Research Article

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The Exponentiated Generalized-G Poisson Family of Distributions

DOI: 10.1515/eqc-2017-0004
Received January 20, 2017; revised March 27, 2017; accepted March 29, 2017

Abstract: In this article we propose and study a new family of distributions which is defined by using the genesis of the truncated Poisson distribution and the exponentiated generalized-G distribution. Some mathematical properties of the new family including ordinary and incomplete moments, quantile and generating functions, mean deviations, order statistics and their moments, reliability and Shannon entropy are derived. Estimation of the parameters using the method of maximum likelihood is discussed. Although this generalization technique can be used to generalize many other distributions, in this study we present only two special models. The importance and flexibility of the new family is exemplified using real world data.

Keywords: Exponentiated Generalized-G Family, Truncated Poisson Distribution, Order Statistics, Maximum Likelihood Estimation, Goodness-of-Fit

MSC 2010: 60E05, 62P99

1 Introduction

In the last decades, several generalized distributions have been proposed based on different modification methods. These modification methods require the addition of one or more parameters to base model which could provide better adaptability in the modeling of real lifetime data. Modern computing technology has made many of these techniques accessible even if analytical solutions are very complicated. Several continuous univariate-G families have recently appeared. Some notable family includes Marshall–Olkin-G family by Marshall and Olkin [22], exponentiated-G class by R. C. Gupta, P. L. Gupta and R. D. Gupta [17], transmuted exponentiated generalized-G family by Yousof, Afify, Alizadeh, Butt, Hamedani and Ali [36], transmuted geometric-G by Afify, Alizadeh, Yousof, Aryal and Ahmad [1], Kumaraswamy transmuted-G by Afify, Cordeiro, Yousof, Alzaatreh and Nofal [2], Burr X-G by Yousof, Afify, Hamedani and Aryal [37], the odd Lindley-G family of distributions by Silva, Percontini, de Brito, Ramos, Venancio and Cordeiro [33], exponentiated transmuted-G family by Merovci, Alizadeh, Yousof and Hamedani [23], the odd-Burr generalized family by Alizadeh, Cordeiro, Nascimento, Lima and Ortega [5], the transmuted Weibull-G family by Alizadeh, Rasekhi, Yousof and Hamedani [6], the type I half-logistic family by Cordeiro, Alizadeh and Diniz Marinho [11], the complementary generalized transmuted Poisson family by Alizadeh, Yousof, Afify, Cordeiro and Mansoor [7], the Zografos–Balakrishnan odd log-logistic family of distributions by Cordeiro, Alizadeh, Ortega and Serrano [12], logistic-X by Tahir, Cordeiro, Alzaatreh, Mansoor and Zubair [34], a new Weibull-G by Tahir, Zubair, Mansoor, Cordeiro, Alizadeh and Hamedani [35], the generalized odd log-logistic family by Cordeiro, Alizadeh, Ozel, Hosseini, Ortega and Altun [13], the beta odd log-logistic

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generalized family of distributions by Cordeiro, Alizadeh, Tahir, Mansoor, Bourguignon and Hamedani [14], beta transmuted-H by Afify, Yousof and Nadarajah [3], generalized transmuted-G by Nofal, Afify, Yousof and Cordeiro [30] and beta Weibull-G family by Yousof, Rasekhi, Afify, Ghosh, Alizadeh and Hamedani [38] among others.

In this paper we propose and study a generalized family of distribution using the genesis of Poisson distribution with the following motivation. Suppose that a system has N subsystems functioning independently at a given time where N has zero truncated Poisson (ZTP) distribution with parameter λ. It is the conditional probability distribution of a Poisson-distributed random variable, given that the value of the random variable is not zero. The probability mass function (pmf) of N is given by

$$P(N=n)=\frac{1}{[1-\exp(-\lambda)]}\frac{\exp(-\lambda)\lambda^n}{n!}\quad \text{for } n=1,2,\dots .$$

Note that for ZTP variable the expected value and variance are respectively given by

$$E(N)=\frac{\lambda}{[1-\exp(-\lambda)]}$$

and

$$\operatorname{Var}(N)=\frac{\lambda+\lambda^2}{[1-\exp(-\lambda)]}-\frac{\lambda^2}{[1-\exp(-\lambda)]^2}.$$

Suppose that the failure time of each subsystem has the exponentiated Generalized-G (“EGG(a, b)” for short) distribution defined by the cumulative distribution function (cdf) and probability density function(pdf) given by

$$H(x;a,b,\boldsymbol{\psi})=\{1-[1-G(x;\boldsymbol{\psi})]^a\}^b$$

and

$$h(x;a,b,\boldsymbol{\psi})=abg(x;\boldsymbol{\psi})[1-G(x;\boldsymbol{\psi})]^{a-1}\{1-[1-G(x;\boldsymbol{\psi})]^a\}^{b-1},$$

respectively, where $a>0$ and $b>0$ are two additional shape parameters. Let Y_i denote the failure time of the ith subsystem and let $X=\min\{Y_1,Y_2,\dots,Y_N\}$. Then the conditional cdf of X given N is

$$F(x\mid N)=1-P(X>x\mid N)=1-[1-H(x;a,b,\boldsymbol{\psi})]^N.$$

Therefore, the unconditional cdf of X, as described in [31], can be expressed as

$$F(x;a,b,\lambda,\boldsymbol{\psi})=\frac{1-\exp\{-\lambda\{1-[1-G(x;\boldsymbol{\psi})]^a\}^b\}}{[1-\exp(-\lambda)]}. \tag{1}$$

The cdf in (1) is called the exponentiated generalized G Poisson (“EGGP”) family of distributions. The corresponding pdf is

$$f(x;a,b,\lambda,\boldsymbol{\psi})=\frac{ab\lambda g(x)[1-G(x)]^{a-1}\{1-[1-G(x)]^a\}^{b-1}}{[1-\exp(-\lambda)]\exp\{\lambda\{1-[1-G(x)]^a\}^b\}}. \tag{2}$$

For $b=1$ we have EGP class of distribution and for $a=1$ we have GGP class of distribution both of which are embedded in EGGP class.

Using the power series expansion of $\exp(x)$, we express the pdf in (2) as

$$f(x)=\frac{abg(x)[1-G(x)]^{a-1}}{[1-\exp(-\lambda)]}\sum_{i=0}^{\infty}\frac{(-1)^i\{1-[1-G(x)]^a\}^{b(i+1)-1}}{i!\lambda^{-i-1}}.$$

Using the series expansion $(1-z)^{b-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{j!\Gamma(b-j)}z^j$ the last equation can be expressed as

$$f(x)=\sum_{k=0}^{\infty}t_k\pi_{k+1}(x), \tag{3}$$

where

$$t_k=\frac{ab(-1)^k}{[1-\exp(-\lambda)](k+1)}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\binom{b(i+1)-1}{j}\binom{a(j+1)-1}{k}}{i!\lambda^{-i-1}}$$

and

$$\pi_{k+1}(x) = (k+1)g(x)[G(x)]^k.$$

This is the Exp-G pdf with power parameter $(k+1)$. By integrating (3), we obtain the mixture representation of $F(x)$ as

$$F(x) = \sum_{k=0}^{\infty} t_k \Pi_{k+1}(x), \tag{4}$$

where $\Pi_{k+1}(x)$ is the cdf of the Exp-G family with power parameter $(k+1)$. Equation (4) reveals that the EGGP density function is a linear combination of Exp-G densities. Thus, some structural properties of the new family such as the ordinary and incomplete moments and the generating function can be immediately obtained from well-established properties of the Exp-G distributions.

The properties of Exp-G distributions have been studied by many authors in recent years, see Mudholkar and Srivastava [25] and Mudholkar, Srivastava and Freimer [26] for exponentiated Weibull (EW) distributions, R. C. Gupta, P. L. Gupta and R. D. Gupta [17] for exponentiated Pareto distributions, Gupta and Kundu [18] for exponentiated exponential distributions, Nadarajah and Kotz [29] for the exponentiated-type distributions, Nadarajah [27] for exponentiated Gumbel distributions, Shirke and Kakade [32] for exponentiated log-normal distributions and Nadarajah and Gupta [28] for exponentiated gamma distributions (EGa), among others.

The rest of the paper is outlined as follows. In Section 2 we provide the formulation of EGGP models for two special distributions. Mathematical properties of the EGGP model are discussed in Section 3. In Section 4 we discuss stress-strength models. The order statistics is discussed in Section 5. Parameter estimation procedures using method of maximum likelihood are presented in Section 6. Section 7 provides the application of the two generalized distributions to model real world data. Some concluding remarks are given in Section 8.

2 Special Models

The formulation provided in Section 1 can be used to generalize any classical probability distribution. For illustration purpose we will generalize the following two popular and versatile distributions, namely: the Weibull (W) distribution and the Pareto (Pa) distribution. The parameters of these models are positive real numbers. The pdf and cdf of these distributions are provided in Table 1.

Model	Pdf: $g(x;\psi)$	Cdf: $G(x;\psi)$	Support
W	$\beta\alpha^\beta x^{\beta-1}\exp[-(\alpha x)^\beta]$	$1-\exp[-(\alpha x)^\beta]$	$(0,\infty)$
Pa	$(\frac{\alpha}{x})(\frac{\theta}{x})^\alpha$	$1-(\frac{\theta}{x})^\alpha$	(θ,∞)

Table 1. The pdf and cdf of Weibull and Pareto distributions.

2.1 The EGWP Distribution

The cdf and pdf of the EG-Weibull Poisson (EGWP) distribution are given, respectively, by

$$F(x) = \frac{1-\exp(-\lambda\{1-\exp(-a(\alpha x)^\beta)\}^b)}{[1-\exp(-\lambda)]}$$

and

$$f(x) = \frac{ab\lambda\beta\alpha^\beta x^{\beta-1}}{[1-\exp(-\lambda)]}\,\frac{\exp(-a(\alpha x)^\beta)\{1-\exp(-a(\alpha x)^\beta)\}^{b-1}}{\exp(\lambda\{1-\exp(-a(\alpha x)^\beta)\}^b)}.$$

Plots of the pdf and cdf of the EGWP distribution are displayed in Figure 1 for some parameter values. As we shall see from the graphs EGWP distribution is more flexible compare to classical Weibull distribution.

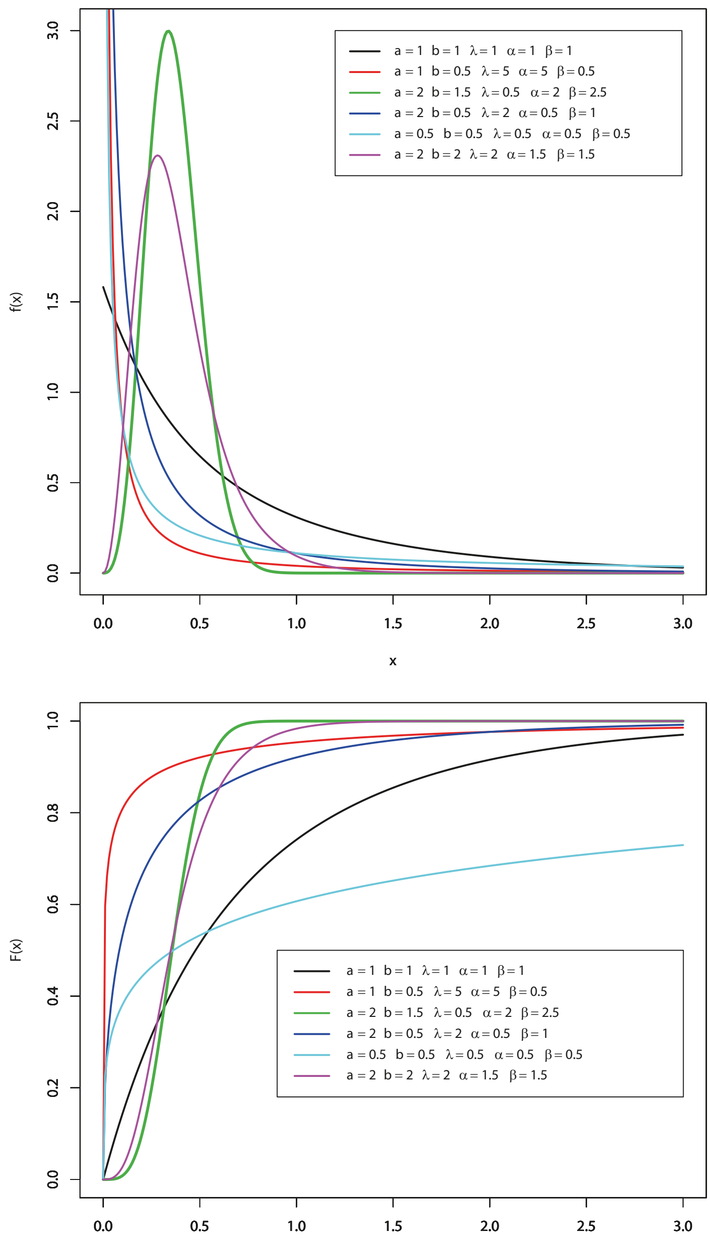

Figure 1. Pdf (top) and cdf (bottom) of EGWP distribution.

2.2 The EGPaP Distribution

The cdf and pdf of the EG-Pareto Poisson(EGPaP) distribution (for $x > \theta$) are, respectively, given by

$$F(x) = \frac{1 - \exp(-\lambda\{1 - (\frac{\theta}{x})^{a\alpha}\}^{b})}{[1 - \exp(-\lambda)]}$$

and

$$f(x) = \frac{ab\lambda\alpha\theta^{\alpha}(\frac{\theta}{x})^{(a-1)\alpha}[1 - (\frac{\theta}{x})^{a\alpha}]^{b-1}}{x^{\alpha+1}[1 - \exp(-\lambda)]\exp(\lambda[1 - (\frac{\theta}{x})^{a\alpha}]^{b})}.$$

Plots of the pdf and cdf of the EGPaP distribution are displayed in Figure 2 for some parameter values. As we shall see from the graphs EGPaP distribution is more flexible compare to the Pareto distribution.

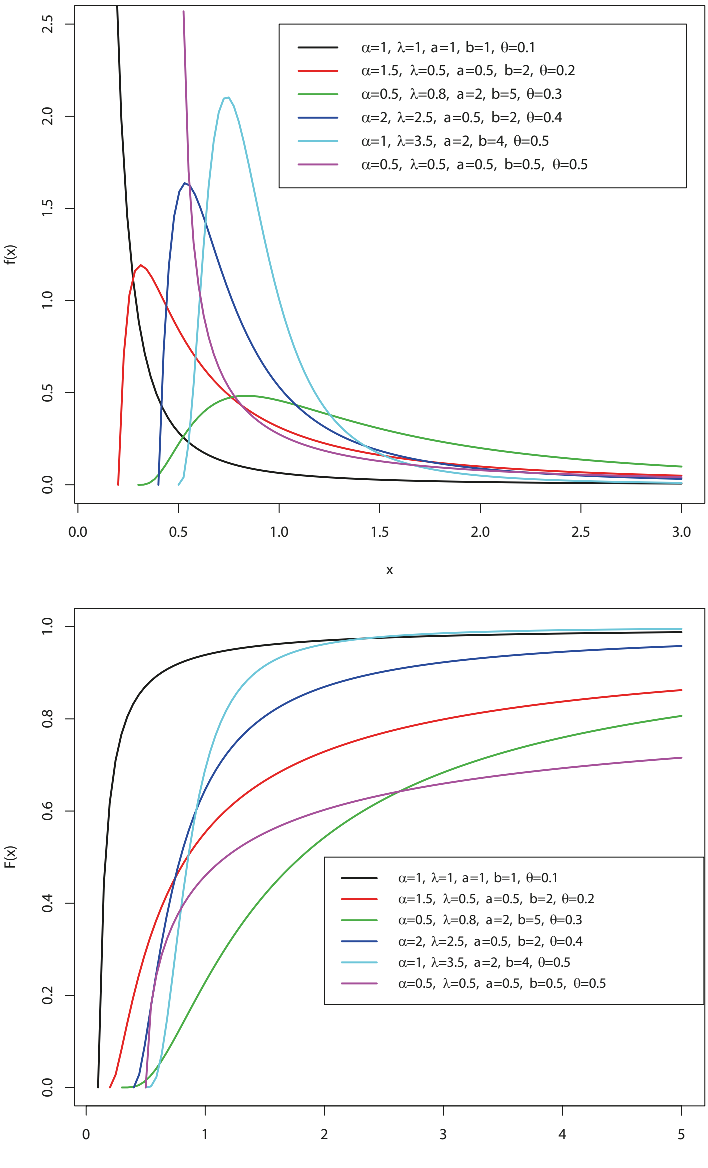

Figure 2. Pdf (top) and cdf (bottom) of EGPaP distribution

3 Mathematical Properties

In this section we provide some structural and mathematical properties of the EGGP distribution including the quantile function, moments, entropy measure, residual and revered residual life.

3.1 Quantile Function

The quantile function of a distribution is the real solution of $F(x_q) = q$ for $0 \le q \le 1$. The quantile function is obtained by inverting equation (1) provided that closed form expression for $Q_G(q) = G^{-1}(q)$ is available. Setting

$$F(x) = \frac{1 - \exp\{-\lambda\{1 - [1 - G(x; \boldsymbol{\psi})]^a\}^b\}}{[1 - \exp(-\lambda)]} = q,$$

we have

$$G(x) = 1 - [1 - \{-\lambda^{-1}\log(1 - (1 - \exp(-\lambda))q)\}^{1/b}]^{1/a}.$$

Therefore

$$x = G^{-1}(1 - [1 - \{-\lambda^{-1}\log(1 - (1 - \exp(-\lambda))q)\}^{1/b}]^{1/a}).$$

We can use the inversion method to simulate random numbers from a given distribution. For example, we can simulate random numbers X from EGWP distribution by

$$1 - \exp(-(\alpha x)^{\beta}) = 1 - [1 - \{-\lambda^{-1}\log(1 - (1 - \exp(-\lambda))U)\}^{1/b}]^{1/a},$$

which implies

$$x = \frac{1}{\alpha}\left\{-\frac{1}{a}\log[1 - \{-\lambda^{-1}\log(1 - (1 - \exp(-\lambda))U)\}^{1/b}]\right\}^{1/\beta},$$

where U has uniform distribution on $(0, 1)$.

3.2 General Properties

The rth ordinary moment of X is given by $\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$. Using (2), we obtain

$$\mu'_r = \sum_{k=0}^{\infty} t_k E(Y_{k+1}^r). \tag{5}$$

Henceforth, Y_{k+1} denotes the Exp-G distribution with power parameter $(k + 1)$, where

$$E(Y_{k+1}^r) = (k+1)\int_{-\infty}^{\infty} x^r g(x; \boldsymbol{\psi})G(x; \boldsymbol{\psi})^k\,dx,$$

which can be computed numerically in terms of the baseline quantile function (qf) $Q_G(u; \boldsymbol{\psi}) = G^{-1}(u; \boldsymbol{\psi})$ as

$$E(Y_{k+1}^r) = (k+1)\int_0^1 Q_G(u; \boldsymbol{\psi})^r u^k\,du.$$

Setting $r = 1$ in (5), we have the mean of X. The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships:

$$\text{Skewness}(X) = \frac{E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3}{[\text{Var}(X)]^{\frac{3}{2}}}$$

and

$$\text{Kurtosis}(X) = \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)[E(X)]^2 - 3[E(X)]^4}{[\text{Var}(X)]^2},$$

where $E(X^2) = \sum_{j=0}^{\infty} t_k E(Y_k^2)$ and $\text{Var}(X) = E(X^2) - [E(X)]^2$. The nth central moment of X, say M_n, are given by

$$M_n = E(X - \mu)^n = \sum_{h=0}^{n}(-1)^h\binom{n}{h}(\mu'_1)^n\mu'_{n-h}.$$

The cumulants (κ_n) of X follow recursively from

$$\kappa_n = \mu'_n - \sum_{r=0}^{n-1}\binom{n-1}{r-1}\kappa_r\mu'_{n-r},$$

where $\kappa_1 = \mu'_1$, $\kappa_2 = \mu'_2 - \mu'^2_1$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu'^3_1$, etc. The skewness and kurtosis measures also can be calculated from the ordinary moments using well-known relationships. The moment generating function (mgf) of X, say $M_X(t) = E(e^{tX})$, is given by

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'_r = \sum_{k,r=0}^{\infty}\frac{t^r t_k}{r!}E(Y_{k+1}^r).$$

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well. The sth incomplete moments, say $\varphi_s(t)$, is given by $\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)\,dx$. Using equation (3), we obtain

$$\varphi_s(t) = \sum_{k=0}^{\infty} t_k \int_{-\infty}^{t} x^s \pi_{k+1}(x)\,dx. \tag{6}$$

The first incomplete moment of the EGGP family, $\varphi_1(t)$, can be obtained by setting $s = 1$ in (6). Another application of the first incomplete moment is related to mean residual life and mean waiting time given by $m_1(t;\boldsymbol{\psi}) = (1 - \varphi_1(t))/R(t;\boldsymbol{\psi}) - t$ and $M_1(t;\boldsymbol{\psi}) = t - (\varphi_1(t)/F(t;\boldsymbol{\psi}))$, respectively. The amount of scatteredness in a population is evidently measured to some extent by the totality of deviations from the mean and median. The mean deviations about the mean $[\delta_\mu(X) = E(|X-\mu_1'|)]$ and about the median $[\delta_\mu(X) = E(|X-M|)]$ of X can be, used as measures of spread in a population, expressed by

$$\delta_\mu(X) = \int_0^\infty |X - \mu_1'| f(x)\,dx = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$$

and

$$\delta_M(X) = \int_0^\infty |X - M| f(x)\,dx = \mu_1' - 2\varphi_1(M),$$

respectively, where $\mu_1' = E(X)$ comes from (5), $F(\mu_1')$ is simply calculated, $\varphi_1(\mu_1')$ is the first incomplete moment and M is the median of X. The mean deviations about the mean $[\delta_1 = E(|X - \mu_1'|)]$ and about the median $[\delta_2 = E(|X - M|)]$ of X are given by $\delta_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(X)$, $M = \text{Median}(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily calculated from (1) and $\varphi_1(t)$ is the first incomplete moment given by (6) with $s = 1$.

Now, we provide two ways to determine δ_1 and δ_2. First, a general equation for $\varphi_1(t)$ can be derived from (3) as

$$\varphi_1(t) = \sum_{k=0}^{\infty} t_k J_{k+1}(x),$$

where

$$J_{k+1}(x) = \int_{-\infty}^{t} x\pi_{k+1}(x)\,dx$$

is the first incomplete moment of the Exp-G distribution. A second general formula for $\varphi_1(t)$ is given by

$$\varphi_1(t) = \sum_{k=0}^{\infty} t_k V_{k+1}(t),$$

where

$$V_{k+1}(t) = (k+1) \int_0^{G(t)} Q_G(u) u^k\,du$$

can be computed numerically. These equations for $\varphi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $\mathrm{B}(\pi) = \varphi_1(q)/(\pi\mu_1')$ and $L(\pi) = \varphi_1(q)/\mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is the qf of X at π. For the EGWP model we have the following results for $r > -\beta$ and $s > -\beta$:

$$\mu_r' = \sum_{k,h=0}^{\infty} t_k \frac{(k+1)(-1)^h}{\alpha^r (h+1)^{(r+\beta)/\beta}} \binom{k}{h} \Gamma\left(1 + \frac{r}{\beta}\right)$$

and

$$\varphi_s(t) = \sum_{k,h=0}^{\infty} t_k \frac{(k+1)(-1)^h}{\alpha^s (h+1)^{(s+\beta)/\beta}} \binom{k}{h} \Gamma\left(1 + \frac{s}{\beta}, \left(\frac{\alpha}{t}\right)^{\beta}\right).$$

3.3 Probability Weighted Moments

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The (s, r)th PWM of X following the EGGP family, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\{X^s F(X)^r\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x)\, dx.$$

Using equations (2) and (3), we can write

$$f(x)F(x)^r = \sum_{k=0}^{\infty} w_k \pi_{k+1}(x),$$

where

$$w_k = \frac{ab}{[1-\exp(-\lambda)]^{r+1}(k+1)} \sum_{w,i,j=0}^{\infty} \frac{(-1)^{w+i+j+k}\binom{r}{w}\binom{b(i+1)-1}{j}\binom{a(1+j)-1}{k}}{i!\lambda^{-i-1}(1+w)^{-i}}.$$

Then the (s, r)th PWM of X can be expressed as

$$\rho_{s,r} = \sum_{k=0}^{\infty} w_k E(Y_{k+1}^s).$$

3.4 Entropy Measures

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_\delta(X) = (1-\delta)^{-1} \log \int_{-\infty}^{\infty} f(x)^\delta\, dx, \quad \delta > 0 \text{ and } \delta \neq 1.$$

Using the power series expansion, the pdf in (2) can be expressed as

$$f(x)^\delta = \sum_{k=0}^{\infty} m_k g(x)^\delta [G(x)]^k,$$

where

$$m_k = \frac{a^\delta b^\delta}{[1-\exp(-\lambda)]^\delta} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+k}\binom{b(i+\delta)-\delta}{j}\binom{a(j+\delta)-\delta}{k}}{i!\lambda^{-\delta-i}\delta^{-i}}.$$

Therefore, the Rényi entropy of the EGGP family is given by

$$I_\delta(X) = (1-\delta)^{-1} \log\left\{\sum_{k=0}^{\infty} m_k \int_{-\infty}^{\infty} g(x)^\delta [G(x)]^k\, dx\right\}.$$

The q-entropy, say $H_q(X)$, can be obtained as

$$H_q(X) = (q-1)^{-1} \log\left\{1 - \left[\sum_{k=0}^{\infty} m_k^* \int_{-\infty}^{\infty} g(x)^q [G(x)]^k\, dx\right]\right\},$$

where

$$m_k^* = \frac{a^q b^q}{[1-\exp(-\lambda)]^q} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+k}\binom{b(i+q)-q}{j}\binom{a(j+q)-q}{k}}{i!\lambda^{-q-i}q^{-i}}, \quad q > 0,\ q \neq 1.$$

The Shannon entropy of a random variable X, say SI, is defined by

$$\mathrm{SI} = E\{-[\log f(X)]\}.$$

It is the special case of the Rényi entropy when $\delta \uparrow 1$.

3.5 Residual Life and Reversed Residual Life

The nth moment of the residual life, say $m_n(t) = E[(X-t)^n \mid X > t]$, $n = 1, 2, \ldots$, uniquely determines $F(x)$. The nth moment of the residual life of X is given by

$$m_n(t) = \frac{1}{1-F(t)} \int_t^{\infty} (x-t)^n \, dF(x).$$

Therefore

$$m_n(t) = \frac{1}{1-F(t)} \sum_{k=0}^{\infty} t_k^* \int_t^{\infty} x^r \pi_{k+1}(x) \, dx, \tag{7}$$

where $t_k^* = t_k \sum_{r=0}^{n} \binom{n}{r}(-t)^{n-r}$. The nth moment of the reversed residual life, say $M_n(t) = E[(t-X)^n \mid X \le t]$ for $t > 0$ and $n = 1, 2, \ldots$, uniquely determines $F(x)$. We obtain

$$M_n(t) = \frac{1}{F(t)} \int_0^{t} (t-x)^n \, dF(x).$$

Then the nth moment of the reversed residual life of X becomes

$$M_n(t) = \frac{1}{F(t)} \sum_{k=0}^{\infty} t_k^{**} \int_0^{t} x^r \pi_{k+1}(x) \, dx, \tag{8}$$

where $t_k^{**} = t_k \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}$. Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $m_1(t) = E[(X-t) \mid X > t]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n = 1$ in equation (7). The mean inactivity time (MIT) or mean waiting time (MWT), also called the mean reversed residual life function, is given by $M_1(t) = E[(t-X) \mid X \le t]$, and it represents the waiting time elapsed since the failure of an item on the condition that this failure had occurred in $(0, t)$. The MIT of the EGGP family of distributions can be obtained easily by setting $n = 1$ in equation (8). For the EGWP model we have the following results: For $n > -\beta$

$$m_n(t) = \frac{1}{1-F(t)} \sum_{k,h=0}^{\infty} t_k^* \frac{(k+1)(-1)^h}{\alpha^n (h+1)^{(n+\beta)/\beta}} \binom{k}{h} \Gamma\left(1 + \frac{n}{\beta}, \left(\frac{\alpha}{t}\right)^{\beta}\right)$$

and

$$M_n(t) = \frac{1}{F(t)} \sum_{k,h=0}^{\infty} t_k^{**} \frac{(k+1)(-1)^h}{\alpha^n (h+1)^{(n+\beta)/\beta}} \binom{k}{h} \Gamma\left(1 + \frac{n}{\beta}, \left(\frac{\alpha}{t}\right)^{\beta}\right).$$

4 Stress-Strength Models

The stress-strength model is the most widely used approach for reliability estimation. This model is used in many applications of physics and engineering, such as strength failure and system collapse. In stress-strength modeling,

$$R = \Pr(X_2 < X_1) = \int_0^{\infty} f(x_1) F(x_2) \, dx$$

is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1 (see [20]). The system fails if and only if the applied stress is greater than its strength and the component will function satisfactorily whenever $X_1 > X_2$. Moreover, R can be considered as a measure of system performance and naturally arises in electrical and electronic systems. Another interpretation can be that the reliability, say R, of the system is the probability that the system is strong enough to overcome the stress imposed on it. Let X_1

and X_2 be two independent random variables have EGGP$(x; a_1, b_1, \lambda_1, \boldsymbol{\psi})$ and EGGP$(x; a_2, b_2, \lambda_2, \boldsymbol{\psi})$ distributions. The reliability R is given by

$$R = \int_0^\infty f_1(x; a_1, b_1, \lambda_1, \boldsymbol{\psi}) F_2(x; a_2, b_2, \lambda_2, \boldsymbol{\psi})\,dx.$$

Then

$$R = \sum_{k,w=0}^\infty \Omega_{k,w},$$

where

$$\Omega_{k,w} = \frac{a_1 a_2 b_1 b_2 (-1)^{k+w}}{[1-\exp(-\lambda_1)][1-\exp(-\lambda_2)](w+1)(k+w+2)} \times \sum_{i,j,m,h=0}^\infty \frac{(-1)^{i+j+m+h}\binom{b_1(i+1)-1}{j}\binom{b_2(m+1)-1}{h}\binom{a_1(j+1)-1}{k}\binom{a_2(h+1)-1}{w}}{i!m!\lambda_1^{-i-1}\lambda_2^{-m-1}}.$$

5 Order Statistics

Let $X_1, \dots, X_n$ be a random sample from the EGGP family of distributions and let $X_{(1)}, \dots, X_{(n)}$ be the corresponding order statistics. The pdf of the ith order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \tag{9}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function. Substituting (1) and (2) into equation (9), we get

$$f(x)F(x)^{j+i-1} = \sum_{k=0}^\infty d_k \pi_{k+1}(x),$$

where

$$d_k = \frac{ab}{[1-\exp(-\lambda)]^{j+i}(k+1)} \sum_{w,m,h=0}^\infty \frac{(-1)^{w+m+h+k}\binom{j+i-1}{w}\binom{b(m+1)-1}{h}\binom{a(h+1)-1}{k}}{m!\lambda^{-m-1}(1+w)^{-m}}.$$

Moreover, the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \frac{(-1)^j\binom{n-i}{j}}{\mathrm{B}(i, n-i+1)} \sum_{k=0}^\infty d_k \pi_{k+1}(x),$$

therefore, the density function of the EGGP order statistics is a mixture of EG densities. Based on the last equation, we note that the properties of $X_{i:n}$ follow from those properties of Y_{k+1}. For example, the moments of $X_{i:n}$ can be expressed as

$$E(X_{i:n}^q) = \sum_{j=0}^{n-i} \frac{(-1)^j\binom{n-i}{j}}{\mathrm{B}(i, n-i+1)} \sum_{k=0}^\infty d_k E(Y_{k+1}^q). \tag{10}$$

For the EGWP model we have

$$E(X_{i:n}^q) = \sum_{k,h=0}^\infty \sum_{j=0}^{n-i} \frac{(k+1)(-1)^{j+h}\binom{n-i}{j}\binom{k}{h} d_k}{\alpha^q \mathrm{B}(i, n-i+1)(h+1)^{(q+\beta)/\beta}} \Gamma\Big(1+\frac{q}{\beta}\Big).$$

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments in equation (10), we can derive explicit expressions for the L-moments of X. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r}\sum_{d=0}^{r-1}(-1)^d\binom{r-1}{d}E(X_{r-d:r}), \quad r \geq 1.$$

The first four L-moments are given by

$$\begin{aligned}
\lambda_1 &= E(X_{1:1}),\\
\lambda_2 &= \frac{1}{2}E(X_{2:2}-X_{1:2}),\\
\lambda_3 &= \frac{1}{3}E(X_{3:3}-2X_{2:3}+X_{1:3}),\\
\lambda_4 &= \frac{1}{4}E(X_{4:4}-3X_{3:4}+3X_{2:4}-X_{1:4}).
\end{aligned}$$

One can simply obtain the L-moments λ_r for X from (10) with $q = 1$.

6 Estimation

Let $X_1, \dots, X_n$ be a random sample from the EGGP distribution with parameters λ, a, b and $\boldsymbol{\psi}$. Further, let $\boldsymbol{\Psi} = (a, b, \lambda, \boldsymbol{\psi}^\top)^\top$ be a $((p+3)\times 1)$ parameter vector, where $\boldsymbol{\psi}$ is a $(p \times 1)$ baseline parameter vector. For determining the MLE of $\boldsymbol{\Psi}$, we have the log-likelihood function

$$\begin{aligned}
\ell = n\log a + n\log b + n\log\lambda - n\log[1-\exp(-\lambda)] + \sum_{i=1}^{n}\log g(x_i;\boldsymbol{\psi})\\
+ (a-1)\sum_{i=1}^{n}\log\overline{G}(x_i;\boldsymbol{\psi}) + (b-1)\sum_{i=1}^{n}\log s_i - \lambda\sum_{i=1}^{n}s_i^b,
\end{aligned}$$

where $\overline{G}(x_i;\boldsymbol{\psi}) = 1 - G(x_i;\boldsymbol{\psi})$, $s_i = 1 - \overline{G}(x_i;\boldsymbol{\psi})^a$. The components of the score vector

$$\boldsymbol{U}(\boldsymbol{\Psi}) = \frac{\partial\ell}{\partial\boldsymbol{\Psi}} = \left(\frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\boldsymbol{\psi}}\right)^\top$$

are given by

$$\begin{aligned}
U_a &= \frac{n}{a} + \sum_{i=1}^{n}\log\overline{G}(x_i;\boldsymbol{\psi}) + (b-1)\sum_{i=1}^{n}\frac{p_i}{s_i} - b\lambda\sum_{i=1}^{n}p_i s_i^{b-1},\\
U_b &= \frac{n}{b} + \sum_{i=1}^{n}\log s_i - \lambda\sum_{i=1}^{n}\frac{\log s_i}{s_i^{-b}},\\
U_\lambda &= \frac{n}{\lambda} + \frac{n\exp(-\lambda)}{1-\exp(-\lambda)} - \sum_{i=1}^{n}s_i^b
\end{aligned}$$

and (for $r = 1, 2, \dots, p$)

$$U_{\boldsymbol{\psi}_r} = \sum_{i=1}^{n}\frac{g'(x_i;\boldsymbol{\psi})}{g(x_i;\boldsymbol{\psi})} - (a-1)\sum_{i=1}^{n}\frac{G'(x_i;\boldsymbol{\psi})}{\overline{G}(x_i;\boldsymbol{\psi})} + (b-1)\sum_{i=1}^{n}\frac{q_i}{s_i} - b\lambda\sum_{i=1}^{n}\frac{q_i}{s_i^{1-b}},$$

where

$$\begin{aligned}
g'(x_i;\boldsymbol{\psi}) &= \frac{\partial g(x_i;\boldsymbol{\psi})}{\partial\boldsymbol{\psi}_r},\\
G'(x_i;\boldsymbol{\psi}) &= \frac{\partial G(x_i;\boldsymbol{\psi})}{\partial\boldsymbol{\psi}_r},\\
p_i &= -\frac{\log\overline{G}(x_i;\boldsymbol{\psi})}{\overline{G}(x_i;\boldsymbol{\psi})^{-a}},\\
q_i &= \frac{aG'(x_i;\boldsymbol{\psi})}{\overline{G}(x_i;\boldsymbol{\psi})^{1-a}}.
\end{aligned}$$

Setting the nonlinear system of equations $U_a = U_b = U_\lambda = 0$ and $U_{\boldsymbol{\psi}} = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\boldsymbol{\Psi}} = (\hat{a}, \hat{b}, \hat{\lambda}, \widehat{\boldsymbol{\psi}}^\intercal)^\intercal$. To solve these equations, it is usually more convenient to use nonlinear optimization methods, such as the quasi-Newton algorithm, to numerically maximize ℓ. For interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\boldsymbol{\Psi}) = \{\frac{\partial^2 \ell}{\partial r \partial s}\}$ (for $r, s = a, b, \lambda, \boldsymbol{\psi}$), whose elements can be computed numerically. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\boldsymbol{\Psi}}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\boldsymbol{\Psi}})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\boldsymbol{\Psi}})$ is the total observed information matrix evaluated at $\widehat{\boldsymbol{\Psi}}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method. The elements of $J(\boldsymbol{\Psi})$ are given in Appendix A.

7 Applications

In many statistical applications, the interest is centered on estimating the parameters and evaluate the goodness-of-fit of the model to analyze the data on hand. In this section, we provide the effectiveness of the EGGP distribution by means of modeling two different data sets choosing two special models discussed in Section 2. These data sets have been used by several authors to show the applicability of other competing models. We also provide a formative evaluation of the goodness-of-fit of the models and make comparisons with other distributions. The measures of goodness-of-fit, including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson–Darling (A^*), Cramér–von Mises (W^*) and Kolmogrov–Smirnov (KS) statistics, are computed to compare the fitted models. The statistics $A*$ and W^* are described in detail by Chen and Balakrishnan [9]. In general, the smaller the values of these statistics, the better the fit to the data. One can employ the Likelihood Ratio Test (LRT) to contrast the adaptability of the EGGP distribution over the other distributions. The required computations are carried out in the R language.

Example 1: Cancer Patient Data. This data set describes the remission times (in months) of a random sample of 128 bladder cancer patients studied by Lee and Wang [21]. For these data, we compare the fit of the EGWP with the other five parameter distributions which has been generalized using the Weibull genesis. We compare the fits of the EGWP with the generalized transmuted-W (GTW) distribution (Nofal, Afify, Yousof and Cordeiro [30]), the McDonald Weibull (McW) distribution (Cordeiro, Hashimoto and Ortega [15]), the modified beta Weibull (MBW) distribution (Khan [19]) and the transmuted additive Weibull (TAW) distribution (Elbatal and Aryal [16]) with the corresponding densities given by (for $x > 0$)

McW: $$f(x) = \frac{\beta\lambda\alpha^\beta}{B(a/\lambda, b)} x^{\beta-1} e^{-(\alpha x)^\beta} [1 - e^{-(\alpha x)^\beta}]^{a-1} \{1 - (1 - e^{-(\alpha x)^\beta})^\lambda\}^{b-1},$$

MBW: $$f(x) = \frac{\beta\alpha^{-\beta}\lambda^a}{B(a/\lambda, b)} x^{\beta-1} \mathrm{e}^{-b(\frac{x}{\alpha})^\beta} [1 - \mathrm{e}^{-(\frac{x}{\alpha})^\beta}]^{a-1} \{1 - (1 - \lambda)[1 - \mathrm{e}^{-(\frac{x}{\alpha})^\beta}]^\lambda\}^{-a-b},$$

GTW: $$f(x) = \frac{\beta\alpha^\beta x^{\beta-1}}{\mathrm{e}^{(\alpha x)^\beta} [1 - \mathrm{e}^{-(\alpha x)^\beta}]^{1-a}} \{a(1 + \lambda) - \lambda(a + b)[1 - \mathrm{e}^{-(\alpha x)^\beta}]^b\},$$

TAW: $$f(x) = \frac{(\alpha b x^{b-1} + a\beta x^{\beta-1})}{e^{(\alpha x^b + a x^\beta)}} \{1 - \lambda + 2\lambda e^{-(\alpha x^b + a x^\beta)}\}.$$

The parameters of the above densities are all positive real numbers except $|\lambda| \le 1$ for the GTW and TAW distributions. The statistics of the fitted models are presented in Table 2 and the MLEs and the corresponding standard errors are given in Table 3. We note from Table 2 that the EGWP gives the lowest values for the AIC, BIC, CAIC, HQIC, A^* and W^* statistics as compared to the other generalizations of the Weibull distribution. Therefore, we conclude that the EGWP distribution yields the best fit to model the remission times of bladder cancer patients.

Example 2: Flood Data. This data set describes the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada for the years 1958–1984. These data were analyzed by many

	Goodness-of-fit criteria					
Model	**AIC**	**CAIC**	**HQIC**	**BIC**	**W***	**A***
EGWP	829.448	829.939	835.242	843.708	0.0227	0.1505
GTW	831.347	831.839	837.141	845.607	0.0469	0.3058
McW	831.680	832.172	837.474	845.94	0.0504	0.3299
MBW	838.027	838.519	843.821	852.288	0.1068	0.7207
TAW	838.478	838.97	844.272	852.739	0.1129	0.7033

Table 2. The AIC, CAIC, HQIC, BIC, W^* and A^* statistics for cancer patient data.

	Estimates				
Model	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
EGWP	0.4202	1.5857	0.1540	0.9237	3.7776
	(2.3728)	(0.7778)	(0.9355)	(0.3111)	(2.4207)
GTW	2.7965	0.0128	0.2991	0.6542	0.002
	(1.117)	(7.214)	(0.151)	(0.121)	(1.769)
McW	4.0633	2.6036	0.1192	0.5582	0.0393
	(2.111)	(2.452)	(0.109)	(0.178)	(0.202)
MBW	57.4167	19.3859	10.1502	0.1632	2.0043
	(37.317)	(13.490)	(22.437)	(0.044)	(0.789)
TAW	0.00003	1.0065	0.1139	0.9722	−0.1630
	(0.0061)	(0.035)	(0.032)	(0.125)	(0.280)

Table 3. MLEs and their standard errors (in parenthesis) for the cancer patient data.

authors including Choulakian and Stephenes [10], Akinsete, Famoye and Lee [4], Nadarajah [27], Merovci and Puka [24], Bourguignon, Silva, Zea and Cordeiro [8], among others. We compare the fits of EGPaP with the Kumarswamy Pareto (KwP) distribution, the beta Pareto (BP) distribution, the transmuted Pareto (TP) distribution, the exponentiated Pareto (EP) distribution and the Pareto(P) distribution whose pdf are given by

$$\text{KwP:} \quad f(x; a, b, \alpha, \theta) = \frac{ab\alpha\theta^{\alpha}}{x^{\alpha+1}}\left[1 - \left(\frac{\theta}{x}\right)^{\alpha}\right]^{a-1}\left\{1 - \left[1 - \left(\frac{\theta}{x}\right)^{\alpha}\right]^{a}\right\}^{b-1},$$

$$\text{BP:} \quad f(x; a, b, \alpha, \theta) = \frac{1}{\mathrm{B}(a, b)}\frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}\left[1 - \left(\frac{\theta}{x}\right)^{\alpha}\right]^{a-1}\left(\frac{\theta}{x}\right)^{\alpha(b-1)},$$

$$\text{TP:} \quad f(x; \alpha, \theta, \lambda) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}\left[1 - \lambda + 2\lambda\left(\frac{\theta}{x}\right)^{\alpha}\right].$$

The parameters of the above densities are all positive real numbers except $|\lambda| \le 1$ for TP distribution. The MLEs and corresponding standard errors are given in Table 4 and the statistics of the fitted models are presented in Table 5. We note from Table 5 that the EGPaP gives the lowest values for the AIC, CAIC, BIC, HQIC and KS statistics as compared to the other generalizations of the Pareto distribution. Therefore, the EGPaP distribution yields the best fit to model the exceedances of flood peaks.

Fitted pdf, cdf and QQ-plots for both data are provided in Figure 3. It can be observed that the EGWP distribution is appropriate to model the cancer patient data and the EGPaP distribution is appropriate to model the flood peak exceedance data.

8 Conclusions

In this study, we have introduced the so-called exponentiated generalized G-Poisson family of distribution. Some mathematical properties of the new family including ordinary and incomplete moments, quantile and generating functions, mean deviations, order statistics and their moments, reliability and Shannon entropy

Model	$\widehat{a}$	$\widehat{b}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\theta}$
EGPaP	6.5163	4.9880	20.4148	0.0264	0.1
	(2.2125)	(0.8487)	(8.9005)	(0.0088)	–
KwP	2.8553	85.8468	–	0.0528	0.1
	(0.3371)	(60.4213)	–	(0.0185)	–
BP	3.1473	85.7508	–	0.0088	0.1
	(0.4993)	(0.0001)	–	(0.0015)	–
TP	1	1	−0.952	0.3490	0.1
	–	–	(0.089)	(0.072)	–
EP	2.8797	1	–	0.4241	0.1
	(0.4911)	–	–	(0.0463)	–
P	1	1	–	0.2438	0.1
	–	–	–	(0.0287)	–

Table 4. Estimated parameters and their standard errors for Wheaton river data.

	Statistics					
Model	$-\ell(\cdot, x)$	AIC	CAIC	BIC	HQIC	KS
EGPaP	255.131	520.262	521.171	531.645	524.794	0.1428
KwP	271.200	548.400	548.753	555.230	551.119	0.1700
BP	283.700	573.400	573.753	580.230	576.119	0.1747
TP	286.201	576.402	576.575	580.954	578.214	0.2870
EP	287.300	578.600	578.774	583.153	580.413	0.1987
P	303.100	608.200	608.257	610.477	609.106	0.3324

Table 5. The AIC, CAIC, BIC, HQIC and KS test statistics of Wheaton river data.

are derived. Although this generalization technique can be used to generalize many other distributions, for illustration purposes we have chosen the Weibull distribution and the Pareto distribution as base distributions. The importance and flexibility of the new family are illustrated by means of two different examples, one for each generalized family. We hope that this study will serve as a reference and help to advance future research in the subject area.

A Appendix

The elements of the observed matrix $J(\boldsymbol{\Psi})$ are given below:

$$U_{aa} = -\frac{n}{a^2} - (b-1)\sum_{i=1}^{n}\left\{\frac{\overline{G}(x_i;\boldsymbol{\psi})^a}{s_i[\log \overline{G}(x_i;\boldsymbol{\psi})]^{-2}} + \frac{p_i^2}{s_i^2}\right\} - b(b-1)\lambda\sum_{i=1}^{n}\frac{s_i^{b-2}}{p_i^{-2}} + b\lambda\sum_{i=1}^{n}\frac{s_i^{b-1}\overline{G}(x_i;\boldsymbol{\psi})^a}{[\log \overline{G}(x_i;\boldsymbol{\psi})]^{-2}},$$

$$U_{ab} = \sum_{i=1}^{n}\frac{p_i}{s_i} - \lambda\sum_{i=1}^{n}p_i s_i^{b-1}(1 + b\log s_i),$$

$$U_{a\lambda} = -b\sum_{i=1}^{n}\frac{p_i}{s_i^{1-b}},$$

$$U_{a\boldsymbol{\psi}} = -\sum_{i=1}^{n}\frac{G'(x_i;\boldsymbol{\psi})}{\overline{G}(x_i;\boldsymbol{\psi})} + (b-1)\sum_{i=1}^{n}\frac{s_i m_i - p_i q_i}{s_i^2} - b\lambda\sum_{i=1}^{n}\frac{m_i + (b-1)p_i q_i s_i^{-1}}{s_i^{1-b}},$$

$$U_{bb} = -\frac{n}{b^2} - \lambda\sum_{i=1}^{n}\frac{(\log s_i)^2}{s_i^{-b}},$$

$$U_{b\lambda} = -\sum_{i=1}^{n}\frac{\log s_i}{s_i^{-b}}$$

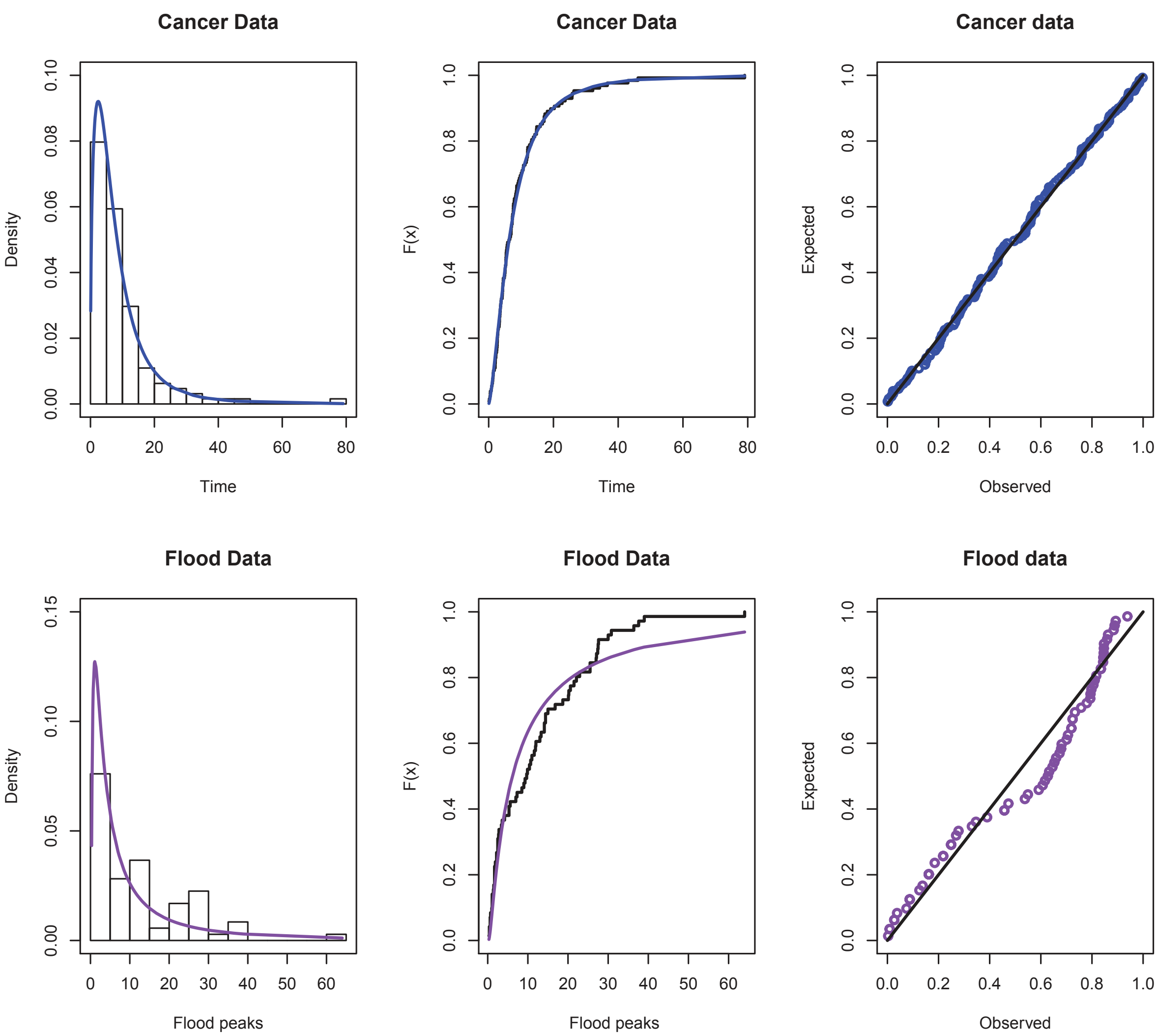

Figure 3. Top: Fitted pdf (left), cdf (center) and QQ-plots (right)of the EGWP distribution. Bottom: Fitted pdf (left), cdf (center) and QQ-plots (right) of the EGPaP distribution.

and

$$U_{\lambda\boldsymbol{\psi}} = -b\sum_{i=1}^{n}\frac{q_i}{s_i^{1-b}},$$

$$U_{b\boldsymbol{\psi}} = \sum_{i=1}^{n}\frac{q_i}{s_i} - \lambda\sum_{i=1}^{n}\frac{\{1+b\log s_i\}}{q_i^{-1}s_i^{1-1}},$$

$$U_{\lambda\lambda} = -\frac{n}{\lambda^2} + \frac{ne^{-\lambda}(1-e^{-\lambda})+ne^{-2\lambda}}{(1-e^{-\lambda})^2},$$

$$U_{\boldsymbol{\psi}_r\boldsymbol{\psi}_l} = \sum_{i=1}^{n}\frac{g(x_i;\boldsymbol{\psi})g''(x_i;\boldsymbol{\psi}) - [h'(x_i;\boldsymbol{\psi})]^2}{g(x_i;\boldsymbol{\psi})^2} + (b-1)\sum_{i=1}^{n}\frac{w_i}{s_i} - (b-1)\sum_{i=1}^{n}\frac{q_i^2}{s_i^2}$$
$$-(a-1)\sum_{i=1}^{n}\frac{\overline{G}(x_i;\boldsymbol{\psi})G''(x_i;\boldsymbol{\psi}) + [G'(x_i;\boldsymbol{\psi})]^2}{\overline{G}(x_i;\boldsymbol{\psi})^2} - b\lambda\sum_{i=1}^{n}[(b-1)q_i^2 s_i^{b-2} + w_i s_i^{b-1}],$$

where

$$g''(x_i;\boldsymbol{\psi}) = \left[\frac{\partial^2 g(x_i;\boldsymbol{\psi})}{\partial\boldsymbol{\psi}_r\partial\boldsymbol{\psi}_l}\right], \quad G''(x_i;\boldsymbol{\psi}) = \left[\frac{\partial^2 G(x_i;\boldsymbol{\psi})}{\partial\boldsymbol{\psi}_r\partial\boldsymbol{\psi}_l}\right]$$

and

$$m_i = \overline{G}(x_i; \boldsymbol{\psi})^a \overline{G}(x_i; \boldsymbol{\psi})^{-1} G'(x_i; \boldsymbol{\psi}) + \frac{a \log \overline{G}(x_i; \boldsymbol{\psi}) G'(x_i; \boldsymbol{\psi})}{\overline{G}(x_i; \boldsymbol{\psi})^{1-a}},$$

$$w_i = a\{\overline{G}(x_i; \boldsymbol{\psi})^{a-1} G''(x_i; \boldsymbol{\psi}) - (a-1)[G'(x_i; \boldsymbol{\psi})]^2 \overline{G}(x_i; \boldsymbol{\psi})^{a-2}\}.$$

Acknowledgment: The authors are grateful to the editor and anonymous reviewer for their constructive comments and valuable suggestions which certainly improved the presentation and quality of the paper.

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