The explicit Laplace transform for the Wishart process*

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Abstract

We derive the explicit formula for the joint Laplace transform of the Wishart process and its time integral which extends the original approach of Bru (1991). We compare our methodology with the alternative results given by the variation of constants method, the linearization of the Matrix Riccati ODE's and the Runge-Kutta algorithm. The new formula turns out to be fast and accurate.

Keywords: Affine processes, Wishart process, ODE, Laplace Transform.

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1 Introduction

In this paper we propose an analytical approach for the computation of the moment generating function for the Wishart process which has been introduced by Bru (1991), as an extension of square Bessel processes (Pitman and Yor (1982), Revuz and Yor (1994)) to the matrix case. Wishart processes belong to the class of affine processes and they generalise the notion of positive factor insofar as they are defined on the set of positive semidefinite real $d \times d$ matrices, denoted by S_d^+.

Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual assumptions and a $d \times d$ matrix Brownian motion B (i.e. a matrix whose entries are independent Brownian motions under $\mathbb{P}$), a Wishart process on S_d^+ is governed by the SDE

$$dS_t = \sqrt{S_t} dB_t Q + Q^\top dB_t^\top \sqrt{S_t} + \left(MS_t + S_t M^\top + \alpha Q^\top Q\right) dt, \quad S_0 \in S_d^+, \quad t \geq 0 \qquad (1)$$

where $Q \in GL_d$ (the set of invertible real $d \times d$ matrices), $M \in M_d$ (the set of real $d \times d$ matrices) with all eigenvalues on the negative half plane in order to ensure stationarity, and where the (Gindikin) real parameter $\alpha \geq d-1$ grants the positive semi definiteness of the process, in analogy with the Feller condition for the scalar case (Bru (1991)). In the dynamics above $\sqrt{S_t}$ denotes the square root in matrix

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sense. Existence and uniqueness results for the solution of (1) may be found in Bru (1991) under parametric restrictions and in Mayerhofer et al. (2011) in full generality. We denote by $WIS_d(S_0, \alpha, M, Q)$ the law of the Wishart process $(S_t)_{t\geq 0}$. The starting point of the analysis was given by considering the square of a matrix Brownian motion $S_t = B_t^\top B_t$, while the generalization to the particular dynamics (1) was introduced by looking at squares of matrix Ornstein-Uhlenbeck processes (see Bru (1991)).

Bru proved many interesting properties of this process, like non-collision of the eigenvalues (when $\alpha \geq d+1$) and the additivity property shared with square Bessel processes. Moreover, she computed the Laplace transform of the Wishart process and its integral (the *Matrix Cameron-Martin formula* using her terminology), which plays a central role in the applications:

$$\mathbb{E}_{S_0}^{\mathbb{P}}\left[\exp\left\{-Tr\left[wS_t + \int_0^t vS_s ds\right]\right\}\right], \tag{2}$$

where Tr denotes the trace operator and w, v are symmetric matrices for which the expression (2) is finite. Bru found an explicit formula for (2) (formula (4.7) in Bru (1991)) under the assumption that the symmetric diffusion matrix Q and the mean reversion matrix M commute.

Positive (semi)definite matrices arise in finance in a natural way and the nice analytical properties of affine processes on S_d^+ opened the door to new interesting affine models that allow for non trivial correlations among positive factors, a feature which is precluded in classic (linear) state space domains like $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^m$ (see Duffie et al. (2003)). Not surprisingly, the last years have witnessed the birth of a whole branch of literature on applications of affine processes on S_d^+. The first proposals were formulated in Gourieroux et al. (2005), Gourieroux and Sufana (2003), Gourieroux and Sufana (2005), Gourieroux (2006) both in discrete and continuous time. Applications to multifactor volatility and stochastic correlation can be found in Da Fonseca et al. (2008), Da Fonseca et al. (2007b), Da Fonseca et al. (2011), Da Fonseca et al. (2007a), Da Fonseca and Grasselli (2011), Buraschi et al. (2010), and Buraschi et al. (2008) both in option pricing and portfolio management. These contributions consider the case of continuous path Wishart processes. As far as jump processes on S_d^+ are concerned we recall the proposals by Barndorff-Nielsen and Stelzer (2007), Muhle-Karbe et al. (2012) and Pigorsch and Stelzer (2009). Leippold and Trojani (2010) and Cuchiero et al. (2011) consider jump-diffusions models in this class, while Grasselli and Tebaldi (2008) investigate processes lying in the more general symmetric cones state space domain, including the interior of the cone S_d^+ (see also the recent developements in Cuchiero (2011)).

The main contribution of this paper consists in relaxing the commutativity assumption made in Bru (1991) and proving that it is possibile to characterize explicitly the joint distribution of the Wishart process and its time integral for a general class of (even not symmetric) mean-reversion and diffusion matrices satisfying the assumptions above. The proof of our general Cameron Martin formula is in line with that of theorem 2" in Bru and we will provide a step-by-step derivation.

The paper is organized as follows: in section 2 we prove our main result, which extends the original approach by Bru. In section 3 we briefly review some other existing methods which have been employed in the past literature for the computation of the Laplace transform: the variation of constant, the linearization and the Runge-Kutta method. The first two methods provide analytical solutions, so they should be considered as *competitors* of our new methodology. We show that the variation of constants method is unfeasible for real-life computations, hence the truly analytic competitor is the linearization procedure. After that, we present some applications of our methodology to various settings: a multifactor stochastic volatiliy model, a stochastic correlation model, a short rate model and finally we present a new approach for the computation of a solution to the Algebraic Riccati equation. Finally, in the Appendix we extend our formula to the case where the Gindikin term $\alpha Q^\top Q$ is replaced by a general symmetric matrix b satisfying $b-(d-1)Q^\top Q \in S_d^+$ according to Cuchiero et al. (2011).

2 The Matrix Cameron-Martin Formula

2.1 Statement of the result

In this section we proceed to prove the main result of this paper. We report a formula completely in line with the Matrix Cameron-Martin formula given by Bru (1991).

Theorem 1 *Let $S \in WIS_d(S_0, \alpha, M, Q)$ be the Wishart process solving (1), assume*

$$M^\top \left(Q^\top Q\right)^{-1} = \left(Q^\top Q\right)^{-1} M, \tag{3}$$

let $\alpha \geq d+1$ and define the set of convergence of the Laplace transform

$$\mathcal{D}_t = \left\{ w, v \in S_d : \mathbb{E}^{\mathbb{P}}_{S_0}\left[\exp\left\{-Tr\left[wS_t + \int_0^t vS_s ds\right]\right\}\right] < +\infty \right\}.$$

Then for all $u, v \in \mathcal{D}_t$ the joint moment generating function of the process and its integral is given by:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}_{S_0}\left[\exp\left\{-Tr\left[wS_t + \int_0^t vS_s ds\right]\right\}\right] \\
&= \det\left(e^{-Mt}\left(\cosh(\sqrt{\bar{v}}t) + \sinh(\sqrt{\bar{v}}t)k\right)\right)^{\frac{\alpha}{2}} \\
&\times \exp\left\{Tr\left[\left(\frac{Q^{-1}\sqrt{\bar{v}}kQ^{\top^{-1}}}{2} - \frac{\left(Q^\top Q\right)^{-1}M}{2}\right)S_0\right]\right\},
\end{aligned}$$

where the matrices $k, \bar{v}, \bar{w}$ are given by

$$\begin{aligned}
k &= -\left(\sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}t) + \bar{w}\sinh(\sqrt{\bar{v}}t)\right)^{-1}\left(\sqrt{\bar{v}}\sinh(\sqrt{\bar{v}}t) + \bar{w}\cosh(\sqrt{\bar{v}}t)\right), \\
\bar{v} &= Q\left(2v + M^\top Q^{-1}Q^{\top^{-1}}M\right)Q^\top, \\
\bar{w} &= Q\left(2w - \left(Q^\top Q\right)^{-1}M\right)Q^\top.
\end{aligned} \tag{4}$$

Remark 2 *In the previous formulation we recognize the exponential affine shape with respect to the state variable S:*

$$\mathbb{E}^{\mathbb{P}}_{S_0}\left[\exp\left\{-Tr\left[wS_t + \int_0^t vS_s ds\right]\right\}\right] = \exp\left\{-\phi(t) - Tr\left[\psi(t)S_0\right]\right\},$$

where the functions ψ and ϕ are given by:

$$\psi(t) = \frac{\left(Q^\top Q\right)^{-1}M}{2} - \frac{Q^{-1}\sqrt{\bar{v}}kQ^{\top^{-1}}}{2}, \tag{5}$$

$$\phi(t) = -\frac{\alpha}{2}\log\left(\det\left(e^{-Mt}\left(\cosh(\sqrt{\bar{v}}t) + \sinh(\sqrt{\bar{v}}t)k\right)\right)\right). \tag{6}$$

Remark 3 *The derivation of Theorem 1 involves a change of probability measure that will be illustrated in the sequel. This change of measure introduces a lack of symmetry which does not allow to derive a* *fully general formula. However, under the assumption* (3) *we are able to span a large class of processes. To be more precise, in the two dimensional case, let:*

$$\left(Q^\top Q\right)^{-1} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad M = \begin{pmatrix} x & y \\ z & t \end{pmatrix},$$

then condition (3) *can be expressed as:*

$$bx + cz = ay + tb,$$

meaning that we can span a large class of parameters, thus going far beyond the commutativity assumption $QM = MQ$ *for* $Q \in S_d, M \in S_d^-$ *as in Bru (1991).*

2.2 Proof of Theorem 1

We will prove the theorem in several steps. We first consider a simple Wishart process with $M = 0$ and $Q = I_d$, defined under a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$. The second step will be given by the introduction of the volatility matrix Q, using an invariance result. Finally, we will prove the extension for the full process by relying on a measure change from $\tilde{\mathbb{P}}$ to $\mathbb{P}$. Under this last measure, the Wishart process will be defined by the dynamics (1).

As a starting point we fix a probability measure $\tilde{\mathbb{P}}$ such that $\tilde{\mathbb{P}} \approx \mathbb{P}$. Under the measure $\tilde{\mathbb{P}}$ we consider a matrix Brownian motion $\hat{B} = (\hat{B}_t)_{t \geq 0}$, which allows us to define the process $\Sigma_t \in WIS_d(S_0, \alpha, 0, I_d)$, i.e. a process that solves the following matrix SDE:

$$d\Sigma_t = \sqrt{\Sigma_t} d\hat{B}_t + d\hat{B}_t^\top \sqrt{\Sigma_t} + \alpha I_d dt. \tag{7}$$

For this process, relying on Pitman and Yor (1982) and Bru (1991), we are able to calculate the Cameron-Martin formula. For the sake of completeness we report the result in Bru (1991), which constitutes an extension of the methodology introduced in Pitman and Yor (1982).

Proposition 4 *(Bru (1991) Proposition 5 p.742) If* $\mathbf{\Phi} : \mathbb{R}_+ \to S_d^+$ *is continuous, constant on* $[t, \infty[$ *and such that its right derivative (in the distribution sense)* $\mathbf{\Phi}'_d : \mathbb{R}_+ \to S_d^-$ *is continuous, with* $\mathbf{\Phi}_d(0) = I_d$*, and* $\mathbf{\Phi}'_d(t) = 0$*, then for every Wishart process* $\Sigma_t \in WIS_d(S_0, \alpha, 0, I_d)$ *we have:*

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}Tr\left[\int_0^t \mathbf{\Phi}''_d(s)\mathbf{\Phi}_d^{-1}(s)\Sigma_s ds\right]\right\}\right] = \left(\det \mathbf{\Phi}_d(t)\right)^{\alpha/2} \exp\left\{\frac{1}{2}Tr\left[\Sigma_0 \mathbf{\Phi}_d^+(0)\right]\right\},$$

where

$$\mathbf{\Phi}_d^+(0) := \lim_{t \searrow 0} \mathbf{\Phi}'_d(t).$$

As a direct application we obtain the following Proposition, whose standard proof is omitted.

Proposition 5 *Let* $\Sigma \in WIS_d(S_0, \alpha, 0, I_d)$*, then*

$$\begin{aligned}\mathbb{E}\left[\exp\left\{-\frac{1}{2}Tr\left[w\Sigma_t + \int_0^t v\Sigma_s ds\right]\right\}\right] &= \det\left(\cosh\left(\sqrt{v}t\right) + \sinh\left(\sqrt{v}t\right)k\right)^{\frac{\alpha}{2}} \\ &\times \exp\left\{\frac{1}{2}Tr\left[\Sigma_0\sqrt{v}k\right]\right\},\end{aligned} \tag{8}$$

where k *is given by*

$$k = -\left(\sqrt{v}\cosh(\sqrt{v}t) + w\sinh(\sqrt{v}t)\right)^{-1}\left(\sqrt{v}\sinh(\sqrt{v}t) + w\cosh(\sqrt{v}t)\right).$$

Invariance under transformations. We define the transformation $S_t = Q^\top \Sigma_t Q$, which is governed by the SDE:

$$dS_t = \sqrt{S_t} d\tilde{B}_t Q + Q^\top d\tilde{B}_t^\top \sqrt{S_t} + \alpha Q^\top Q dt, \tag{9}$$

where the process $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$ defined by $d\tilde{B}_t = \sqrt{S_t}^{-1} Q^\top \sqrt{\Sigma} d\hat{B}_t$ is easily proved to be a Brownian motion under $\tilde{\mathbb{P}}$.
From Bru (1991), we know the Laplace transform of the process S:

$$\begin{aligned}
\mathbb{E}_{S_0}^{\tilde{\mathbb{P}}}\left[e^{-Tr[uS_t]}\right] &= \mathbb{E}_{(Q^\top)^{-1} S_0 Q^{-1}}^{\tilde{\mathbb{P}}}\left[e^{-Tr\left[uQ^\top \Sigma Q\right]}\right] \\
&= \mathbb{E}_{\Sigma_0}^{\tilde{\mathbb{P}}}\left[e^{-Tr\left[\left(QuQ^\top\right)\Sigma\right]}\right] \\
&= \left(\det\left(I_d + 2tQuQ^\top\right)\right)^{-\frac{\alpha}{2}} \times \\
&\quad \exp\left\{-Tr\left[S_0 Q^{-1}\left(I_d + 2tQuQ^\top\right)^{-1} Qu\right]\right\}.
\end{aligned}$$

Using the Taylor expansion $(I + A)^{-1} = I - A + A^2 - A^3 + ...$, we have

$$\begin{aligned}
Q^{-1}\left(I + 2tQuQ^\top\right)^{-1} Q &= Q^{-1}\left(I_d - 2tQuQ^\top + 4t^2\left(QuQ^\top\right)\left(QuQ^\top\right) - \dots\right) Q \\
&= I_d - 2tuQ^\top Q + 4t^2\left(uQ^\top Q\right)\left(uQ^\top Q\right) - \dots,
\end{aligned}$$

then, using Sylvester's law of inertia,

$$\det\left(I_d + AB\right) = \det\left(I_d + BA\right),$$

we obtain

$$\det\left(I_d + 2tQuQ^\top\right) = \det\left(I_d - 2tuQ^\top Q\right).$$

Inclusion of the drift - Girsanov transformation. The final step consists in introducing a measure change from $\tilde{\mathbb{P}}$, where the process has no mean reversion, to the measure $\mathbb{P}$ that will allow us to consider the general process governed by the dynamics in equation (1). We now define a matrix Brownian motion under the probability measure $\mathbb{P}$ as follows:

$$B_t = \tilde{B}_t - \int_0^t \sqrt{S_s} M^\top Q^{-1} ds = \tilde{B}_t - \int_0^t H_s ds.$$

The Girsanov transformation is given by the following stochastic exponential (see e.g. Donati-Martin et al. (2004)):

$$\begin{aligned}
\left.\frac{\partial \mathbb{P}}{\partial \tilde{\mathbb{P}}}\right|_{\mathcal{F}_t} &= \exp\left\{\int_0^t Tr\left[H^\top d\tilde{B}_s\right] - \frac{1}{2}\int_0^t Tr\left[HH^\top\right] ds\right\} \\
&= \exp\left\{\int_0^t Tr\left[Q^{-1^\top} M \sqrt{S_s} d\tilde{B}_s\right] - \frac{1}{2}\int_0^t Tr\left[S_s M^\top Q^{-1} Q^{-1^\top} M\right] ds\right\}.
\end{aligned}$$

We concentrate on the stochastic integral term, which may be rewritten as

$$\int_0^t Tr\left[\left(Q^\top Q\right)^{-1} M \sqrt{S_s} d\tilde{B}_s Q\right]$$

which, under the parametric restriction (3), can be expressed as

$$\frac{1}{2}\int_0^t Tr\left[\left(Q^\top Q\right)^{-1} M\left(\sqrt{S_s}d\tilde{B}_sQ + Q^\top d\tilde{B}_s^\top\sqrt{S_s}\right)\right] \tag{10}$$

and then we can write

$$\frac{1}{2}\int_0^t Tr\left[\left(Q^\top Q\right)^{-1} M\left(dS_s - \alpha Q^\top Q ds\right)\right].$$

In summary, the stochastic exponential may be written as

$$\left.\frac{\partial\mathbb{P}}{\partial\tilde{\mathbb{P}}}\right|_{\mathcal{F}_t} = \exp\left\{\frac{\left(Q^\top Q\right)^{-1}M}{2}\left(S_t - S_0 - \alpha Q^\top Qt\right) - \frac{1}{2}\int_0^t Tr\left[S_s M^\top Q^{-1}Q^{-1^\top}M\right]ds\right\}.$$

Mayerhofer (2012) shows that under the assumption $\alpha \geq d+1$ (which is a sufficient condition ensuring that the process does not hit the boundary of the cone S_d^+) the stochastic exponential is a true martingale.

Derivation of the Matrix Cameron-Martin formula. We finally consider the process under $\mathbb{P}$:

$$dS_t = \sqrt{S_t}dB_tQ + Q^\top dB_t^\top\sqrt{S_t} + \left(MS_t + S_tM^\top + \alpha Q^\top Q\right)dt.$$

Recall that under $\tilde{\mathbb{P}}$, we have

$$dS_t = \sqrt{S_t}d\tilde{B}_tQ + Q^\top d\tilde{B}_t^\top\sqrt{S_t} + \alpha Q^\top Q dt,$$

then $\Sigma_t = Q^{-1^\top}S_tQ^{-1}$ solves

$$d\Sigma_t = \sqrt{\Sigma_t}d\hat{B}_t + d\hat{B}_t^\top\sqrt{\Sigma_t} + \alpha I_d dt.$$

We are now ready to apply the change of measure along the following steps:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}_{S_0}\left[\exp\left\{-\frac{1}{2}Tr\left[wS_t + \int_0^t vS_s ds\right]\right\}\right] \\
&= \mathbb{E}^{\tilde{\mathbb{P}}}_{S_0}\left[\exp\left\{-\frac{1}{2}Tr\left[wS_t + \int_0^t vS_s ds\right] - \frac{\alpha}{2}tTr\left[M\right]\right.\right. \\
&- Tr\left[\frac{\left(Q^\top Q\right)^{-1}M}{2}S_0\right] + Tr\left[\frac{\left(Q^\top Q\right)^{-1}M}{2}S_t\right] \\
&\left.\left.-\frac{1}{2}\int_0^t Tr\left[S_sM^\top Q^{-1}Q^{-1^\top}M\right]ds\right\}\right] \\
&= \exp\left\{-\frac{\alpha}{2}tTr\left[M\right] - Tr\left[\frac{\left(Q^\top Q\right)^{-1}M}{2}S_0\right]\right\} \\
&\times \mathbb{E}^{\tilde{\mathbb{P}}}_{S_0}\left[\exp\left\{-\frac{1}{2}Tr\left[\left(w - \left(Q^\top Q\right)^{-1}M\right)S_t\right.\right.\right. \\
&\left.\left.\left.+\int_0^t\left(v + M^\top Q^{-1}Q^{-1^\top}M\right)S_s ds\right]\right\}\right].
\end{aligned}$$

But $S_t = Q^\top \Sigma_t Q$, then:

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}_{S_0}\left[\exp\left\{-\frac{1}{2}Tr\left[wS_t+\int_0^t vS_s ds\right]\right\}\right]\\&=\exp\left\{-\frac{\alpha}{2}tTr\left[M\right]-Tr\left[\frac{\left(Q^\top Q\right)^{-1}M}{2}S_0\right]\right\}\\&\times\mathbb{E}^{\tilde{\mathbb{P}}}_{Q^{\top^{-1}}S_0Q^{-1}}\left[\exp\left\{-\frac{1}{2}Tr\left[Q\left(w-\left(Q^\top Q\right)^{-1}M\right)Q^\top\Sigma_t\right.\right.\right.\\&\left.\left.\left.+\int_0^t Q\left(v+M^\top Q^{-1}Q^{-1^\top}M\right)Q^\top\Sigma_s ds\right]\right\}\right].\end{aligned}$$

The expectation may be computed via a direct application of formula (8) and after some standard algebra we get the result of Theorem 1, with the obvious substitutions $v \to 2v$ and $w \to 2w$.

3 Alternative existing methods

3.1 Variation of Constants Method

Since the process is affine, it is possible to reduce the PDE associated to the computation of (2) to a non linear (matrix Riccati) ODE.

Proposition 6 *Let $S_t \in WIS_d(S_0, \alpha, M, Q)$ be the Wishart process defined by (1), then*

$$\mathbb{E}^{\mathbb{P}}_{S_0}\left[\exp\left\{-Tr\left[wS_t+\int_0^t vS_s ds\right]\right\}\right]=\exp\left\{-\phi(t)-Tr\left[\psi(t)S_0\right]\right\},$$

where the functions ψ and ϕ satisfy the following system of ODE's.

$$\frac{d\psi}{dt}=\psi M+M^\top\psi-2\psi Q^\top Q\psi+v \quad \psi(0)=w, \tag{11}$$

$$\frac{d\phi}{dt}=Tr\left[\alpha Q^\top Q\psi(t)\right] \quad \phi(0)=0. \tag{12}$$

Proof. See Cuchiero et al. (2011). ■

The idea underlying the variation of constants method is that in order to compute the solution of the system of matrix ODE's (11), (12), it is sufficient to find a particular solution to the equation for ψ, since the solution for ϕ will be obtained via direct integration. We denote by ψ' a solution to the algebraic Riccati equation defined below. In order to grant the boundness of (2) for all t, we assume $\lambda\left(M-2Q^\top Q\psi'\right)<0, \forall\lambda\in\sigma(M-2Q^\top Q\psi')$, where $\lambda(C), \sigma(C)$ denote respectively an eigenvalue and the spectrum of a matrix C. We will proceed in two steps: first, we will solve the equation for $v=0$ and then provide the most general form. To this aim, we first introduce the following standard lemma.

Lemma 7 *Let $\psi' \in S_d$ be a symmetric solution to the algebraic Riccati equation:*

$$\psi' M+M^\top\psi'-2\psi' Q^\top Q\psi'+v=0, \tag{13}$$

where $\lambda\left(M-2Q^\top Q\psi'\right)<0, \forall\lambda\in\sigma(M-2Q^\top Q\psi')$. Then the function $Z(t)=\psi(t)-\psi'$ solves the following matrix ODE:

$$\frac{dZ}{dt}=Z(t)M'+M'^\top Z(t)-2Z(t)Q^\top QZ(t), \tag{14}$$

with $Z(0)=w'$, $w'=w-\psi'$, $M'=M-2Q^\top Q\psi'$.

The next step is the computation of the solution for $Z(t)$, which is given by the following:

Lemma 8 *The solution to the equation* (14) *is given by:*

$$Z(t) = e^{M'^{\top}t}\left(w'^{-1} + \int_0^t e^{M's}Q^{\top}Qe^{M'^{\top}s}ds\right)^{-1} e^{M't}.$$

Proof. Consider the function $f(t)$ defined by $Z(t) = e^{M'^{\top}t}f(t)e^{M't}$. By differentiating this expression we get:

$$\frac{dZ}{dt} = M'^{\top}Z(t) + Z(t)M' + e^{M'^{\top}t}\frac{df(t)}{dt}e^{M't}.$$

A comparison with (14) gets

$$e^{M'^{\top}t}\frac{df(t)}{dt}e^{M't} = -2e^{M'^{\top}t}f(t)e^{M't}Q^{\top}Qe^{M'^{\top}t}f(t)e^{M't},$$

which implies

$$\begin{aligned}\frac{df(t)}{dt} &= -2f(t)e^{M't}Q^{\top}Qe^{M'^{\top}t}f(t) \quad f(0) = w' \\ -f(t)^{-1}\frac{df(t)}{dt}f(t)^{-1} &= 2e^{M't}Q^{\top}Qe^{M'^{\top}t}.\end{aligned}$$

From Faraut and Korànyi (1994) we know that this is equivalent to

$$\frac{df(t)^{-1}}{dt} = 2e^{M't}Q^{\top}Qe^{M'^{\top}t} \quad f(0) = w'^{-1}.$$

Direct integration of this ODE and substitution of the solution in the identity defining Z yields the desiderd result. ■

If we combine the two lemmas above, we obtain the solution for the functions ψ and ϕ. This is stated in the following proposition.

Proposition 9 *The solutions for $\psi(t), \phi(t)$ in Proposition 6 are given by:*

$$\begin{aligned}\psi(t) &= \psi' + e^{\left(M^{\top}-2\psi'Q^{\top}Q\right)t}\Bigg[(w-\psi')^{-1} \\ &\quad + 2\int_0^t e^{\left(M-2Q^{\top}Q\psi'\right)s}Q^{\top}Qe^{\left(M^{\top}-2\psi'Q^{\top}Q\right)s}ds\Bigg]^{-1} e^{\left(M-2Q^{\top}Q\psi'\right)t}, \quad (15)\\ \phi(t) &= Tr\left[\alpha Q^{\top}Q\int_0^t \psi(s)ds\right], \quad (16)\end{aligned}$$

where ψ' is a symmetric solution to the algebraic Riccati equation (13).

Remark 10 *In this subsection we followed closely Gourieroux and Sufana (2005), who solved the Riccati ODE (11) by using the variation of constants method (see also Gourieroux et al. (2005), Gourieroux and Sufana (2003), Gourieroux and Sufana (2005)). This is also equivalent to the procedure followed by Ahdida and Alfonsi (2010) and Mayerhofer (2010) who found the Laplace transform of the Wishart process alone (i.e. corresponding to $v = 0$ in (2)). The variation of constants method represents the first solution provided in literature for the solution of the matrix ODE's (11) and (12), and despite its theoretical simplicity, it turns out to be very time consuming, as we will show later in the numerical exercise.*

3.2 Linearization of the Matrix Riccati ODE

The second approach we consider is the one proposed by Grasselli and Tebaldi (2008), who used the Radon lemma in order to linearize the matrix Riccati ODE (11) (see also Levin (1959), Jong and Zhou (1999) and Anderson and Moore (1971)). As usual we are interested in the computation of the moment generating function of the process and of the integrated process, hence we look at the system of equations (11) and (12).

Proposition 11 *(Grasselli and Tebaldi (2008)) The functions $\psi(t), \phi(t)$ in Proposition 6 are given by*

$$\psi(t) = \left(w\psi_{12}(t) + \psi_{22}(t)\right)^{-1}\left(w\psi_{11}(t) + \psi_{21}(t)\right),$$
$$\phi(t) = \frac{\alpha}{2} Tr\left[log\left(w\psi_{12}(t) + \psi_{22}(t)\right) + M^\top t\right],$$

where

$$\begin{pmatrix} \psi_{11}(t) & \psi_{12}(t) \\ \psi_{21}(t) & \psi_{22}(t) \end{pmatrix} = exp\left\{t\begin{pmatrix} M & 2Q^\top Q \\ v & -M^\top \end{pmatrix}\right\}.$$

3.3 Runge-Kutta Method

The Runge-Kutta method is a classical approach for the numerical solution of ODE's. For a detailed treatment, see e.g. Quarteroni et al. (2000). If we want to solve numerically the system of equations (11) and (12), the most commonly used Runge-Kutta scheme is the fourth order one:

$$\begin{aligned} \psi(t_{n+1}) &= \psi(t_n) + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right), \\ t_{n+1} &= t_n + h, \\ k_1 &= g(t_n, \psi(t_n)), \\ k_2 &= g(t_n + \frac{1}{2}h, \psi(t_n) + \frac{1}{2}hk_1), \\ k_3 &= g(t_n + \frac{1}{2}h, \psi(t_n) + \frac{1}{2}hk_2), \\ k_4 &= g(t_n + h, \psi(t_n) + hk_3), \end{aligned}$$

where the function g is given by:

$$g(t_n, \psi(t_n)) = g(\psi(t_n)) = \psi(t_n)M + M^\top\psi(t_n) - 2\psi(t_n)Q^\top Q\psi(t_n) + v.$$

3.4 Comparison of the methods

A formal numerical analysis of the various methods is beyond the scope of this paper. Anyhow, we would like to stress some important points, which we believe are sufficient to highlight the importance of our new methodology. First of all we compare the results of the four different methods. We consider different time horizons $t \in [0, 0.3]$ and use the following values for the parameters:

$$S_0 = \begin{pmatrix} 0.0120 & 0.0010 \\ 0.0010 & 0.0030 \end{pmatrix}; \quad Q = \begin{pmatrix} 0.141421356237310 & -0.070710678118655 \\ 0 & 0.070710678118655 \end{pmatrix};$$
$$M = \begin{pmatrix} -0.02 & -0.02 \\ -0.01 & -0.02 \end{pmatrix}; \quad \alpha = 3;$$
$$v = \begin{pmatrix} 0.1000 & 0.0400 \\ 0.0400 & 0.1000 \end{pmatrix}; \quad w = \begin{pmatrix} 0.1100 & 0.0300 \\ 0.0300 & 0.1100 \end{pmatrix}.$$

The value for Q was obtained along the following steps: given a matrix $A \in S_d^+$ such that $AM = M^\top A$, we compute its inverse and let Q be obtained from a Cholesky factorization of this inverted matrix. Table (1) shows the value of the moment generating function for different values of the time horizon t. The four methods lead to values which are very close to each other, and this constitutes a first test proving that the new methodology produces correct results.

The next important point that we should consider is the execution speed. In order to obtain a good degree of precision for the variation of constants method, we were forced to employ a fine integration grid. This results in a poor performance of this method in terms of speed. In Figure (3) we compare the time spent by the three analytical methods for the calculation of the moment generating function. As t gets larger, the execution time for the variation of constants method grows exponentially, whereas the time required by the linearization and the new methodology is the same.

The Runge-Kutta method is a numerical solution to the problem, so the real competitors of our methodology are the variation of constants and the linearization method. As we saw above, the variation of constant method is quite cumbersome. This is because in order to implement the variation of constants methods we have to solve numerically an algebraic Riccati equation, then we have to perform a first numerical integration in order to determine ψ and another numerical integration to compute ϕ. The above procedure is obviously time consuming, hence we believe that this method is not suitable for applications, in particular in a calibration setting.

Finally, we would like to compare the linearization of the Riccati ODE to the new methodology. In terms of precision and execution speed the two methodologies seem to provide the same performance, up to the fourteenth digit. This shows that, under the parametric restriction of Theorem 1 our methodology represents a valid alternative.

4 Applications

4.1 Pricing of derivatives

The knowledge of the functional form of the Laplace transform represents an important tool for the application of a stochastic model in mathematical finance. In the following, we will provide two examples of asset pricing models whose Laplace transform is of exponentially affine form and such that our previous results may be applied. The first one is the model proposed by Da Fonseca et al. (2008) which describes the evolution of a single asset, whose instantaneous volatility is modelled by means of a Wishart process. The second is the model introduced in Da Fonseca et al. (2007b), where the evolution of a vector of assets is described by a vector-valued SDE where the Wishart process models the instantaneous variance-covariance matrix of the assets.

4.1.1 A stochastic volatility model

In this subsection we consider the model proposed in Da Fonseca et al. (2008) and we derive the explicit Laplace transform of the log-price using our new methodology. As a starting point, we report the dynamics defining the model:

$$
\begin{aligned}
\frac{dX_t}{X_t} &= Tr\left[\sqrt{S_t}\left(dW_t R^\top + dB_t\sqrt{I_d - RR^\top}\right)\right],\\
dS_t &= \left(\alpha Q^\top Q + MS_t + S_t M^\top\right)dt + \sqrt{S_t}dW_t Q + Q^\top dW_t^\top \sqrt{S_t},
\end{aligned}
$$

where X_t denotes the price of the underlying asset, and the Wishart process acts as a multifactor source of stochastic volatility. W and B are independent matrix Brownian motions and the matrix R parametrizes all possible correlation structures preserving the affinity. This model is a generalization of the (multi-)Heston model, see Heston (1993) and Christoffersen et al. (2009), and it offers a very rich structure for the modelization of stochastic volatilities as the factors governing the instantaneous variance are non-trivially correlated. It is easy to see that the log-price Y is given as

$$
dY = -\frac{1}{2}Tr\left[S_t\right]dt + Tr\left[\sqrt{S_t}\left(dW_t R^\top + dB_t\sqrt{I_d - RR^\top}\right)\right].
$$

We are interested in the Laplace transform of the log-price, i.e.

$$
\varphi_t(\tau, -\omega) = \mathbb{E}\left[e^{-\omega Y_T}\left|\mathcal{F}_t\right.\right], \quad \tau := T - t.
$$

This expectation satisfies a backward Kolmogorov equation, see Da Fonseca et al. (2008) for a detailed derivation. Since the process $S = (S_t)_{0\le t\le T}$ is affine, we make a guess of a solution of the form

$$
\varphi_t(\tau, -\omega) = \exp\left\{-\omega \ln X_t - \phi(\tau) - Tr\left[\psi(\tau)S_t\right]\right\}.
$$

By substituting it into the PDE, we obtain the system of ODE's

$$
\begin{aligned}
\frac{d\psi}{d\tau} &= \psi\left(M - \omega Q^\top R^\top\right) + \left(M^\top - \omega RQ\right)\psi - 2\psi Q^\top Q\psi - \frac{\omega^2+\omega}{2}I_d, &(17)\\
\psi(0) &= 0, &(18)\\
\frac{d\phi}{d\tau} &= Tr\left[\alpha Q^\top Q\psi(\tau)\right], &(19)\\
\phi(0) &= 0. &(20)
\end{aligned}
$$

If we look at the first ODE, we recognize the same structure as in (11): instead of M and v we have respectively $M - \omega Q^\top R^\top$ and $-\frac{\omega^2+\omega}{2}I_d$. This means that we can rewrite the solution for ψ, using Remark 2, as

$$
\begin{aligned}
\psi(\tau) &= \frac{\left(Q^\top Q\right)^{-1}\left(M - \omega Q^\top R^\top\right)}{2} - \frac{Q^{-1}\sqrt{\bar{v}}kQ^{\top^{-1}}}{2},\\
\phi(\tau) &= -\frac{\alpha}{2}\log\left(\det\left(e^{-\left(M-\omega Q^\top R^\top\right)\tau}\left(\cosh(\sqrt{\bar{v}}\tau) + \sinh(\sqrt{\bar{v}}\tau)k\right)\right)\right),\\
\bar{v} &= Q\left(2\left(-\frac{\omega^2+\omega}{2}I_d\right) + \left(M^\top - \omega RQ\right)Q^{-1}Q^{\top^{-1}}\left(M - \omega Q^\top R^\top\right)\right)Q^\top,\\
\bar{w} &= Q\left(-\left(Q^\top Q\right)^{-1}\left(M - \omega Q^\top R^\top\right)\right)Q^\top,\\
k &= -\left(\sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}\tau) + \bar{w}\sinh(\sqrt{\bar{v}}\tau)\right)^{-1}\left(\sqrt{\bar{v}}\sinh(\sqrt{\bar{v}}\tau) + \bar{w}\cosh(\sqrt{\bar{v}}\tau)\right).
\end{aligned}
$$

Condition (3) in this setting has the following form

$$
\left(M - \omega Q^\top R^\top\right)^\top\left(Q^\top Q\right)^{-1} = \left(Q^\top Q\right)^{-1}\left(M - \omega Q^\top R^\top\right). \tag{21}
$$

For fixed ω we can express the condition above via the following system

$$
\begin{cases}
M^\top\left(Q^\top Q\right)^{-1} = \left(Q^\top Q\right)^{-1}M,\\
RQ\left(Q^\top Q\right)^{-1} = \left(Q^\top Q\right)^{-1}Q^\top R^\top.
\end{cases} \tag{22}
$$

4.1.2 A stochastic correlation model

In this subsection we consider the model introduced in Da Fonseca et al. (2007b). This model belongs to the class of multi-variate affine volatility models, for which many interesting theoretical results have been presented in Cuchiero (2011). In this framework we consider a vector of prices together with a stochastic variance-covariance matrix.

$$\begin{aligned} dX_t &= Diag(X_t)\sqrt{S_t}\left(dW_t\rho + \sqrt{1-\rho^\top\rho}dB_t\right), \\ dS_t &= \left(\alpha Q^\top Q + MS_t + S_tM^\top\right)dt + \sqrt{S_t}dW_tQ + Q^\top dW_t^\top\sqrt{S_t}, \end{aligned}$$

where now the vector Brownian motion $Z = W_t\rho + \sqrt{1-\rho^\top\rho}B_t$ is correlated with the matrix Brownian motion W through the correlation vector ρ. Using the same arguments as before, we compute the joint conditional Laplace transform of the vector of the log-prices $Y_T = \log(X_T)$

$$\varphi_t(\tau, -\omega) = \mathbb{E}\left[e^{-\omega^\top Y_T}|\mathcal{F}_t\right], \quad \tau := T-t.$$

The affine property allows us to write the associated system of matrix Riccati ODE's (see Da Fonseca et al. (2007b) for more details), which is given as

$$\begin{aligned} \frac{d\psi}{d\tau} &= \psi\left(M - Q^\top\rho\omega^\top\right) + \left(M^\top - \omega\rho^\top Q\right)\psi - 2\psi Q^\top Q\psi \\ &\quad - \frac{1}{2}\left(\sum_{i=1}^d \omega_i e_{ii} + \omega^\top\omega\right)I_d, && (23) \\ \psi(0) &= 0, && (24) \\ \frac{d\phi}{d\tau} &= Tr\left[\alpha Q^\top Q\psi(\tau)\right], && (25) \\ \phi(0) &= 0. && (26) \end{aligned}$$

We recognize the same structure as in Equations (12) and (11) where instead of M and v, we now have $M - Q^\top\rho\omega^\top$ and $-\frac{1}{2}\left(\sum_{i=1}^d \omega_i e_{ii} + \omega^\top\omega\right)I_d$ respectively. Consequently, using Remark 2, we can compute the solution as

$$\begin{aligned} \psi(\tau) &= \frac{\left(Q^\top Q\right)^{-1}\left(M - Q^\top\rho\omega^\top\right)}{2} - \frac{Q^{-1}\sqrt{\bar{v}}kQ^{\top^{-1}}}{2}, \\ \phi(\tau) &= -\frac{\alpha}{2}\log\left(\det\left(e^{-\left(M-Q^\top\rho\omega^\top\right)\tau}\left(\cosh(\sqrt{\bar{v}}\tau) + \sinh(\sqrt{\bar{v}}\tau)k\right)\right)\right), \\ \bar{v} &= Q\left(2\left(-\frac{1}{2}\left(\sum_{i=1}^d \omega_i e_{ii} + \omega^\top\omega\right)I_d\right) + \left(M^\top - \omega\rho Q\right)Q^{-1}Q^{\top^{-1}}\left(M - Q^\top\rho\omega^\top\right)\right)Q^\top, \\ \bar{w} &= Q\left(-\left(Q^\top Q\right)^{-1}\left(M - Q^\top\rho\omega^\top\right)\right)Q^\top, \\ k &= -\left(\sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}\tau) + \bar{w}\sinh(\sqrt{\bar{v}}\tau)\right)^{-1}\left(\sqrt{\bar{v}}\sinh(\sqrt{\bar{v}}\tau) + \bar{w}\cosh(\sqrt{\bar{v}}\tau)\right). \end{aligned}$$

Condition (3) is rephrased in this setting as follows

$$\left(M^\top - \omega\rho^\top Q\right)\left(Q^\top Q\right)^{-1} = \left(Q^\top Q\right)^{-1}\left(M - Q^\top\rho\omega^\top\right), \tag{27}$$

which may be expressed as

$$\begin{cases} M^\top\left(Q^\top Q\right)^{-1} = \left(Q^\top Q\right)^{-1}M, \\ \omega\rho^\top Q^{\top^{-1}} = Q^{-1}\rho\omega^\top. \end{cases} \tag{28}$$

4.1.3 A short rate model

Our methodology for the computation of the Laplace transform may be directly employed to provide a closed form formula for the price of zero coupon bonds when the short rate is driven by a Wishart process. The Wishart short rate model has been studied in Gourieroux and Sufana (2003), Grasselli and Tebaldi (2008), Buraschi et al. (2008), Chiarella et al. (2010) and Gnoatto (2012). The short rate is modeled as

$$r_t = a + Tr\left[vS_t\right], \tag{29}$$

where $a \in \mathbb{R}_{\geq 0}$, v is a symmetric positive definite matrix and $S = (S_t)_{t\geq 0}$ is the Wishart process. Standard arbitrage arguments allow us to claim that the price of a zero coupon bond at time t with time to maturity $\tau := T - t$, denoted by $P_t(\tau)$, is given by the following expectation

$$\begin{aligned} P_t(\tau) :&= \mathbb{E}\left[e^{-\int_t^T a+Tr[vX_u]du} | \mathcal{F}_t\right] \\ &= \exp\left\{-\phi(\tau) - Tr\left[\psi(\tau)X_t\right]\right\}, \end{aligned} \tag{30}$$

where the associated ODE are

$$\frac{\partial \phi}{\partial \tau} = Tr\left[\alpha Q^\top Q \psi(\tau)\right] + a, \quad \phi(0) = 0, \tag{31}$$

and

$$\frac{\partial \psi}{\partial \tau} = \psi(\tau)M + M^\top \psi(\tau) - 2\psi(\tau)Q^\top Q\psi(\tau) + v, \quad \psi(0) = 0. \tag{32}$$

We can employ again the Cameron-Martin formula and write the solution to the system as follows

$$\psi(\tau) = \frac{\left(Q^\top Q\right)^{-1} M}{2} - \frac{Q^{-1}\sqrt{\bar{v}}kQ^{\top^{-1}}}{2}, \tag{33}$$

$$\phi(\tau) = -\frac{\alpha}{2}\log\left(\det\left(e^{-M\tau}\left(\cosh(\sqrt{\bar{v}}\tau) + \sinh(\sqrt{\bar{v}}\tau)k\right)\right)\right) + a\tau. \tag{34}$$

4.2 A solution to the algebraic Riccati equation

As as application we look at the problem of computing a solution to the algebraic Riccati equation (ARE). This equation is well known from control theory. We will construct a solution to the ARE by comparing the solution of the system of differential equations in Proposition 6 obtained according to our new methodology and the variation of constant approach. For convenience, rewrite the system of ODE's (11) (12) as follows

$$\frac{d\psi}{dt} = \mathcal{R}\left(\psi\right), \quad \psi(0) = w, \tag{35}$$

$$\frac{d\phi}{dt} = \mathcal{F}\left(\psi\right), \quad \phi(0) = 0. \tag{36}$$

An ARE is given by

$$\mathcal{R}\left(\psi'\right) = 0. \tag{37}$$

As before, we denote by ψ' a solution to this equation. We begin by reporting a useful lemma, whose standard proof is omitted.

Lemma 12 *Let $A \in M_d$. Assume $\Re(\lambda(A)) < 0, \quad \forall \lambda \in \sigma(A)$, then*

$$\lim_{\tau\to\infty} e^{A\tau} = 0 \in M_{d\times d}. \tag{38}$$

As a nice consequence, we have this result.

Lemma 13 *Let $O \in S_d^+$, define*

$$\sinh(O\tau) = \frac{e^{O\tau} - e^{-O\tau}}{2}, \quad \cosh(O\tau) = \frac{e^{O\tau} + e^{-O\tau}}{2} \tag{39}$$

and

$$\tanh(O\tau) = (\cosh(O\tau))^{-1} \sinh(O\tau), \quad \coth(O\tau) = (\sinh(O\tau))^{-1} \cosh(O\tau), \tag{40}$$

then

$$\lim_{\tau\to\infty} \tanh(O\tau) = \lim_{\tau\to\infty} \coth(O\tau) = I_d. \tag{41}$$

Proof.

$$\lim_{\tau\to\infty} \tanh(O\tau) = \lim_{\tau\to\infty} \left(I_d + e^{-2O\tau}\right)^{-1} \left(I_d - e^{-2O\tau}\right) = I_d. \tag{42}$$

The second equality follows along the same lines. ■

Let us recall some well known results from control theory. We refer to the review article by Kucera (1973). Let us write $v = C^\top C$. We introduce the following notions.

- The pair (M, Q) is said to be stabilizable if $\exists$ a matrix L such that $M + QL$ is stable, i.e. all eigenvalues are negative.
- The pair (C, M) is said to be detectable if $\exists$ a matrix F stuch that $FC + A$ is stable.

We introduce again the matrix $M - 2Q^\top Q\psi'$ and call it the closed loop system matrix. A classical result is the following.

Theorem 14 *Stabilizability of (M, Q) and detectability of (C, M) is necessary and sufficient for the matrix Riccati ODE to have a unique non-negative solution which makes the closed loop system matrix stable.*

Now, looking at the variation of constant approach we can prove the next result.

Corollary 15

$$\lim_{\tau\to\infty} \tilde{\psi}(\tau) = \psi'. \tag{43}$$

Proof. Under the assumptions of Theorem 14, we have $\lambda\left(M - 2Q^\top Q\psi'\right) < 0, \forall\lambda \in \sigma\left(M - 2Q^\top Q\psi'\right)$ hence we know that the integral in (15), the solution for ψ, is convergent, moreover, from Lemma 12, we know that $e^{\left(M-2Q^\top Q\psi'\right)\tau} \searrow 0$ as $\tau \to \infty$, hence the proof is complete. ■

This last corollary tells us that the function ψ tends to a stability point of the Riccati ODE. This allows us to claim that, as $\tau \to \infty$, we have $\mathcal{R}\left(\psi(\tau)\right) \searrow 0$. A nice consequence of this fact is that we are able to provide a new representation for ψ', which constitutes another application of the Cameron-Martin approach.

Proposition 16 *The value of ψ' in Corollary 15 admits the following representation*

$$\psi' = \frac{Q^{-1}\sqrt{v}Q^{\top^{-1}}}{2} + \frac{\left(Q^\top Q\right)^{-1} M}{2}. \tag{44}$$

Proof. On the basis of (5), we want to compute

$$\lim_{\tau\to\infty} -\frac{Q^{-1}\sqrt{\bar{v}}kQ^{\top^{-1}}}{2} + \frac{\left(Q^\top Q\right)^{-1}M}{2}. \tag{45}$$

To perform the computation, it is sufficient to calculate

$$\begin{aligned}
&\lim_{\tau\to\infty} k \\
&= \lim_{\tau\to\infty} -\left(\sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}\tau) + \bar{w}\sinh(\sqrt{\bar{v}}\tau)\right)^{-1}\left(\sqrt{\bar{v}}\sinh(\sqrt{\bar{v}}\tau) + \bar{w}\cosh(\sqrt{\bar{v}}\tau)\right) \\
&= \lim_{\tau\to\infty} -\left(\cosh\sqrt{\bar{v}}\tau\right)^{-1}\left(\sqrt{\bar{v}} + \bar{w}\tanh\sqrt{\bar{v}}\tau\right)^{-1}\left(\sqrt{\bar{v}} + \bar{w}\coth\sqrt{\bar{v}}\tau\right)\sinh\sqrt{\bar{v}}\tau. \qquad (46)
\end{aligned}$$

From Lemma 13 we know that both $\tanh$ and $\coth$ tend to I_d as $\tau \to \infty$, hence we conclude that

$$\lim_{\tau\to\infty} k = -I_d \tag{47}$$

and so we obtain the final claim:

$$\lim_{\tau\to\infty} \tilde{\psi}(\tau) = \frac{Q^{-1}\sqrt{\bar{v}}Q^{\top^{-1}}}{2} + \frac{\left(Q^\top Q\right)^{-1}M}{2}. \tag{48}$$

Finally, from Corollary 15, we know that $\lim_{\tau\to\infty}\tilde{\psi}(\tau) = \psi'$, hence the claim. ■

5 Conclusions

In this paper we derived a new explicit formula for the joint Laplace transform of the Wishart process and its time integral based on the original approach of Bru (1991). Our methodology leads to a truly explicit formula that does not involve any additional integration (like the highly time consuming variation of constants method) or blocks of matrix exponentials (like the linearization method) at the price of a simple condition on the parameters. We showed some examples of applications in the context of multifactor and multi-variate stochastic volatility.

In the Appendix B we consider a generalization of our formula where the Gindikin parameter α has a generic matrix form (see Cuchiero et al. (2011)). As a further extension we would like to point out that our formula may also be employed for the computation of the Laplace transform of a particular class of jump-diffusion models in S_d^+ with constant jump coefficients, using the terminology of Cuchiero et al. (2011). The resulting formulae for this class are available upon request.

A A generalization of the Wishart process

Affine processes on positive semidefinite matrices have been classified in full generality by Cuchiero et al. (2011). In this reference, a complete set of sufficient and necessary conditions providing a full characterization of this family of processes is derived. These conditions are the S_d^+ analogue of the concept of admissibility for the state space $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^m$, which has been studied by Duffie et al. (2003) under the assumption of regularity of the process, and by Keller-Ressel (2008), who proved that regularity is a consequence of the stochastic continuity requirement in the definition of affine process. Cuchiero et al. (2011) showed that an admissible drift can be considerably different from the one considered in (1). In particular, the Gindikin condition $\alpha > d - 1$ can be generalized, thus leading to a generalized Wishart process with dynamics[1] with respect to $\mathbb{P}$:

$$dS_t = \sqrt{S_t} dB_t Q + Q^\top dB_t^\top \sqrt{S_t} + \left(MS_t + S_t M^\top + b\right) dt, \quad S_0 = s_0 \in S_d^+, \tag{49}$$

where the symmetric matrix b satisfies

$$b - (d-1) Q^\top Q \in S_d^+. \tag{50}$$

We will denote by $WIS_d(s_0, b, M, Q)$ the law of the Wishart process $(S_t)_{t\geq 0}$ that satisfies (49).

In this Appendix we will find the explicit Cameron Martin formula for the more general specification (49). We will first characterize the distribution function of the process $WIS_d(s_0, b, 0, I_d)$ at a fixed time through the Laplace transform and then we will proceed with the analogue of Theorem 1.

A.1 Laplace transform of $WIS_d(\Sigma_0, \tilde{b}, 0, I_d)$

We fix a probability measure $\tilde{\mathbb{P}}$ such that $\tilde{\mathbb{P}} \approx \mathbb{P}$. Under the measure $\tilde{\mathbb{P}}$ we consider a matrix Brownian motion $\hat{B} = (\hat{B}_t)_{t\geq 0}$ that will allow us to define the process Σ_t having law $WIS_d(\Sigma_0, \tilde{b}, 0, I_d), \tilde{b} \in S_d^+$, that is a process which solves the following SDE:

$$d\Sigma_t = \sqrt{\Sigma_t} d\hat{B}_t + d\hat{B}_t^\top \sqrt{\Sigma_t} + \tilde{b} dt, \quad \Sigma_0 \in S_d^+,$$

where the drift term $\tilde{b}$ satisfies the following condition:

$$\tilde{b} - (d-1) I_d \in S_d^+.$$

We may characterize the distribution of this process by means of its Laplace Transform.

Theorem 17 *Let Σ be a generalized Wishart process in $WIS_d(\Sigma_0, \tilde{b}, 0, I_d)$, then the distribution of Σ_t, for fixed t, under $\tilde{\mathbb{P}}$, is given by its Laplace transform:*

$$\mathbb{E}_{\Sigma_0}^{\tilde{\mathbb{P}}} \left[e^{-Tr[u\Sigma_t]} \right] = \det \left(e^{\tilde{b} \log (I_d + 2tu)^{-\frac{1}{2}}} \right) e^{-Tr\left[(I_d+2tu)^{-1} u \Sigma_0\right]},$$

for all $u \in S_d$ such that $(I_d + 2tu)$ is nonsingular.

Proof. We know that this process belongs to the class of affine processes on S_d^+, which means that we may write the Laplace Transform in the following way:

$$\varphi(t, \Sigma_0) = \mathbb{E}_{\Sigma_0}^{\tilde{\mathbb{P}}} \left[e^{-Tr[u\Sigma_t]} \right] = \exp \left\{ -\phi(t) - Tr \left[\psi(t) \Sigma_0 \right] \right\}.$$

We proceed along the lines of Bru (1991) and look for the solution for the Laplace transform by means of the associated backward Kolmogorov equation.

$$\frac{d}{dt} \varphi(t, \Sigma_0) = Tr \left[b D\varphi(t, \Sigma_0) + 2 D\varphi(t, \Sigma_0)^2 \right], \quad \varphi(0, \Sigma_0) = e^{-Tr[u\Sigma_0]},$$

[1]Cuchiero et al. (2011) consider a matrix jump diffusion dynamics with a more general structure for the drift parameter involving a linear operator that cannot be written a priori in the matrix form $MS_t + S_t M^\top$. In this appendix we restrict ourselves to the continuous path version with the usual matrix drift. We also emphasize that in view of applications the specification (1) is highly preferable since it is more parsimonius in terms of parameters: this is a delicate and crucial issue when calibrating any model.

where D denotes the matrix differential operator whose ij-th element is given by

$$\frac{\partial}{\partial \Sigma_{ij}}.$$

Upon substitution of the exponentially affine guess we obtain the following system of ODE's, which is a simplified version of (11), (12):

$$\begin{aligned}\frac{d\psi}{dt} &= -2\psi(t)^2, \quad \psi(0) = u,\\ \frac{d\phi}{dt} &= Tr\left[\tilde{b}\psi(t)\right], \quad \psi(0) = 0.\end{aligned}$$

The solution for $\psi(t)$ is:

$$\psi(t) = (I_d + 2tu)^{-1} u,$$

while the solution for $\phi(t)$ is obtained via direct integration:

$$\begin{aligned}\phi(t) &= Tr\left[\tilde{b}\int_0^t (I_d + 2su)^{-1} uds\right]\\ &= Tr\left[\tilde{b}\frac{1}{2}\log(I_d + 2tu)\right].\end{aligned}$$

Noting that

$$e^{-\phi(t)} = \det\left(e^{\tilde{b}\log(I_d+2tu)^{-\frac{1}{2}}}\right),$$

we obtain the result. ■

A.2 Cameron Martin formula for the process $WIS_d(s_0, b, M, Q)$

We now consider, under the measure $\mathbb{P}$, the process governed by the SDE

$$dS_t = \sqrt{S_t}dB_tQ + Q^\top dB_t^\top\sqrt{S_t} + \left(MS_t + S_tM^\top + b\right)dt, \quad S_0 = s_0 \in S_d^+,$$

with $b \in S_d^+$ satisfying (50).
In this subsection we compute the joint moment generating function of the process and its time integral.

Theorem 18 *The joint Laplace transform of the generalized Wishart process $S \in WIS_d(s_0, b, M, Q)$ and its time integral is given by:*

$$\mathbb{E}_{s_0}^{\mathbb{P}}\left[\exp\left\{-Tr\left[wS_t + \int_0^t vS_sds\right]\right\}\right] = \exp\left\{-\phi(t) - Tr\left[\psi(t)s_0\right]\right\},$$

where the functions ϕ and ψ are given by:

$$\begin{aligned}\psi(t) &= \frac{\left(Q^\top Q\right)^{-1}M}{2} - \frac{Q^{-1}\sqrt{\bar{v}}kQ^{\top^{-1}}}{2},\\ \phi(t) &= Tr\left[b\frac{\left(Q^\top Q\right)^{-1}M}{2}\right]t\\ &+ \frac{1}{2}Tr\left[\left(Q^\top\right)^{-1}b\left(Q\right)^{-1}\log\left(\sqrt{\bar{v}}^{-1}\left(\sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}t) + \bar{w}\sinh(\sqrt{\bar{v}}t)\right)\right)\right],\end{aligned}$$

with k given by:

$$k=-\left(\sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}t)+\bar{w}\sinh(\sqrt{\bar{v}}t)\right)^{-1}\left(\sqrt{\bar{v}}\sinh(\sqrt{\bar{v}}t)+\bar{w}\cosh(\sqrt{\bar{v}}t)\right)$$

and $\bar{v},\bar{w}$ are defined as follows:

$$\begin{aligned}\bar{v}&=Q\left(2v+M^\top Q^{-1}Q^{\top^{-1}}M\right)Q^\top,\\ \bar{w}&=Q\left(2w-\left(Q^\top Q\right)^{-1}M\right)Q^\top.\end{aligned}$$

Proof. The proof will be based on the previous discussion in Theorem 1 on the standard process with a scalar Gindikin parameter. First of all we have that the invariance under transformation may still be used.

We consider the process $\Sigma_t\in WIS_d(\Sigma_0,\left(Q^\top\right)^{-1}bQ^{-1},0,I_d)$, i.e. a process solving the following matrix SDE:

$$d\Sigma_t=\sqrt{\Sigma_t}d\hat{B}_t+d\hat{B}_t^\top\sqrt{\Sigma_t}+\left(Q^\top\right)^{-1}bQ^{-1}dt.$$

We define the following quantities: $S_t=Q^\top\Sigma_tQ$, $\Sigma_t=\left(Q^\top\right)^{-1}S_tQ^{-1}$. Under this transformation the process under $\tilde{\mathbb{P}}$ is governed by the SDE:

$$dS_t=\sqrt{S_t}d\tilde{B}_tQ+Q^\top d\tilde{B}_t^\top\sqrt{S_t}+bdt,$$

where the process $\tilde{B}=(\tilde{B}_t)_{t\geq0}$ defined by $\sqrt{S_t}^{-1}Q^\top\sqrt{\Sigma_t}d\hat{B}_t$ is a matrix Brownian motion under $\tilde{\mathbb{P}}$. From Bru (1991) we have:

$$\begin{aligned}\mathbb{E}_{s_0}^{\tilde{\mathbb{P}}}\left[e^{-Tr[uS_t]}\right]&=\mathbb{E}^{\tilde{\mathbb{P}}}_{(Q^\top)^{-1}s_0Q^{-1}}\left[e^{-Tr\left[uQ^\top\Sigma_tQ\right]}\right]\\&=\mathbb{E}^{\tilde{\mathbb{P}}}_{\Sigma_0}\left[e^{-Tr\left[\left(QuQ^\top\right)\Sigma_t\right]}\right].\end{aligned}$$

Again, by relying on a Taylor expansion and Sylvester's rule of inertia, we obtain the following closed-form formula for the Laplace transform:

$$\varphi(t,s_0)=\det\left(e^{\left(Q^\top\right)^{-1}bQ^{-1}\log\left(I_d+2tuQ^\top Q\right)^{-\frac{1}{2}}}\right)e^{-Tr\left[\left(I_d+2tuQ^\top Q\right)^{-1}us_0\right]}.$$

Now we consider the process under $\mathbb{P}$:

$$dS_t=\sqrt{S_t}dB_tQ+Q^\top dB_t^\top\sqrt{S_t}+\left(MS_t+S_tM^\top+b\right)dt,$$

where $dB_t=d\tilde{B}_t-\sqrt{S_t}M^\top Q^{-1}dt$, with an associated Girsanov kernel satisfying (3) as before. The system of Riccati ODE's satisfied by the joint Laplace transform of the process S_t under $\mathbb{P}$ is given by:

$$\begin{aligned}\frac{d\psi}{dt}&=\psi M+M^\top\psi-2\psi Q^\top Q\psi+v,\quad\psi(0)=w,\\ \frac{d\phi}{dt}&=Tr\left[b\psi(t)\right],\quad\phi(0)=0.\end{aligned}$$

We realize that the ODE for ψ is the same as in (11) , whose solution is known from Remark 2. We can recover the solution also for ϕ upon a direct integration. We show the calculation in detail:

$$\begin{aligned}\frac{d\phi}{dt}&=Tr\left[b\psi(t)\right]\\&=Tr\left[b\left(\frac{\left(Q^\top Q\right)^{-1}M}{2}-\frac{Q^{-1}\sqrt{\bar{v}}kQ^{\top^{-1}}}{2}\right)\right].\end{aligned}$$

Integrating the ODE yields

$$\phi(t) = Tr\left[b\frac{\left(Q^\top Q\right)^{-1} M}{2}\right] t - \frac{1}{2} Tr\left[\left(Q^\top\right)^{-1} bQ^{-1}\sqrt{\bar{v}}\int_0^t kds\right].$$

We concentrate on the integral appearing in the second term:

$$\int_0^t kds \quad = \int_0^t -\left(\sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}s) + \bar{w}\sinh(\sqrt{\bar{v}}s)\right)^{-1}\left(\sqrt{\bar{v}}\sinh(\sqrt{\bar{v}}s) + \bar{w}\cosh(\sqrt{\bar{v}}s)\right) ds.$$

Define $f(s) = \sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}s) + \bar{w}\sinh(\sqrt{\bar{v}}s)$ and let us differentiate it:

$$\frac{df}{ds} = \left(\sqrt{\bar{v}}\sinh(\sqrt{\bar{v}}s) + \bar{w}\cosh(\sqrt{\bar{v}}s)\right)\sqrt{\bar{v}},$$

hence we can write

$$\begin{aligned}
&-\frac{1}{2}Tr\left[\left(Q^\top\right)^{-1} bQ^{-1}\sqrt{\bar{v}}\int_0^t kds\right] \\
&= \frac{1}{2}Tr\left[\left(Q^\top\right)^{-1} bQ^{-1}\left(\log\left(\sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}t) + \bar{w}\sinh(\sqrt{\bar{v}}t)\right) - \log\left(\sqrt{\bar{v}}\right)\right)\right] \\
&= \frac{1}{2}Tr\left[\left(Q^\top\right)^{-1} bQ^{-1}\log\left(\sqrt{\bar{v}}^{-1}\left(\sqrt{\bar{v}}\cosh(\sqrt{\bar{v}}t) + \bar{w}\sinh(\sqrt{\bar{v}}t)\right)\right)\right]
\end{aligned}$$

and the proof is complete. ■

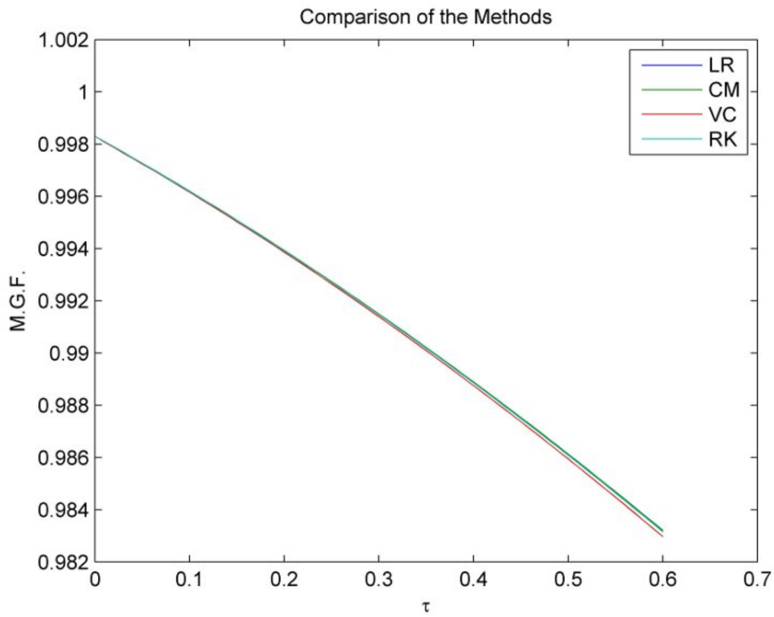

Figure 1: In this image we plot the value of the joint moment generating function of the Wishart process and its time integral for different time horizons τ. All four methods are considered. It should be noted that the variation of constants method requires a very fine integration grid in order to produce precise values that can be compared with the results of the other methods.

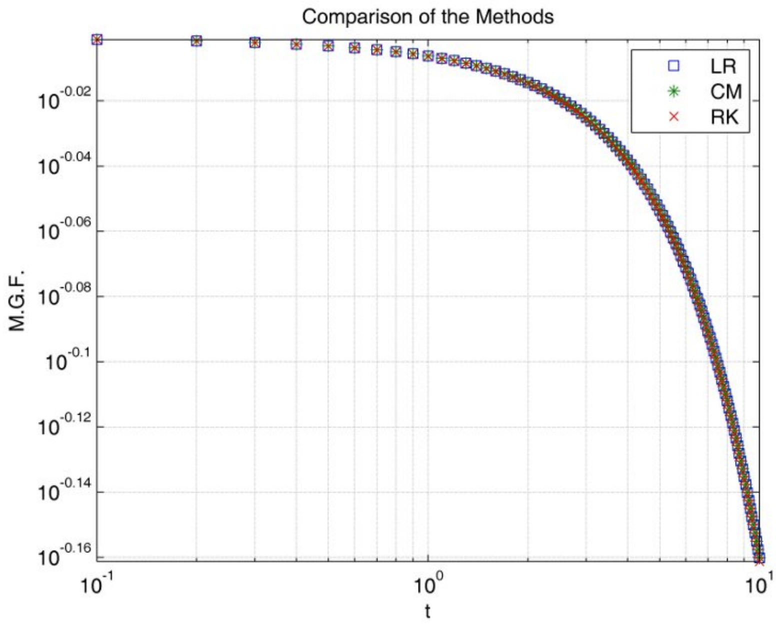

Figure 2: In this case we exclude the variation of constants method and treat a larger time horizon.

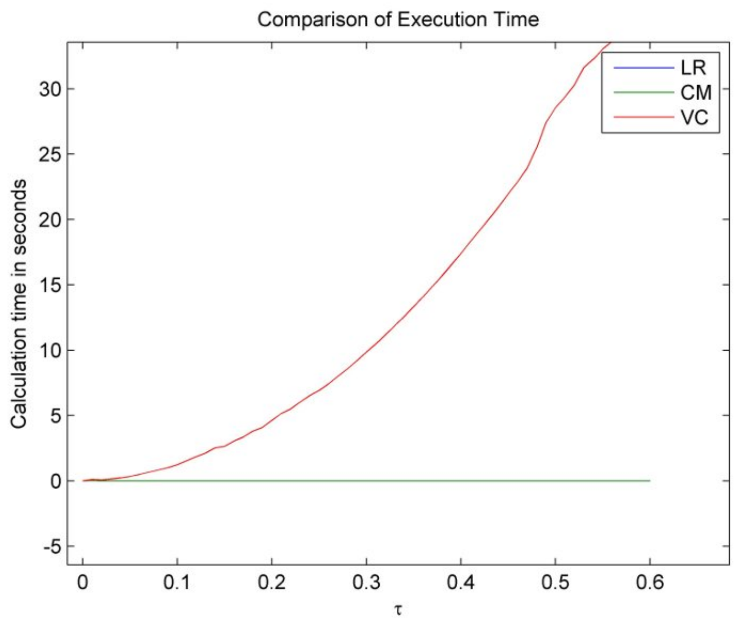

Figure 3: In this image we plot the time spent by the three analytical methods to compute the joint moment generating function of the Wishart process and its time integral for different time horizons.

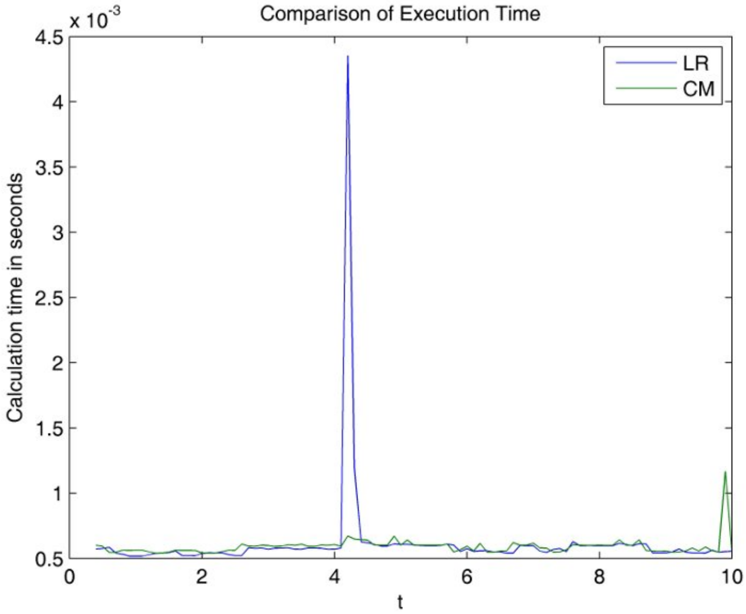

Figure 4: We restrict the previous comparison of execution times to the linearization method and the Cameron-Martin formula.

Time Horizon	Lin.	C.-M.	Var. Const.	R.-K.
0	0.998291461216988	0.998291461216988	0.998291461216988	0.998291461216988
0.1	0.997303305375919	0.997303305375919	0.997306285702955	0.997271605593416
0.2	0.996253721242885	0.996253721242885	0.996258979717961	0.996190498718109
0.3	0.995143124879428	0.995143124879428	0.995142369361912	0.995048563313279
0.4	0.993971944944528	0.993971944944528	0.993956917727425	0.993846234580745
0.5	0.992740622447456	0.992740622447456	0.992703104707601	0.992583959952442
0.6	0.991449610496379	0.991449610496380	0.991381426685627	0.991262198836806
0.7	0.990099374042951	0.990099374042951	0.989992396217013	0.989881422361235
0.8	0.988690389623073	0.988690389623073	0.988536541704748	0.988442113110838
0.9	0.987223145094070	0.987223145094070	0.987014407067708	0.986944764863693
1.0	0.985698139368470	0.985698139368470	0.985426551402640	0.985389882322825
1.1	0.984115882144609	0.984115882144608	0.983773548640023	0.983777980845113
1.2	0.982476893634278	0.982476893634278	0.982055987194167	0.982109586167352
1.3	0.980781704287638	0.980781704287638	0.980274469607849	0.980385234129674
1.4	0.979030854515581	0.979030854515582	0.978429612191836	0.978605470396549
1.5	0.977224894409802	0.977224894409802	0.976522044659620	0.976770850175581
1.6	0.975364383460752	0.975364383460752	0.974552409757698	0.974881937934301
1.7	0.973449890273708	0.973449890273707	0.972521362891742	0.972939307115188
1.8	0.971481992283166	0.971481992283166	0.970429571748981	0.970943539849112
1.9	0.969461275465768	0.969461275465768	0.968277715917132	0.968895226667421
2.0	0.967388334051965	0.967388334051964	0.966066486500228	0.966794966212865
2.1	0.965263770236630	0.965263770236631	0.963796585731653	0.964643364949586
2.2	0.963088193888842	0.963088193888842	0.961468726584740	0.962441036872358
2.3	0.960862222260992	0.960862222260992	0.959083632381238	0.960188603215284
2.4	0.958586479697485	0.958586479697484	0.956642036397998	0.957886692160160
2.5	0.956261597343174	0.956261597343174	0.954144681472186	0.955535938544691
2.6	0.953888212851758	0.953888212851759	0.951592319605355	0.953136983570760
2.7	0.951466970094322	0.951466970094322	0.948985711566685	0.950690474512938
2.8	0.948998518868209	0.948998518868209	0.946325626495722	0.948197064427429
2.9	0.946483514606424	0.946483514606425	0.943612841504911	0.945657411861629
3.0	0.943922618087738	0.943922618087738	0.940848141282233	0.943072180564490

Table 1: This table visualizes in more detail the numerical values for the joint moment generating function plotted in Figure (1).

Time horizon	Lin.	C.M.	R.-K.
0	0.998291461216988	0.998291461216988	0.998291461216988
0.1	0.997303305375919	0.997303305375919	0.997271605593416
0.2	0.996253721242885	0.996253721242885	0.996190498718109
0.3	0.995143124879428	0.995143124879428	0.995048563313279
0.4	0.993971944944528	0.993971944944528	0.993846234580745
0.5	0.992740622447456	0.992740622447456	0.992583959952442
1.0	0.985698139368470	0.985698139368470	0.985389882322825
2.0	0.967388334051965	0.967388334051964	0.966794966212865
3.0	0.943922618087738	0.943922618087738	0.943072180564490
4.0	0.915938197508059	0.915938197508059	0.914862207389661
5.0	0.884120166104796	0.884120166104796	0.882852196560219
10.0	0.691634000576684	0.691634000576684	0.689897813632122
100.0	0.000001636282753	0.000001636282753	0.000001629036716

Table 2: In this table we do not include the results for the variation of constants method. This allows us to look at a longer time horizon and appreciate the precision of the new methodology also in this case.