which provides a fundamental representation for $E_\theta T$, and Lemma 6.

Let

$$\tau = \min(T, m) \qquad (m = 2^{10}, 2^{10}+1, \cdots) \text{ and}$$
$$F_n = \mathscr{B}(x_1, \cdots, x_n) \qquad (n = 0, 1, \cdots)\,.$$

LEMMA 1. *For each* $\theta > 0$, $E_\theta T < \infty$.

PROOF. Since θ_{n+1} is F_n-measurable, $E_\theta(\theta_{n+1}x_{n+1} \mid F_n) = \theta_{n+1}E_\theta(x_{n+1} \mid F_n) = \theta_{n+1}\theta$, so $\{\sum_1^n \theta_k(x_k - \theta), F_n, 1 \leqq n < \infty\}$ is a martingale. Since τ is a bounded stopping time, it follows from Wald's lemma for martingales (cf. Theorem 2.3 of [2]) that

$$(10) \qquad E_\theta(\textstyle\sum_1^\tau \theta_k x_k) = \theta E_\theta(\sum_1^\tau \theta_k)\,,$$

and hence by algebra that

$$(11) \qquad E_\theta(\tau) = 2\theta^{-2}E_\theta(\textstyle\sum_1^\tau (\theta_k x_k - \frac{1}{2}\theta_k^2)) + \theta^{-2}E_\theta(\sum_1^\tau (\theta_k - \theta)^2)\,.$$

Let $0 < \eta < \theta$. It follows from Lemma 2 below that for all m

$$(12) \qquad E_\theta(\textstyle\sum_1^\tau (\theta_k x_k - \frac{1}{2}\theta_k^2)) - c \leqq \left(\theta + \dfrac{\varphi(\theta)}{\Phi(\theta)}\right)[\theta + E_\theta(\sup_n (\theta_n - \theta)^+)] < \infty$$

(where $\Phi(x) = \int_{-\infty}^x \varphi(u)\,du$), and from Lemma 3 that

$$(13) \qquad E_\theta(\textstyle\sum_1^\tau (\theta_k - \theta)^2) \leqq \eta^2 E_\theta \tau + O(1)$$

as $m \to \infty$. Upon letting $m \to \infty$, we obtain Lemma 1 from (11), (12), (13), and the monotone convergence theorem.

LEMMA 2. *For each* $\theta > 0$ *and all* m

$$(14) \qquad \int_{\{T \leqq m\}} (\textstyle\sum_1^T (\theta_k x_k - \frac{1}{2}\theta_k^2))\,dP_\theta - c$$
$$\leqq \left(\theta + \dfrac{\varphi(\theta)}{\Phi(\theta)}\right)[\theta + E_\theta(\sup_n (\theta_n - \theta)^+)] < \infty\,.$$

PROOF. If z is normally distributed with mean 0 and variance 1 and $h(t) = E(z + t \mid z > -t) = t + (\varphi(t)/\Phi(t))$, it may be verified by direct computation that $h'(t) = \text{Var}(z \mid z > -t) > 0$, so h is increasing. The left-hand side of (14) is

$$\leqq \int_{\{T \leqq m\}} \theta_T E_\theta(x_T - \tfrac{1}{2}\theta_T - \theta_T^{-1}(c - \textstyle\sum_1^{T-1} [\theta_k x_k - \tfrac{1}{2}\theta_k^2]) \mid x_1, \cdots, x_{T-1}, T)\, dP_\theta ,$$

which is majorized by

$$\int_{\{T \leqq m\}} \theta_T \sup_{r \geqq 0} h(\theta - r)\, dP_\theta \leqq h(\theta) E_\theta \theta_\tau \leqq (\theta + \varphi(\theta)/\Phi(\theta))(\theta + E_\theta(\theta_\tau - \theta)^+) .$$

This proves the first inequality of (14). Now observe that for $x > 0$

$$P_\theta\{\sup_n (\theta_n - \theta)^+ > x\} \leqq P_\theta\{\textstyle\sum_1^\infty (\theta_n - \theta)^4 > x^4\} \leqq x^{-4} \textstyle\sum_1^\infty E_\theta(\theta_{n+1} - \theta)^4 ,$$

which together with (9) shows that $E_\theta(\sup_n (\theta_n - \theta)^+) < \infty$, completing the proof.

LEMMA 3. *For each* $\theta > 0$, $\eta > 0$, $i = 1$ *or* 2

$$E_\theta(\textstyle\sum_1^\tau |\theta_n - \theta|^i) \leqq \eta^i E_\theta \tau + O(1) \qquad \textit{as } m \to \infty .$$

PROOF.

$$\begin{aligned} &E_\theta(\textstyle\sum_1^\tau |\theta_n - \theta|^i) \\ (15) \qquad &\leqq |\theta_1 - \theta|^i + \textstyle\sum_{n=1}^\infty \int_{\{\tau > n\}} |\theta_{n+1} - \theta|^i \, dP_\theta \\ &\leqq |\theta_1 - \theta|^i + \textstyle\sum_{n=1}^\infty [\int_{\{\tau > n, |\theta_{n+1} - \theta| > \eta\}} |\theta_{n+1} - \theta|^i \, dP_\theta + \eta^i P_\theta\{\tau > n\}] \\ &\leqq |\theta_1 - \theta|^i + \eta^i E_\theta \tau + \textstyle\sum_{n=1}^\infty \int_\eta^\infty x^i P_\theta\{|\theta_{n+1} - \theta| \in dx\} . \end{aligned}$$

Integrating by parts, and using the fourth moment Markov inequality and (9), gives

$$\begin{aligned} \int_\eta^\infty x^i P_\theta\{|\theta_{n+1} - \theta| \in dx\} &= i \int_\eta^\infty x^{i-1} P_\theta\{|\theta_{n+1} - \theta| > x\}\, dx + \eta^i P_\theta\{|\theta_{n+1} - \theta| > \eta\} \\ &\leqq \text{const. } n^{-2}(\int_\eta^\infty x^{i-5}\, dx + \eta^{i-4}) , \end{aligned}$$

which together with (15) completes the proof.

LEMMA 4.

$$E_\theta(\textstyle\sum_1^T |\theta_k - \theta|^i) < \infty , \qquad E_\theta(\textstyle\sum_1^T \theta_k^i) < \infty \qquad (\theta > 0,\ i = 1, 2) .$$

PROOF. The proof follows at once from Lemmas 1 and 3 and the monotone convergence theorem.

LEMMA 5. *For each* $\theta > 0$,

$$E_\theta T = \theta^{-2} E_\theta(\textstyle\sum_1^T (\theta_k - \theta)^2) + 2\theta^{-2} E_\theta(\textstyle\sum_1^T (\theta_k x_k - \tfrac{1}{2}\theta_k^2)) .$$

PROOF. From Lemma 4 and Theorem 2.3 of [2] we obtain (10) with T in place of τ. Lemma 5 now follows by algebra.

LEMMA 6. *For any* $\theta > 0$, *stopping time* Q, *positive integers* $k_0 < k_1$, *and* $a > 0$

$$P_\theta\{k_0 < Q \leqq k_1\} \leqq 1 - \Phi(a) + \exp\left[\frac{\theta^2 k_1}{2} + a\theta k_1^{\frac{1}{2}}\right] P_0\{k_0 < Q \leqq k_1\} .$$

PROOF.

(16) $$P_\theta\{k_0 < Q \leqq k_1\} \leqq P_\theta\{s_{k_1} - k_1\theta \geqq ak_1^{\frac{1}{2}}\} + P_\theta\{k_0 < Q \leqq k_1, s_{k_1} < k_1\theta + ak_1^{\frac{1}{2}}\}.$$

Since $A = \{k_0 < Q \leqq k_1, s_{k_1} < k_1\theta + ak_1^{\frac{1}{2}}\} \in F_{k_1}$, we have

$$P_\theta(A) = \int_A \exp\left[\theta s_{k_1} - \frac{\theta^2 k_1}{2}\right] dP_0 \leqq \exp\left[\frac{\theta^2 k_1}{2} + a\theta k_1^{\frac{1}{2}}\right] P_0(A),$$

which together with (16) completes the proof.

3. Proof of Theorem 1. Before proceeding with the proof of Theorem 1, note that if we put $\theta_n = \theta'(n = 1, 2, \cdots)$, then T defined by (8) becomes the "optimal" stopping rule $N(\theta', b)$ of Section 1. Moreover, putting $T = N(\theta, b)$ in Lemma 5 and (14), we see that $E_\theta N(\theta, b) = 2c\theta^{-2}(1 + o(1))$ as $\theta \downarrow 0$. Hence for arbitrary $\{\theta_k\}$, from Lemma 5, we find that $E_\theta T$ can be separated into two components: one, to the extent that $\sum_1^T (\theta_k x_k - \frac{1}{2}\theta_k^2)$ is about equal to c, is approximately $2c\theta^{-2} \approx E_\theta N(\theta, b)$; the other, $\theta^{-2}E_\theta(\sum_1^T (\theta_k - \theta)^2)$, arises from our ignorance of which of the possible values $\theta > 0$ is the correct one. Since, roughly speaking, T must be at least as large under P_θ as $N(\theta, b)$, we should expect to find for $\theta_{n+1} = n^{-1}s_n^+$ and small θ that

$$\theta^{-2}E_\theta(\textstyle\sum_1^T (\theta_k - \theta)^2) \gtrsim \theta^{-2} \sum_1^{[2c\theta^{-2}]} E_\theta(\theta_k - \theta)^2 \approx \theta^{-2} \sum_1^{[2c\theta^{-2}]} (2k)^{-1} \approx \theta^{-2} \log \theta^{-1},$$

which dominates the component $2c\theta^{-2}$ as $\theta \downarrow 0$. This heuristic analysis is made precise in Theorem 1. The key to the proof is Lemma 8, the proof of which is a refinement of the proof of Lemma 3.

Thus, in addition to the assumptions of Section 2, let $\theta_{n+1} = n^{-1}s_n^+$, so that

(17) $$\theta_{n+1} - \theta = (n^{-1}s_n - \theta)I_{\{s_n \geqq 0\}} - \theta I_{\{s_n < 0\}}$$

and for $\theta > 0$

(18) $$E_\theta(\theta_{n+1} - \theta)^2 \leqq n^{-1}, \qquad E_\theta(\theta_{n+1} - \theta)^4 \leqq 2n^{-2}.$$

In particular, (9) is satisfied so that $E_\theta T < \infty$ for all $\theta > 0$.

LEMMA 7. *If $n = n_\theta \uparrow \infty$ as $\theta \downarrow 0$, then*

$$\limsup_{\theta \downarrow 0} P_\theta\{T > n\} \leqq P_0\{T = \infty\}.$$

PROOF. Let $n_0 < n$. Then

$$P_\theta\{T \leqq n\} \geqq P_\theta\{T \leqq n_0\} = \int_{\{T \leqq n_0\}} \exp[\theta s_{n_0} - \theta^2 n_0/2]\, dP_0.$$

Letting $\theta \downarrow 0$, we have by Fatou's lemma

$$\liminf_{\theta \downarrow 0} P_\theta\{T \leqq n\} \geqq P_0\{T \leqq n_0\} \uparrow P_0\{T < \infty\} \qquad \text{as } n_0 \uparrow \infty.$$

LEMMA 8. *For each $\delta > 0$*

$$E_\theta(\textstyle\sum_1^T (\theta_k - \theta)^2) \leqq P_0\{T = \infty\} \log \theta^{-1} + \delta^2\theta^2 E_\theta T + o(\log \theta^{-1}) \qquad \text{as } \theta \downarrow 0.$$

PROOF. Let $\varepsilon, \delta > 0$ and $\eta = \delta\theta$. Let $n_0 < n_1 < n_2 = [\varepsilon^2\theta^{-2}]$. Then

$$E_\theta(\sum_1^T (\theta_n - \theta)^2) = (\theta_1 - \theta)^2 + \sum_1^\infty \int_{\{T>n\}} (\theta_{n+1} - \theta)^2 \, dP_\theta \leqq \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 ,$$

where

$$\Sigma_1 = \sum_0^{n_1} E_\theta(\theta_{n+1} - \theta)^2 ,$$
$$\Sigma_2 = \sum_{n_1+1}^{n_2} \int_{\{T>n_0\}} E_\theta[(\theta_{n+1} - \theta)^2 \mid F_{n_0}] \, dP_\theta ,$$
$$\Sigma_3 = \sum_{n=n_2+1}^\infty \int_{\{T>n, |\theta_{n+1}-\theta|>\eta\}} (\theta_{n+1} - \theta)^2 \, dP_\theta ,$$
$$\Sigma_4 = \sum_{n=n_2+1}^\infty \int_{\{T>n, |\theta_{n+1}-\theta|\leqq\eta\}} (\theta_{n+1} - \theta)^2 \, dP_\theta .$$

By (18)

(19) $\quad \Sigma_1 = O(\log n_1) = o(\log \theta^{-1})$ if, for example, $n_1 = O(\log \theta^{-1})$,

and

(20) $$\Sigma_4 \leqq \eta^2 \sum_1^\infty P_\theta\{T > n\} \leqq \delta^2\theta^2 E_\theta T .$$

Observing that the $O(n^{-2})$ in (9) is uniform in θ for the choice $\theta_{n+1} = n^{-1}s_n^+$, and using integration by parts and the fourth moment Markov inequality as in the proof of Lemma 3, we see that

(21) $$\Sigma_3 \leqq \text{const.} \, (\sum_{n_2+1}^\infty n^{-2})\eta^{-2} \leqq \text{const.}/\delta^2\varepsilon^2 .$$

We now turn to the estimation of Σ_2. Using (17) and the Schwarz inequality we obtain, since $s_n - n\theta = (s_{n_0} - n_0\theta) + (s_n - s_{n_0} - (n - n_0)\theta)$,

(22) $$E_\theta[(\theta_{n+1} - \theta)^2 \mid F_{n_0}] \leqq n^{-2}(s_{n_0} - \theta n_0)^2 + 2n^{-2}|s_{n_0} - \theta n_0|(n - n_0)^{\frac{1}{2}} + \theta^2 + n^{-2}E_\theta(I_{\{s_n\geqq 0\}}(s_n - s_{n_0} - \theta(n - n_0))^2 \mid F_{n_0}) .$$

Now $\{s_n - s_{n_0} \geqq -s_{n_0}\} \subset \{s_n - s_{n_0} \geqq -(\theta + 1)n_0\} \cup \{s_{n_0} > (\theta + 1)n_0\}$. Thus, putting $g = (\theta + 1)n_0(n_1 - n_0)^{-\frac{1}{2}} + \theta(n_2 - n_0)^{\frac{1}{2}}$, and recalling that $\int_x^\infty y^2\varphi(y)\,dy = 1 - \Phi(x) + x\varphi(x) \leqq 1 - \Phi(x)$ for $x \leqq 0$, we see that the last term of (22) is for all $n_1 < n \leqq n_2$

$$\leqq n^{-2}(E_\theta[I_{\{s_n-s_{n_0}\geqq-(\theta+1)n_0\}}(s_n - s_{n_0} - \theta(n - n_0))^2] + I_{\{s_{n_0}>(\theta+1)n_0\}}(n - n_0))$$
$$\leqq n^{-1}(1 - \Phi(-g) + I_{\{s_{n_0}>(\theta+1)n_0\}}) .$$

Hence after some computation we obtain from (22)

(23) $$\Sigma_2 \leqq [P_\theta\{T > n_0\}(1 - \Phi(-g)) + 1 - \Phi(n_0^{\frac{1}{2}})] \log n_2 + \text{const.}$$

Now $\theta n_2^{\frac{1}{2}} \to \varepsilon$ as $\theta \downarrow 0$, so if n_0 and n_1 become infinite as $\theta \downarrow 0$ in such a way that $n_0(n_1 - n_0)^{-\frac{1}{2}} \to 0$, then $g \to \varepsilon$, and hence by (23) and Lemma 7

$$\Sigma_2 \leqq P_0\{T = \infty\}(1 - \Phi(-\varepsilon))2 \log \theta^{-1} + o(\log \theta^{-1}) .$$

This together with (19), (20), and (21) completes the proof since ε is arbitrary.

LEMMA 9. *Let Q be an arbitrary stopping time for $x_1, x_2, \cdots$ and let $n_0 < n_2 = [K\theta^{-2}]$, where K may depend on θ but is bounded as $\theta \downarrow 0$. Then* $\lim_{n_0\to\infty} \limsup_{\theta\downarrow 0} P_\theta\{n_0 < Q \leqq n_2\} = 0$.

PROOF. Putting $k_0 = n_0$, $k_1 = n_2$ in Lemma 6 yields the proof immediately since $a > 0$ is arbitrary.

LEMMA 10.

$$E_\theta\left(\sum_1^T (\theta_n - \theta)^2\right) \geqq P_0\{T = \infty\} \log \theta^{-1}(1 + o(1)) \qquad as \ \theta \downarrow 0 .$$

PROOF. Let $\varepsilon > 0$, $n_0 < n_1 < n_2 = [\varepsilon^2\theta^{-2}]$. Then

$$\begin{aligned}
&E_\theta\left(\sum_1^T (\theta_n - \theta)^2\right) \\
(24) \qquad &\geqq \sum_{n_1+1}^{n_2} \int_{\{T>n\}} (\theta_{n+1} - \theta)^2 \, dP_\theta \\
&= \sum_{n_1+1}^{n_2} \left[\int_{\{T>n_0\}} E_\theta((\theta_{n+1} - \theta)^2 \mid F_{n_0}) \, dP_\theta - \int_{\{n_0<T\leqq n\}} (\theta_{n+1} - \theta)^2 \, dP_\theta\right] .
\end{aligned}$$

By the Schwarz inequality and (18), for all $n_1 < n \leqq n_2$,

$$\begin{aligned}
\int_{\{n_0<T\leqq n\}} (\theta_{n+1} - \theta)^2 \, dP_\theta &\leqq (P_\theta\{n_0 < T \leqq n_2\} E_\theta(\theta_{n+1} - \theta)^4)^{\frac{1}{2}} \\
&\leqq 2 \cdot n^{-1}(P_\theta\{n_0 < T \leqq n_2\})^{\frac{1}{2}} ,
\end{aligned}$$

and hence

$$(25) \qquad \sum_{n_1+1}^{n_2} \int_{\{n_0<T\leqq n\}} (\theta_{n+1} - \theta)^2 \, dP_\theta \leqq \text{const.}\ (P_\theta\{n_0 < T \leqq n_2\})^{\frac{1}{2}} \left(\log \frac{1}{\theta}\right) .$$

Arguing as in the proof of Lemma 8, we see that for all $n_1 < n \leqq n_2$

$$\begin{aligned}
&E_\theta((\theta_{n+1} - \theta)^2 \mid F_{n_0}) \\
&\qquad \geqq -2n^{-\frac{3}{2}}|s_{n_0} - \theta n_0| + n^{-2}(n - n_0) I_{\{s_{n_0} \geqq (\theta-1)n_0\}} \Phi(-n_0(n_1 - n_0)^{-\frac{1}{2}}) .
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{n=n_1+1}^{n_2} \int_{\{T>n_0\}} E_\theta((\theta_{n+1} - \theta)^2 \mid F_{n_0}) \, dP_\theta \\
(26) \qquad &\geqq \left(\log \frac{n_2}{n_1}\right) \Phi(-n_0(n_1 - n_0)^{-\frac{1}{2}}) P_\theta\{T > n_0, s_{n_0} \geqq (\theta - 1)n_0\} - \text{const.} \\
&\geqq \log \frac{n_2}{n_1} \cdot \Phi(-n_0(n_1 - n_0)^{-\frac{1}{2}})(P_\theta\{T > n_0\} - \Phi(-n_0^{\frac{1}{2}})) - \text{const.}
\end{aligned}$$

Letting $\theta \downarrow 0$, $n_1 \uparrow \infty$, and $n_0 \uparrow \infty$ (in the indicated order) we obtain the Lemma from (24), (25), (26), and Lemma 9.

Turning now to the proof of Theorem 1, we see by Lemmas 5 and 10 that

$$E_\theta T \geqq (\theta^{-2} \log \theta^{-1}) P_0\{T = \infty\}(1 + o(1)) \qquad \text{as } \theta \downarrow 0 .$$

To prove the reverse inequality, note that by Lemmas 5 and 8

$$\begin{aligned}
E_\theta T \leqq (\theta^{-2} \log \theta^{-1}) P_0\{T = \infty\} &+ \delta^2 E_\theta T + 2\theta^{-2} E_\theta\left(\sum_1^T (\theta_k x_k - \tfrac{1}{2}\theta_k^2)\right) \\
&+ o(\theta^{-2} \log \theta^{-1}) \qquad \text{as } \theta \downarrow 0 .
\end{aligned}$$

Letting $m \to \infty$ in (14) shows that (12) holds with T in place of τ. For $\theta_{n+1} = n^{-1}s_n^+$, $\theta_{n+1} - \theta \leqq (n^{-1}s_n - \theta)^+$, so the right-hand side of (12) is bounded as $\theta \downarrow 0$. Since δ is arbitrary this completes the proof of Theorem 1.

A similar argument proves

THEOREM 3. *If* $\theta_{n+1} = (n + 1)^{-1}s_n + (n + 1)^{-\frac{1}{2}}\varphi((n + 1)^{-\frac{1}{2}}s_n)/\Phi((n + 1)^{-\frac{1}{2}}s_n)$, *then*

$$E_\theta T \sim (2P_0\{T = \infty\} \log \theta^{-1})/\theta^2 \qquad as \ \theta \downarrow 0 .$$

To test the hypothesis $\theta < 0$ against the alternative $\theta > 0$ ($\theta = 0$ being excluded) with uniformly small error probabilities, let $\theta_1 = 0$, $\theta_{n+1} = n^{-1}s_n$ ($n \geqq 1$), let T be defined by (8), and if $T = n$ decide $\theta > 0$ if and only if $s_T > 0$. Then for $\theta < 0$ the arguments in [10] show that $P_\theta\{T < \infty, s_T > 0\} \leqq P_0\{T < \infty, s_T > 0\} = \frac{1}{2} \cdot P_0\{T < \infty\} \leqq \frac{1}{2} \cdot e^{-c}$. Similarly for $\theta > 0$, $P_\theta\{T < \infty, s_T < 0\} \leqq \frac{1}{2} \cdot e^{-c}$, so $P_\theta(\text{error}) \leqq \frac{1}{2} \cdot e^{-c}$ for all $\theta \neq 0$. The method of proof of Theorem 1 shows that

$$E_\theta T \sim (2P_0\{T = \infty\} \log \theta^{-1})/\theta^2 \qquad \text{as } \theta \to 0 .$$

A different application of such "two-sided" stopping rules is given in Section 9.

4. Proof of Theorem 2. We keep the assumptions of Section 2 but set

$$\theta_{n+1} = n^{-1}s_n I_{\{s_n \geqq n^{\frac{1}{2}} a_n\}}, \qquad \text{where } a_n^2 = 2 \log_2^+ n + 3 \log_3^+ n .$$

By direct computation

$$(27) \qquad E_\theta(\theta_{n+1} - \theta)^2 = n^{-1}(1 - \Phi(a_n - \theta n^{\frac{1}{2}}) + (a_n - \theta n^{\frac{1}{2}})\varphi(a_n - \theta n^{\frac{1}{2}}) + \theta^2\Phi(a_n - \theta n^{\frac{1}{2}})$$

and

$$(28) \qquad E_\theta(\theta_{n+1} - \theta)^4 \leqq 2n^{-2} + \theta^4\Phi(a_n - \theta n^{\frac{1}{2}}) .$$

In particular (9) is satisfied, so $E_\theta T < \infty$ for all $\theta > 0$ and Lemma 5 holds.

Lemma 11 resembles Lemma 8, but the details of its proof are somewhat different.

LEMMA 11. *For each* $\delta > 0$, $\lambda > 1$

$$E_\theta(\textstyle\sum_1^T (\theta_n - \theta)^2) \leqq 2\lambda P_0\{T = \infty\}(\log_2 \theta^{-1}) + \delta^2\theta^2 E_\theta T + o(\log_2 \theta^{-1}) \qquad \textit{as } \theta \downarrow 0 .$$

PROOF. Let $\delta > 0$, $\eta = \delta\theta$, $n_1 = [(\theta^2 \log_2 \theta^{-1})^{-1}]$, $\lambda > 1$, and $n_2 = [2\lambda\theta^{-2}\log_2 \theta^{-1}]$. As in the proof of Lemma 8

$$(29) \qquad E_\theta(\textstyle\sum_1^T (\theta_n - \theta)^2) = (\theta_1 - \theta)^2 + \sum_1^\infty \int_{\{T > n\}} (\theta_{n+1} - \theta)^2 \, dP_\theta \leqq \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 ,$$

where

$$\Sigma_1 = \textstyle\sum_{n=0}^{n_1} E_\theta(\theta_{n+1} - \theta)^2 ,$$
$$\Sigma_2 = \textstyle\sum_{n=n_1+1}^{n_2} \int_{\{T > n_1\}} E_\theta((\theta_{n+1} - \theta)^2 \mid F_{n_1}) \, dP_\theta ,$$
$$\Sigma_3 = \textstyle\sum_{n=n_2+1}^{\infty} \int_{\{T > n, |\theta_{n+1} - \theta| > \eta\}} (\theta_{n+1} - \theta)^2 \, dP_\theta ,$$
$$\Sigma_4 = \textstyle\sum_{n=n_2+1}^{\infty} \int_{\{T > n, |\theta_{n+1} - \theta| \leqq \eta\}} (\theta_{n+1} - \theta)^2 \, dP_\theta .$$

As before

$$(30) \qquad \Sigma_4 \leqq \delta^2\theta^2 E_\theta T .$$

Also by simple computations for $n > n_1$

$$E_\theta((\theta_{n+1} - \theta)^2 \mid F_{n_1}) \leqq n^{-2}(s_{n_1} - \theta n_1)^2 + \theta^2 + n^{-2}(n - n_1) + 2n^{-2}|s_{n_1} - \theta n_1|(n - n_1)^{\frac{1}{2}} ,$$

so

$$\int_{\{T > n_1\}} E_\theta((\theta_{n+1} - \theta)^2 \mid F_{n_1}) \, dP_\theta \leqq n^{-1} + \theta^2 P_\theta\{T > n_1\} + 2n^{-\frac{3}{2}}n_1^{\frac{1}{2}} .$$

Hence by Lemma 7 as $\theta \downarrow 0$

(31) $$\Sigma_2 \leqq n_2\theta^2 P_\theta\{T > n_1\} + \log \frac{n_2}{n_1} + 5 \leqq 2\lambda(\log_2 \theta^{-1})P_0\{T = \infty\}(1 + o(1)) .$$

For $10 \leqq n \leqq n_1$, $\theta \leqq e^{-e}$, we have $\theta^2 \leqq (n \log_2 \theta^{-1})^{-1} \leqq 2(n \log_2 n)^{-1}$, so $(a_n - \theta n^{\frac{1}{2}})^2 \geqq a_n^2 - 2\theta n^{\frac{1}{2}} a_n \geqq 2 \log_2 n + 3 \log_3 n -$ const. Hence by (27)

(32) $$\Sigma_1 = O(\textstyle\sum^{n_1} (n \log n \log_2 n)^{-1}) = O(\log_3 \theta^{-1}) = o(\log_2 \theta^{-1}) .$$

Using integration by parts and the fourth moment Markov inequality as in the proofs of Lemmas 3 and 8, we obtain from (28)

(33) $$\Sigma_3 \leqq \text{const. } \eta^{-2} (n_2^{-1} + \textstyle\sum_{n=n_2+1}^{\infty} \theta^4 \Phi(a_n - \theta n^{\frac{1}{2}})) .$$

Let

$$f(x) = (2 \log_2 x + 3 \log_3 x)^{\frac{1}{2}} - \theta x^{\frac{1}{2}} \qquad (x \geqq n_2) .$$

Then for all θ sufficiently small, $f(n_2) < -1$ and $f'(x) < 0$ for all $x \geqq n_2$. So from the estimate $\Phi(y) \leqq \varphi(y)$ $(y \leqq -1)$ we obtain

(34) $$\theta^2 \textstyle\sum_{n_2+1}^{\infty} \Phi(a_n - \theta n^{\frac{1}{2}}) \leqq \theta^2 \int_{n_2}^{\infty} \varphi(f(x))\, dx .$$

Let $\gamma > 0$ be so small that $\lambda - \gamma > 1$. Then for θ small and $t \geqq n_2$

$$\theta t^{\frac{1}{2}}(2(\lambda - \gamma) \log_2 t)^{-\frac{1}{2}} \geqq \theta n_2^{\frac{1}{2}}(2(\lambda - \gamma) \log_2 n_2)^{-\frac{1}{2}} \geqq 1 ,$$

so

$$(2 \log_2 t + 3 \log_3 t)^{\frac{1}{2}} \leqq \theta t^{\frac{1}{2}}(\lambda - \gamma)^{-\frac{1}{2}}(1 + o(1)) \leqq \theta t^{\frac{1}{2}}(1 - \varepsilon)$$

for some $\varepsilon > 0$. Hence $f(t) \leqq -\varepsilon\theta t^{\frac{1}{2}}$ $(t \geqq n_2)$ and

(35) $$\theta^2 \int_{n_2}^{\infty} \varphi(f(x))\, dx \leqq \theta^2 \int_{n_2}^{\infty} \varphi(\varepsilon\theta t^{\frac{1}{2}})\, dt = O\left[\exp\left(-\frac{\varepsilon^2\theta^2 n_2}{2}\right)\right].$$

From (33), (34), and (35) we see that

(36) $$\Sigma_3 = o(1) \qquad \text{as } \theta \downarrow 0 .$$

The lemma follows from (29), (30), (31), (32), and (36).

LEMMA 12. *Let* $0 < \lambda < 1$, $n_1 = [\theta^{-2}]$, $n_2 = [2\lambda\theta^{-2} \log_2 \theta^{-1}]$. *If Q is any stopping rule for* $x_1, x_2, \cdots$ *such that for some* $\varepsilon < (1 - \lambda)/3$ *and all θ sufficiently small*

(37) $$\{n_1 < Q \leqq n_2\} \subset \{|s_n| \geqq (2(1 - \varepsilon)n \log_2 n)^{\frac{1}{2}} \text{ for some } n_1 \leqq n \leqq n_2\} ,$$

then $P_\theta\{n_1 < Q \leqq n_2\} \to 0$ *as* $\theta \to 0$.

PROOF. From Lemma 6 with $k_0 = n_1$ and $k_1 = n_2$ we obtain

(38) $$P_\theta\{n_1 < Q \leqq n_2\} \leqq 1 - \Phi(a) + (\log n_1)^{\lambda + o(1)} P_0\{n_1 < Q \leqq n_2\} .$$

Since a is arbitrary and $2n_1 \log_2 n_1 \geqq n_2$, by (37) and (38) it suffices to prove

(39) $$P_0\{|s_n| \geqq (2(1 - \varepsilon)n \log_2 n)^{\frac{1}{2}} \text{ for some } i \leqq n \leqq 2i \log_2 i\} = O((\log i)^{-1+3\varepsilon}) \qquad \text{as } i \to \infty .$$

The proof of (39) is by standard calculations. Let $r > 1$ be such that

$(1-\varepsilon)r^{-1} > 1-2\varepsilon$ and put $r_k = [r^k]$. Since

$$P_0\{\max_{1\le k\le m}|s_k| \geqq x\} \leqq 2P_0\{|s_m| \geqq x\} \qquad (x>0,\ m=1,2,\cdots)$$

and $1-\Phi(x) \leqq \varphi(x)$ $(x \geqq 1)$, we see that the left-hand side of (39) is majorized by

$$\begin{aligned}\textstyle\sum_{i\le r_k\le 2i\log_2 i} & P_0\{\max_{r_k\le n\le r_{k+1}}|s_n| \geqq (2(1-\varepsilon)r_k\log_2 r_k)^{\frac{1}{2}}\}\\ &\leqq 4\textstyle\sum_{i\le r_k\le i\log i}[1-\Phi((2(1-\varepsilon)r_{k+1}^{-1}r_k\log_2 r_k)^{\frac{1}{2}})]\\ &\leqq 4\textstyle\sum_{i\le r_k\le i\log i}(\log r_k)^{-1+2\varepsilon} \leqq 8\dfrac{\log_2 i}{\log r}\left(\dfrac{1}{\log i}\right)^{1-2\varepsilon} = O(\log i)^{-1+3\varepsilon}\,.\end{aligned}$$

It is now easy to complete the proof of Theorem 2. Arguing as in the proof of Theorem 1 but replacing Lemma 8 by Lemma 11 gives

$$\limsup_{\theta\downarrow 0}\theta^2(\log_2\theta^{-1})^{-1}E_\theta T \leqq 2\lambda(1-\delta^2)^{-1}P_0\{T=\infty\} \to 2P_0\{T=\infty\}$$

as $\lambda\downarrow 1$ and $\delta\downarrow 0$. To prove the reverse inequality let $n_0 < n_1 = [\theta^{-2}]$ and for arbitrary $\lambda < 1$ let $n_2 = [2\lambda\theta^{-2}\log_2\theta^{-1}]$. Then

$$\begin{aligned}(40)\qquad E_\theta T &\geqq n_2P_\theta\{T>n_2\} = n_2(P_\theta\{T>n_0\} - P_\theta\{n_0<T\leqq n_2\})\\ &= n_2(P_\theta\{T>n_0\} - P_\theta\{n_0<T\leqq n_1\} - P_\theta\{n_1<T\leqq n_2\})\,.\end{aligned}$$

As $\theta\downarrow 0$ (n_0 fixed) $P_\theta\{T>n_0\} = \int_{\{T>n_0\}}\exp(\theta s_{n_0} - \theta^2 n_0/2)\,dP_0 \to P_0\{T>n_0\} \geqq P_0\{T=\infty\}$. Moreover, since $\{n_1<T\leqq n_2\} \subset \{\theta_n \neq 0 \text{ for some } n_1<n\leqq n_2\} = \{s_n \geqq n^{\frac{1}{2}}a_n \text{ for some } n_1\leqq n<n_2\}$, by Lemma 12 $P_\theta\{n_1<T\leqq n_2\}\to 0$ as $\theta\downarrow 0$. Hence by Lemma 9, letting $\theta\downarrow 0$, then $n_0\uparrow\infty$, we obtain from (40)

$$E_\theta T \geqq 2\lambda\theta^{-2}\log_2\theta^{-1}(P_0\{T=\infty\} - o(1))\,,$$

which completes the proof, since $\lambda<1$ is arbitrary.

5. On Farrell's theorem. The fundamental result of [4], which provides a standard of comparison for $E_\theta T$ as $\theta\downarrow 0$, implies that for any stopping rule M for the sequence $x_1, x_2, \cdots$ such that $P_0\{M=\infty\}>0$

$$(41)\qquad \limsup_{\theta\downarrow 0}\theta^2(\log_2\theta^{-1})^{-1}E_\theta M \geqq 2P_0\{M=\infty\}$$

and

$$(42)\qquad \lim_{\theta\downarrow 0}\theta^2E_\theta M = \infty\,.$$

(An example at the end of this section shows that lim sup cannot be replaced by lim inf in (41).)

We now show that the method of proof of Theorem 2 provides a considerably simplified proof of Farrell's theorem. With slight modifications our method also works in the more general context of an arbitrary exponential family considered by Farrell.

Putting M in place of T and $n_2 = [K\theta^{-2}]$ in the first line of (40) and appealing to Lemma 9 gives $\liminf_{\theta\downarrow 0}\theta^2E_\theta M \geqq KP_0\{M=\infty\}$. This proves (42), since K is arbitrary.

The proof of (41) involves a preliminary argument which goes back to Weiss

[12]. From considerations of sufficiency and monotonicity it is possible to show that there exists a sequence $\{\beta_n\}$ such that the stopping rule

$$(43)\qquad \begin{aligned} M' &= \text{first} \quad n \geqq 1 \quad \text{such that} \quad s_n \geqq \beta_n \\ &= \infty \quad \text{if} \quad s_n < \beta_n \quad \text{for all} \quad n \end{aligned}$$

satisfies $P_0\{M' < \infty\} = P_0\{M < \infty\}$ and $P_\theta\{M' > n\} \leqq P_\theta\{M > n\}$ for all $n = 1, 2, \cdots$ and $\theta > 0$. In particular, $E_\theta M' \leqq E_\theta M$ for all $\theta > 0$, and hence it suffices to prove (41) with M' in place of M.

The proof of Theorem 2 shows that to prove (41) for M' it suffices to prove (in the notation of Lemma 12)

$$(44)\qquad \liminf_{\theta \downarrow 0} P_\theta\{n_1 < M' \leqq n_2\} = 0\,,$$

and hence by (43) and the proof of Lemma 12 to show for each $\varepsilon > 0$, for infinitely many $\theta \downarrow 0$

$$\beta_n \geqq (2(1-\varepsilon)n \log_2 n)^{\frac{1}{2}} \qquad \text{for all} \quad n_1 \leqq n \leqq 2n_1 \log_2 n_1\,.$$

If this is not the case, then for any subsequence $\{i_k\}$ of positive integers there exists a k_0 such that for all $k \geqq k_0$ $\beta_n < (2(1-\varepsilon)n\log_2 n)^{\frac{1}{2}}$ for some $n = n_k$, $i_k \leqq n_k \leqq 2i_k \log_2 i_k$. Putting $i_k = [k^k]$ and observing that $i_{k+1}/i_k \log_2 i_k \to \infty$ as $k \to \infty$, one may argue exactly as in the standard proof of the law of the iterated logarithm (eg. [5], page 192) to show that

$$P_0\{s_{n_k} \geqq \beta_{n_k} \text{ i.o.}\} \geqq P_0\{s_{n_k} \geqq (2(1-\varepsilon)n_k \log_2 n_k)^{\frac{1}{2}} \text{ i.o.}\} = 1\,,$$

contradicting the hypothesis that $P_0\{M' < \infty\} < 1$.

It is easy to give examples to show that lim sup cannot be replaced by lim inf in (41). The following example gives a stopping rule M for which equality holds in (41) and for which for a certain sequence of positive numbers $\mu_i \to 0$, $\mu_i^2 E_{\mu_i} M$ diverges arbitrarily slowly as $i \to \infty$.

Let $0 < \alpha < 1$, and let $\{b_i\}$ be an increasing sequence of positive numbers such that $\sum b_i^{-1} \leqq \alpha/2$. For any strictly decreasing function $h\colon (0, \infty) \to (0, \infty)$ such that $h(0+) = +\infty$, define $\mu_i = h^{-1}(2 \log b_i)$. Since $b_i \uparrow \infty$, $\mu_i \downarrow 0$ as $i \to \infty$. If

$$\begin{aligned} T_i &= \text{first} \quad n \geqq 1 \quad \text{such that} \quad b_i^{-1} \exp(\mu_i s_n - \mu_i^2 n/2) \geqq 1 \\ &= \infty \quad \text{if no such} \quad n \quad \text{occurs,} \end{aligned}$$

then T_i is of the form (8) with $b_i = b$ and $\theta_n \equiv \mu_i (n = 1, 2, \cdots)$, and hence by Lemmas 2 and 5 $E_{\mu_i} T_i \sim \mu_i^{-2} 2 \log b_i = \mu_i^{-2} h(\mu_i)$ as $i \to \infty$. Putting $\theta_{n+1} = n^{-1} s_n I_{\{s_n \geqq n^{\frac{1}{2}} a_n\}}$, define

$$\begin{aligned} M = \text{first} \quad n \geqq 1 \quad \text{such that} \quad & \textstyle\sum_{i=1}^\infty b_i^{-1} \exp(\mu_i s_n - \mu_i^2 n/2) \\ & + \frac{\alpha}{2} \exp(\textstyle\sum_1^n (\theta_k x_k - \frac{1}{2}\theta_k^2)) \geqq 1 \\ = \infty \quad \text{if no such} \quad n \quad & \text{occurs.} \end{aligned}$$

By the results of [10], $P_\theta\{M < \infty\} \leqq \sum b_i^{-1} + \alpha/2 \leqq \alpha$ for all $\theta \leqq 0$, and since

$M \leqq T_i$ $(i = 1, 2, \cdots) E_{\mu_i} M \leqq E_{\mu_i} T_i \sim \mu_i^{-2} h(\mu_i)$ as $i \to \infty$. Moreover, for $T = T(c)$ defined by (8) with θ_n as in Theorem 2, we have $M \leqq T(c)$ provided $c \geqq \log 2/\alpha$. The proofs of Lemmas 2, 5, and 11 remain valid with M in place of T and consequently

$$\limsup_{\theta \downarrow 0} \theta^2 (\log_2 \theta^{-1})^{-1} E_\theta M \leqq 2P_0\{M = \infty\},$$

i.e., equality holds in (41).

6. Comparison with the method of mixtures. It is interesting to compare Theorems 1–3 with results obtained by the method of mixtures in [8] and [9]. The most revealing comparison occurs in a continuous time formulation of the problem, in which the sequence $s_n = x_1 + \cdots + x_n$ of partial sums of independent normally distributed random variables is replaced by a standard Wiener process $X(t)$, $0 \leqq t < \infty$, with drift θ per unit time. The stopping rule defined by (8) becomes

$$T = \inf\{t : \textstyle\int_0^t \theta(s)\, dX(s) - \frac{1}{2} \int_0^t \theta^2(s)\, ds \geqq c\}, \tag{45}$$

where $\int_0^t \theta(s)\, dX(s)$ is an Itô stochastic integral. (See [6] for the definitions and fundamental results of the Itô stochastic calculus.) By a passage to the limit from the discrete case we have $P_\theta\{T < \infty\} \leqq e^{-c}$ for all $\theta \leqq 0$.

In the method of mixtures, for any probability measure F on $(0, \infty)$ we define for $t \geqq 0$

$$f(x, t) = \textstyle\int_0^\infty \exp(xy - y^2 t/2)\, dF(y) \tag{46}$$

and

$$T' = \inf\{t : t > 0, f(X(t), t) \geqq b = e^c\}. \tag{47}$$

It was shown in [9] that $P_\theta\{T' < \infty\} \leqq e^{-c}$ for all $\theta \leqq 0$ (indeed equality holds when $\theta = 0$).

According to Itô's lemma (cf. [6], page 32), any sufficiently smooth function $u(X(t), t)$ can be expressed as a stochastic integral according to the formula

$$u(X(t), t) = u(0, 0) + \textstyle\int_0^t u_x(X(s), s)\, dX(s) + \int_0^t (u_t + \frac{1}{2} u_{xx})(X(s), s)\, ds. \tag{48}$$

By applying (48) to $u(x, t) = \log f(x, t)$ and observing by differentiation under the integral in (46) that $f_t + \frac{1}{2} f_{xx} = 0$ $(-\infty < x < \infty, 0 < t < \infty)$, it may be shown, at least whenever $\int_0^\infty y\, dF(y) < \infty$, that

$$\log f(X(t), t) = \int_0^t \frac{f_x}{f} (X(s), s)\, dX(s) - \frac{1}{2} \int_0^t \left(\frac{f_x}{f}\right)^2 (X(s), s)\, ds. \tag{49}$$

Hence (47) is of the form (45) with

$$\theta(t) = (f_x/f)(X(t), t).$$

Now a second differentiation under the integral shows that $(f_x/f)(X(t), t)$ is the posterior mean of θ conditional on $X(t)$, when θ has the prior distribution F.

Hence in a continuous time formulation a test of power one given by the method of mixtures is a special case of the class of tests studied in this paper, obtained by letting $\theta(t)$ be the posterior mean of θ conditional on $X(t)$ when θ has as its prior distribution the given mixing measure.

For example, the sequence $\{\theta_n\}$ of Theorem 3 is the sequence of posterior means for a half-normal prior distribution, and although there is no exact correspondence with the method of mixtures in discrete time the asymptotic formula of Theorem 3 is the same as that obtained for the case of a half-normal mixture (in either discrete or continuous time).

Consideration of the measure

$$dF(y) = \delta\, dy/y(\log y^{-1})(\log_2 y^{-1})^{1+\delta}\,, \qquad 0 < y < e^{-e} \tag{50}$$
$$= 0 \quad \text{elsewhere}$$

leads to an interesting comparison of the two methods. It follows from results of [4] and [9] that for T' defined by (47) with F given by (50)

$$E_\theta T' \sim (2P_0\{T' = \infty\} \log_2 \theta^{-1})/\theta^2 \tag{51}$$

as $\theta \downarrow 0$. (The same asymptotic formula holds for the discrete time method of mixtures.) Hence by the correspondence of T defined by (45) and T' defined by (47) we have

$$E_\theta T \sim (2P_0\{T = \infty\} \log_2 \theta^{-1})/\theta^2 \tag{52}$$

when $\theta(t)$ is the posterior mean of θ for the prior distribution (50). However, we are unable to prove (52) directly by the methods of Theorems 1–3. Indeed we cannot prove (52) in the discrete time problem where there is no exact correspondence with the method of mixtures, although the result must certainly be true.

It was proved in [9] that for T' defined by (47), $P_0\{T' < \infty\} = e^{-c}$, and it is interesting to ask how generally such an equality holds for T defined by (45). The process

$$z(t) = \exp\left(\int_0^t \theta(s)\, dX(s) - \tfrac{1}{2}\int_0^t \theta^2(s)\, ds\right)$$

is in general a P_0-supermartingale ([6], page 25), but it is easy to show that

$$\{z(T \wedge t),\, 0 \leqq t < \infty\}$$

is a P_0-martingale having continuous sample paths. Hence by Lemma 1 of [9], to prove

$$P_0\{T < \infty\} = e^{-c}\,, \tag{53}$$

it suffices to prove

$$P_0\{T = \infty,\, \lim_{t\to\infty} z(t) > 0\} = 0\,. \tag{54a}$$

It follows from results of [6], page 29 or [7], page 148 that under P_0, $\lim_{t\to\infty} z(t) = 0$ if and only if $\int_0^\infty \theta^2(s)\, ds = \infty$, and hence to prove (54a) it suffices to show

$$P_0\{T = \infty,\, \int_0^\infty \theta^2(s)\, ds < \infty\} = 0\,. \tag{54b}$$

For the process $\theta(t) = 0$, for $t < 1$, $= t^{-1}X^+(t)$, for $t \geqq 1$, corresponding to Theorem 1, it is easy to see that (54a) holds. For a more interesting example let

$$\begin{aligned}(55) \qquad \theta(t) &= 0\,, & 0 \leqq t < 1\\ &= t^{-1}X(t)I_{\{X(t)\geqq t^{\frac{1}{2}}a(t)\}}\,, & 1 \leqq t < \infty\,,\end{aligned}$$

where $a(t) = (2\log_2^+ t + 3\log_3^+ t)^{\frac{1}{2}}$ as in Theorem 3. Since the function taking t into $(\log_2 t)/t$ is ultimately decreasing, for $\theta(t)$ given by (55) and large t

$$\begin{aligned}(56) \qquad \textstyle\int_0^t \theta^2(s)\,ds &\geqq \textstyle\int_3^t s^{-1}a^2(s)I_{\{X(s)\geqq s^{\frac{1}{2}}a(s)\}}\,ds\\ &\geqq t^{-1}(\log_2 t)\lambda\{s\colon 3 < s \leqq t,\ X(s) \geqq {}^{\frac{1}{2}}a(s)\}\,,\end{aligned}$$

where λ denotes Lebesgue measure. Since the left-hand side of (56) is increasing in t, to prove (54b) it suffices to show

$$(57) \qquad P_0\{\limsup_{t\to\infty} t^{-1}(\log_2 t)\lambda\{s\colon 3 < s \leqq t,\ X(s) \geqq s^{\frac{1}{2}}a(s)\} = \infty\} = 1\,.$$

((57) is related to results of V. Strassen [11]. It would be interesting to provide a unified formulation and proof.)

To prove (57) let a and α be arbitrary positive numbers, let

$$r_k = \exp(\alpha k/\log k)\,, \qquad \text{and let} \quad n_k = r_k(1 + 4a/\log_2 r_k)\,.$$

Then $(r_{k+1} - r_k)/r_k \sim \alpha/\log k$ and $(n_k - r_k)/r_k \sim 4a/\log k$, so

$$(58) \qquad r_{k+1} > n_k$$

if α is sufficiently large.

Let $A_k = \{X(r_k) \geqq n_k^{\frac{1}{2}}a(n_k)\}$ and $B_k = \{\lambda\{t\colon r_k \leqq t \leqq n_k,\ X(t) \geqq n_k^{\frac{1}{2}}a(n_k)\} \geqq an_k/\log_2 n_k\}$. Note that for large k

$$r_k = n_k(1 + 4a/\log_2 r_k)^{-1} \leqq n_k(1 - 2a/\log_2 n_k)\,,$$

so $n_k - r_k \geqq 2an_k/\log_2 n_k$ and hence $B_k \neq \emptyset$. Also

$$A_k B_k \subset \{\lambda\{t\colon 3 \leqq t \leqq n_k,\ X(t) \geqq t^{\frac{1}{2}}a(t)\} \geqq an_k/\log_2 n_k\}\,,$$

and hence, since a is arbitrary, it suffices to prove

$$(59) \qquad P_0(A_k B_k \ \text{i.o.}) = 1\,.$$

From a standard proof of the general law of the iterated logarithm (e.g. [3]) it is easy to see that

$$(60) \qquad P_0(A_k \ \text{i.o.}) = 1\,.$$

Let τ_i denote the ith value r_k for which A_k occurs. By (60) $P\{\tau_i < \infty\} = 1$ for all $i = 1, 2, \cdots$. Let

$$B_i' = \textstyle\bigcup_{k=1}^{\infty} (\{\tau_i = r_k\} \cap B_k)$$

and $F(t) = \mathscr{B}(X(s), s \leqq t)$. Then by (58)

$$B_i' \cap \{\tau_{i+1} = r_j\} = \textstyle\bigcup_{k=1}^{j-1} \{\tau_i = r_k,\ \tau_{i+1} = r_j\} \cap B_k \in F(r_j)\,,$$

and hence $B_i' \in F(\tau_{i+1})$, where for any stopping time τ, $F(\tau) = \{A: A \cap \{\tau \leqq t\} \in F(t)$ for all $t\}$. Moreover, it is obvious from considerations of symmetry that

$$P_0(B_i' \mid F(\tau_i)) \geqq \tfrac{1}{2}$$

and hence by Lévy's form of the Borel-Cantelli Lemma (cf. [2], page 26) $P_0(B_i', \text{ i.o.}) = 1$, and hence (59) holds.

Another application of (57) is given in Section 9.

7. Behavior of $E_\theta T$ as $c \to \infty$. There are several reasons for being interested in the behavior of $E_\theta T$ for small θ. (a) Tests of power one are of particular interest when detection of a small positive value of θ is important. (b) Farrell's theorem provides an asymptotic standard of comparison as $\theta \downarrow 0$. (c) The connection between the behavior of $E_\theta T$ for small θ and the law of the iterated logarithm gives the subject added mathematical interest. However, the proofs of Theorems 1 and 2 show that the asymptotic expressions (6) and (7) cannot be regarded as even crude approximations for $E_\theta T$ unless θ is quite small. Lemma 5 expresses $E_\theta T$ as the sum of two quantities one of which is about $2c\theta^{-2}$ (Lemma 2) while the other grows more rapidly than θ^{-2} as $\theta \downarrow 0$. However, simple arithmetic shows that for typical values of c, say $3 \leqq c \leqq 4.5$, the term $2c\theta^{-2}$ is larger than the asymptotic approximations given in (6) and (7) unless θ is quite small.

In this section we present an asymptotic analysis of $E_\theta T$ as $c \to \infty$ in which the term $2c\theta^{-2}$ is dominant. (The proof given permits θ to approach 0 as $c \to \infty$, which allows for interesting comparisons with Theorems 1 and 2. We omit the details.) Similar analysis of the method of mixtures and related questions of optimality are being studied by Mr. M. Pollak.

THEOREM 4. *Let $f_\theta(x) = \varphi(x - \theta)$ and assume that* (9) *holds for each $\theta > 0$. Let $T = T(c)$ be defined by* (8). *Then for each $\theta > 0$*

$$E_\theta T = \theta^{-2}(2c + \textstyle\sum_{n=0}^{[2c\theta^{-2}]} E_\theta(\theta_{n+1} - \theta)^2 + 2R(c, \theta) + o(1)) \quad \text{as } c \to \infty\,, \tag{61}$$

where

$$0 < R(c, \theta) \leqq (\theta + \varphi(\theta)/\Phi(\theta))(\theta + o(1)) \qquad \text{as } c \to \infty\,. \tag{62}$$

For particular choices of the sequence $\{\theta_n\}$ it is possible to obtain analytic expressions for the sum on the right-hand side of (61). For example, for $\theta_{n+1} = n^{-1}s_n^+$ $(n = 1, 2, \cdots, \theta_1 = 0)$, we have

$$E_\theta(\theta_{n+1} - \theta)^2 = n^{-1}(1 - \Phi(-\theta n^{\frac{1}{2}}) - \theta n^{\frac{1}{2}}\varphi(\theta n^{\frac{1}{2}})) + \theta^2\Phi(-\theta n^{\frac{1}{2}})$$

and hence after summation by parts

$$\begin{aligned} \textstyle\sum_{n=0}^{[2c\theta^{-2}]} E_\theta(\theta_{n+1} - \theta)^2 &= \log(2c\theta^{-2}) + \gamma + \theta^2 + o(1) \\ &\quad - \textstyle\sum_{n=2}^{\infty} \{\Phi(\theta n^{\frac{1}{2}}) - \Phi(\theta(n-1)^{\frac{1}{2}})\} \sum_{k=1}^{n} k^{-1} \\ &\quad - \textstyle\sum_{n=1}^{\infty} \{\theta n^{-\frac{1}{2}}\varphi(\theta n^{\frac{1}{2}}) - \theta^2\Phi(-\theta n^{\frac{1}{2}})\}\,, \end{aligned} \tag{63}$$

where γ is Euler's constant. The two-sided testing problem mentioned at the end of Section 3 is analytically more tractable: for $\theta_{n+1} = n^{-1}s_n$ $(n = 1, 2, \cdots, \theta_1 = 0)$, we obtain the first line of (63).

The proof of Theorem 4 is preceded by several lemmas. Let $0 < \lambda_1 < 1 < \lambda_2 < \infty$ and $n_i = [2\lambda_i c\theta^{-2}]$.

LEMMA 13. *For any* $r > 0$, $P_\theta\{T \leqq n_1\} = O(c^{-r})$ *uniformly in* θ *as* $c \to \infty$.

PROOF. From Lemma 6 with $k_0 = 0$, $k_1 = n_1$ we obtain

$$P_\theta\{T \leqq n_1\} \leqq 1 - \Phi(a) + \exp[\lambda_1 c + a(2\lambda_1 c)^{\frac{1}{2}}]P_0\{T \leqq n_1\}.$$

Putting $a = (2(r+1)\log c)^{\frac{1}{2}}$, using the estimate $1 - \Phi(a) \leqq \varphi(a)$ $(a \geqq 1)$, and recalling that $P_0\{T \leqq n_1\} \leqq P_0\{T < \infty\} \leqq e^{-c}$ completes the proof.

LEMMA 14.

$$E_\theta(\textstyle\sum_1^T (\theta_n - \theta)^2) \geqq \sum_0^{n_1} E_\theta(\theta_{n+1} - \theta)^2 + o(1) \qquad \text{as } c \to \infty.$$

PROOF. Since

$$\begin{aligned} E_\theta(\textstyle\sum_1^T (\theta_n - \theta)^2) &\geqq \textstyle\sum_{n=0}^{n_1} \int_{\{T>n\}} (\theta_{n+1} - \theta)^2\, dP_\theta \\ &= \textstyle\sum_{n=0}^{n_1} E_\theta(\theta_{n+1} - \theta)^2 - \sum_1^{n_1} \int_{\{T\leqq n\}} (\theta_{n+1} - \theta)^2\, dP_\theta, \end{aligned}$$

it suffices to show that $\sum_1^{n_1} \int_{\{T\leqq n\}} (\theta_{n+1} - \theta)^2\, dP_\theta \to 0$.
By (9), the Schwarz inequality, and Lemma 13, we have

$$\textstyle\sum_1^{n_1} \int_{\{T\leqq n\}} (\theta_{n+1} - \theta)^2\, dP_\theta \leqq (P_\theta\{T \leqq n_1\})^{\frac{1}{2}} \text{ const.} (\log n_1) \to 0 \qquad \text{as } c \to \infty.$$

LEMMA 15.

$$E_\theta(\textstyle\sum_1^T (\theta_k x_k - \frac{1}{2}\theta_k^2)) = c + R(c, \theta),$$

where $R(c, \theta)$ *satisfies* (62).

PROOF. From the definition of T and the proof of Lemma 2 we see that for all c

$$0 < E_\theta(\textstyle\sum_1^T (\theta_k x_k - \frac{1}{2}\theta_k^2)) - c \leqq \left(\theta + \dfrac{\varphi(\theta)}{\Phi(\theta)}\right)(\theta + E_\theta(\theta_T - \theta)^+),$$

and hence it suffices to show

$$E_\theta(\theta_T - \theta)^+ \to 0 \qquad \text{as } c \to \infty. \tag{64}$$

By the Schwarz inequality and (9) we have

$$\begin{aligned} \textstyle\int_{\{T\leqq n_1\}} (\theta_T - \theta)^+\, dP_\theta &\leqq (P_\theta\{T \leqq n_1\} \textstyle\sum_0^{n_1} E_\theta(\theta_{n+1} - \theta)^2)^{\frac{1}{2}} \\ &\leqq \text{const.} (P_\theta\{T \leqq n_1\} \log n_1)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \textstyle\int_{\{T>n_1\}} (\theta_T - \theta)^+\, dP_\theta &\leqq (\textstyle\int_{\{T>n_1\}} (\theta_T - \theta)^4\, dP_\theta)^{\frac{1}{4}} \\ &\leqq (\textstyle\sum_{n_1+1}^\infty E_\theta(\theta_{n+1} - \theta)^4)^{\frac{1}{4}} \leqq \text{const.}\, n_1^{-\frac{1}{4}}, \end{aligned}$$

which together with Lemma 13 prove (64).

LEMMA 16.

$$E_\theta(\textstyle\sum_1^T (\theta_n - \theta)^2) \leqq \sum_{n=0}^{n_2} E_\theta(\theta_{n+1} - \theta)^2 + o(1) \qquad \text{as } c \to \infty.$$

PROOF. As in the proofs of Lemmas 8 and 11 we have for arbitrary $\eta > 0$

$$E_\theta(\sum_1^T (\theta_k - \theta)^2) \leqq \sum_0^{n_2} E_\theta(\theta_{n+1} - \theta)^2 + \eta^2 \sum_{n=n_2+1}^\infty P_\theta\{T > n\} + \sum_{n=n_2+1}^\infty \int_{\{T>n, |\theta_{n+1}-\theta|>\eta\}} (\theta_{n+1} - \theta)^2 \, dP_\theta . \tag{65}$$

Estimating the right-hand side of (65) as in previous arguments we obtain from Lemmas 5 and 15 the preliminary result

$$E_\theta T \sim 2c\theta^{-2} \qquad \text{as } c \to \infty . \tag{66}$$

By Lemma 13, for arbitrary $\delta > 0$ $P_\theta\{T \leqq (1 - \delta)2c\theta^{-2}\} \to 0$ as $c \to \infty$, which together with (66) and some elementary analysis shows that

$$P_\theta\{T \geqq n_2\} \to 0 \tag{67}$$

and

$$\sum_{n=n_2+1}^\infty P_\theta\{T > n_2\} \leqq \int_{\{T>n_2\}} T \, dP_\theta = o(n_2) \qquad \text{as } c \to \infty . \tag{68}$$

By the Hölder inequality, integration by parts, the fourth moment Markov inequality, and (9) we see that

$$\int_{\{T>n, |\theta_{n+1}-\theta|>\eta\}} (\theta_{n+1} - \theta)^2 \, dP_\theta \leqq (P_\theta\{T > n\})^{\frac{1}{3}} (\int_{\{|\theta_{n+1}-\theta|>\eta\}} |\theta_{n+1} - \theta|^3 \, dP_\theta)^{\frac{2}{3}} \leqq (P_\theta\{T > n\})^{\frac{1}{3}} (\text{const. } \eta^{-1} n^{-2})^{\frac{2}{3}} ,$$

and hence

$$\sum_{n=n_2+1}^\infty \int_{\{T>n, |\theta_{n+1}-\theta|>\eta\}} (\theta_{n+1} - \theta)^2 \, dP_\theta \leqq \text{const. } (P_\theta\{T > n_2\})^{\frac{1}{3}} (\eta^{-\frac{2}{3}} n_2^{-\frac{1}{3}}) . \tag{69}$$

Putting $\eta = (c\theta^{-2})^{-\frac{1}{2}}$, we complete the proof by appealing to (65), (67), (68), and (69).

Turning now to the proof of Theorem 4, we obtain from Lemmas 5, 14, 15, and 16 the double inequality

$$\sum_0^{n_1} E_\theta(\theta_{n+1} - \theta)^2 + o(1) \leqq \theta^2 E_\theta T - 2(c + R(c, \theta)) \leqq \sum_0^{n_2} E_\theta(\theta_{n+1} - \theta)^2 + o(1) \tag{70}$$

as $c \to \infty$, where $R(c, \theta)$ satisfies (62). By (9) the extreme terms of (61) differ by at most const. $\log(\lambda_2/\lambda_1)$ for c sufficiently large. Since $n_1 < 2c\theta^{-2} < n_2$ we obtain (61) from (70) by choosing $\lambda_i (i = 1, 2)$ sufficiently close to one.

8. Exponential families f_θ. In this section we discuss extending the results of Sections 2–5 to non-normal exponential families of probability densities. Most of the difficulties encountered involve questions of the speed of convergence in the central limit theorem and may be solved by straightforward but tedious application of Taylor's theorem or by an appeal to known results on the rate of normal approximation. We only sketch a general approach and leave the details to the interested reader. An application to the case of Bernoulli variables is given in Section 8.

Assume that

$$f_\theta(x) = \exp(\theta x - \psi(\theta)) , \qquad \theta \in \Omega , \tag{71}$$

where Ω is some open interval of real numbers, and let $\theta_0 \in \Omega$. Let $\mu_\theta = E_\theta x_k$ and $\sigma_\theta^2 = E_\theta x_k^2 - \mu_\theta^2$. Without loss of generality it may be assumed that

$$\mu_{\theta_0} = 0 .$$

By putting $\nu_0(dx) = f_{\theta_0}(x)\nu(dx)$, so that under P_θ the random variables $x_1, x_2, \cdots$ have relative to ν_0 the probability density function $f_\theta(x)/f_{\theta_0}(x) = \exp[(\theta - \theta_0)x - (\psi(\theta) - \psi(\theta_0))]$, we may by relabelling the parameter space Ω assume that

$$\theta_0 = 0 \qquad \text{and} \qquad \psi(\theta_0) = 0 .$$

It is easy to see that ψ is infinitely differentiable and

$$\mu_\theta = \psi'(\theta) \qquad \text{and} \qquad \sigma_\theta^2 = \psi''(\theta) . \tag{72}$$

In particular, except for the degenerate case in which x_k is almost surely constant for all θ, $\psi'' > 0$ and hence ψ' is strictly increasing. We denote the inverse of ψ' by ψ_1, so

$$\mu = \psi'(\theta) \qquad \text{if and only if} \quad \theta = \psi_1(\mu) .$$

For the important special case

$$P_\theta\{x_k = 1\} = p = e^{2\theta}(e^{2\theta} + 1)^{-1} , \qquad P_\theta\{x_k = -1\} = q = (e^{2\theta} + 1)^{-1} , \tag{73}$$

we see that $\nu_0\{1\} = \nu_0\{-1\} = \frac{1}{2}$ and f_θ satisfies (71) with

$$\psi(\theta) = \log((e^\theta + e^{-\theta})/2) . \tag{74}$$

For any sequence $\{\theta_n\}$, by (5)

$$\begin{aligned} T &= \text{first} \quad n \geqq 1 \quad \text{such that} \quad \textstyle\sum_1^n (\theta_k x_k - \psi(\theta_k)) \geqq c = \log b \\ &= \infty \quad \text{if no such} \quad n \quad \text{occurs.} \end{aligned} \tag{75}$$

As before we put $\tau = \min(T, m)$. Then equation (10) becomes

$$E_\theta(\textstyle\sum_1^\tau \theta_n x_n) = \psi'(\theta) E_\theta(\sum_1^\tau \theta_n) , \tag{76}$$

and hence in place of (11) we have

$$\begin{aligned} (\theta\psi'(\theta) - \psi(\theta))E_\theta \tau &= E_\theta(\textstyle\sum_1^\tau [\theta_n x_n - \psi(\theta_n)]) \\ &\quad + E_\theta(\textstyle\sum_1^\tau [\psi(\theta_n) - \psi(\theta) - \psi'(\theta)(\theta_n - \theta)]) . \end{aligned} \tag{77}$$

To proceed with something specific in mind we shall discuss proving an analogue of Theorem 1. Theorem 2 can be similarly generalized, but Theorem 4 involves several important differences, some of which are already apparent in (77).

In analogy with Theorem 1 we would like to define the sequence $\{\theta_n\}$ by letting θ_{n+1} be the solution θ of the equation $\psi'(\theta) = n^{-1}s_n^+$, to wit by (72), $\theta_{n+1} = \psi_1(n^{-1}s_n^+)$. However, this solution may not exist in Ω (e.g. the Bernoulli family (73) if $s_n = n$). A simple modification which works quite generally (it suffices that $\Omega = (-\infty, \infty)$, but this is by no means necessary) is to put

$$\theta_{n+1} = \psi_1((n+1)^{-1}s_n^+) \qquad (n = 1, 2, \cdots) . \tag{78}$$

Expanding ψ_1 and $\psi \circ \psi_1$ about $\mu_\theta = \psi'(\theta)$ according to Taylor's theorem, we obtain by (71) and (72)

$$(\psi_1(x) - \theta)^2 = (x - \mu_\theta)^2/\sigma_\theta^4 + r(x, \mu_\theta) \tag{79}$$

and

$$\psi \circ \psi_1(x) - \psi(\theta) - (\psi_1(x) - \theta)\psi'(\theta) = (x - \mu_\theta)^2/2\sigma_\theta^2 + t(x, \mu_\theta) . \tag{80}$$

We assume that

$$\textstyle\sum_1^\infty E_\theta|r((n + 1)^{-1}s_n^+, \mu_\theta)| < \infty \tag{81}$$

and

$$\textstyle\sum_1^\infty E_\theta|t((n + 1)^{-1}s_n^+, \mu_\theta)| < \infty , \tag{82}$$

and that the convergence indicated by (81) and (82) is uniform for small positive values of θ. Since $r((n + 1)^{-1}s_n^+, \mu_\theta)$ and $t((n + 1)^{-1}s_n^+, \mu_\theta)$ are stochastically of order $n^{-\frac{3}{2}}$ (uniformly for small positive θ), the assumptions (81) and (82) should hold quite generally. That they hold for the Bernoulli family (73) is an easy consequence of the Schwarz inequality.

For θ_{n+1} as in (78), using (80) and (82), the analysis of the second term on the right-hand side of (77) along the lines of Lemma 3 reduces to an analysis of the quadratic term of (80), which proceeds as in the normal case. A similar reduction occurs if we replace τ by T in (77) and argue as in Lemmas 8 and 10.

The analysis of Lemma 2 depends heavily on the normal distribution, but is much sharper than Theorems 1 and 2 require. By the Schwarz inequality and Wald's lemma

$$\begin{aligned} & E_\theta(\textstyle\sum_1^\tau [\theta_k x_k - \psi(\theta_k)]) - c \\ & \quad \leqq E_\theta(\theta_\tau x_\tau^+) = E_\theta(\theta_\tau - \theta)x_\tau^+ + \theta E_\theta x_\tau^+ \\ & \quad \leqq [E_\theta((\theta_\tau - \theta)x_\tau)^2]^{\frac{1}{2}} + [\theta^2 E_\theta(x_\tau^2)]^{\frac{1}{2}} \\ & \quad \leqq [E_\theta(\textstyle\sum_1^\tau (\theta_k - \theta)^2 x_k^2)]^{\frac{1}{2}} + [\theta^2 E_\theta(\textstyle\sum_1^\tau x_k^2)]^{\frac{1}{2}} \\ & \quad = [(\sigma_\theta^2 + \mu_\theta^2)E_\theta(\textstyle\sum_1^\tau (\theta_k - \theta)^2)]^{\frac{1}{2}} + [(\sigma_\theta^2 + \mu_\theta^2)\theta^2 E_\theta \tau]^{\frac{1}{2}} . \end{aligned} \tag{83}$$

Using (79) and (81) one may show that as $m \to \infty$ the first term on the right-hand side of (83) is of the order of magnitude of the square root of the second term on the right-hand side of (77). If we replace τ by T in (77) and (83) and let $\theta \downarrow 0$, we obtain the same result, and this argument replaces Lemma 2 in the proof of Theorem 1. Theorem 4, however, requires sharper bounds and hence is much more dependent on the underlying distribution. In specific cases it is usually possible to obtain these bounds, but no general argument along the lines of Lemma 2 works.

Arguments dependent on the basic inequality of Lemma 6 go through almost unchanged. Hence if the sequence $\{\theta_n\}$ is defined by (78), and (81) and (82) hold, then (6) holds. If (78), (81), and (82) are satisfied with s_n^+ replaced by $s_n I_{\{s_n \geq n^{\frac{1}{2}} a_n\}}$, then (7) holds.

Similar results hold for two-sided stopping rules as indicated at the end of Section 3.

9. Application to gambling theory. In [1] Breiman discusses a class of gambling problems of which the simplest and perhaps most interesting special case is the following. A gambler has an initial fortune $g_0 = 1$. At times $n = 1, 2, \cdots$ he can bet any proportion λ_n of his current fortune g_{n-1} on heads and any proportion $\bar{\lambda}_n \leqq 1 - \lambda_n$ on tails. A coin having a fixed probability $p \neq \frac{1}{2}$ for heads and $q = 1 - p$ for tails is tossed. If heads occurs the gambler's fortune increases by the amount bet on heads and decreases by the amount bet on tails, similarly if tails occurs. Symbolically

$$(84) \qquad \begin{aligned} g_n &= g_{n-1}(1 + \lambda_n - \bar{\lambda}_n) && \text{if heads occurs} \\ &= g_{n-1}(1 + \bar{\lambda}_n - \lambda_n) && \text{if tails occurs.} \end{aligned}$$

It is easy to see that without loss of generality we can assume that $\lambda_n + \bar{\lambda}_n = 1$, and putting $y_n = 1$ or 0 according as the nth toss is heads or tails, we obtain from (84) the fundamental relation

$$(85) \qquad g_n = 2g_{n-1}\lambda_n^{y_n}(1 - \lambda_n)^{1-y_n} .$$

A gambling strategy is a sequence $\{\lambda_n\}$, where for each $n = 1, 2, \cdots, \lambda_n$ is a function of $y_1, \cdots, y_{n-1}$ taking values in [0, 1]. Since $p \neq \frac{1}{2}$ by assumption there exist many strategies $\{\lambda_n\}$ for which $\lim_{n\to\infty} g_n = +\infty$ with probability one, so that for any $b > 1$ the time $T = T(b)$ at which the gambler's fortune first exceeds b is a finite valued random variable.

Assuming that p is known to the gambler, Breiman poses the problem of finding that strategy for which the expectation of $T(b)$ is a minimum, and he shows that as $b \to \infty$ this expectation is asymptotically minimized in a certain sense by setting

$$(86) \qquad \lambda_n = p \qquad \text{for all} \quad n = 1, 2, \cdots .$$

(A similar result holds as $p \to \frac{1}{2}$, b fixed.)

If p is *unknown* (as we henceforth assume), it is impossible to use the strategy (86), but it is natural to mimic it by using for λ_n an estimator of p based on $y_1, \cdots, y_{n-1}$.

To see the connection of this problem with the statistical problem discussed in the rest of this paper, note that if $\{\lambda_n\}$ satisfies (86), then by (85) $g_n = 2^n p^{t_n} q^{n-t_n}$, where we have put $t_n = \sum_1^n y_k$. But this is just the likelihood ratio $\prod_1^n (f_p(y_k)/f_{\frac{1}{2}}(y_k))$, where

$$(87) \qquad f_p(y) = p^y(1 - p)^{1-y} \qquad (y = 0, 1),$$

and hence the time at which the gambler's fortune first exceeds the amount b is just the stopping rule $N(p, b)$ defined in Section 1. And if p is unknown, using for λ_n an estimate of p based on $y_1, \cdots, y_{n-1}$ is equivalent to the general test procedure described in Section 1 and studied throughout this paper.

By putting $x_k = 2y_k - 1$ and $\theta = 2^{-1} \log p/q$, we obtain a sequence $x_1, x_2, \cdots$ of independent random variables with density function of the form (68), where ψ is defined by (74). Hence Theorems 1-3 (as modified in Section 7) provide asymptotic approximations for the expected time at which the gambler's fortune first exceeds a given amount. For example, for $\lambda_{n+1} = (n+1)^{-1}(\frac{1}{2} + t_n)$, so that $\theta_{n+1} = \tanh^{-1}(s_n/(n+1))$, we have

$$E_\theta T \sim 2P_0\{T = \infty\}\theta^{-2}\log\theta^{-1} \qquad (\theta \to 0)$$
$$\sim 2P_0\{T = \infty\}(2p-1)^{-2}\log(2p-1)^{-1} \qquad (p \to \tfrac{1}{2}) . \tag{88}$$

For

$$\lambda_{n+1} = (n+1)^{-1}(\tfrac{1}{2} + t_n)I_{\{|2t_n - n| \geqq n^{\frac{1}{2}}a_n\}} + \tfrac{1}{2}I_{\{|2t_n - n| < n^{\frac{1}{2}}a_n\}} \tag{89}$$

or equivalently $\theta_{n+1} = \tanh^{-1}[(n+1)^{-1}s_n]I_{\{|s_n| \geqq n^{\frac{1}{2}}a_n\}}$, with a_n as in Theorem 2, we obtain

$$E_\theta T \sim (2P_0\{T = \infty\}\log_2\theta^{-1})/\theta^2 \qquad (\theta \to 0)$$
$$\sim (2P_0\{T = \infty\}\log_2(2p-1)^{-1})/(2p-1)^2 \qquad (p \to \tfrac{1}{2}) , \tag{90}$$

and by Farrell's theorem this is the smallest possible rate of divergence of $E_\theta T$ as $\theta \to 0$.

For $\theta = 0$, for every strategy (λ_n), the process $\{g_n, n = 0, 1, \cdots\}$ is a non-negative martingale and hence

$$P_0\{T < \infty\} \leqq b^{-1} . \tag{91}$$

By a slight modification of the strategy (89) we can achieve equality in (91) and hence minimize the coefficient of $\log_2\theta^{-1}/\theta^2$ on the right-hand side of (90). First observe that by slight changes in the argument presented at the end of Section 6, for the strategy (89)

$$P_0\{g_n \to 0\} = 1 . \tag{92}$$

We now modify the strategy (89) by adding the proviso that if betting λ_n on heads for the nth toss might cause our fortune g_n to exceed b, we bet a smaller (if $\lambda_n > \frac{1}{2}$) or larger (if $\lambda_n < \frac{1}{2}$) amount, so that our fortune g_n may equal, but may not exceed b. If we denote the resulting strategy and sequence of fortunes by (λ_n^*) and (g_n^*), and let

$$T^* = \text{first} \quad n \geqq 1 \quad \text{such that} \quad g_n^* \geqq b$$
$$= \infty \quad \text{if no such} \quad n \quad \text{occurs,}$$

then $P_0\{T^* = \infty, \lambda_n^* \neq \lambda_n \text{ i.o.}\} = 0$. Hence by (92) $P_0\{T^* = \infty, \lim_{n\to\infty} g_n^* > 0\} = 0$ and since obviously $P_0\{T^* < \infty, g_{T^*}^* > b\} = 0$, it follows from Lemma 1 of [9] that $P_0\{T^* < \infty\} = b^{-1}$.

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