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*Research Article*

# The Exponential Diophantine Equation $2^x + b^y = c^z$

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Let $b$ and $c$ be fixed coprime odd positive integers with $\min\{b, c\} > 1$. In this paper, a classification of all positive integer solutions $(x, y, z)$ of the equation $2^x + b^y = c^z$ is given. Further, by an elementary approach, we prove that if $c = b + 2$, then the equation has only the positive integer solution $(x, y, z) = (1, 1, 1)$, except for $(b, x, y, z) = (89, 13, 1, 2)$ and $(2^r - 1, r + 2, 2, 2)$, where $r$ is a positive integer with $r \geq 2$.

## 1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let $a, b, c$ be fixed coprime positive integers with $\min\{x, y, z\} > 1$. In recent years, the solutions $(x, y, z)$ of the equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N} \tag{1}$$

have been investigated in many papers (see [1–3] and its references). In this paper we deal with (1) for the case that $a = 2$. Then (1) can be rewritten as

$$2^x + b^y = c^z, \quad x, y, z \in \mathbb{N}, \tag{2}$$

where $b$ and $c$ are fixed coprime odd positive integers with $\min\{b, c\} > 1$. We will give a classification of all solutions $(x, y, z)$ of (2) as follows.

**Theorem 1.** *Every solution $(x, y, z)$ of (2) satisfies one of the following types:*

(i) $(b, c, x, y, z) = (7, 3, 5, 2, 4)$;

(ii) $(b, c, x, y, z) = (2^r - 1, 2^r + 1, r + 2, 2, 2)$, *where $r$ is a positive integer with $r \geq 2$;*

(iii) $(b, c, x, y, z) = (5, 3, 1, 2, 3)$;

(iv) $(b, c, x, y, z) = (11, 5, 2, 2, 3)$;

(v) $2 \mid y$ *and* $z = 1$;

(vi) $(b, c, x, y, z) = (17, 71, 7, 3, 2)$;

(vii) $x = 1$, $y > 1$, $2 \nmid y$ *and* $2 \mid z$;

(viii) $x > 1$, $y = 1$, $2 \mid z$ *and* $2^x < b^{50/13}$;

(ix) $2 \nmid yz$.

*Recently, Miyazaki and Togbé [4] showed that if $b \geq 5$ and $c = b + 2$, then (2) has only the solution $(x, y, z) = (1, 1, 1)$, except for $(b, x, y, z) = (89, 13, 1, 2)$. However, there are some exceptional cases missing from the result of [4]. In this paper, by an elementary approach, we prove the following result.*

**Corollary 2.** *If $c = b + 2$, then (2) has only the solution $(x, y, z) = (1, 1, 1)$, except for $(b, x, y, z) = (89, 13, 1, 2)$ and $(2^r - 1, r + 2, 2, 2)$, where $r$ is a positive integer with $r \geq 2$.*

## 2. Preliminaries

**Lemma 3** (see [5, Formula 1.76]). *For any positive integer $n$ and any complex numbers $\alpha$ and $\beta$, one has*

$$\alpha^n + \beta^n = \sum_{i=0}^{[n/2]} \begin{bmatrix} n \\ i \end{bmatrix} (\alpha + \beta)^{n-2i} (-\alpha\beta)^i, \tag{3}$$

*where $[n/2]$ is the integer part of $n/2$;*

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(n-i-1)!n}{(n-2i)!i!}, \quad i = 0, \ldots, \left[\frac{n}{2}\right] \tag{4}$$

*are positive integers.*

**Lemma 4** (see [6]). *Let $A$ and $B$ be coprime odd positive integers with* $\min\{A, B\} > 1$. *If the equation*

$$Au^2 - Bv^2 = 2, \quad u, v \in \mathbb{N} \tag{5}$$

*has solutions $(u, v)$, then it has a unique solution $(u_1, v_1)$ such that $u_1\sqrt{A} + v_1\sqrt{B} \le u\sqrt{A} + v\sqrt{B}$, where $(u, v)$ through all solutions of* (5). *The solution $(u_1, v_1)$ is called the least solution of* (5). *Every solution $(u, v)$ of* (5) *can be expressed as*

$$\frac{u\sqrt{A} + v\sqrt{B}}{\sqrt{2}} = \left(\frac{u_1\sqrt{A} + v_1\sqrt{B}}{\sqrt{2}}\right)^n, \quad n \in \mathbb{N},\ 2 \nmid n. \tag{6}$$

*Further, by* (6), *we have $u_1 \mid u$ and $v_1 \mid v$.*

**Lemma 5.** *Equation* (5) *has no solutions $(u, v)$ such that $u > u_1$, $v > v_1$, and every prime divisor of $u/u_1$ and $v/v_1$ divides $A$ and $B$, respectively.*

*Proof.* We now assume that $(u, v)$ is a solution of (5) satisfying the hypothesis. Since $u > u_1$, by Lemma 4, the $(u_n, v_n)$ is all solutions of (5). Let

$$\alpha = \frac{u_1\sqrt{A} + v_1\sqrt{B}}{\sqrt{2}}, \qquad \beta = \frac{u_1\sqrt{A} - v_1\sqrt{B}}{\sqrt{2}}. \tag{7}$$

We get

$$\frac{u_n}{u_1} = \frac{\alpha^n - (-\beta)^n}{\alpha - (-\beta)}, \qquad \frac{v_n}{v_1} = \frac{\alpha^n - (\beta)^n}{\alpha - (\beta)}, \tag{8}$$

where $n$ is odd. Numbers $\alpha$ and $\beta$ are such that $(\alpha, -\beta)$ satisfy $x^2 - \sqrt{2Bv_1^2}x + 1 = 0$ and $(\alpha, \beta)$ satisfy $x^2 - \sqrt{2Au_1^2}x + 1 = 0$. Thus, $\{u_n/u_1\}_{n\ge1}$ and $\{v_n/v_1\}_{n\ge1}$ are the odd indexed subsequences of the two Lehmer sequences of roots $(\alpha, -\beta)$ and $(\alpha, \beta)$. Their discriminants are $(\alpha + \beta)^2 = 2Au_1^2$ and $(\alpha - \beta)^2 = 2Bv_1^2$, respectively. Saying that all prime factors of $u_n/u_1$ divide $A$ implies that all primes of the $n$th term of a Lehmer sequence divide its discriminant. The same is true for $v_n/v_1$. Hence, $u_n/u_1$ and $v_n/v_1$ are terms of a Lehmer sequence of real roots lacking primitive divisors. By Table 2 in [7], this is possible only for $n = 3, 5$. Even more, in the present case,

$$\frac{\left(\alpha^2\right)^n - \left(\beta^2\right)^n}{\alpha^2 - \beta^2} = \frac{u_n v_n}{u_1 v_1} \tag{9}$$

is the $n$th term of the Lucas sequence of positive real roots $(\alpha^2, \beta^2)$ whose all prime factors divide its discriminant $(\alpha^2 - \beta^2)^2 = 4ABu_1^2v_2^2$, and by Table 1 in [7] this is possible for $n$ odd only if $n = 3$ or $n = 5$. Furthermore, when $n = 5$, we must have $\alpha^2 = (1 + \sqrt{5})/2$, but this is not possible since $\alpha = \sqrt{(1 + \sqrt{5})/2}$ is not of the form $(u_1\sqrt{A} + v_1\sqrt{B})/\sqrt{2}$ for some positive integers $A > 1$, $B > 1$, $u_1$ and $v_1$. So, only $n = 3$ is possible. Now by some simple numerical computation for $u_3$ and $v_3$, we see that it is not possible that all prime factors of $u_3$ and all prime factors of $v_3$ divide $B$. Thus, Lemma 5 is proved. □

**Lemma 6** (see [8]). *The equation*

$$X^2 + 7 = 2^{n+2}, \quad X, n \in \mathbb{N} \tag{10}$$

*has only the solutions $(X, n) = (1, 1), (3, 2), (5, 3), (11, 5)$, and* $(181, 13)$.

**Lemma 7** (see [9]). *Let $D$ be an odd positive integer with $D > 1$. If $(X, n)$ is a solution of the equation*

$$X^2 - D = 2^n, \quad X, n \in \mathbb{N}, \tag{11}$$

*then $2^n < D^{50/13}$.*

**Lemma 8** (see [10, 11]). *The equation*

$$X^2 + 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \ \gcd(X, Y) = 1, \ n \ge 3 \tag{12}$$

*has only the solutions $(X, Y, m, n) = (5, 3, 1, 3), (7, 3, 5, 4)$, and* $(11, 5, 2, 3)$.

**Lemma 9** (see [12]). *The equation*

$$X^2 - 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \ \gcd(X, Y) = 1, \quad Y > 1, \quad m > 1, \quad n \ge 3 \tag{13}$$

*has only the solution $(X, Y, m, n) = (71, 17, 7, 3)$.*

**Lemma 10** (see [13]). *The equation*

$$X^m - Y^n = 1, \quad X, Y, m, n \in \mathbb{N}, \ \min\{X, Y, m, n\} > 1 \tag{14}$$

*has only the solution $(X, Y, m, n) = (3, 2, 2, 3)$.*

## 3. Proof of Theorem

Let $(x, y, z)$ be a solution of (2). If $2 \mid y$ and $2 \mid z$, then we have $x \ge 3$, $c^{z/2} + b^{y/2} = 2^{x-1}$, and $c^{z/2} - b^{y/2} = 2$. It follows that

$$c^{z/2} = 2^{x-2} + 1, \qquad b^{y/2} = 2^{x-2} - 1. \tag{15}$$

Applying Lemma 10 to (15), we can only obtain the solutions of types (i) and (ii).

If $2 \mid y$ and $2 \nmid z$, then we have

$$\left(b^{y/2}\right)^2 + 2^x = c^z, \quad 2 \nmid z. \tag{16}$$

Applying Lemma 8 to (16), we can only get the solutions of types (iii), (iv), and (v).

Similarly, if $2 \nmid y$ and $2 \mid z$, using Lemmas 7 and 9, then we can only obtain the solutions of types (vi), (vii), and (viii). Finally, if $2 \nmid yz$, then the solutions are of type (ix). Thus, the theorem is proved.

## 4. Proof of Corollary

Since $c = b + 2$, (2) can be rewritten as

$$2^x + b^y = (b + 2)^z, \quad x, y, z \in \mathbb{N}. \tag{17}$$

Let $(x, y, z)$ be a solution of (17). By the theorem, (17) has only the solutions

$$(b, x, y, z) = (2^r - 1, r + 2, 2, 2), \quad r \in \mathbb{N}, \ r \geq 2 \tag{18}$$

satisfying $2 \mid y$ and $2 \mid z$.

If $x = 1$, $2 \nmid y$ and $2 \mid z$, then from (17) we get

$$\begin{aligned} 2 + b^y &= 2 + ((b + 1) - 1)^y \\ &= 1 + (b + 1) \sum_{i=1}^{y} (-1)^{i-1} \binom{y}{i} (b + 1)^{i-1} \\ &= 1 + (b + 1) \sum_{j=1}^{z} \binom{z}{j} (b + 1)^{j-1} \\ &= ((b + 1) + 1)^z = (b + 2)^z, \end{aligned} \tag{19}$$

whence we obtain

$$y \equiv z \,(\text{mod}\,(b + 1)). \tag{20}$$

But, since $2 \mid b+1$ and $2 \nmid y-z$, congruence (20) is impossible.

If $x > 1$, $2 \nmid y$ and $2 \mid z$, by the theorem, then we have $y = 1$ and $2^x < b^{50/13}$. Hence, by (17), we get

$$b^2 < (b + 2)^2 \leq (b + 2)^z = 2^x + b < b^{50/13} + b. \tag{21}$$

Since $b \geq 3$ and $2 \mid z$, we see from (21) that $z = 2$. Substituting it into (17), we have $b^2 + 3b - 4(2^{x-2} - 1) = 0$ and

$$b = \frac{1}{2} \left( -3 + \sqrt{2^{x+2} - 7} \right). \tag{22}$$

By (22), we get

$$b = \frac{1}{2} (X - 3), \qquad x = n, \tag{23}$$

where $(X, n)$ is a solution of (10). Since $2 \nmid b$ and $b > 1$, by Lemma 6, we can only have $(X, n) = (181, 13)$ and

$$(b, x, y, z) = (89, 13, 1, 2), \tag{24}$$

by (23).

If $2 \nmid yz$, then $b^{y-1} \equiv (b + 2)^{z-1} \equiv 1 (\text{mod } 8)$. Hence, by (17), we get $2^x \equiv (b + 2)^z - b^y \equiv (b + 2) - b \equiv 2 (\text{mod } 8)$ and $x = 1$. It implies that the equation

$$(b + 2) u^2 - bv^2 = 2, \quad u, v \in \mathbb{N} \tag{25}$$

has the solution

$$(u, v) = \left( (b + 2)^{(z-1)/2}, b^{(y-1)/2} \right). \tag{26}$$

Notice that the least solution of (25) is $(u_1, v_1) = (1, 1)$; $y$ and $z$ satisfy either $y = z = 1$ or $\min\{y, z\} > 1$. Applying Lemma 5 to (26), we only obtain that $(u, v) = (1, 1)$ and

$$(x, y, z) = (1, 1, 1). \tag{27}$$

Thus, (17) has only the solutions (18), (24), and (27). The corollary is proved.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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