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THE EXPONENTIAL DECAY OF GLOBAL SOLUTIONS TO THE GENERALIZED LANDAU EQUATION NEAR MAXWELLIANS

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Abstract. Global-in-time classical solutions near Maxwellians are constructed for the generalized Landau equation in a periodic box for $\gamma \geq -2$. The exponential decay of such a solution is also obtained.

1. Introduction. In this paper, we consider the following generalized Landau equation

$$[\partial_t + v \cdot \nabla_x]F = Q[F, F], \qquad F(0, x, v) = F_0(x, v) \tag{1.1}$$

where $F(t, x, v)$ is the spatially periodic distribution function for the particles at time $t \geq 0$, with spatial coordinates $x = (x_1, x_2, x_3) \in [-\pi, \pi]^3 = T^3$ and velocity $v = (v_1, v_2, v_3) \in R^3$. The collision between particles is given by the following generalized Landau operator,

$$\begin{aligned} Q[F, G] &= \nabla_v \cdot \{\int_{R^3} \phi(v - v')[F(v')\nabla_v G(v) - G(v)\nabla_v F(v')]dv'\} \\ &= \partial_i \int_{R^3} \phi^{ij}(v - v')[F(v')\partial_j G(v) - G(v)\partial_j F(v')]dv'. \end{aligned}$$

where $\phi^{ij} = \{\delta_{ij} - v_i v_j/|v|^2\}|v|^{\gamma+2}$, and $\gamma \geq -3$. The original Landau collision operator for the Coulombic interaction corresponds to the case $\gamma = -3$.

The conservation of the mass, momentum as well as energy, can be formulated as $(i = 1, 2, 3)$

$$\frac{d}{dt}\int_{T^3 \times R^3} F(t) = \frac{d}{dt}\int_{T^3 \times R^3} v_i F(t) = \frac{d}{dt}\int_{T^3 \times R^3} |v|^2 F(t) \equiv 0.$$

We study classical solutions for (1.1) near a global Maxwellian $\mu = e^{-|v|^2}$. We define the standard perturbation $f(t, x, v)$ to μ as $F = \mu + \mu^{1/2} f$. It is well known that

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$Q[\mu,\mu]=0$. By expanding $Q[\mu+\mu^{1/2}g_1,\mu+\mu^{1/2}g_2]$, we define

$$Q[\mu+\mu^{1/2}g_1,\mu+\mu^{1/2}g_2]\equiv Q[\mu,\mu]+\mu^{1/2}\{Kg_1+Ag_2+\Gamma[g_1,g_2]\}.$$

The system (1.1) for $f(t,x,v)$ becomes

$$[\partial_t+v\cdot\nabla_x]f+Lf=\Gamma[f,f],\qquad f(0,x,v)=f_0(x,v), \tag{1.2}$$

where $L=-A-K$. Notice that A, K and Γ are defined in the same way as in [5], namely, $\sigma^i=\phi^{ij}*[v_j\mu]$, $\sigma^{ij}=\phi^{ij}*\mu$,

$$Ag=\mu^{-1/2}\partial_i\{\mu^{1/2}\sigma^{ij}[\partial_j g+v_jg]\},\qquad Kg=-\mu^{-1/2}\partial_i\{\mu[\phi^{ij}*\{\mu^{1/2}[\partial_j g+v_jg]\}]\},$$
$$\Gamma[g_1,g_2]=\partial_i[\{\phi^{ij}*[\mu^{1/2}g_1]\}\partial_j g_2]-\{\phi^{ij}*[v_i\mu^{1/2}g_1]\}\partial_j g_2$$
$$-\partial_i[\{\phi^{ij}*[\mu^{1/2}\partial_j g_1]\}g_2]+\{\phi^{ij}*[v_i\mu^{1/2}\partial_j g_1]\}g_2.$$

In the case when initially $F_0(x,v)$ has the same mass, momentum, and energy as Maxwellian μ, we can rewrite the conservation laws as $(i=1,2,3)$,

$$\int_{T^3\times R^3} f\mu^{1/2}=\int_{T^3\times R^3} v_if\mu^{1/2}=\int_{T^3\times R^3}|v|^2f\mu^{1/2}=0. \tag{1.3}$$

We shall use $(\cdot,\cdot)$ to denote the standard L^2 inner product in T^3, $T^3\times R^3$ and $\|\cdot\|$ to denote the corresponding L^2 norms. Let the multi-indices α and β be $\alpha=[\alpha_0,\alpha_1,\alpha_2,\alpha_3]$ and $\beta=[\beta_1,\beta_2,\beta_3]$ with $|\alpha|=\sum_{k=0}^3\alpha_k$ and $|\beta|=\sum_{k=1}^3\beta_k$. We define $\partial^\alpha_\beta\equiv\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\partial_{v_1}^{\beta_1}\partial_{v_2}^{\beta_2}\partial_{v_3}^{\beta_3}$. If each component of β is not greater than that of $\overline{\beta}$'s, we denote it by $\beta\le\overline{\beta}$. We define $\beta<\overline{\beta}$ if $\beta\le\overline{\beta}$, and $|\beta|<|\overline{\beta}|$. We also denote $\binom{\beta}{\overline{\beta}}$ by $C^{\beta}_{\overline{\beta}}$.

We introduce a weight function of v as $\omega=\omega(v)=[1+|v|]^{\gamma+2}$. We denote the weighted L^2 norm as $|g|^2_{2,\theta}=\int_{R^3}\omega^{2\theta}g^2dv$, $\|g\|^2_\theta=\int_{R^3\times T^3}\omega^{2\theta}g^2dxdv$, where $\|\cdot\|_0=\|\cdot\|$.

We define the weighted norm and the high order energy norm as

$$|g|^2_{\sigma,\theta}=\int_{R^3}\omega^{2\theta}[\sigma^{ij}\partial_ig\partial_jg+\sigma^{ij}v_iv_jg^2]dv,$$
$$\|g\|^2_{\sigma,\theta}=\int_{R^3\times T^3}\omega^{2\theta}[\sigma^{ij}\partial_ig\partial_jg+\sigma^{ij}v_iv_jg^2]dxdv,$$
$$E(f(t))\equiv\frac12|||f|||^2(t)+\int_0^t|||f|||^2_\sigma(s)ds,$$
$$E(f_0)=E(f(0))\equiv\sum_{|\alpha|+|\beta|\le N}\|\partial^\alpha_\beta f_0\|^2,$$

where $|\cdot|_{\sigma,0}=|\cdot|_\sigma$, $\|\cdot\|_{\sigma,0}=\|\cdot\|_\sigma$ and

$$|||f|||(t)\equiv\sum_{|\alpha|+|\beta|\le N}\|\partial^\alpha_\beta f(t)\|,\qquad |||f|||_\sigma(t)\equiv\sum_{|\alpha|+|\beta|\le N}\|\partial^\alpha_\beta f(t)\|_\sigma.$$

Throughout this article, $N\ge 8$. The main result is as follows:

THEOREM 1.1. Let $\gamma\ge-2$. Assume that $f_0(x,v)$ satisfies (1.3), and $F_0(x,v)=\mu+\mu^{1/2}f_0(x,v)\ge 0$. There is an $C_0>0$ and $M>0$ such that if $E(f_0)\le M$, then there exists a unique global solution $f(t,x,v)$ to (1.2) with $F(t,x,v)=\mu+\mu^{1/2}f(t,x,v)\ge 0$

and $\sup_{0\leq s\leq\infty} E(f(s)) \leq C_0 E(f_0)$. Moreover, there are constants $C_1 > 0$ and $\delta^* > 0$ such that

$$|||f|||(t) \leq C_1 E^{1/2}(f_0) e^{-\delta^* t}.$$

There have been some investigations about the dynamical problems of the Landau system [1], [2], [3], [5], [9], [10], [11], [12], [13], [14]. It is shown in [5] that the global classical solution of the Landau equation near Maxwellians with $\gamma \geq -3$ can be obtained by the energy method. Guo proves in [8] that under the hard sphere condition, the global in time solution of the Vlasov-Maxwell-Boltzmann system can be obtained. Desvillettes and Villani [2], [3] construct a global classical solution to the spatially homogeneous Landau equation with $0 < \gamma \leq 1$, which converges towards equilibrium exponentially. It is shown in [10, 14] that the smooth solution of the spatially homogeneous Landau equation with $-3 < \gamma < 0$ converges to a global Maxwellian super-algebraically.

Motivated by an idea in [8], we establish a global in time classical solution to the generalized Landau equation near Maxwellians for $\gamma \geq -2$. And we also obtain the exponential decay of such a solution, that is, the global classical solution F of (1.1) converges to a global Maxwellian μ in some Sobolev space exponentially. Compared to the previous work ([2], [3], [10], [14]]), it should be pointed out that the Landau equation discussed in the present paper is dependent on space variables, the global solutions here converge towards the equilibrium exponentially and the Landau collision operator concerned includes both hard potentials and soft potentials, i.e., $\gamma \geq -2$. To obtain the exponential decay of the global solution, we are not able to use the argument developed in [5, 6, 7] to construct the crucial positivity of the linearized Landau operator L. Instead, we have revised the methods in [8] to obtain it. It seems that global in time classical solutions to the Landau equation near Maxwellians for $-3 \leq \gamma < -2$ can also be established by this approach. But we could not obtain the exponential decay of the global classical solutions to the Landau equation near Maxwellians for $-3 \leq \gamma < -2$. Very recently, there has been new progress made by Strain and Guo [9] who obtain almost exponential decay of the global solutions for a large class of kinetic equations near Maxwellians, including the Landau equation for $-3 \leq \gamma < -2$.

2. The proof of Theorem 1.1. In this section we first give some lemmas which can be found in [5].

LEMMA 2.1. Let $|\beta| > 0$, $|\alpha| + |\beta| \leq N$. Then for small $\eta > 0$, there exists $C > 0$ and $C_\eta > 0$ such that

$$-(\partial_\beta[Ag], \partial_\beta g) \geq \|\partial_\beta g\|_\sigma^2 - \eta \sum_{|\beta_1|\leq|\beta|} \|\partial_{\beta_1} g\|_\sigma^2 - C_\eta \|\mu g\|^2, \tag{2.1}$$

$$|(\partial_\beta[Kg_1], \partial_\beta g_2)| \leq \{\eta \sum_{|\beta_1|\leq|\beta|} \|\partial_{\beta_1} g_1\|_\sigma + C_\eta \|\mu g_1\|\} \|\partial_\beta g_2\|_\sigma, \tag{2.2}$$

$$\begin{aligned}(\partial_\beta^\alpha \Gamma[g_1, g_2], \partial_\beta^\alpha g_3) \leq C[\{&\sum_{|\alpha_1|+|\beta_1|\leq N} \|\partial_{\beta_1}^{\alpha_1} g_1\|\}\{\sum_{|\alpha_1|+|\beta_1|\leq N} \|\partial_{\beta_1}^{\alpha_1} g_2\|_\sigma\}\\ &+\{\sum_{|\alpha_1|+|\beta_1|\leq N} \|\partial_{\beta_1}^{\alpha_1} g_1\|_\sigma\}\{\sum_{|\alpha_1|+|\beta_1|\leq N} \|\partial_{\beta_1}^{\alpha_1} g_2\|\}] \|\partial_\beta^\alpha g_3\|_\sigma.\end{aligned} \tag{2.3}$$

Since $L \geq 0$ and $(Lg, g) = 0$ if and only if $g(v) = \{a + b \cdot v + c|v|^2\}\mu^{1/2}$ where $a, c \in R$ and $b \in R^3$, we denote the orthogonal basis for $\{1, v, |v|^2\}\mu^{1/2}$ in the same way as in [4] $\{e_1, e_2, e_3, e_4, e_5\}$ and we define a projection P in $L^2(R^3)$ for any fixed x as $Pg(x, v) = \sum(g(x, .), e_j)e_j$.

Lemma 2.2. Let $\chi(v)$ be a smooth function so that $\{|\chi| + |\nabla\chi| + |\nabla^2\chi|\} \leq C\mu(v/4)$, then

$$\|\int \partial^\alpha\Gamma[g_1, g_2]\chi dv\| \leq C\{\sum_{|\alpha_1|\leq N} \|\partial^{\alpha_1} g_1\|\}\{\sum_{|\alpha_1|\leq N} \|\partial^{\alpha_1} g_2\|_\sigma\}, \tag{2.4}$$

$$(Lg, g) \geq \delta|\{I - P\}g|_\sigma^2, \tag{2.5}$$

$$|g|_{\sigma,\theta}^2 \geq c|\omega^\theta[1 + |v|]^{\frac{\gamma+2}{2}} g|_2^2. \tag{2.6}$$

By a straightforward modification of the argument used in [5], we have the following local existence result for the Landau equation.

Lemma 2.3. For any sufficiently small $M_0 > 0$, there exists $T^* > 0$ such that if $E(f_0) \leq M_0/2$, then there is a unique classical solution $f(t, x, v)$ to (1.2) in $[0, T^*) \times T^3 \times R^3$ such that $\sup_{0\leq t\leq T^*} E(f(t)) \leq M_0$ and $E(f(t))$ is continuous over $[0, T^*)$. If $F_0(x, v) = \mu + \mu^{1/2} f_0 \geq 0$, then $F(t, x, v) = \mu + \mu^{1/2} f(t, x, v) \geq 0$. Furthermore, the conservation law (1.3) holds for all $0 < t < T^*$ if they are valid initially at $t = 0$.

We shall first establish the positivity of the linearized operator L for every small amplitude solution $f(t, x, v)$ to (1.2), and then prove the main result, Theorem 1.1. We know that P is a projection from $L^2(R^3)$ to the null space of the linearized operator L. Thus, for any fixed (t, x), a function $g(t, x, v)$ can be decomposed uniquely as

$$g(t, x, v) = \{Pg\}(t, x, v) + \{I - P\}g(t, x, v).$$

Split f as $f(t, x, v) = \{Pf\}(t, x, v) + \{I - P\}f(t, x, v)$ in the Landau equation (1.2). Thus, we have

$$[\partial_t + v \cdot \nabla_x]Pf = l(\{I - P\}f) + h(f), \tag{2.7}$$

where

$$l(\{I - P\}f) \equiv -[\partial_t + v \cdot \nabla_x + L]\{I - P\}f, \qquad h(f) \equiv \Gamma[f, f]. \tag{2.8}$$

Lemma 2.4. Let $\partial^\alpha = \partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$. It can be shown that $\partial^\alpha Pf = P\partial^\alpha f$. Furthermore, there exists $C > 1$ such that for any $f \in C_c^\infty(R \times T^3 \times R^3)$,

$$\frac{1}{C}\|\partial^\alpha Pf\|_\sigma^2 \leq \|\partial^\alpha a\|^2 + \|\partial^\alpha b\|^2 + \|\partial^\alpha c\|^2 \leq C\|\partial^\alpha Pf\|^2. \tag{2.9}$$

Proof. A direct computation implies $\partial^\alpha Pf = P\partial^\alpha f$. We substitute $\|\cdot\|_\sigma$ with $P\partial^\alpha f = \partial^\alpha a(t, x)\mu^{1/2} + \partial^\alpha b(t, x) \cdot v\mu^{1/2} + \partial^\alpha c(t, x)|v|^2\mu^{1/2}$. Using $\sigma^{ij} \leq C[1 + |v|]^{\gamma+2}$ and the exponential decay of e_j, we can obtain the first half of (2.9) by a direct computation. The second half of (2.9) can be obtained by the fact that $|\partial^\alpha a|^2 + |\partial^\alpha b|^2 + |\partial^\alpha c|^2$ is bounded by $C\int |\partial^\alpha Pf|^2 dv$ for any (t, x), since a, b and c are the coefficients of a basis to the null space of L. We then deduce (2.9) by a further integration over x.

We now derive the macroscopic equations for Pf's coefficients, a, b and c. Recalling equation (2.7) and (2.8), we further use $Pf = a(t,x)\mu^{1/2} + \sum_{j=1}^{3} \partial^\alpha b_j(t,x)v_j\mu^{1/2} + \partial^\alpha c(t,x)|v|^2\mu^{1/2}$ to expand the entries of the left-hand side of (2.7) as

$$\sum_i \Big[v_i\partial^i c|v|^2 + [\partial^0 c + \partial^i b_i]v_i^2 + \sum_{j>i}[\partial^i b_j + \partial^j b_i]v_iv_j + [\partial^0 b_i + \partial^i a]v_i + \partial^0 a\Big]\mu^{1/2}, \quad (2.10)$$

where $\partial^0 = \partial_t$, $\partial^j = \partial_{x_j}$ and $\partial^i = \partial_{x_i}$. This is an expansion to the left-hand side of (2.7), for fixed (t,x), with respect to the basis of $\mu^{1/2}$, $v_i\mu^{1/2}$, $v_i^2\mu^{1/2}$, $v_iv_j\mu^{1/2}$ and $|v|^2v_i\mu^{1/2}$ where $1 \le i \ne j \le 3$. We denote an orthogonal basis for this 13-dimensional space by ϵ_j, $1 \le j \le 13$ as in [4]. Expand the right-hand side of (2.7) with respect to the same basis, and compare with their coefficients on both sides. Then we have

$$\begin{aligned}
&(1) \quad \nabla_x c = l_c + h_c,\\
&(2) \quad \partial^0 c + \partial^i b_i = l_i + h_i,\\
&(3) \quad \partial^0 a = l_a + h_a,\\
&(4) \quad \partial^i b_j + \partial^j b_i = l_{ij} + h_{ij}, \quad i \ne j,\\
&(5) \quad \partial^0 b_i + \partial^i a = l_{bi} + h_{bi},
\end{aligned}$$

where $\partial^0 = \partial_t$ and $\partial^j = \partial_{x_j}$. Here $l_c(t,x)$, $l_i(t,x)$, $l_{ij}(t,x)$, $l_{bi}(t,x)$ and $l_a(t,x)$ are the corresponding coefficients of such an expansion of the linear term $l(\{I-P\}f)$, and $h_c(t,x)$, $h_i(t,x)$, $h_{ij}(t,x)$, $h_{bi}(t,x)$ and $h_a(t,x)$ are the corresponding coefficients of the same expansion of the higher order term $h(f)$. Let

$$[\mu^{1/2}, v_i\mu^{1/2}, v_i^2\mu^{1/2}, v_iv_j\mu^{1/2}, |v|^2v_i\mu^{1/2}]A_{13\times 13} = [\epsilon_j^*],$$

with $detA \ne 0$. We know that for any fixed (t,x), $l_c(t,x)$, $l_i(t,x)$, $l_{ij}(t,x)$, $l_{bi}(t,x)$ and $l_a(t,x)$, which are the coefficients of the projection of $\{I-P\}f$, we have the form

$$\sum_{i,n=1}^{13} \lambda^{ij}\lambda^{in}\int_{R^3} l(\{I-P\}f)\cdot\epsilon_j(v)dv.$$

The same is true after we take ∂^α. Let $|\alpha| \le N-1$. By (2.8), we have that

$$\begin{aligned}
&\|\int(-[\partial_t + v\cdot\nabla_x]\{I-P\}\partial^\alpha f)\cdot\epsilon_n(v)dv\|^2\\
&\quad\le \int|\epsilon_n(v)|dv \times \int_{R^3\times T^3}|\epsilon_n(v)|(|\{I-P\}\partial^0\partial^\alpha f|^2 + |v|^2|\{I-P\}\nabla_x\partial^\alpha f|^2)dxdv\\
&\quad\le C[\|\{I-P\}\partial^0\partial^\alpha f\| + \|\{I-P\}\nabla_x\partial^\alpha f\|]^2,\\
&\int L\{I-P\}\partial^\alpha f\epsilon_n(v)dv = \int(-A-K)\{I-P\}\partial^\alpha f\epsilon_n(v)dv.
\end{aligned}$$

Recalling the expressions of A, K and ϵ_n, integration by parts and the Schwartz inequality will result in

$$\|\int L\{I-P\}\partial^\alpha f\epsilon_n(v)dv\|^2 \le C\|\{I-P\}\partial^\alpha f\|^2.$$

Here $\partial^0 = \partial_t$, and we have also used the facts $\partial^\alpha\{I-P\}f = \{I-P\}\partial^\alpha f$ and the exponential decay of $\epsilon_n(v)$. Thus, we have

$$\sum_{|\alpha|\le N-1}\{\|\partial^\alpha l_c\| + \|\partial^\alpha l_i\| + \|\partial^\alpha l_{ij}\| + \|\partial^\alpha l_{bi}\| + \|\partial^\alpha l_a\|\} \le C\sum_{|\alpha|\le N}\|\{I-P\}\partial^\alpha f\|. \quad (2.11)$$

Notice that $h(f) = \Gamma[f, f]$ and $\partial^\alpha h_c$, $\partial^\alpha h_i$, $\partial^\alpha h_{ij}$, $\partial^\alpha h_{bi}$, $\partial^\alpha h_a$ are of the form

$$\sum_{i,n=1}^{13} \lambda^{ij}\lambda^{in} \int_{R^3} \partial^\alpha \Gamma[f, f] \cdot \epsilon_j(v) dv,$$

where λ^{ij} and λ^{in} are the entries of the matrix A. If $|\alpha| \le N$ and $\sum_{|\alpha|\le N} \|\partial^\alpha f(t)\|^2 \le M_0$ for some $M_0 > 0$, then we can apply (2.4) in Lemma 2.2 to get

$$\begin{aligned}\|\int \partial^\alpha \Gamma[f, f] \cdot \epsilon_n(v) dv\| &\le \sum C_\alpha^{\alpha_1} \|\int \Gamma[\partial^{\alpha_1} f, \partial^{\alpha-\alpha_1} f] \cdot \epsilon_n(v) dv\| \\ &\le C\{\sum_{|\alpha_1|\le N} \|\partial^{\alpha_1} f\|\}\{\sum_{|\alpha_1|\le N} \|\partial^{\alpha_1} f\|_\sigma\} \le CM_0^{1/2} \sum_{|\alpha|\le N} \|\partial^\alpha f\|_\sigma.\end{aligned}$$

Thus, we have

$$\sum_{|\alpha|\le N} \{\|\partial^\alpha h_c\| + \|\partial^\alpha h_i\| + \|\partial^\alpha h_{ij}\| + \|\partial^\alpha h_{bi}\| + \|\partial^\alpha h_a\|\} \le CM_0^{1/2} \sum_{|\alpha|\le N} \|\partial^\alpha f\|_\sigma. \quad (2.12)$$

THEOREM 2.5. Let $f(t, x, v)$ be a classical solution to (1.2) satisfying (1.3). There exists $M_0 > 0$ and $\delta_0 = \delta_0(M_0) > 0$ such that if

$$\sum_{|\alpha|\le N} \|\partial^\alpha f(t)\|^2 \le M_0, \quad (2.13)$$

then

$$\sum_{|\alpha|\le N} (L\partial^\alpha f(s), \partial^\alpha f(s)) \ge \delta_0 \sum_{|\alpha|\le N} \|\partial^\alpha f(s)\|_\sigma^2.$$

Proof. Recall that from (2.5) in Lemma 2.2

$$(L\partial^\alpha f, \partial^\alpha f) \ge \delta\|\{I - P\}\partial^\alpha f\|_\sigma^2.$$

It thus suffices to show that if (2.13) is valid for some small $M_0 > 0$, then there is a constant $C > 0$ such that

$$\sum_{|\alpha|\le N} \|P\partial^\alpha f(t)\|_\sigma \le C \sum_{|\alpha|\le N} \|\{I - P\}\partial^\alpha f(t)\|_\sigma.$$

We notice that by Lemma 2.4,

$$\sum_{|\alpha|\le N} \|P\partial^\alpha f(t)\|_\sigma \le C \sum_{|\alpha|\le N} \{\|\partial^\alpha a\| + \|\partial^\alpha b\| + \|\partial^\alpha c\|\}.$$

Thus, we only need to prove that

$$\sum_{|\alpha|\le N} \{\|\partial^\alpha a\| + \|\partial^\alpha b\| + \|\partial^\alpha c\|\} \le C \sum_{|\alpha|\le N} \|\{I - P\}\partial^\alpha f(t)\|_\sigma. \quad (2.14)$$

We first estimate $\nabla\partial^\alpha b$. Let $|\alpha|\le N-1$. We take ∂^j of (2) and (4) to get

$$\begin{aligned}\triangle\partial^\alpha b_i &= \sum_j \partial^{jj}\partial^\alpha b_i = \sum_{j\neq i}\partial^{jj}\partial^\alpha b_i + \partial^{ii}\partial^\alpha b_i = \sum_{j\neq i}[-\partial^{ji}\partial^\alpha b_j + \partial^j\partial^\alpha l_{ij} + \partial^j\partial^\alpha h_{ij}]\\ &+[\partial^i\partial^\alpha l_i + \partial^i\partial^\alpha h_i - \partial^0\partial^i\partial^\alpha c] = \sum_{j\neq i}[\partial^0\partial^i\partial^\alpha c - \partial^i\partial^\alpha l_j - \partial^i\partial^\alpha h_j] - \partial^0\partial^i\partial^\alpha c\\ &+\sum_{j\neq i}[\partial^j l_{ij} + \partial^j h_{ij}] + \partial^i\partial^\alpha l_i + \partial^i\partial^\alpha h_i = \partial^0\partial^i\partial^\alpha c\\ &-\sum_{i\neq j}[\partial^i\partial^\alpha l_j + \partial^i\partial^\alpha h_j - \partial^j\partial^\alpha l_{ij} - \partial^j\partial^\alpha h_{ij}] + \partial^i\partial^\alpha l_i + \partial^i\partial^\alpha h_i\\ &= -\partial^{ii}\partial^\alpha b_i + 2[\partial^i\partial^\alpha l_i + \partial^i\partial^\alpha h_i] - \sum_{i\neq j}[\partial^i\partial^\alpha l_j + \partial^i\partial^\alpha h_j - \partial^j\partial^\alpha l_{ij} - \partial^j\partial^\alpha h_{ij}].\end{aligned}$$

We multiply with $\partial^\alpha b_i$ to get

$$\begin{aligned}\|\nabla\partial^\alpha b_i\| &\le C\sum_{|\alpha|\le N-1}\{\|\partial^\alpha l_i\| + \|\partial^\alpha h_i\| + \|\partial^\alpha l_j\| + \|\partial^\alpha h_j\| + \|\partial^\alpha l_{ij}\| + \|\partial^\alpha h_{ij}\|\}\\ &\le CM_0^{1/2}\sum_{|\alpha|\le N}\|\partial^\alpha f\|_\sigma + C\sum_{|\alpha|\le N}\|\{I-P\}\partial^\alpha f\|, \end{aligned} \tag{2.15}$$

where we have used (2.11) and (2.12). We will leave the proof of the purely temporal derivatives of $\partial^\alpha b_i(t,x)$ with $\alpha=[\alpha_0,0,0,0]$ and $|\alpha|\le N$ to the end.

Next we will estimate the derivatives of $c(t,x)$. From (1) and (2), we have

$$\|\partial^0\partial^\alpha c\| \le \|\partial^i\partial^\alpha b_i\| + \|\partial^\alpha l_i\| + \|\partial^\alpha h_i\|, \qquad \|\nabla\partial^\alpha c\| \le \|\partial^\alpha h_c\| + \|\partial^\alpha l_c\|.$$

Thus, for $|\alpha|\le N-1$, we have, from (2.11), (2.12) and (2.15), that

$$\|\partial^0\partial^\alpha c\| + \|\nabla\partial^\alpha c\| \le CM_0^{1/2}\sum_{|\alpha|\le N}\|\partial^\alpha f\|_\sigma + C\sum_{|\alpha|\le N}\|\{I-P\}\partial^\alpha f\|. \tag{2.16}$$

From the Poincaré inequality, we easily see that $\|c\|\le C[\|\nabla c\| + |\int c dx|]$. From the conservation laws in (1.3), we know that $\int_{T^3} b dx = 0$ and

$$|\int_{T^3} a dx| + |\int_{T^3} c dx| = 0.$$

Thus, the term $\|c\|$ is controlled by the right-hand side of (2.16).

Now we consider $a(t,x)$. Let $|\alpha|\le N-1$. By (3), we have

$$\|\partial^0\partial^\alpha a\| \le \|\partial^\alpha l_a\| + \|\partial^\alpha h_a\|. \tag{2.17}$$

By (2.11) and (2.12), for $|\alpha|\le N-1$, $\|\partial^0\partial^\alpha a\|$ is bounded by the right-hand side of (2.16). We now consider the spatial derivatives of $a(t,x)$. Let $|\alpha|\le N-1$ and $\alpha=[0,\alpha_1,\alpha_2,\alpha_3]\neq 0$. By taking ∂^i of (5) and summing over i, we get

$$-\triangle\partial^\alpha a = \nabla\cdot\partial^0\partial^\alpha b - \sum_i \partial^i\partial^\alpha[l_{bi} + h_{bi}].$$

Multiplying the above equation with $\partial^\alpha a$ and integrating over T^3, we get

$$\|\nabla\partial^\alpha a\| \le \|\partial^0\partial^\alpha b\| + \sum_i[\|\partial^\alpha[l_{bi} + h_{bi}]\|]. \tag{2.18}$$

It is clear that the right-hand side of (2.18) is bounded by the right-hand side of (2.16). Furthermore, by the the Poincaré inequality, we easily know $\|a\| \leq C[\|\nabla a\| + |\int adx|]$. Thus, the right-hand side of it is bounded by the right-hand side of (2.16). We thus complete the estimate for $a(t,x)$.

Finally, we estimate the purely temporal derivatives of $b(t,x)$ and $\partial^\alpha b(t,x)$ with $\alpha = [\alpha_0, 0, 0, 0]$. If $|\alpha| \leq 2$, we use the Poincaré inequality to get

$$\|\partial^\alpha b_i\| \leq C\|\nabla \partial^\alpha b_i\| + |\partial^\alpha \int b_i(t,x)dx|.$$

By (2.15) and $\int b_i(t,x)dx = 0$, it suffices to estimate the term $\partial^\alpha b_i$. For the higher purely temporal derivative $\partial^\alpha b(t,x)$ with $|\alpha| \geq 3$, we take $\partial^{\alpha-1}$ of (5) to get

$$\|\partial^\alpha b_i\| = \| - \partial^i \partial^{\alpha-1} a + \partial^{\alpha-1}[l_{bi} + h_{bi}]\| \leq \|\partial^i \partial^{\alpha-1} a\| + \|\partial^{\alpha-1} l_{bi}\| + \|\partial^{\alpha-1} h_{bi}\|.$$

By (2.18) and (2.11), (2.12), we easily know that the right-hand side of the above inequality is bounded by the right-hand side of (2.16).

Therefore, we have, by the above estimates, that

$$\sum_{|\alpha|\leq N} \{\|\partial^\alpha a\| + \|\partial^\alpha b\| + \|\partial^\alpha c\|\} \leq CM_0^{1/2} \sum_{|\alpha|\leq N} \|\partial^\alpha f\|_\sigma + C \sum_{|\alpha|\leq N} \|\{I-P\}\partial^\alpha f\|.$$

The first term of the right-hand side of the above inequality can be neglected for M_0 small. This is because we have by Lemma 2.4 that

$$\begin{aligned}\|\partial^\alpha f\|_\sigma^2 &\leq \|\{I-P\}\partial^\alpha f\|_\sigma^2 + \|P\partial^\alpha f\|_\sigma^2 \\ &\leq \|\{I-P\}\partial^\alpha f\|_\sigma^2 + C[\|\partial^\alpha a\| + \|\partial^\alpha b\| + \|\partial^\alpha c\|]^2.\end{aligned}$$

By (2.6) with $\gamma \geq -2$, we know that (2.14) holds.

In the following we extend the local in time solution in Lemma 2.3 to the global in time solution. We need to prove the following theorem.

THEOREM 2.6. Let $f(t,x,v)$ be the unique solution constructed in Lemma 2.3 which satisfies the conservation law (1.3). Let the small amplitude (2.13) be valid. Then, for any given $0 \leq m \leq N$ and $|\beta| \leq m$, there are constants $C_{|\beta|} > 0$, $C_m^* > 0$ and $\delta_m > 0$ such that

$$\sum_{|\beta|\leq m, |\alpha|+|\beta|\leq N} \Big[C_{|\beta|}\frac{d}{dt}\|\partial_\beta^\alpha f(t)\|^2 + \delta_m \|\partial_\beta^\alpha f(t)\|_\sigma^2\Big] \leq C_m^* E^{1/2}(f(t))|||f|||_\sigma^2(t). \tag{2.19}$$

Proof. We use an induction over m, the order of the v-derivatives. For $m = 0$, by taking the pure ∂^α of (1.2), we obtain

$$[\partial_t + v\cdot\nabla + L]\partial^\alpha f = \partial^\alpha \Gamma[f,f]. \tag{2.20}$$

Multiplying (2.20) by $\partial^\alpha f$ and integrating over $T^3 \times R^3$, we obtain, by Theorem 2.5 and (2.3) in Lemma 2.1 with $g_1 = g_2 = g_3 = f$, that

$$\sum_{|\alpha|\leq N} \Big[\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f(t)\|^2 + \delta_0 \|\partial^\alpha f(t)\|_\sigma^2\Big] \leq CE^{1/2}(f(t))|||f|||_\sigma^2(t).$$

This concludes the case for $m = 0$ with $C_0 = 1/2$ and $C_0^* = C$.

Now assume the theorem is valid for m. For $|\beta| = m+1$, taking ∂^α_β ($|\beta| \neq 0$) of (1.2), we obtain,

$$[\partial_t + v\cdot\nabla_x]\partial^\alpha_\beta f + \partial_\beta L[\partial^\alpha f] + \sum_{\beta_1\neq 0} \partial_{\beta_1} v\cdot\nabla_x \partial^\alpha_{\beta-\beta_1} f = \partial^\alpha_\beta \Gamma[f,f]. \tag{2.21}$$

For any $\eta > 0$, applying Lemma 2.1 and $\|\mu g\| \leq C\|g\|_\sigma$, and then integrating over T^3, we deduce

$$\begin{aligned}
-\big(\partial_\beta A[\partial^\alpha f], \partial^\alpha_\beta f\big) &\geq \|\partial^\alpha_\beta f\|^2_\sigma - \eta \sum_{|\beta_1|\leq|\beta|} \|\partial^\alpha_{\beta_1} f\|^2_\sigma - C_\eta \|\partial^\alpha f\|^2_\sigma,\\
-\big(\partial_\beta K[\partial^\alpha f], \partial^\alpha_\beta f\big) &\geq -\{\eta \sum_{|\beta_1|\leq|\beta|} \|\partial^\alpha_{\beta_1} f\|_\sigma + C_\eta \|\mu \partial^\alpha f\|\} \|\partial^\alpha_\beta f\|_\sigma\\
&\geq -\eta \sum_{|\beta_1|\leq|\beta|} \|\partial^\alpha_{\beta_1} f\|^2_\sigma - \eta\|\partial^\alpha_\beta f\|^2_\sigma - C_\eta\|\partial^\alpha f\|^2_\sigma.
\end{aligned}$$

Thus, we have, for any $\eta > 0$,

$$(\partial_\beta L[\partial^\alpha f], \partial^\alpha_\beta f) \geq \|\partial^\alpha_\beta f\|^2_\sigma - \eta \sum_{|\beta'|\leq|\beta'|} \|\partial^\alpha_{\beta'} f\|^2_\sigma - C_\eta \|\partial^\alpha f\|^2_\sigma.$$

For any $\eta > 0$, we have

$$\sum_{\beta_1\neq 0} (\partial_{\beta_1} v\cdot\nabla_x \partial^\alpha_{\beta-\beta_1} f, \partial^\alpha_\beta f) \leq \eta\|\partial^\alpha_\beta f\|^2 + C_\eta \sum_{|\beta_1|=1} \|\nabla_x \partial^\alpha_{\beta-\beta_1} f\|^2.$$

By (2.3), we easily see that

$$(\partial^\alpha_\beta \Gamma[f,f], \partial^\alpha_\beta f) \leq C E^{1/2}(f(t)) |||f|||^2_\sigma(t).$$

We thus have, by collecting terms and summing over $|\beta| = m+1$ and $|\alpha| + |\beta| \leq N$,

$$\begin{aligned}
&\sum_{|\beta|=m+1, |\alpha|+|\beta|\leq N} \Big[\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f(t)\|^2 + \|\partial^\alpha_\beta f(t)\|^2_\sigma\Big]\\
&\leq \sum_{|\beta|=m+1, |\alpha|+|\beta|\leq N} \Big[\sum_{|\beta|=m+1} 2\eta \|\partial^\alpha_\beta f\|^2_\sigma + 3C_\eta \sum_{|\beta|\leq m, |\alpha|+|\beta|\leq N} \|\partial^\alpha_\beta f\|^2\\
&\qquad + C E^{1/2}(f(t)) |||f|||^2_\sigma(t)\Big]\\
&\leq Z_{m+1}\Big[\sum_{|\beta|=m+1, |\alpha|+|\beta|\leq N} 2\eta\|\partial^\alpha_\beta f\|^2_\sigma + 3C_\eta \sum_{|\beta|\leq m, |\alpha|+|\beta|\leq N} \|\partial^\alpha_\beta f\|^2\\
&\qquad + C E^{1/2}(f(t)) |||f|||^2_\sigma(t)\Big],
\end{aligned}$$

where Z_{m+1} denotes the number of all possible (α, β) such that $|\beta| \leq m+1$ and $|\alpha|+|\beta| \leq N$. Choose $\eta = \frac{1}{4Z_{m+1}}$, then there is a constant $C(Z_{m+1}) > 0$ such that

$$\begin{aligned}
&\sum_{|\beta|=m+1, |\alpha|+|\beta|\leq N} \Big[\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f(t)\|^2 + \frac{1}{2}\|\partial^\alpha_\beta f(t)\|^2_\sigma\Big]\\
&\leq C(Z_{m+1})\Big[\sum_{|\beta|\leq m, |\alpha|+|\beta|\leq N} \|\partial^\alpha_\beta f\|^2_\sigma + E^{1/2}(f(t)) |||f|||^2_\sigma(t)\Big].
\end{aligned} \tag{2.22}$$

We may assume $C(Z_{m+1}) > 1$. We multiply (2.22) by $\frac{\delta_m}{2C(Z_{m+1})}$ and add it to (2.19) for $|\beta| \le m$ to get

$$\sum_{|\beta|=m+1,|\alpha|+|\beta|\le N} \Big[\frac{\delta_m}{4C(Z_{m+1})}\frac{d}{dt}\|\partial^\alpha_\beta f(t)\|^2 + \frac{\delta_m}{4C(Z_{m+1})}\|\partial^\alpha_\beta f(t)\|^2_\sigma\Big]$$
$$+ \sum_{|\beta|\le m,|\alpha|+|\beta|\le N} \Big[C_{|\beta|}\frac{d}{dt}\|\partial^\alpha_\beta f(t)\|^2 + \delta_m\|\partial^\alpha_\beta f(t)\|^2_\sigma\Big]$$
$$\le \frac{\delta_m}{2}\sum_{|\beta|\le m,|\alpha|+|\beta|\le N}\|\partial^\alpha_\beta f(t)\|^2_\sigma + [C^*_m + \frac{\delta_m}{2}]E^{1/2}(f(t))|||f|||^2_\sigma(t).$$

It is clear the first term on the right-hand side is absorbed by the last term on the left. We thus conclude the theorem by letting

$$C_{m+1} = \frac{\delta_m}{4C(Z_{m+1})}, \quad \delta_{m+1} = \frac{\delta_m}{4C(Z_{m+1})} \le \frac{\delta_m}{2}, \quad C^*_{m+1} = C^*_m + \frac{\delta_m}{2}.$$

Proof of Theorem 1.1. We first fix $M_0 \le 1$ such that both Lemma 2.3 and Theorem 2.6 are valid. For such a M_0, we let $m = N$ in (2.19) and define

$$y(t) \equiv \sum_{|\alpha|+|\beta|\le N}[C_{|\beta|}\|\partial^\alpha_\beta f(t)\|^2]. \tag{2.23}$$

We choose a constant $C_1 > 1$ such that for any $t \ge 0$,

$$\frac{1}{C_1}[y(t) + \frac{\delta_N}{2}\int_0^t |||f|||^2_\sigma(s)]ds \le E(f(t)) \le C_1[y(t) + \frac{\delta_N}{2}\int_0^t |||f|||^2_\sigma(s)ds].$$

Recall constant C^*_N in (2.19). We define $M \equiv \min\{\frac{\delta_N^2}{8C^{*2}_N C_1^2}, \frac{M_0}{2C_1^2}\}$, and choose initial data so that $E(f_0) \le M < M_0$. From Lemma 2.3, we may denote $T > 0$ so that

$$T = \sup_t\{t : E(f(t)) \le 2C_1^2 M\} > 0.$$

Notice that $E(f(t)) \le 2C_1^2 M \le M_0$ for $0 \le t \le T$, by the definition of M. Thus, the small amplitude assumption (2.13) is valid. We apply Theorem 2.6 and the definition of M and T to get, for $0 \le t \le T$, that

$$y'(t) + \delta_N|||f|||^2_\sigma(t) \le C^*_N E^{1/2}(f(t))|||f|||^2_\sigma(t) \le \frac{\delta_N}{2}|||f|||^2_\sigma(t). \tag{2.24}$$

Therefore, an integration over $0 \le t \le s < T$ yields

$$E(f(s)) \le C_1[y(s) + \frac{\delta_N}{2}\int_0^s |||f|||^2_\sigma(\tau)d\tau]$$
$$\le C_1 y(0) \le C_1^2 E(f(0)) \le C_1^2 M < 2C_1^2 M.$$

Since $E(f(s))$ is continuous in s, this implies $E(f(T)) \le C_1^2 M$ if $T < \infty$. Thus, we get a contradiction to the definition of T. Hence $T = \infty$. It is easily seen from the above inequality that such a global solution satisfies $E(f(t)) \le C_1^2 E(f_0)$ for all $t \ge 0$. By (2.24), we have for $t \ge 0$,

$$y'(t) + \frac{\delta_N}{2}|||f|||^2_\sigma(t) \le 0.$$

We easily know $C_3|||f|||(t) \leq y(t) \leq C_2|||f|||_\sigma(t)$ by (2.24) and (2.6) with $\gamma \geq -2$. Thus, there is $C_4 > 0$ and $\delta^* > 0$ such that

$$|||f|||(t) \leq C_4 E^{1/2}(f_0)e^{-\delta^* t}.$$

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