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The extended linear complementarity problem[*]

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The Extended Linear Complementarity Problem[1]

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Abstract.

In this paper we define the Extended Linear Complementarity Problem (ELCP), an extension of the well-known Linear Complementarity Problem (LCP). We study the general solution set of an ELCP and we present an algorithm to find all its solutions. Finally we show that the ELCP can be used to solve some important problems in the max algebra.

1 Introduction

1.1 Overview

In this paper we propose the Extended Linear Complementarity Problem (ELCP), an extension of the Linear Complementarity Problem, which is one of the fundamental problems in mathematical programming. We shall present an algorithm to find all solutions of an ELCP. The core of this algorithm is formed by an adaptation of Motzkin's double description method for solving sets of linear inequalities [10]. Our algorithm yields a description of the complete solution set of an ELCP by central and extreme rays. In that way it provides a geometrical insight in the solution set of the ELCP and related problems.
The formulation of the ELCP arose from our work in the study of discrete event systems. We shall briefly indicate how the ELCP can be used to solve a system of multivariate polynomial equalities and inequalities in the max algebra, the framework that we use to model a class of discrete event systems. This allows us to solve many other problems in the max algebra such as matrix decompositions, state space transformations, minimal state space realization of single input single output discrete event systems and so on [7, 8].

1.2 Notations

If a is a vector then a_i represents the ith component of a. If A is an m by n matrix then the entry on the ith row and the jth column is denoted by a_{ij}. We use $A_{.j}$ to represent the jth column of A. The n by n identity matrix is denoted by I_n and the m by n zero matrix by $O_{m,n}$.

1.3 The Linear Complementarity Problem

One of the possible formulations of the Linear Complementarity Problem (LCP) is the following [4]:

Given a matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, find two vectors $w, z \in \mathbb{R}^n$ such that

$$w \geqslant 0 \ ,$$
$$z \geqslant 0 \ ,$$
$$w = q + Mz \ ,$$
$$z^T w = 0 \ ,$$

or show that no such vectors w and z exist.

The LCP has numerous applications such as linear and quadratic programming problems, the bimatrix game problem, the market equilibrium problem, the optimal invariant capital stock problem, the optimal stopping problem, etc. [4]. This makes the LCP one of the fundamental problems of mathematical programming.

2 The Extended Linear Complementarity Problem

In this section we introduce the Extended Linear Complementarity Problem (ELCP) and we present an algorithm to find all its solutions. We also give a geometric characterization of the solution set of a general ELCP.

2.1 Problem formulation

Consider the following problem:

Given two matrices $A \in \mathbb{R}^{p \times n}, B \in \mathbb{R}^{q \times n}$, two vectors $c \in \mathbb{R}^p, d \in \mathbb{R}^q$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a vector $x \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0 \ , \tag{1}$$
$$Ax \ \geqslant \ c \ , \tag{2}$$
$$Bx \ = \ d \ , \tag{3}$$

or show that no such vector exists.

We call this problem the Extended Linear Complementarity Problem since it is an extension of the Linear Complementarity Problem: if we set

$$x = \begin{bmatrix} w \\ z \end{bmatrix}, \ A = I_{2n}, \ B = [\, I_n \ -M \,], \ c = O_{2n,1}, \ d = q, \ m = n \ ,$$

and $\phi_j = \{j, j+n\}$ for $j \in \{1, 2, \ldots, n\}$ in the formulation of the ELCP we get an LCP. So the LCP can be considered as a particular case of the ELCP. In [6] we have shown that the Vertical LCP of Cottle and Dantzig [3] and the Generalized LCP of De Moor [5] are also specific cases of an ELCP. Therefore the ELCP can be viewed as a unifying framework for the LCP and its various extensions.

Equation (1) represents the *complementarity condition*. One possible interpretation of this condition is the following: since $Ax \geqslant c$, condition (1) is equivalent to

$$\prod_{i \in \phi_j} (Ax - c)_i = 0 \ , \quad \forall j \in \{1, 2, \ldots, m\} \ .$$

So we could say that each set ϕ_j corresponds to a subgroup of inequalities of $Ax \geqslant c$ and that in each group at least one inequality should hold with equality:

$$\forall j \in \{1, 2, \ldots, m\} \ : \ \exists i \in \phi_j \text{ such that } (Ax - c)_i = 0 \ .$$

2.2 The homogeneous ELCP

In order to solve the ELCP we have to homogenize it: we introduce a scalar $\alpha \geqslant 0$ and define

$$u = \begin{bmatrix} x \\ \alpha \end{bmatrix}, \ P = \begin{bmatrix} A & -c \\ O_{n,1} & 1 \end{bmatrix} \text{ and } Q = [\, B \ -d \,] \ .$$

Then we get a homogeneous ELCP of the following form:

Given two matrices $P \in \mathbb{R}^{p \times n}, Q \in \mathbb{R}^{q \times n}$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a non-trivial vector $u \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pu)_i = 0 \ , \tag{4}$$

$$Pu \ \geqslant \ 0 \ , \tag{5}$$

$$Qu \ = \ 0 \ , \tag{6}$$

or show that no such vector u exists.

In the next section we present an algorithm to solve this homogeneous ELCP. Afterwards we shall extract the solutions of the original inhomogeneous ELCP.

2.3 The ELCP algorithm

When we consider the homogeneous ELCP we see that we have a system of homogeneous linear equalities and inequalities subject to a complementarity condition.

The solution set of the system of homogeneous linear inequalities and equalities $(5) - (6)$ is a polyhedral cone $\mathcal{P}$ and can be described using two sets of rays: a set of central rays $\mathcal{C}$ and a

set of extreme rays $\mathcal{E}$. A set of central rays can be considered as a basis for the largest linear subspace of the polyhedral cone $\mathcal{P}$. If $c \in \mathcal{C}$ then $Pc = 0$, and if $e \in \mathcal{E}$ then $Pe \neq 0$.

A vector u is a solution of $(5) - (6)$ if and only if it can be written as

$$u = \sum_{c_i \in \mathcal{C}} \alpha_i \, c_i \; + \; \sum_{e_i \in \mathcal{E}} \beta_i \, e_i \;\; , \tag{7}$$

with $\alpha_i \in \mathbb{R}$ and $\beta_i \geqslant 0$.

To calculate $\mathcal{C}$ and $\mathcal{E}$ we use an iterative algorithm that is an adaptation and extension of the double description method of Motzkin [10]. During the iteration we already remove rays that do not satisfy the partial complementarity condition since such rays cannot yield solutions of the ELCP. In the kth step the partial complementarity condition is defined as follows:

$$\prod_{i \in \phi_j} (Pu)_i = 0, \qquad \forall j \in \{1, 2, \ldots, m\} \text{ such that } \phi_j \subset \{1, 2, \ldots, k\} \;\; . \tag{8}$$

For $k \geqslant p$ the partial complementarity condition (8) coincides with the full complementarity condition (4). This leads to the following algorithm:

Algorithm 1 : Calculation of the central and extreme rays.

Initialization:

- $\mathcal{C}_0 := \{c_i \mid c_i = (I_n)_{.i} , \text{ for } i = 1, 2, \ldots, n\}$
- $\mathcal{E}_0 := \emptyset$

Iteration:
     for $k := 1, 2, \ldots, p + q$,

- Calculate the intersection of the current polyhedral cone (described by $\mathcal{C}_{k-1}$ and $\mathcal{E}_{k-1}$) with the half-space or hyperplane determined by the kth inequality or equality. This yields a new polyhedral cone described by $\mathcal{C}_k$ and $\mathcal{E}_k$.
- Remove the rays that do not satisfy the partial complementarity condition.

Result: $\mathcal{C} := \mathcal{C}_{p+q}$ and $\mathcal{E} := \mathcal{E}_{p+q}$

Not every combination of the form (7) satisfies the complementarity condition. Although every linear combination of the central rays satisfies the complementarity condition, not every positive combination of the extreme rays satisfies the complementarity condition. Therefore we introduce the concept of cross-complementarity:

Definition 2.1 (Cross-complementarity)
Let $\mathcal{E}$ be a set of extreme rays of an homogeneous ELCP. A subset $\mathcal{E}_s$ of $\mathcal{E}$ is cross-complementary if every combination of the form

$$u = \sum_{e_i \in \mathcal{E}_s} \beta_i \, e_i \;\; ,$$

with $\beta_i \geqslant 0$, satisfies the complementarity condition.

In [6] we have proven that in order to check whether a set $\mathcal{E}_s$ is cross-complementary it suffices to test only one strictly positive combination of the rays in $\mathcal{E}_s$, e.g. the combination with $\forall i : \beta_i = 1$. Now we can determine Γ, the set of cross-complementary sets of extreme rays:

$$\Gamma = \{\, \mathcal{E}_s \mid \mathcal{E}_s \text{ is cross-complementary} \,\} \ \ .$$

Algorithm 2 : Determination of the cross-complementary sets of extreme rays.

Initialization:

- $\Gamma := \emptyset$
- Construct the cross-complementarity graph $\mathcal{G}$ with a node n_i for each ray $e_i \in \mathcal{E}$ and an edge between nodes n_k and n_l if the set $\{e_k, e_l\}$ is cross-complementary.
- $\mathcal{S} := \{e_1\}$

Depth-first search in $\mathcal{G}$:

- Select a new node n^{new} that is connected by an edge with all the nodes that correspond to rays of $\mathcal{S}$ and add the corresponding ray e^{new} to the test set: $\mathcal{S}^{\mathrm{new}} := \mathcal{S} \cup \{e^{\mathrm{new}}\}$.
- **if** $\mathcal{S}^{\mathrm{new}}$ is cross-complementary
  then $\mathcal{S} := \mathcal{S}^{\mathrm{new}}$
  Select a new node and add it to the test set.
  else If $\mathcal{S}$ is not a subset of one of the sets that are already in Γ, then add $\mathcal{S}$ to Γ: $\Gamma := \Gamma \cup \{\mathcal{S}\}$. Go back to the last point where a choice was made.
- Continue until all possible choices have been considered.

Result: Γ

Once $\mathcal{C}$, $\mathcal{E}$ and Γ are determined, the solution set of the homogeneous ELCP is given by the following theorem:

Theorem 2.2 *When a set of central rays $\mathcal{C}$, a set of extreme rays $\mathcal{E}$ and the set of cross-complementary sets of extreme rays Γ are given then u is a solution of the homogeneous ELCP if and only if there exists a set $\mathcal{E}_s \in \Gamma$ such that u can be written as*

$$u = \sum_{c_i \in \mathcal{C}} \alpha_i\, c_i \ + \ \sum_{e_i \in \mathcal{E}_s} \beta_i\, e_i \ , \tag{9}$$

with $\alpha_i \in \mathbb{R}$ and $\beta_i \geqslant 0$.

Finally we have to extract the solutions of the original ELCP, i.e. we have to retain solutions of the form (9) that have an α component equal to 1 ($u_\alpha = 1$). We transform the sets $\mathcal{C}$, $\mathcal{E}$ and Γ as follows:

- If $c \in \mathcal{C}$ then $c_\alpha = 0$. We drop the α component and put the result in the set $\mathcal{X}^{\mathrm{cen}}$, a set of central rays.

- If $e \in \mathcal{E}$ then there are two possibilities:

– If $e_\alpha = 0$ then we drop the α component and put the result in the set $\mathcal{X}^{\text{inf}}$, a set of vertices at infinity.

– If $e_\alpha > 0$ then we normalize e such that $e_\alpha = 1$. Next we drop the α component and put the result in the set $\mathcal{X}^{\text{fin}}$, a set of finite vertices.

- For each set $\mathcal{E}_s \in \Gamma$ we construct the set of corresponding vertices at infinity $\mathcal{X}_s^{\text{inf}}$ and the set of corresponding finite vertices $\mathcal{X}_s^{\text{fin}}$. If $\mathcal{X}_s^{\text{fin}} \neq \emptyset$ then we add the pair $\left\{ \mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}} \right\}$ to Λ, the set of pairs of cross-complementary subsets of vertices.

Now we can characterize the solution set of the ELCP:

Theorem 2.3 *When $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and Λ are given, then x is a solution of the ELCP if and only if there exists a pair $\left\{ \mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}} \right\} \in \Lambda$ such that*

$$x = \sum_{x_k \in \mathcal{X}^{\text{cen}}} \lambda_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k + \sum_{x_k \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k \ ,$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ and $\sum_k \mu_k = 1$.

This leads to:

Theorem 2.4 *The general solution set of an ELCP consists of the union of faces of a polyhedron.*

For a more detailed and precise description of the algorithms and for the proofs of the theorems of this section the interested reader is referred to [6], where also a worked example can be found.

2.4 The computational complexity of the ELCP

Our ELCP algorithm yields the entire solution set of the ELCP. This sometimes leads to large computation times and storage space requirements especially if the number of variables and equations is large. Therefore it might be interesting to develop algorithms that only find one solution. However, the following theorem shows that the ELCP is intrinsically a computationally hard problem:

Theorem 2.5 *The general ELCP is an NP-hard problem.*

Proof: The decision problem that corresponds to the ELCP belongs to NP: a nondeterministic algorithm can guess a vector x and then check in polynomial time whether x satisfies the complementarity condition and the system of linear equalities and inequalities. Chung [2] has proven that the decision problem that corresponds to the LCP is in general an NP-complete problem. The LCP is a special case of the ELCP and therefore the decision problem that corresponds to the ELCP is also NP-complete. This means that in general the ELCP is NP-hard.

So the ELCP can probably not be solved in polynomial time (unless the class P would coincide with the class NP). The interested reader is referred to [9] for an extensive treatment of NP-completeness.

3 A link between the max algebra and the ELCP

The formulation of the ELCP arose from our research on discrete event systems, examples of which are flexible manufacturing systems, traffic networks and telecommunications networks. Normally the behavior of discrete event systems is highly nonlinear. However, when the order of the events is known or fixed some of these systems can be described by a *linear* description in the max algebra [1].

The basic operations of the max algebra are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$x \oplus y \;=\; \max(x, y) \;, \tag{10}$$

$$x \otimes y \;=\; x + y \;. \tag{11}$$

The reason for choosing these symbols is that many results from linear algebra can be translated to the max algebra simply by replacing $+$ by $\oplus$ and $\times$ by $\otimes$. The max-algebraic power is defined as follows:

$$x^{\otimes^a} \;=\; a \cdot x \;. \tag{12}$$

Many important problems in the max algebra can be reformulated as a set of max-algebraic polynomial equalities and inequalities in the max algebra:

Given a set of integers $\{m_k\}$ and three sets of real numbers $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $i \in \{1, \ldots, m_k\}$, $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, p_1, p_1+1, \ldots, p_1+p_2\}$, find a vector $x \in \mathbb{R}^n$ that satisfies

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \,, \qquad \text{for } k = 1, 2, \ldots, p_1 \,,$$

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} \leqslant b_k \,, \qquad \text{for } k = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2 \,,$$

or show that no such vector x exists.

Note that the exponents can be negative or real. In [8] we have proven the following theorem:

Theorem 3.1 *A set of multivariate polynomial equalities and inequalities in the max algebra is equivalent to an extended linear complementarity problem.*

We shall illustrate this by an example:

Example 3.2
Consider the following set of multivariate polynomial equalities and inequalities:

$$3 \otimes x_1^{\otimes^3} \otimes x_2^{\otimes^{-1}} \otimes x_4^{\otimes^{-2}} \;\oplus\; x_1^{\otimes^{-1}} \otimes x_3^{\otimes^3} \;\oplus\; 7 \otimes x_2 \otimes x_3^{\otimes^2} \;=\; 3 \tag{13}$$

$$2 \otimes x_1 \otimes x_3^{\otimes^2} \;\oplus\; (-1) \otimes x_1 \otimes x_4^{\otimes^{-2}} \;=\; 4 \tag{14}$$

$$4 \otimes x_1 \otimes x_3^{\otimes^{-4}} \;\leqslant\; 6 \;. \tag{15}$$

Let us first consider the first term of (13). Using definitions (11) and (12) we find that this term is equivalent to

$$3 \,+\, 3x_1 \,+\, (-1)x_2 \,+\, (-2)x_4 \;.$$

The other terms of (13) can be transformed to linear algebra in a similar way. Each term has to be smaller than 3 and at least one of them has to be equal to 3. So we get a group of three inequalities in which at least one inequality should hold with equality. If we also include expressions (14) and (15) we get the following ELCP:

Given

$$
A = \begin{bmatrix} -3 & 1 & 0 & 2 \\ 1 & 0 & -3 & 0 \\ 0 & -1 & -2 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & 2 \\ -1 & 0 & 4 & 0 \end{bmatrix} \text{ and } c = \begin{bmatrix} 0 \\ -3 \\ 4 \\ -2 \\ -5 \\ -2 \end{bmatrix} ,
$$

find a column vector $x \in \mathbb{R}^4$ such that

$$
(Ax - c)_1 \, (Ax - c)_2 \, (Ax - c)_3 + (Ax - c)_4 \, (Ax - c)_5 = 0 \text{ and } Ax \geqslant c .
$$

The ELCP algorithm yields the vertices of Table 1 and the pairs of subsets of Table 2. There are no central rays.
Any arbitrary solution of the set of multivariate polynomial equalities and inequalities can now be expressed as

$$
x = \sum_{x_k^i \in \mathcal{X}_s^{\text{inf}}} \kappa_k x_k^i + \sum_{x_k^f \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k^f ,
$$

with $s \in \{1, \ldots, 6\}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ and $\sum_k \mu_k = 1$.

Many problems in the max algebra such as matrix decompositions, transformation of state space models, construction of matrices with a given characteristic polynomial, minimal state space realization and so on, can be reformulated as a set of multivariate max-algebraic polynomial equalities and inequalities [8]. These problems are equivalent to an ELCP and can thus be solved using the ELCP algorithm. In general their solution set consists of the union of faces of a polyhedron.

4 Conclusions and Future Research

In this paper we have introduced the Extended Linear Complementarity Problem (ELCP) and sketched an algorithm to find all its solutions. Since this algorithm yields all solutions, it provides a geometrical insight in the solution set of an ELCP and other problems that can be reduced to an ELCP. However, we are not always interested in obtaining all solutions of an ELCP. Therefore our further research efforts will concentrate on algorithms that yield only one solution. Although we have shown that in general the ELCP is NP-hard, it may be interesting to determine which subclasses of the ELCP can be solved with a polynomial time algorithm.

We have also demonstrated that the ELCP can be used to solve a set of max-algebraic polynomial equalities and inequalities and related problems. Therefore the ELCP is a powerful mathematical tool for solving max-algebraic problems.

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Set	$\mathcal{X}^{\text{inf}}$		$\mathcal{X}^{\text{fin}}$					
Ray	x_1^{i}	x_2^{i}	x_1^{f}	x_2^{f}	x_3^{f}	x_4^{f}	x_5^{f}	x_6^{f}
x_1	0	0	0	-4.125	2	-18	-18	-3.2
x_2	-2	0	-6	-3.25	-4	-31	6	-1.4
x_3	0	0	1	-0.375	0	-5	-5	-1.3
x_4	1	1	3	-4.5625	5	-11.5	-11.5	-4.1

Table 1: The vertices of the ELCP of Example 3.2.

s	$\mathcal{X}_s^{\text{inf}}$	$\mathcal{X}_s^{\text{fin}}$
1	$\{x_1^{\text{i}}, x_2^{\text{i}}\}$	$\{x_1^{\text{f}}\}$
2	$\{x_1^{\text{i}}\}$	$\{x_1^{\text{f}}, x_3^{\text{f}}\}$
3	$\{x_2^{\text{i}}\}$	$\{x_1^{\text{f}}, x_3^{\text{f}}\}$
4	$\{\}$	$\{x_2^{\text{f}}, x_4^{\text{f}}, x_5^{\text{f}}\}$
5	$\{\}$	$\{x_2^{\text{f}}, x_4^{\text{f}}, x_6^{\text{f}}\}$
6	$\{\}$	$\{x_2^{\text{f}}, x_5^{\text{f}}, x_6^{\text{f}}\}$

Table 2: The cross-complementary sets of the ELCP of Example 3.2.

List of footnotes

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