

## The Exponential Lindley Odd Log-Logistic-G Family: Properties, Characterizations and Applications

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A new family of distributions called the exponential Lindley odd log-logistic G family is introduced and studied. The new generator generalizes three newly defined G families and also defines two new G families. We provide some mathematical properties of the new family. Characterizations based on truncated moments as well as in terms of the hazard function are presented. The maximum likelihood is used for estimating the model parameters. We assess the performance of the maximum likelihood estimators in terms of biases and mean squared errors by means of a simulation study. Finally, the usefulness of the family is illustrated by means of three real data sets. The new model provides consistently better fits than other competitive models for these data sets.

*Keywords:* Exponential Lindley family; odd log-logistic family; mixture distribution family; maximum likelihood; moments; characterizations; generalized distribution family; mixture family.

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### 1. Introduction

Recently, several families of continuous uni-variate distributions have been constructed by extending common families of continuous models. These generalized distributions give more flexibility by adding one or more parameters to the baseline model. Many classes can be cited such as the Marshall-Olkin-G family by Marshall and Olkin [25], transmuted exponentiated generalized-G family by Yousof et al. [34], Burr X-G by Yousof et al. [35], type I half-logistic family by Cordeiro et al. [12], Zografos-Balakrishnan odd log-logistic family of distributions by Cordeiro et al. [13], a new generalized two-sided family of distributions by Korkmaz and Genç [22], generalized odd log-logistic family by Cordeiro et al. [10], odd-Burr generalized family by Alizadeh et al. [4],

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beta Weibull G by Yousof et al. [36], exponentiated generalized-G Poisson family by Aryal and Yousof [8], type I general exponential class by Hamedani et al. [20] and beta transmuted-H by Afify et al. [2] among others.

Recently, Gleaton and Lynch [17] introduced a class of distributions called the odd log-logistic family with one extra shape parameter  $\alpha > 0$  with cumulative distribution function (cdf)

$$H(x; \xi) = \frac{G(x; \xi)^\alpha}{G(x; \xi)^\alpha + \bar{G}(x; \xi)^\alpha},$$

where  $G(x; \xi)$  is the baseline cdf depending on the vector parameter  $\xi$  and  $\bar{G}(x; \xi) = 1 - G(x; \xi)$  and  $\xi$  the parameter vector of associated parameters from baseline distribution. Alzaatreh et al. [5] defined the  $T - X$  family with cdf

$$F(x) = \int_a^{W[G(x; \xi)]} r(t) dt, \quad (1.1)$$

where  $r(t)$  is the probability density function (pdf) of the random variable of  $T \in [a, b]$  for  $-\infty < a < b < \infty$ ,  $G(x; \xi)$  is cdf of the baseline random variable  $X$  and  $W[G(x)]$  is a function of the baseline cdf which satisfies the following conditions: i)  $W[G(x)] \in [a, b]$ , ii)  $W[G(x)]$  is differentiable and monotonically non-decreasing, iii)  $\lim_{x \rightarrow -\infty} W[G(x)] = a$  and  $\lim_{x \rightarrow \infty} W[G(x)] = b$ . The pdf of the  $T - X$  family is given by

$$f(x) = \left\{ \frac{\partial}{\partial x} W[G(x; \xi)] \right\} r(W[G(x; \xi)]). \quad (1.2)$$

On the other hand, an extension of the Lindley distribution, denoted by exponential Lindley (EL), was introduced by Gomez et al. [18]. The cdf and pdf of this extension are given, respectively, by

$$G_{EL}(x; \beta, \theta) = 1 - \frac{(\theta + \beta + \theta \beta x)}{\theta + \beta} e^{-\theta x} \quad (1.3)$$

and

$$g_{EL}(x; \beta, \theta) = \frac{\theta^2 (1 + \beta x)}{\theta + \beta} e^{-\theta x} \quad (1.4)$$

where  $x > 0$ ,  $\beta > 0$  and  $\theta > 0$ . Clearly, distribution of this extension is exponential distribution for  $\beta = 0$  and is Lindley distribution for  $\beta = 1$ . Also, this distribution is more flexible than exponential and Lindley distributions.

The goal of this paper is to introduce a new flexible and wider family of the distributions based on T-X family using the EL as the generator. The paper is organized as follows. The new family is introduced in Section 2. Some members and sub-families of the new family are introduced in Section 3. In Section 4, the series expansions for cdf and pdf of the new family are presented. In Section 5, some of its mathematical properties are derived. Section 6 deals with some characterizations of the new family. In Section 7, the maximum likelihood method is presented. In Section 8, a simulation study is performed to evaluate the efficiency of the Maximum Likelihood method. In Section 9, we illustrate the importance of the new family by means of three applications to real data sets. The paper is concluded in Section 10.

## 2. The new family

We define the new family by taking  $W[G(x; \xi)] = -\log[1 - H(x; \xi)]$  and  $r(t) = g_{EL}(t; \beta, \theta)$ . From (1.1), (1.2), (1.3) and (1.4), we introduce cdf and pdf of the new family as

$$F(x, \alpha, \beta, \theta, \xi) = 1 - \frac{\bar{G}(x)^{\alpha\theta}}{(\theta + \beta) [G(x)^\alpha + \bar{G}(x)^\alpha]^\theta} \left\{ \theta + \beta - \theta\beta \log \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right] \right\} \quad (2.1)$$

and

$$f(x, \alpha, \beta, \theta, \xi) = \frac{\alpha\theta^2}{\theta + \beta} \frac{g(x)\bar{G}(x)^{\alpha\theta-1}G(x)^{\alpha-1}}{[G(x)^\alpha + \bar{G}(x)^\alpha]^{\theta-1}} \left\{ 1 - \beta \log \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right] \right\} \quad (2.2)$$

respectively, where  $G(x) = G(x; \xi)$  is the baseline cdf,  $g(x) = g(x; \xi)$  is the baseline pdf,  $\alpha > 0$ ,  $\beta > 0$  and  $\theta > 0$  are three extra parameters. We call it exponential Lindley odd log-logistic family and denote it by ELOLL-G( $\alpha, \beta, \theta, \xi$ ). Further, the ELOLL-G family is in fact a mixture of the exponential-odd log-logistic-G family (EOLL), whose cdf is  $1 - \left( \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right)^\theta$  and gamma(2,  $\theta$ )-odd log-logistic-G family (GaOLL), whose cdf is  $1 - \left[ 1 - \log \left( \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right) \right]^\theta \left( \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right)^\theta$ , with mixing rate  $\theta / (\theta + \beta)$  and  $\beta / (\theta + \beta)$  respectively. So, we express the ELOLL-G family as

$$\text{ELOLL-G}(\alpha, \beta, \theta, \xi) = \frac{\theta}{\theta + \beta} \text{EOLL-G}(\alpha, \theta, \xi) + \frac{\beta}{\theta + \beta} \text{GaOLL-G}(\alpha, \theta, \xi). \quad (2.3)$$

As a result, we can say that the ELOLL-G family is a mixture family. The hazard rate function (hrf) of this family is given by,

$$h(x, \alpha, \beta, \theta, \xi) = \frac{\alpha\theta^2 g(x) G(x)^{\alpha-1} \left( 1 - \beta \log \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right] \right)}{(\bar{G}(x) [G(x)^\alpha + \bar{G}(x)^\alpha]) \left( \theta + \beta - \theta\beta \log \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right] \right)}. \quad (2.4)$$

The quantile function (qf) of ELOLL-G is given as follows: if  $U$  has a uniform  $U(0, 1)$  the solution of the non-linear equation  $x = Q(U) = Q_G \left( T_U^{1/\alpha} \left[ T_U^{1/\alpha} + (1 - T_U)^{1/\alpha} \right]^{-1} \right)$  has random number on the ELOLL-G( $\alpha, \beta, \theta, \xi$ ), where  $Q_G(\cdot) = G^{-1}(\cdot)$  is the qf of the baseline distribution,  $Q_{EL}(u)$  is the solution of (1.3) that is the qf of the EL distribution and  $T_U = (1 - e^{-Q_{EL}(U)})$ . Hence, if  $U$  is a uniform random variable on  $U(0, 1)$ , then  $X = Q(U)$  has the ELOLL-G distribution. Also, the mixture representation in (2.3) can be used to generate random number from the ELOLL-G distribution.

## 3. Some special cases of the ELOLL-G family

The ELOLL-G family includes important sub-families. We mention these sub-families in Table (1). Also, the (2.1) and (2.2) will be most tractable when  $g(x)$  and  $G(x)$  have simple analytic forms. Here, we provide three special models of the ELOLL-G family. These special models generalize some well-known distributions reported in the literature. We note that since the cdf, pdf and hrf of any ELOLL-G distribution will be easily determined by (2.1), (2.2) and (2.4), we don't give them in following sub-sections.

Table 1: Sub families of the ELOLL-G family.

$\alpha$	$\beta$	$\theta$	Sub-family	References
1	—	—	Exponential Lindley-G	New
1	1	—	Lindley-G	Çakmakyapan and Ozel [9]
1	0	—	Lehmann Type II-G or Proportional hazard rate model	Gupta et al. [19]
—	0	—	Odd Burr-G or Lehmann Type II odd log-logistic-G	Alizadeh et al. [4]
—	1	—	Lindley odd log-logistic-G	New
1	0	1	$G(x)$	-

### 3.1. The ELOLL-normal distribution

We define the ELOLL-normal (ELOLL-N) distribution from (2.2) by taking  $G(x; \mu, \sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right)$  and  $g(x; \mu, \sigma) = \sigma^{-1} \phi\left(\frac{x-\mu}{\sigma}\right)$  with  $\xi = (\mu \in \mathfrak{R}, \sigma > 0)$ , where  $\phi(\cdot)$  and  $\Phi(\cdot)$  are the pdf and cdf of the standard normal distribution, respectively. Its pdf is denoted by  $\text{ELOLL-N}(\Theta)$  where  $\Theta = (\alpha, \beta, \theta, \mu, \sigma)$ . We plot this pdf and its hrf in Figure 1. From Figure 1, we see that the pdf shapes of the ELOLL-N are skewed and bi-modal. Also, its hrf are increasing or firstly unimodal and then increasing. The ordinary normal distribution acquires these properties.

### 3.2. The ELOLL-Weibull distribution

We now consider the Weibull distribution as a baseline distribution with pdf  $g(x; \lambda, \gamma) = \gamma \lambda^\gamma x^{\gamma-1} e^{-(\lambda x)^\gamma}$  and cdf  $G(x; \lambda, \gamma) = 1 - e^{-(\lambda x)^\gamma}$  for  $x, \lambda, \gamma > 0$ . Its pdf is denoted by  $\text{ELOLL-W}(\Theta)$  where  $\Theta = (\alpha, \beta, \theta, \lambda, \gamma)$ . Its pdf and hrf for selected parameter values are displayed in Figure 2. From Figure 2, we see that pdf and hrf of the ELOLL-W model have various shapes. The pdf shapes are decreasing, unimodal, bi-modal, firstly decreasing then unimodal shaped, U-shaped. Also, its hrf shapes are increasing, decreasing, unimodal, bathtub-shaped and firstly unimodal and then increasing shaped. So, we can say that ELOLL-W distribution can be useful for modelling data.

### 3.3. The ELOLL-Lomax distribution

The pdf and cdf of the Lomax distribution with scale parameter  $\lambda > 0$  and shape parameter  $k > 0$  are given by  $g(x, \lambda, k) = \frac{k}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-k-1}$  and  $G(x, \lambda, k) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-k}$  respectively. Its pdf is denoted by  $\text{ELOLL-Lx}(\Theta)$  where  $\Theta = (\alpha, \beta, \theta, \lambda, k)$ . Plots of its pdf and hrf for selected parameter values are displayed in Figure 3.

## 4. Expansions for cdf and pdf

In this Section, we will obtain cdf and pdf expansions of the ELOLL-G family using Taylor series of  $\log(1-z) = -\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)}, |z| < 1$  and others series expansions. Then equation (2.1) can be written

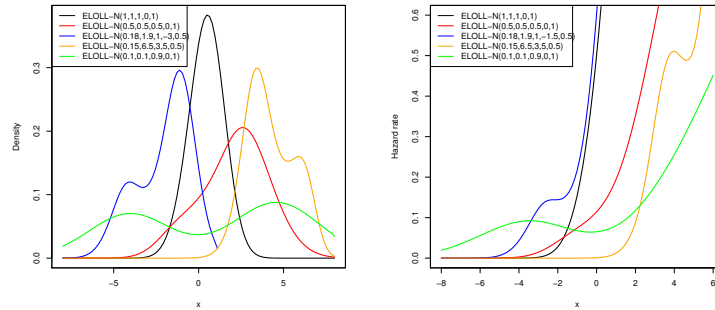


Fig. 1: Plots of the pdf and hrf of the ELOLL-N distributions

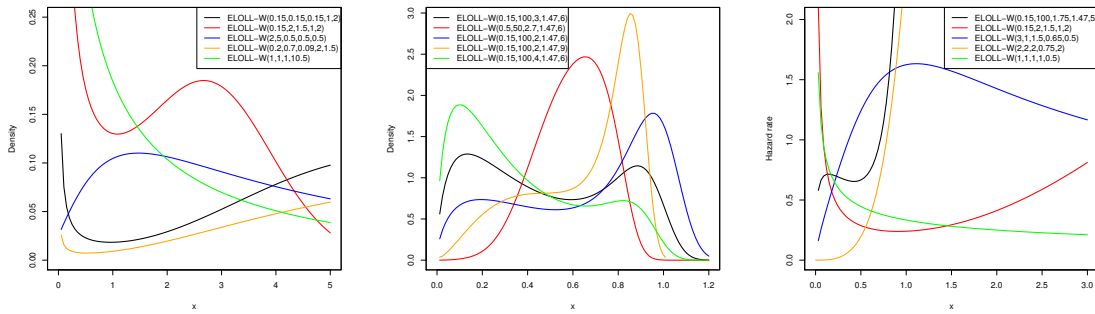


Fig. 2: Plots of the pdf and hrf of the ELOLL-W distributions

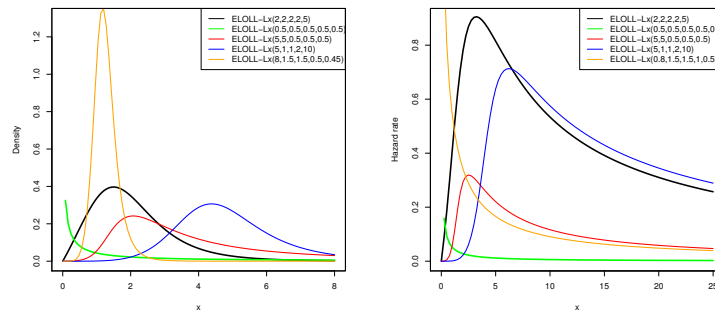


Fig. 3: Plots of the pdf and hrf of the ELOLL-Lx distributions

as

$$F(x) = 1 - \frac{\overbrace{\bar{G}(x)^{\alpha\theta}}^A}{\underbrace{[G(x)^\alpha + \bar{G}(x)^\alpha]^\theta}_B} - \sum_{i=0}^{\infty} \frac{\overbrace{\theta\beta \sum_{j=0}^{\infty} (-1)^j \binom{\alpha\theta}{j} G(x)^{\alpha(i+1)+j}}^D}{\underbrace{(\theta + \beta)(i+1) [G(x)^\alpha + \bar{G}(x)^\alpha]^{\theta+i+1}}_C}$$

where the quantities  $A, B, C$  and  $D$  are defined in Appendix A. The last equation can be written as

$$F(x) = 1 - \left[ \frac{\sum_{k=0}^{\infty} d_k G(x)^k}{\sum_{k=0}^{\infty} b_k G(x)^k} + \sum_{i=0}^{\infty} \frac{\theta \beta \sum_{k=0}^{\infty} p_k G(x)^k}{(\theta + \beta)(i+1) \sum_{k=0}^{\infty} h_k G(x)^k} \right],$$

where  $b_k$  and  $h_k$  are defined in the Appendix A. Then

$$F(x) = 1 - \left[ \sum_{k=0}^{\infty} \Upsilon_k G(x)^k + \sum_{i=0}^{\infty} \frac{\theta \beta}{(\theta + \beta)(i+1)} \sum_{k=0}^{\infty} \Omega_k G(x)^k \right],$$

where  $\Upsilon_0 = \frac{d_0}{b_0}$  and  $\Omega_0 = \frac{p_0}{h_0}$  for  $k \geq 1$  and

$$\Upsilon_k = \frac{1}{b_0} \left[ d_k + \frac{1}{b_0} \sum_{r=1}^k b_r \Upsilon_{k-r} \right] \text{ and } \Omega_k = \frac{1}{h_0} \left[ p_k + \frac{1}{h_0} \sum_{r=1}^k h_r \Omega_{k-r} \right],$$

then

$$F(x) = 1 - \sum_{k=0}^{\infty} \zeta_k \Pi_k(x) = \sum_{k=0}^{\infty} v_k \Pi_k(x), \tag{4.1}$$

where  $\Psi_k = \sum_{i=0}^{\infty} \frac{\theta \beta}{(\theta + \beta)(i+1)} \Omega_k$ ,  $v_0 = 1 - \zeta_0 = 1 - (\Upsilon_0 + \Psi_0)$  and for  $k \geq 1$  we have  $v_0 = -\zeta_k$  and  $\Pi_k(x)$  is the cdf of the exponentiated-G (exp-G) family with power parameter ( $\gamma$ ). Equation (4.1) reveals that the density of ELOLL-G family is a linear combination of exp-G densities. So, some mathematical properties of this family can be determined from those of the exp-G distribution. By differentiating (4.1), we obtain the same mixture representation

$$f(x) = \sum_{k=0}^{\infty} v_{k+1} \pi_{k+1}(x), \tag{4.2}$$

where  $\pi_{\gamma}(x) = \gamma g(x) G(x)^{\gamma-1}$  represents the exponentiated-G family density with power parameter  $\gamma > 0$ . The equations (4.1) and (4.2) are the main results of this section.

### 5. Properties

The  $r^{th}$  moment of  $X$ , say  $\mu'_r$ , follows from (4.2) as

$$\mu'_r = E(X^r) = \sum_{k=0}^{\infty} v_{k+1} E(Y_{k+1}^r).$$

Henceforth,  $Y_{\gamma}$  denotes the exp-G distribution with power parameter ( $\gamma$ ). For  $\gamma > 0$ , we have  $E(Y_{\gamma}^r) = \gamma \int_{-\infty}^{\infty} x^r g(x; \xi) G(x; \xi)^{\gamma-1} dx$ , which can be computed numerically in terms of the baseline quantile function (qf)  $Q_G(u; \xi) = G^{-1}(u; \xi)$  as  $E(Y_{\gamma}^r) = \gamma \int_0^1 Q_G(u; \xi)^r u^{\gamma-1} du$ . The  $n^{th}$  central moment of  $X$ , say  $M_n$ , is given by

$$M_n = E(X - \mu'_1)^n = \sum_{r=0}^n \binom{n}{r} (-\mu'_1)^{n-r} E(X^r), = \sum_{r=0}^n \sum_{k=0}^{\infty} v_{k+1} \binom{n}{r} \binom{n}{r} (-\mu'_1)^{n-r} E(X^r).$$

Here, we provide two formulae for the mgf  $M_X(t) = E(e^{tX})$  of  $X$ . Clearly, the first one can be derived from equation (5) as  $M_X(t) = \sum_{k=0}^{\infty} v_{k+1} M_{k+1}(t)$ , where  $M_{k+1}(t)$  is the mgf of  $Y_{k+1}$ . Hence,

$M_X(t)$  can be determined from the exp-G generating function. A second formula for  $M_X(t)$  follows from (4.2) as  $M_X(t) = \sum_{k=0}^{\infty} v_{k+1} \zeta(t, k)$ , where  $\zeta(t, k) = \int_0^1 \exp[t Q_G(u)] u^k du$  and  $Q_G(u)$  is the qf corresponding to  $G(x; \phi)$ , i.e.,  $Q_G(u) = G^{-1}(u; \phi)$ . For the ELOLL-W model we get

$$\mu'_r = E(X^r) = \sum_{k,h=0}^{\infty} v_{k+1} \frac{k+1}{\lambda^r} b_h^{(k+1)} \Gamma\left(1 + \frac{r}{\gamma}\right), r > -\gamma \text{ with } b_h^{(d)} = \frac{(-1)^h}{(h+1)^{\frac{r+\gamma}{\gamma}}} \binom{d-1}{h}, d > 0.$$

Using the series expansion  $(1-z)^a = \sum_{i=0}^{\infty} \binom{a}{i} (-z)^i$  for  $|z| < 1$ , one can expand  $(k+1) \beta \alpha^\beta x^{\beta-1} \exp[-(\alpha x)^\beta] \{1 - \exp[-(\alpha x)^\beta]\}^k$  as

$$f(x) = (k+1) \sum_{h=0}^{\infty} \binom{k}{h} \frac{(-1)^h}{h+1} f_{(h+1)^{1/\gamma} \lambda}(x), \tag{5.1}$$

where  $f_{(h+1)^{1/\gamma} \lambda}(\cdot)$  denotes the pdf of a two-parameter Weibull distribution with  $\lambda$  replaced by  $(h+1)^{1/\gamma} \lambda$ . So, whenever possible, (5.1) can be used to derive moment generating function of the ELOLL-W model from those of a two-parameter Weibull distribution. Consider  ${}_p\Psi_q(\cdot)$  as the complex parameter Wright generalized hypergeometric function with  $p$  numerator and  $q$  denominator parameters (Kilbas *et al.* [21], Equation (1.9)) defined by the series

$${}_p\Psi_q \left[ \begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix}; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^q \Gamma(\beta_j + B_j n)} \frac{z^n}{n!}, \text{ for } z \in \beta \tag{5.2}$$

where  $\alpha_j, \beta_k \in B_k, A_j, B_k \neq 0, j = \overline{1, p}, k = \overline{1, q}$  and the series converges for  $1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j > 0$ , compare with Mathai and Saxena [26] and Srivastava *et al.* [31]. This function was originally introduced by Wright [33]. Let  $X$  be a random variable having the pdf (3.2). Following similar algebraic developments of Nadarajah *et al.* [28], we can write the moment generating function (mgf) of the ELOLL-W model

$$M_X(t) = \sum_{k,h=0}^{\infty} v_{k+1} \frac{(-1)^h}{h+1} {}_1\Psi_0 \left[ \begin{matrix} (1, 1/\gamma) \\ - \end{matrix}; t \{(h+1)^{1/\gamma} \lambda\}^{-1} \right]. \tag{5.3}$$

Hypergeometric functions are included as in-built functions in most popular algebraic mathematical software packages, so the special function in (5.2) and hence (5.3) can be easily evaluated by the software packages Maple, Matlab and Mathematica using known procedures.

Suppose  $X_1, \dots, X_n$  is a random sample from an ELOLL-G distribution. Let  $X_{i:n}$  denote the  $i^{th}$  order statistic. The pdf of  $X_{i:n}$  can be expressed as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}.$$

Following similar algebraic developments of Nadarajah *et al.* [29], we can write the density function of  $X_{i:n}$  as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} b_{r,k} \pi_{r+k+1}(x), \tag{5.4}$$

where  $b_{r,k} = \frac{n!(r+1)(i-1)!v_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!j!}$ ,  $v_{r+1}$  is given in Section 3 and the quantities  $f_{j+i-1,k}$  can be determined with  $f_{j+i-1,0} = v_0^{j+i-1}$  and recursively for  $k \geq 1$

$$f_{j+i-1,k} = (k v_0)^{-1} \sum_{m=1}^k [m(j+i) - k] v_m f_{j+i-1,k-m}.$$

Equation (5.4) reveals that the pdf of the ELOLL-G order statistics is a linear combination of exp-G density functions.

## 6. Characterizations

This section deals with various characterizations of ELOLL distribution. These characterizations are presented in two directions: (i) based on truncated moments and (ii) in terms of the hazard function. It should be noted that characterization (i) can be employed also when the cdf does not have a closed form. We present our characterizations (i) and (ii) in two subsections.

### 6.1. Characterizations based on truncated moments

Our first characterization employs a theorem due to Glanzel [16], see Theorem 1 of Appendix B. The result, however, holds also when the interval  $H$  is not closed since the condition of Theorem 1 is on the interior of  $H$ .

**Proposition 1.6** Let  $X : \Omega \rightarrow \mathbb{R}$  be a continuous random variable and let  $q_1(x) = [G(x)^\alpha + \bar{G}(x)^\alpha]^{-2} \left\{ 1 - \beta \log \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right] \right\}^{-1}$  and  $q_2(x) = q_1(x) \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right]$  for  $x \in \mathbb{R}$ . The random variable  $X$  belongs to the family (2.2) if and only if the function  $\eta$  defined in Theorem 1 has the form

$$\eta(x) = \frac{\theta}{(\theta + 1)} \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right], \quad x \in \mathbb{R}.$$

Proof. Let  $X$  be a random variable with pdf (2.2), then

$$(1 - F(x)) E[q_1(x) | X \geq x] = \frac{\theta}{(\theta + \beta)} \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right]^\theta, \quad x \in \mathbb{R},$$

and

$$(1 - F(x)) E[q_2(x) | X \geq x] = \frac{\theta^2}{(\theta + 1)(\theta + \beta)} \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right]^{\theta+1}, \quad x \in \mathbb{R},$$

and finally

$$\eta(x) q_1(x) - q_2(x) = -\frac{1}{(\theta + 1)} q_1(x) \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right] < 0, \quad \text{for } x \in \mathbb{R}.$$

Conversely, if  $\eta$  is given as above, then



$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{\alpha\theta g(x)G(x)^{\alpha-1}}{\bar{G}(x)[G(x)^\alpha + \bar{G}(x)^\alpha]}, \quad x \in \mathbb{R},$$

and hence

$$s(x) = \log \left\{ \left[ \frac{\bar{G}(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha} \right]^\theta \right\}, \quad x \in \mathbb{R}.$$

Now, in view of Theorem 1,  $X$  has density (2.2).

**Corollary 1.6.** Let  $X : \Omega \rightarrow \mathbb{R}$  be a continuous random variable and let  $q_1(x)$  be as in Proposition 1. 6. The pdf of  $X$  is (2.2) pdf if and only if there exist functions  $q_2$  and  $\eta$  defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{\alpha\theta g(x)G(x)^{\alpha-1}}{\bar{G}(x)[G(x)^\alpha + \bar{G}(x)^\alpha]}, \quad x \in \mathbb{R}.$$

The general solution of the above differential equation is

$$\eta(x) = \left\{ 1 + \frac{G(x)^\alpha}{\bar{G}(x)^\alpha} \right\} \left[ \int \frac{\alpha\theta g(x)G(x)^{\alpha-1}\bar{G}(x)^{\alpha-1}}{[G(x)^\alpha + \bar{G}(x)^\alpha]^2} (q_1(x))^{-1} q_2(x) dx + D \right],$$

where  $D$  is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 1. 6. with  $D = 0$ . However, it should be also mentioned that there are other triplets  $(q_1, q_2, \eta)$  satisfying the conditions of Theorem 1.

## 6.2. Characterization in terms of the hazard function

It is known that the hazard function,  $h_F$ , of a twice differentiable distribution function,  $F$ , satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establishes a non-trivial characterization of ELOLL distribution for  $\beta = 0$ , in terms of the hazard function which is not of the above trivial form.

**Proposition 2.6** Let  $X : \Omega \rightarrow \mathbb{R}$  be a continuous random variable. Then, for  $\beta = 0$ ,  $X$  has pdf (2.2) if and only if its hazard function  $h_F(x)$  satisfies the differential equation

$$h'_F(x) - \frac{g'(x)}{g(x)}h_F(x) = \alpha\theta g(x) \frac{d}{dx} \left\{ \frac{G(x)^{\alpha-1}}{\bar{G}(x)[G(x)^\alpha + \bar{G}(x)^\alpha]} \right\}, \quad x \in \mathbb{R}.$$

Proof. If  $X$  has pdf (2.2), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx} \left\{ (g(x))^{-1} h_F(x) \right\} = \alpha \theta \frac{d}{dx} \left\{ \frac{G(x)^{\alpha-1}}{\bar{G}(x) [G(x)^\alpha + \bar{G}(x)^\alpha]} \right\}$$

or

$$h_F(x) = \alpha \theta g(x) \left\{ \frac{G(x)^{\alpha-1}}{\bar{G}(x) [G(x)^\alpha + \bar{G}(x)^\alpha]} \right\}, \quad x \in \mathbb{R},$$

which is the hazard function of (2.2).

### 7. Maximum Likelihood Estimation (MLE) and Inference

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties and can be used for constructing confidence intervals and also for test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. Here, we consider the estimation of the unknown parameters of the new family from complete samples only by maximum likelihood. Let  $x_1, \dots, x_n$  be a random sample from ELOLL-G model with a  $(q+3) \times 1$  parameter vector  $\Xi = (\alpha, \beta, \theta, \xi)^\top$ , where  $\xi$  is a  $q \times 1$  baseline parameter vector. The log-likelihood function for  $\Xi$  can be expressed as

$$\begin{aligned} \ell_n(\Xi) = & n \log(\alpha) + 2n \log(\theta) - n \log(\theta + \beta) + \sum_{i=1}^n \log g(x_i, \xi) + (\alpha\theta - 1) \sum_{i=1}^n \log \bar{G}(x_i, \xi) \\ & + (\alpha - 1) \sum_{i=1}^n \log G(x_i, \xi) + (1 - \theta) \sum_{i=1}^n \log(t_i) + \sum_{i=1}^n \log \left\{ 1 - \beta \log \left[ \frac{\bar{G}(x_i, \xi)^\alpha}{t_i} \right] \right\}, \end{aligned}$$

where  $t_i = G(x_i, \xi)^\alpha + \bar{G}(x_i, \xi)^\alpha$ . The log-likelihood function can be maximized by solving the following nonlinear normal equations which are available if needed. To solve these equations, it is more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize  $\ell(\Xi)$ .

The likelihood ratio (LR) statistic can be used for comparing the some sub-models of ELOLL-G model. For example, the LR statistic can be used to discriminate between the ELOLL-Lomax and L-Lomax since they are nested models, which is equivalently to test  $H_0 : \alpha = \beta = 1$ . The LR statistic reduces to  $w = 2[\ell(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\beta}) - \ell(1, 1, \tilde{\sigma}, \tilde{\beta})]$ , where  $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\beta})$  are the unrestricted MLEs and  $(1, 1, \tilde{\sigma}, \tilde{\beta})$  are the restricted estimates under  $H_0$ . The statistic  $w$  is asymptotically (as  $n \rightarrow \infty$ ) distributed as  $\chi_k^2$ , where  $k$  is difference of two parameter vectors of nested models. For example,  $k = 2$  for above hypothesis test.

### 8. A simulation study

In this subsection, we perform the simulation study using the exponential Lindley odd log-logistic exponential distribution (ELOLL-E) which is the generalization of the exponential distribution with cdf  $F(x; \lambda) = 1 - \exp(-\lambda x), x > 0, \lambda > 0$ . To see the performance of MLE's of this distributions, we generate 1,000 samples of sizes 20, 50 and 100 from ELOLL-E using inverse of the its cdf for  $\theta = 2$ . The results of the simulation are reported in Table (2). From this Table, we observe that the

estimates approach true values as the sample size increases whereas the standard deviations of the estimates decrease.

Table 2: Empirical means and standard deviations (in paranthesis) for selected parameters values of ELOLL-E distribution

Parameters	$n = 20$	$n = 50$	$n = 100$
$\alpha, \beta, \lambda$	$\hat{\alpha}, \hat{\beta}, \hat{\lambda}$	$\hat{\alpha}, \hat{\beta}, \hat{\lambda}$	$\hat{\alpha}, \hat{\beta}, \hat{\lambda}$
1,1,1	1.0899,1.1654,1.0906 (0.2581),(0.9818),(0.3488)	1.0170,1.0452,1.0189 (0.1379),(0.8293),(0.2274)	1.0111,1.0031,0.9847 (0.0848),(0.7345),(0.1670)
2,2,2	2.1705,2.3459,1.9854 (0.4130),(1.9485),(0.2642)	2.0673,1.9774,1.9883 (0.2550),(0.7811),(0.1546)	2.0361,1.9851,1.9916 (0.1795),(0.5796),(0.1736)
2,1,0.5	2.1053,1.0154,0.5046 (0.3937),(0.3690),(0.0603)	2.0586,1.0128,0.4996 (0.2405),(0.3371),(0.0424)	2.0117,1.0085,0.5009 (0.1653),(0.2717),(0.0286)
1,0.5,2	1.0147,0.7305,2.1798 (0.2305),(0.8415),(0.7783)	1.0062,0.6800,2.0887 (0.1214),(0.6699),(0.4054)	0.9945,0.6546,2.0635 (0.0912),(0.5922),(0.4099)
1,2,0.5	1.0834,1.8239,0.5047 (0.2335),(0.6285),(0.1195)	1.0284,1.9398,0.5030 (0.1259),(0.4712),(0.0764)	1.0210,1.9620,0.5017 (0.0872),(0.4550),(0.0530)
2,0.5,0.5	2.0795,0.5410,0.5105 (0.3949),(0.2886),(0.0661)	2.0380,0.5515,0.5035 (0.2302),(0.6220),(0.0445)	2.0156,0.5280,0.5033 (0.1683),(0.5280),(0.0295)
0.75,5,2	0.8201,4.8903,2.1585 (0.1484),(0.7185),(0.5310)	0.7895,4.8956,1.9550 (0.1029),(0.3244),(0.3808)	0.7619,4.9821,1.9942 (0.0659),(0.2595),(0.2433)
10,10,10	10.679,10.0500,10.0436 (2.0187),(2.1032),(0.2139)	10.1903,10.0244,10.0041 (1.1108),(1.5571),(0.1350)	10.1852,9.9983,9.9934 (0.9405),(2.2128),(0.1038)

### 9. Data analysis

In this section, we introduce three applications using well-known data sets to show the flexibility and applicability of the proposed models over other models. First, we describe the data sets and we fit some distributions to these data sets using MLE method. Then we compare proposed distributions with several member of distribution families. The model selection is applied using the estimated log-likelihood ( $\hat{\ell}$ ), Kolmogorov-Smirnov (K-S) statistics, Akaike information criterion (AIC), Consistent Akaike information criteria (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC). The AIC, CAIC, BIC and HQIC are by given by  $AIC = -2\hat{\ell} + 2p$ ,  $CAIC = -2\hat{\ell} + 2pn(n - k - 1)^{-1}$ ,  $BIC = -2\hat{\ell} + p \log n$  and  $HQIC = -2\hat{\ell} + p \log(\log n)$ , where  $p$  is the number of the estimated model parameters and  $n$  is sample size. When searching the best fit among others to data, the distribution with the smallest AIC, CAIC, BIC, HQIC and K-S values and the biggest log-likelihood and p values of the K-S statistics is chosen. All calculations are obtained by maxLik routine in R programme.

First data set studied by Murthy et al. [27], which represent failure times for a particular wind-shield device. This data set has been analyzed by Cordeiro et al. [14]. Using this data set, we fit the ELOLL-N, Lindley-normal (L-N) (Çakmakyapan and Ozel, [9]), exponential-normal (E-N)

or Lehmann type II exponentiated-normal (Alzaatreh et al., [5]; Cordeiro et al. [15]), McDonald-normal (McN) (Alexander et al. [3]), normal-normal{exponential} (NNE) (Alzaatreh et al. [6]), normal-Cauchy{log-logistic} (NCLL) (Alzaatreh, et al. [7]), logistic-normal (LN) (Tahir et al. [32]), generalized Kumaraswamy-normal (GKw-N) (Cordeiro et al. [11]) and generalized odd log-logistic normal (GOLLN) (Cordeiro, et al. [10]) distributions models. The results of this application are listed in Table 3.

As second data analysis, we analyze the data set studied by Abouammoh et al. [1], which represent the lifetime in days of 40 patients suffering from leukemia from one of the Ministry of Health Hospitals in Saudi Arabia. By using this data set we compare the ELOLL-W distribution with logistic-Weibull (LW) (Tahir et al. [32]), odd-Burr-Weibull (OBW) (Alizadeh et al. [4]), Lindley-Weibull (L-W) (Çakmakyapan and Ozel [9]) and exponential-Weibull (E-W) or ordinary Weibull distributions. The results of this application are listed in Table 4.

The third real data consists of the number of successive failure for the air conditioning system reported of each member in a fleet of 13 Boeing 720 jet airplanes. The pooled data with 213 observations was considered by Proschan [30] and Kuş [23]. For this data, we use the ELOLL-Lx distribution and compare it Zografos-Balakrishan odd-loglogistic-Lomax (ZBOLL-Lx) (Cordeiro et al. [13]), Kumaraswamy Lomax (Kw-Lx) (Lemonte and Cordeiro [24]), beta-Lomax (B-Lx) (Lemonte and Cordeiro [24]) and its sub-models. The results of the application are in Table 5.

Table 3: MLEs of the model parameters for the windshield data, the corresponding standard errors (given in parentheses below estimated parameters) and the AIC, CAIC, BIC, HQIC and K-S values

Model	$\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\mu}, \hat{\sigma}$	$-\hat{\ell}$	AIC, CAIC, BIC, HQIC	K-S (p-v)
ELOLL-N	0.1679, 6.5946, 3.0515, 3.1785, 0.4070 (0.0849), (9.5475), (0.6488), (0.2077), (0.1255)	124.3870	258.7740, 259.5433, 270.9281, 263.6599	0.0463 (0.9937)
McN	125.6276, 6.9936, 0.0299, 4.7966, 2.4578 (0.2334), (0.3941), (0.0131), (0.9119), (0.4754)	128.2184	266.4367, 267.2059, 278.5908, 271.3225	0.0805 (0.6478)
NNE	9.0338, 0.7306, —, 19.3314, 5.9635 (0.4896), (0.0873), —, (0.4711), (0.3606)	128.4202	264.8404, 265.3467, 274.5637, 268.7491	0.0991 (0.3816)
NCLL	14.6194, 0.7789, —, 19.0441, 4.4038 (3.8058), (0.1874), —, (2.6534), (0.9849)	128.1291	264.2582, 264.7645, 273.9814, 268.1669	0.0921 (0.4742)
LN	—, —, 6.1695, 1.1181, 4.0953 —, —, (4.5532), (1.0089), (3.0241)	130.0301	266.0602, 266.3602, 273.3526, 268.9917	0.0805 (0.6482)
GKwN	90.5274, 4.6255, 0.6399, —, 6.3998, 4.0251 (13.5280), (5.0395), (0.3384), (1.7059), (1.2968)	127.9231	265.8463, 266.6155, 278.0003, 270.7321	0.0971 (0.4067)
GOLLN	0.4557, 1.1863, —, 2.5139, 0.6358 (0.2353), (1.2905), —, (0.7508), (0.3306)	127.0450	262.0900, 262.5963, 271.8133, 265.9987	0.0901 (0.5036)
L-N	—, —, 0.1446, 0.258, 0.5530 —, —, (0.0112), (1.3e-5), (6.2e-6)	127.8708	261.7416, 262.0416, 269.0341, 264.6731	0.0775 (0.6946)
E-N	—, —, 0.1382, 1.0954, 0.5583 —, —, (0.0151), (0.0006), (0.0002)	128.0645	262.1289, 262.4289, 269.4214, 265.0604	0.0618 (0.8908)

Table 4: MLEs of the model parameters for the leukemia data, the corresponding standard errors (given in parentheses below estimated parameters) and the AIC, CAIC ,BIC, HQIC and K-S values

Model	$\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{\gamma}$	$-\hat{\ell}$	AIC,CAIC,BIC,HQIC	K-S(p-v.)
ELOLL-W	0.2389, 441.8416, 1.7418, 0.0009, 4.5233 (0.0567), (4.1943), (0.4284), (0.0001), (0.4813)	299.1043	608.2087, 609.9734, 616.6531, 611.2619	0.0651 (0.9958)
LW	-, -, 34.7613, 0.0009, 0.0916 (3.8176), (0.0000), (0.0016)	311.4651	628.9302, 629.5969, 633.9968, 630.7621	0.1618 (0.2459)
OBW	3.0365, -, 28.9395, 0.00015, 0.7735 (1.3087), -, (1.1964), (0.0000), (0.3973)	303.8669	615.7339, 616.8768, 622.4894, 618.1765	0.1163 (0.6520)
L-W	1, 1, 0.8129, 0.0011, 2.3265 (0.9133), (0.0006), (0.4141)	303.6019	613.2038, 613.8704, 618.2704, 615.0357	0.1075 (0.7436)
E-W	1, 0, 336.0406, 0.00008, 2.5775 (4.2136), (0.00002), (0.4103)	304.4153	614.8305, 615.4972, 619.8972, 616.6625	0.1237 (0.5728)

Table 5: MLEs of the model parameters for the Proschan data, the corresponding standard errors (given in parentheses below estimated parameters) and the AIC, CAIC ,BIC, HQIC and K-S values

Model	$\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{k}, \hat{\lambda}$	$-\hat{\ell}$	AIC,CAIC,BIC,HQIC	K-S (p-v.)
ELOLL-Lx	5.1293, 42.3803, 193.8547, 0.0461, 0.0879 (1.4911), (2.1788), (1.1108), (0.0032), (0.1794)	1172.9880	2355.9770, 2356.2670, 2372.7830, 2362.7690	0.0372 (0.9296)
Kw-Lx	1.1322, 234.9295, -, 0.0208, 172.2718 (0.0622), (0.3286), -, (0.0053), (2.1826)	1174.9830	2357.9660, 2358.1590, 2371.4110, 2363.4000	0.0439 (0.8043)
BLx	0.9395, 13.5670, - (0.0053), (2.1826) (0.0794),(0.6603),-, (0.2486),(2.1986)	1176.7130	2361.4260, 2363.7160, 2380.2330, 2370.2180	0.0583 (0.4642)
ZBOLL-Lx	14.9281, 0.4497, -, 0.0605, 0.0013 (1.8577), (0.0878),-, (0.0028), (0.0006)	1177.1000	2362.1990, 2362.3920, 2375.6440, 2367.6330	0.0489 (0.6878)
L-Lx	1, 1, 441.4941, 0.0130, 441.9408 (2.1022), (0.0009), (0.8563)	1175.3800	2356.8750, 2356.8750, 2366.8440, 2360.8350	0.0389 (0.9040)
E-Lx	1, 0, 1346.2550, 0.0045, 469.5059 (2.3463), (0.0003), (1.0493)	1175.3850	2356.7500, 2356.8850, 2366.8540, 2360.8450	0.0390 (0.9021)
OBLx	1.0194, -, 1251.0490, 0.0052, 444.8587 (0.0544),-, (2.1920), (0.0018), (0.6342)	1175.3160	2358.632, 2358.8250, 2372.0780, 2364.0660	0.0403 (0.8796)
EL-Lx	1, 78.9168, 1361.9960, 0.0043, 433.0816 (2.0991), (0.7953), (0.0003), (0.7930)	1175.3780	2358.7560, 2358.9480, 2372.2010, 2364.1900	0.0390 (0.9036)
LOLL-Lx	1.1071, 1.396.4973, 0.0131, 201.9921 (0.0587), -, (2.1042), (0.0035), (2.1207)	1174.9910	2357.9810, 2358.1740, 2371.4260, 2363.4150	0.0433 (0.8181)
Lx	1, 0, 1, 22085.62, 1930929.40 (23.9112), (00000)	1178.3410	2360.6830, 2360.7400, 2367.4060, 2363.4000	0.2346 (00000)

From Tables 3-5, we see that the ELOLL-N, ELOLL-W and ELOLL-Lx models have the lowest AIC, CAIC, BIC, HQIC and K-S values and has the biggest estimated log-likelihood and p-value

**Table 6: LR statistics for three data sets**

Model	Hypothesis	Test statistics	p-value
ELLOL-N vs L-N	$H_0: \alpha = \beta = 1, H_1: H_0$ false	6.9668	0.0307
ELLOL-N vs E-N	$H_0: \alpha = 1, \beta = 0, H_1: H_0$ false	7.3550	0.0253
Model			
ELLOL-W vs L-W	$H_0: \alpha = \beta = 1, H_1: H_0$ false	8.9952	0.01111
ELLOL-W vs E-W	$H_0: \alpha = 1, \beta = 0, H_1: H_0$ false	10.622	0.00494
ELLOL-W vs	$H_0: \beta = 0, H_1: H_0$ false	9.5252	0.00202
Model			
ELLOL-Lx vs L-Lx	$H_0: \alpha = \beta = 1, H_1: H_0$ false	4.7840	0.09144
ELLOL-Lx vs E-Lx	$H_0: \alpha = 1, \beta = 0, H_1: H_0$ false	4.7940	0.09099
ELLOL-Lx vs OBLx	$H_0: \beta = 0, H_1: H_0$ false	4.6560	0.03094
ELLOL-Lx vs EL-Lx	$H_0: \alpha = 1, H_1: H_0$ false	4.7800	0.02879
ELLOL-Lx vs LOLL-Lx	$H_0: \beta = 1, H_1: H_0$ false	4.0060	0.04533
ELLOL-Lx vs Lx	$H_0: \alpha = \theta = 1, \beta = 0, H_1: H_0$ false	10.7060	0.013426

of the K-S statistics among all the fitted models. So they could be chosen as the best models under these criteria.

The histograms of three data sets and the estimated pdfs and cdfs of the application models are displayed in Figures 4. From the Figures 4, we show that the ELOLL-G models provide the good fit to these data sets as compared to other models.

A comparison of the proposed distributions with some of their sub-models using LR statistics is performed in Table 6. From Table 6, we can conclude that the ELOLL-N and ELOLL-W models yield a better fit to these data than the other two and three sub-models respectively. Also, there is no difference among the fits to the current data using the ELLOL-Lx, L-Lx and E-Lx models. In addition, these models provide a better representation of the data than the other sub-models based on the LR test at the 5% significance level.

## 10. Conclusions

A new family of distributions called the exponential Lindley odd log-logistic G family is introduced and studied. The new family generalizes the Lindley-G family [9], Lehmann Type II-G family [19] and odd Burr-G family [4] as well as it introduces new distribution families such as exponential Lindley-G family and Lindley odd log-logistic-G family. We provide some mathematical properties of the new family including ordinary, generating function and order statistics. Characterizations based on truncated moments as well as in terms of the hazard function are presented. The maximum likelihood is used for estimating the model parameters. We give a simulation study for the maximum likelihood estimators by using the special member of new family. Finally, the usefulness of the family is illustrated by means of three real data sets. The new models provide consistently better fits than other competitive models for these data sets.

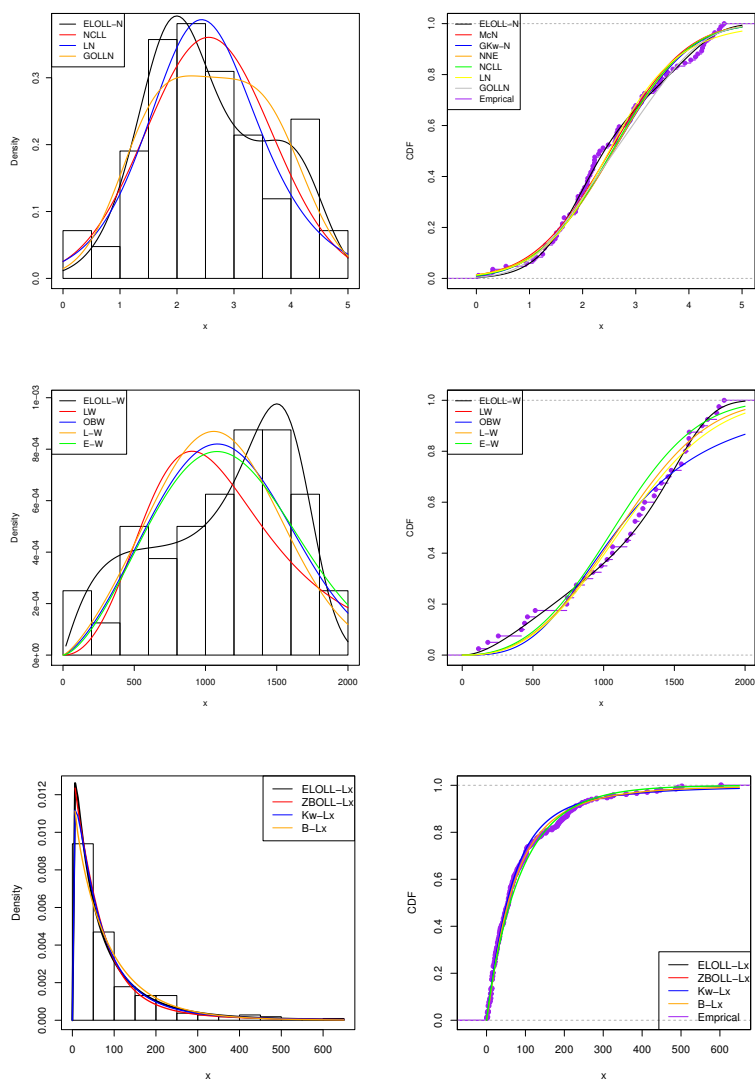


Fig. 4: The fitted pdfs and the fitted cdfs for three data sets

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**Appendix A.**

$$A = \sum_{k=0}^{\infty} \overbrace{(-1)^k \binom{\alpha\theta}{k}}^{d_k} G(x)^k, B = \sum_{k=0}^{\infty} b_k G(x)^k, C = \sum_{k=0}^{\infty} h_k G(x)^k,$$

and

$$D = \sum_{k=0}^{\infty} \overbrace{\sum_{l=k}^{\infty} (-1)^{l+k} \binom{\alpha(i+1)+j}{l} \binom{l}{k}}^{q_k} G(x)^k.$$

Since  $[G(x)^c + \bar{G}(x)^c] = \sum_{m=0}^{\infty} t_m G(x)^m$ , where  $t_m = (-1)^m [\binom{c}{m} + \sum_{w=m}^{\infty} (-1)^w \binom{c}{w} \binom{c}{m}]$ . Then

$$[G(x)^c + \bar{G}(x)^c]^a = \sum_{w=0}^{\infty} \overbrace{\sum_{k=0}^{\infty} \frac{(-1)^{k-w}}{k!} \binom{k}{w} (a)_w}^{f_w} \left( \sum_{m=0}^{\infty} t_m G(x)^m \right)^w,$$

where  $(a)_w = a(a-1)\dots(a-k+1)$  is the descending factorial. For a power series raised to a positive integer  $w \left( \sum_{m=0}^{\infty} t_m u^m \right)^w = \sum_{m=0}^{\infty} \mathbf{m}_{w,m} u^m$ , where the coefficients  $\mathbf{m}_{w,m}$  (for  $m = 1, 2, \dots$ ) are easily determined from the recurrence equation  $\mathbf{m}_{w,m} = (mt_0)^{-1} \sum_{d=1}^m [d(m+1)-m] (t_d) (\mathbf{m}_{w,m-d})$ , where  $\mathbf{m}_{w,0} = t_0^w$ . The coefficient  $\mathbf{m}_{w,m}$  can be calculated from  $\mathbf{m}_{w,0}, \dots, \mathbf{m}_{w,m-1}$  and hence from the quantities  $t_0, \dots, t_m$ . So,  $[G(x)^\alpha + \bar{G}(x)^\alpha]^\theta = \sum_{k=0}^{\infty} b_k G(x)^k$ , where  $b_k = b_k(\alpha, \theta) = \sum_{w=0}^{\infty} f_w \mathbf{m}_{w,k}$ , and  $[G(x)^\alpha + \bar{G}(x)^\alpha]^{\theta+i+1} = \sum_{m=0}^{\infty} h_k G(x)^k$ , where  $h_k = h_k(\alpha, \theta + i + 1) = \sum_{w=0}^{\infty} f_w \mathbf{m}_{w,k}$ .

**Appendix B.**

**Theorem 1.** Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a given probability space and let  $H = [d, e]$  be an interval for some  $d < e$  ( $d = -\infty, e = \infty$  might as well be allowed). Let  $X : \Omega \rightarrow H$  be a continuous random variable with the distribution function  $F$  and let  $q_1$  and  $q_2$  be two real functions defined on  $H$  such that

$$\mathbf{E}[q_2(X) | X \geq x] = \mathbf{E}[q_1(X) | X \geq x] \eta(x), \quad x \in H,$$

is defined with some real function  $\eta$ . Assume that  $q_1, q_2 \in C^1(H)$ ,  $\eta \in C^2(H)$  and  $F$  is twice continuously differentiable and strictly monotone function on the set  $H$ . Finally, assume that the equation  $\eta q_1 = q_2$  has no

real solution in the interior of  $H$ . Then  $F$  is uniquely determined by the functions  $q_1, q_2$  and  $\eta$ , particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)} \right| \exp(-s(u)) du,$$

where the function  $s$  is a solution of the differential equation  $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$  and  $C$  is the normalization constant, such that  $\int_H dF = 1$ .