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# **The extended linear complementarity problem and its applications in analysis and control of discrete-event systems\***

B. De Schutter

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Delft Center for Systems and Control  
Delft University of Technology  
Mekelweg 2, 2628 CD Delft  
The Netherlands  
phone: +31-15-278.24.73 (secretary)  
URL: <https://www.dcsc.tudelft.nl>

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## Chapter 1

# THE EXTENDED LINEAR COMPLEMENTARITY PROBLEM AND ITS APPLICATIONS IN ANALYSIS AND CONTROL OF DISCRETE-EVENT SYSTEMS

Bart De Schutter

*Delft Center for Systems and Control*

*Delft University of Technology*

*Mekelweg 2, 2628CD Delft*

*The Netherlands*

b.deschutter@dcsc.tudelft.nl

**Abstract** In this chapter we give an overview of complementarity problems with a special focus on the extended linear complementarity problem (ELCP) and its applications in analysis and control of discrete-event systems such as traffic signal controlled intersections, manufacturing systems, railway networks, etc. We start by giving an introduction to the (regular) linear complementarity problem (LCP). Next, we discuss some extensions, with a particular emphasis on the ELCP, which can be considered to be the most general linear extension of the LCP. We then discuss some algorithms to compute one or all solutions of an ELCP. Next, we present a link between the ELCP and max-plus equations. This is then the basis for some applications of the ELCP in analysis and model-based predictive control of a special class of discrete-event systems. We also show that — although the general ELCP is NP-hard — the ELCP-based control problem can be transformed into a linear programming problem, which can be solved in polynomial time.

**Keywords:** linear complementarity problem, extended linear complementarity problem, algorithms, control applications, discrete-event systems, max-plus-linear systems

## Introduction

The Linear Complementarity Problem (LCP) is one of the fundamental problems in optimization and mathematical programming (Cottle et al., 1992; Murty, 1988). Several authors have introduced (both linear and nonlinear) extensions of the LCP, and some of these linear extensions will be discussed in more detail below. The importance of the LCP and its generalizations is evidenced by a broad range of applications in the fields of engineering and eco-

nomics such as quadratic programming, determination of Nash equilibriums, nonlinear obstacle problems, and problems involving market equilibriums, invariant capital stock, optimal stopping, contact and structural mechanics, elasto-hydrodynamic lubrication, traffic equilibriums, operation planning in deregulated electricity markets, manufacturing systems, etc. (see the other chapters of this book, the books and overview papers (Cottle et al., 1992; Ferris and Pang, 1997a; Ferris and Pang, 1997b; Ferris et al., 2001; Isac et al., 2002) and the references therein).

Apart from the LCP the focus of this chapter will be on yet another extension of the LCP, which we have called the Extended Linear Complementarity Problem (ELCP) (De Schutter and De Moor, 1995a), and which can in some way be considered as the most general linear extension of the LCP. This problem arose from our research on discrete-event systems (max-plus-linear systems, max-plus-algebraic applications, and min-max-plus systems (De Schutter and De Moor, 1995b; De Schutter and van den Boom, 2000)) and hybrid systems (traffic signal control, and first-order hybrid systems with saturation (De Schutter and De Moor, 1998b; De Schutter, 2000)). Furthermore, the ELCP can also be used in the analysis of several classes of hybrid systems such as piecewise-affine systems (Sontag, 1981; Heemels et al., 2001), max-min-plus-scaling systems (De Schutter and van den Boom, 2001b), and linear complementarity systems (De Schutter and De Moor, 1998a; Heemels et al., 2000).

This chapter is organized as follows: In Section 1.1 we present the LCP and the ELCP, and we discuss how they are related. In Section 1.2 we present some other (linear) generalizations of the LCP, and we show that they can be considered as special cases of the ELCP. Next, we discuss some algorithms to compute one or all solutions of an ELCP in Section 1.3. In Section 1.4 we then explain the relation between systems of max-plus equations and the ELCP, which is the basis for several applications of the ELCP in analysis and control of discrete-event systems, some of which are then discussed in more detail in Section 1.5. We conclude this chapter with a summary.

As this chapter is mainly intended to be an overview the proofs will be reduced to a minimum (with appropriate references to the papers where the full proofs can be found), and only be given in case they are functional.

## 1. Linear Complementarity Problem

### 1.1 Notation

All vectors used in this paper are assumed to be column vectors. The transpose of a vector  $a$  is denoted by  $a^T$ . Furthermore, inequalities for vectors have to be interpreted entrywise. We use  $I_n$  to denote the  $n$  by  $n$  identity matrix, and  $0_{m \times n}$  to denote the  $m$  by  $n$  zero matrix. If the dimensions of the identity matrix or the zero matrix are omitted, they should be clear from the context.

## 1.2 Linear Complementarity Problem (LCP)

One of the possible formulations of the LCP is the following (Cottle et al., 1992):

Given  $M \in \mathbb{R}^{n \times n}$  and  $q \in \mathbb{R}^n$ , find vectors  $w, z \in \mathbb{R}^n$  such that

$$w = Mz + q \quad (1.1)$$

$$w, z \geq 0 \quad (1.2)$$

$$w^T z = 0 \quad (1.3)$$

Note that if  $w$  and  $z$  are solutions of the LCP then it follows from (1.2) and (1.3) that

$$z_i w_i = 0 \quad \text{for } i = 1, \dots, n,$$

i.e., for each  $i$  we have the following conditions: if  $w_i > 0$  then we should have  $z_i = 0$ , and if  $z_i > 0$  then  $w_i = 0$ . So the zero patterns of  $w$  and  $z$  are complementary. Therefore, condition (1.3) is called the *complementarity condition* of the LCP.

For an extensive state-of-the-art overview of the LCP (and related problems) we refer the interested reader to (Cottle et al., 1992; Ferris and Pang, 1997a; Ferris and Pang, 1997b; Ferris et al., 2001; Isac et al., 2002).

## 1.3 Extended Linear Complementarity Problem (ELCP)

The ELCP is defined as follows (De Schutter and De Moor, 1995a):

Given  $A \in \mathbb{R}^{p \times n}$ ,  $B \in \mathbb{R}^{q \times n}$ ,  $c \in \mathbb{R}^p$ ,  $d \in \mathbb{R}^q$ , and  $m$  index sets  $\phi_1, \dots, \phi_m \subseteq \{1, \dots, p\}$ , find  $x \in \mathbb{R}^n$  such that

$$Ax \geq c \quad (1.4)$$

$$Bx = d \quad (1.5)$$

$$\sum_{j=1}^m \prod_{i \in \phi_j} (Ax - c)_i = 0 \quad (1.6)$$

The feasible set of the ELCP (1.4)–(1.6) is defined by

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid Ax \geq c, Bx = d\} \quad .$$

The surplus variable  $\text{surp}(i, x)$  of the  $i$ th inequality of  $Ax \geq c$  is defined as  $\text{surp}(i, x) = (Ax - c)_i$ .

Condition (1.6) represents the *complementarity condition* of the ELCP. One possible interpretation of this condition is the following: Since  $Ax \geq c$ , all the terms in (1.6) are nonnegative. Therefore, (1.6) is equivalent to

$$\prod_{i \in \phi_j} (Ax - c)_i = 0 \quad \text{for } j = 1, \dots, m.$$

So we could say that each set  $\phi_j$  corresponds to a group of inequalities in  $Ax \geq c$ , and that in each group at least one inequality should hold with equality (i.e., the corresponding surplus variable is equal to 0).

The solution set of an ELCP can be characterized as follows (De Schutter and De Moor, 1995a):

**THEOREM 1.1** *In general the solution set  $\mathcal{S}$  of an ELCP consists of the union of faces of a polyhedron.*

This solution set can be represented using four sets:

- a set  $\mathcal{X}^{\text{fin}}$  of finite vertices of  $\mathcal{S}$ ,
- a set  $\mathcal{X}^{\text{ext}}$  of generators for the extreme rays of  $\mathcal{S}$ ,
- a basis  $\mathcal{X}^{\text{cen}}$  for the linear subspace associated with the maximal affine subspace of  $\mathcal{S}$ ,
- and a set  $\Lambda$  of pairs of so-called maximal cross-complementary subsets of  $\mathcal{X}^{\text{ext}}$  and  $\mathcal{X}^{\text{fin}}$  (where each pair corresponds to a face of  $\mathcal{S}$ ).

In Section 1.3.1 we will present an algorithm to compute these sets. Then  $x$  is a solution of the ELCP if and only if there exists an ordered pair  $(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}) \in \Lambda$  such that

$$x = \sum_{x_k^{\text{cen}} \in \mathcal{X}^{\text{cen}}} \lambda_k x_k^{\text{cen}} + \sum_{x_k^{\text{ext}} \in \mathcal{X}_s^{\text{ext}}} \kappa_k x_k^{\text{ext}} + \sum_{x_k^{\text{fin}} \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k^{\text{fin}} \quad (1.7)$$

with  $\lambda_k \in \mathbb{R}$ ,  $\kappa_k \geq 0$ ,  $\mu_k \geq 0$  for all  $k$  and  $\sum_k \mu_k = 1$ .

We can also reverse Theorem 1.1 (De Schutter and De Moor, 1995a):

**THEOREM 1.2** *The union of any arbitrary set of faces of an arbitrary polyhedron can be described by an ELCP.*

**REMARK 1.3** *The complementarity conditions of both the LCP and the ELCP consist of a sum of products. However, in contrast to the ELCP where the products may contain one, two or more factors, the products in complementarity condition of the LCP always contain exactly two factors. Moreover, any variable in the LCP is contained in precisely one index set  $\phi_j$ , while in the ELCP formulation it may be contained in any number of index sets.*

We also have the following complexity result:

**THEOREM 1.4** *In general the ELCP with rational data is an NP-hard problem.*

The proof of this result is based on the fact that in general the LCP with rational data is also NP-hard (Chung, 1989).

## 1.4 The link between the LCP and ELCP

It is easy to verify that the following lemma holds:

LEMMA 1.5 *The LCP is a special case of the ELCP.*

Moreover, we also have a reverse statement (De Schutter et al., 2002):

THEOREM 1.6 *If the surplus variables of the inequalities of an ELCP are bounded (from above<sup>1</sup>) over the feasible set of the ELCP, then the ELCP can be rewritten as an LCP.*

**Proof:** Consider the ELCP (1.4)–(1.6). If there is an equality condition  $Bx = d$  present, then we remove it using the following procedure: we can replace  $Bx = d$  by  $Bx \geq d$ , and impose equality conditions on these inequalities by adding the index sets  $\phi_{m+1} = \{p+1\}, \dots, \phi_{m+q} = \{p+q\}$ . So from now on we consider the following formulation of the ELCP<sup>2</sup>:

$$Ax \geq c \quad (1.8)$$

$$\sum_{i=1}^m \prod_{j \in \phi_i} (Ax - c)_j = 0 \quad (1.9)$$

The proof of the theorem consists of two main steps:

- 1 First, we transform the ELCP into a mixed integer problem to get rid of the ELCP complementarity condition at the cost of introducing some additional binary variables.
- 2 Next, we transform all variables (both binary and real-valued ones) into nonnegative real ones, which will lead to an LCP.

Step 1: Transformation into a mixed integer problem

Define a diagonal matrix  $D^{\text{upp}} \in \mathbb{R}^{p \times p}$  with  $(D^{\text{upp}})_{ii} = d_i^{\text{upp}}$  an upper bound for  $\text{surp}(i, x) = (Ax - c)_i$  over the feasible set  $\mathcal{F}$  of the ELCP. So for each  $i \in \{1, \dots, p\}$  we have  $d_i^{\text{upp}} \geq (Ax - c)_i$  for all  $x \in \mathcal{F}$ . Now consider the following system of equations:

$$\delta \in \{0, 1\}^p, \quad x \in \mathbb{R}^n \quad (1.10)$$

$$0 \leq (Ax - c)_i \leq d_i^{\text{upp}} \delta_i \quad \text{for } i = 1, \dots, p \quad (1.11)$$

$$\sum_{i \in \phi_j} \delta_i \leq \#\phi_j - 1 \quad \text{for } j = 1, \dots, m, \quad (1.12)$$

where  $\#\phi_j$  denotes the number of elements of the set  $\phi_j$ . Problem (1.10)–(1.12) will be called the equivalent *mixed integer linear feasibility problem* (MILFP).

Now we show that the MILFP is equivalent to the ELCP (1.8)–(1.9) in the sense that  $x$  is a solution of the ELCP (1.8)–(1.9) if and only if there exists a  $\delta$  such that  $(x, \delta)$  is a solution of (1.10)–(1.12). Equation (1.8) is implied

by (1.11). Note that (1.10) and (1.12) imply that for each  $j$  at least one of the  $\delta_i$ 's with  $i \in \phi_j$  is equal to 0. If  $\delta_{i'} = 0$ , then it follows from (1.11) that  $(Ax - c)_{i'} = 0$ . This implies that in each index set  $\phi_j$  there is at least one index for which the corresponding surplus variable equals 0. Hence, the complementarity condition (1.9) is also implied by (1.10)–(1.12). So (1.10)–(1.12) imply (1.8)–(1.9), and it is easy to verify that the reverse statement also holds. As a consequence, the MILFP is equivalent to the ELCP.

Define  $S \in \mathbb{R}^{m \times p}$  with  $s_{ji} = 1$  if  $i \in \phi_j$  and  $s_{ji} = 0$  otherwise, and  $t \in \mathbb{R}^m$  with  $t_j = \#\phi_j - 1$ . The MILFP can now be rewritten compactly as

Find  $x \in \mathbb{R}^n$  and  $\delta \in \{0, 1\}^p$  such that

$$0 \leq Ax - c \leq D^{\text{upp}} \delta \quad (1.13)$$

$$S\delta \leq t \quad (1.14)$$

Step 2: Now we transform the MILFP into an LCP.

This will be done in three steps.

- (a) First we transform condition  $\delta \in \{0, 1\}^p$  into the LCP framework. All the variables of an LCP should be real-valued, but the vector  $\delta$  in the MILFP is a binary vector. However, the condition  $\delta_i \in \{0, 1\}$  is equivalent to the set of conditions

$$\delta_i \in \mathbb{R}, \quad \delta_i \geq 0, \quad 1 - \delta_i \geq 0, \quad \delta_i(1 - \delta_i) = 0 \quad .$$

So if we introduce a vector  $v_\delta \in \mathbb{R}^p$  of auxiliary variables, then the condition  $\delta \in \{0, 1\}^p$  is equivalent to

$$\delta, v_\delta \in \mathbb{R}^p, \quad \delta, v_\delta \geq 0, \quad v_\delta = \mathbf{1}_p - \delta, \quad \delta^T v_\delta = 0 \quad ,$$

where  $\mathbf{1}_p$  is a  $p$ -component column vector consisting of all 1's.

- (b) The inequality  $0 \leq Ax - c$  can be adapted to the LCP framework by introducing an auxiliary vector  $v_A \in \mathbb{R}^p$  with  $v_A = Ax - c \geq 0$ . To obtain a complementarity condition for  $v_A$  we introduce  $w_A \in \mathbb{R}^p$  such that  $v_A, w_A \geq 0$  and  $v_A^T w_A = 0$  (Note that we can always take  $w_A = 0$  to get these conditions satisfied). Hence,  $0 \leq Ax - c$  can be rewritten as

$$v_A, w_A \geq 0, \quad v_A = Ax - c, \quad v_A^T w_A = 0 \quad ,$$

with  $v_A, w_A \in \mathbb{R}^p$ . The inequalities  $Ax - c \leq D^{\text{upp}} \delta$  and  $S\delta \leq t$  can be dealt with in a similar way.

- (c) All variables in an LCP are nonnegative whereas this condition is not present in the MILFP. Therefore, we split  $x$  in its positive part  $x^+ = \max(x, 0)$  and its negative part  $x^- = \max(-x, 0)$ . So  $x = x^+ - x^-$  with  $x^+, x^- \geq 0$  and  $(x^+)^T x^- = 0$ . To obtain independent LCP-like complementarity conditions for  $x^+$  and  $x^-$  we introduce additional auxiliary

vectors  $v^+, v^- \in \mathbb{R}^n$  with  $v^+ = x^+$  and  $v^- = x^-$  such that  $(v^-)^T x^+ = 0$  and  $(v^+)^T x^- = 0$  with  $x^+, x^-, v^+, v^- \geq 0$ .

Combining these three steps results in the following equivalent LCP:

$$\underbrace{\begin{bmatrix} v_\delta \\ v^- \\ v^+ \\ v_A \\ v_{D^{\text{upp}}} \\ v_S \end{bmatrix}}_w = \underbrace{\begin{bmatrix} -I_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & A & -A & 0 & 0 & 0 \\ D^{\text{upp}} & -A & A & 0 & 0 & 0 \\ -S & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_M \underbrace{\begin{bmatrix} \delta \\ x^+ \\ x^- \\ w_A \\ w_{D^{\text{upp}}} \\ w_S \end{bmatrix}}_z + \underbrace{\begin{bmatrix} \mathbf{1}_p \\ 0 \\ 0 \\ -c \\ c \\ t \end{bmatrix}}_q \quad (1.15)$$

$$w, z \geq 0 \quad (1.16)$$

$$w^T z = 0, \quad (1.17)$$

with  $w, z \in \mathbb{R}^{3p+2n+m}$ . The solution of the original ELCP can be extracted from the solution of the LCP (1.15)–(1.17) by setting  $x = x^+ - x^-$ .

The introduction of the MILFP in this proof was inspired by the paper (Bemporad and Morari, 1999), in which a class of hybrid systems is discussed consisting of mixed logical dynamic systems, which can be shown to be equivalent to systems with an ELCP-based model description (Heemels et al., 2001).

## 2. Other extensions of the LCP

Several authors have introduced linear and nonlinear extensions and generalizations of the LCP. Some examples of “linear” extensions of the LCP are:

- the Horizontal LCP (Cottle et al., 1992):

Given  $M, N \in \mathbb{R}^{n \times n}$  and  $q \in \mathbb{R}^n$ , find  $w, z \in \mathbb{R}^n$  such that

$$\begin{aligned} w, z &\geq 0 \\ Mz + Nw &= q \\ z^T w &= 0. \end{aligned}$$

- the Vertical LCP (Cottle et al., 1992) (also known as the Generalized LCP of Cottle and Dantzig (Cottle and Dantzig, 1970)):

Let  $M \in \mathbb{R}^{m \times n}$  with  $m \geq n$  and let  $q \in \mathbb{R}^m$ . Suppose that  $M$  and  $q$  are partitioned as follows:

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix},$$



with  $M_i \in \mathbb{R}^{m_i \times n}$  and  $q_i \in \mathbb{R}^{m_i}$  for  $i = 1, \dots, n$  and with  $\sum_{i=1}^n m_i = m$ .

Now find  $z \in \mathbb{R}^n$  such that

$$\begin{aligned} z &\geq 0 \\ q + Mz &\geq 0 \\ z_i \prod_{j=1}^{m_i} (q_i + M_i z)_j &= 0 \quad \text{for } i = 1, \dots, n. \end{aligned}$$

- the Extended LCP of Mangasarian and Pang (Gowda, 1996; Mangasarian and Pang, 1995):

Given  $M, N \in \mathbb{R}^{m \times n}$  and a polyhedral set  $\mathcal{P} \subseteq \mathbb{R}^m$ , find  $x, y \in \mathbb{R}^n$  such that

$$\begin{aligned} x, y &\geq 0 \\ Mx - Ny &\in \mathcal{P} \\ x^T y &= 0. \end{aligned}$$

- the Extended Horizontal LCP of Sznajder and Gowda (Sznajder and Gowda, 1995):

Given  $k + 1$  matrices  $C_0, C_1, \dots, C_k \in \mathbb{R}^{n \times n}$ ,  $q \in \mathbb{R}^n$  and  $k - 1$  vectors  $d_1, d_2, \dots, d_{k-1} \in \mathbb{R}^n$  with positive components, find  $x_0, x_1, \dots, x_k \in \mathbb{R}^n$  such that

$$\begin{aligned} x_0, x_1, \dots, x_k &\geq 0 \\ d_j - x_j &\geq 0 \quad \text{for } j = 1, \dots, k - 1 \\ C_0 x_0 &= q + \sum_{j=1}^k C_j x_j \\ x_0^T x_1 &= 0 \\ (d_j - x_j)^T x_{j+1} &= 0 \quad \text{for } j = 1, \dots, k - 1. \end{aligned}$$

- the Generalized LCP of Eaves (Eaves, 1971):

Given  $n$  positive integers  $m_1, m_2, \dots, m_n$ ,  $n$  matrices  $A_1, A_2, \dots, A_n \in \mathbb{R}^{p \times m_i}$ , and a vector  $b \in \mathbb{R}^p$ , find  $x_1, x_2, \dots, x_n \in \mathbb{R}^{m_i}$  such that

$$\begin{aligned} x_1, x_2, \dots, x_n &\geq 0 \\ \sum_{i=1}^n A_i x_i &\leq b \\ \sum_{i=1}^n \prod_{j=1}^{m_i} (x_i)_j &= 0. \end{aligned}$$

- the Generalized LCP of Ye (Ye, 1993):

Given  $A, B \in \mathbb{R}^{m \times n}$ ,  $C \in \mathbb{R}^{m \times k}$  and  $q \in \mathbb{R}^m$ , find  $x, y \in \mathbb{R}^n$  and  $z \in \mathbb{R}^k$  such that

$$\begin{aligned} x, y, z &\geq 0 \\ Ax + By + Cz &= q \\ x^T y &= 0 . \end{aligned}$$

- the Generalized LCP of De Moor and Vandenberghe (De Moor et al., 1992):

Given  $Z \in \mathbb{R}^{p \times n}$  and  $m$  subsets  $\phi_1, \phi_2, \dots, \phi_m$  of  $\{1, 2, \dots, p\}$ , find  $u \in \mathbb{R}^n$  (with  $u \neq 0$ ) such that

$$\begin{aligned} u &\geq 0 \\ Zu &= 0 \\ \sum_{j=1}^m \prod_{i \in \phi_j} u_i &= 0 . \end{aligned}$$

- the (Extended) Generalized Order LCP of Gowda and Sznajder (Gowda and Sznajder, 1994):

Given  $B_0, B_1, \dots, B_k \in \mathbb{R}^{n \times n}$ , and  $b_0, b_1, \dots, b_k \in \mathbb{R}^n$ , find  $x \in \mathbb{R}^n$  such that

$$(B_0 x + b_0) \wedge (B_1 x + b_1) \wedge \dots \wedge (B_k x + b_k) = 0$$

where  $\wedge$  is the entrywise minimum: if  $x, y \in \mathbb{R}^n$  then  $(x \wedge y)_i = \min(x_i, y_i)$  for  $i = 1, \dots, n$ .

This problem is the Extended Generalized Order LCP. If we take  $B_0 = I_n$  and  $b_0 = 0_{n \times 1}$  we get the (regular) Generalized Order LCP.

- the mixed LCP (Cottle et al., 1992):

Given  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{m \times m}$ ,  $C \in \mathbb{R}^{n \times m}$ ,  $D \in \mathbb{R}^{m \times n}$ ,  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}^m$ , find  $u \in \mathbb{R}^n$  and  $v \in \mathbb{R}^m$  such that

$$\begin{aligned} v &\geq 0 \\ a + Au + Cv &= 0 \\ b + Du + Bv &\geq 0 \\ v^T (b + Du + Bv) &= 0 . \end{aligned}$$

It is quite easy<sup>3</sup> to show (De Schutter and De Moor, 1995a) that all these generalizations are special cases of the ELCP. Furthermore, in (De Schutter and De Moor, 1998a) we have shown that the following extension of the LCP is also a special case of the ELCP:

- the Linear Dynamic Complementarity Problem (Schumacher, 1996), which is defined as follows:

Given  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times k}$ ,  $C \in \mathbb{R}^{k \times n}$  and  $D \in \mathbb{R}^{k \times k}$ , find for a given  $x_0 \in \mathbb{R}^n$  sequences  $\{y_l\}_{l=0}^{n-1}$ ,  $\{u_l\}_{l=0}^{n-1}$  with  $y_l, u_l \in \mathbb{R}^k$  for all  $l$  such that

$$\begin{aligned} y_0 &= Cx_0 + Du_0 \\ y_1 &= CAx_0 + CBu_0 + Du_1 \\ &\vdots \\ y_{n-1} &= CA^{n-1}x_0 + CA^{n-2}Bu_0 + \dots + CBu_{n-2} + Du_{n-1} \end{aligned}$$

and such that for each index  $i \in \{1, 2, \dots, k\}$  at least one of the following statements is true:

$$\begin{aligned} [(y_0)_i \ \dots \ (y_{n-1})_i]^T &= 0 \quad \text{and} \quad [(u_0)_i \ \dots \ (u_{n-1})_i]^T \succeq 0 \\ [(y_0)_i \ \dots \ (y_{n-1})_i]^T &\succeq 0 \quad \text{and} \quad [(u_0)_i \ \dots \ (u_{n-1})_i]^T = 0, \end{aligned}$$

where  $z \succeq 0$  for a vector  $z \in \mathbb{R}^n$  indicates that  $z$  is lexicographically nonnegative, i.e., either  $z_i = 0$  for all  $i$  or the first nonzero component of  $z$  is positive.

Hence, we have

**CONCLUSION** *The ELCP can be considered as a unifying framework for the LCP and its various generalizations.*

The underlying geometrical explanation for the fact that all the generalizations of the LCP mentioned above are particular cases of the ELCP is that they all have a solution set that consists of the union of faces of a polyhedron, and that the union of any arbitrary set of faces of an arbitrary polyhedron can be described by an ELCP (see Theorem 1.2). More generally, if we define a “linear” generalization of the LCP as a problem consisting of an explicit or implicit system of linear (in)equalities in combination with a “general” complementarity condition, i.e., an ELCP-like complementarity condition that constrains the solutions of the problem to lie on the (relative) boundary of the feasible set, then the solution set of this “linear” generalization will consist of the union of faces of a polyhedron, which implies that such a “linear” generalization of the LCP is a special case of the ELCP.

For more information on the generalizations discussed above and for applications and methods to solve these problems the interested reader may consult the references cited above and (Andreani and Martínez, 1998; Ebiefung and Kostreva, 1992; Isac, 1992; Júdice and Vicente, 1994; Mangasarian, 1995; McShane, 1994; Mohan et al., 1996; Murty, 1988; Vandenberghe et al., 1989; Zhang, 1994) and the references therein.

### 3. Algorithms for the ELCP

In this section we present some algorithms to compute all or just one solutions of an ELCP. For algorithms to solve a (regular) LCP we refer to (Bai, 1999; Chen and Mangasarian, 1995; Cottle et al., 1992; Kaliski and Ye, 1993; Kanzow, 1996; Kočvara and Zowe, 1994; Kremers and Talman, 1994; Mehrotra and Stubbs, 1994; Murty, 1988; Pardalos and Resende, 2002; Schäfer, 2004; Sheng and Potra, 1997; Wright, 1994; Yuan and Song, 2003) and the references therein.

#### 3.1 An algorithm to compute all solutions

In order to compute the entire solution set of the ELCP (1.4)–(1.6) we first homogenize the ELCP by introducing a scalar  $\alpha \geq 0$  and defining

$$u = \begin{bmatrix} x \\ \alpha \end{bmatrix}, \quad P = \begin{bmatrix} A & -c \\ 0_{1 \times n} & 1 \end{bmatrix} \quad \text{and} \quad Q = [B - d] .$$

Then we get a homogeneous ELCP of the following form:

Given  $P \in \mathbb{R}^{p \times n}$ ,  $Q \in \mathbb{R}^{q \times n}$  and  $m$  subsets  $\phi_j$  of  $\{1, 2, \dots, p\}$ , find  $u \in \mathbb{R}^n$  (with  $u \neq 0$ ) such that

$$Pu \geq 0 \quad (1.18)$$

$$Qu = 0 \quad (1.19)$$

$$\sum_{j=1}^m \prod_{i \in \phi_j} (Pu)_i = 0 . \quad (1.20)$$

So now we have a system of homogeneous linear equalities and inequalities subject to a complementarity condition. Recall that the complementarity condition (1.20) can also be written as

$$\prod_{i \in \phi_j} (Pu)_i = 0 \quad \text{for } j = 1, \dots, m. \quad (1.21)$$

The solution set of the system of homogeneous linear inequalities and equalities (1.18)–(1.19) is a polyhedral cone  $\mathcal{P}$  and can be described using two sets of generators: a set of central generators  $\mathcal{C}$  and a set of extreme generators  $\mathcal{E}$ . The set  $\mathcal{C}$  can be considered as a basis for the linear subspace of  $\mathcal{P}$ . The generators in  $\mathcal{E}$  generate the extreme rays of  $\mathcal{P}$ . Now  $u$  is a solution of (1.18)–(1.19) if and only if it can be written as

$$u = \sum_{c_k \in \mathcal{C}} \alpha_k c_k + \sum_{e_k \in \mathcal{E}} \beta_k e_k , \quad (1.22)$$

with  $\alpha_k \in \mathbb{R}$  and  $\beta_k \geq 0$ .

To calculate the sets  $\mathcal{C}$  and  $\mathcal{E}$  we use an iterative algorithm that is an adaptation of the double description method of Motzkin (Motzkin et al., 1953).

During the iteration we already remove generators that do not satisfy the (partial) complementarity condition since such rays cannot yield solutions of the ELCP. In the  $k$ th step of the algorithm the partial complementarity condition is defined as follows:

$$\prod_{i \in \phi_j} (Pu)_i = 0 \quad \text{for all } j \text{ such that } \phi_j \subset \{1, 2, \dots, k\} . \quad (1.23)$$

So we only consider those groups of inequalities that have already been processed entirely. For  $k \geq p$  the partial complementarity condition (1.23) coincides with the full complementarity condition (1.21) or (1.20). This leads to the following algorithm:

**Algorithm 1 : Calculation of the central and extreme generators.**

**Initialization:**

- $\mathcal{C}_0 := \{c_i \mid c_i \text{ is the } i\text{th column of } I_n \text{ for } i = 1, \dots, n\}$
- $\mathcal{E}_0 := \emptyset$

**Iteration:**

**for**  $k := 1, 2, \dots, p + q$ ,

- Calculate the intersection of the current polyhedral cone (described by  $\mathcal{C}_{k-1}$  and  $\mathcal{E}_{k-1}$ ) with the half-space or hyperplane determined by the  $k$ th inequality or equality of (1.18)–(1.19). This yields a new polyhedral cone described by  $\mathcal{C}_k$  and  $\mathcal{E}_k$ .
- Remove the generators that do not satisfy the partial complementarity condition.

**Result:**  $\mathcal{C} := \mathcal{C}_{p+q}$  and  $\mathcal{E} := \mathcal{E}_{p+q}$

Not every combination of the form (1.22) satisfies the complementarity condition. Although every linear combination of the central generators satisfies the complementarity condition, not every positive combination of the extreme generators satisfies the complementarity condition. Therefore, we introduce the concept of cross-complementarity:

**DEFINITION 1.7 (Cross-complementarity)** *Let  $\mathcal{E}$  be the set of extreme generators of an homogeneous ELCP. A subset  $\mathcal{E}_s$  of  $\mathcal{E}$  is cross-complementary if every combination of the form*

$$u = \sum_{e_k \in \mathcal{E}_s} \beta_k e_k ,$$

*with  $\beta_k \geq 0$ , satisfies the complementarity condition.*

In (De Schutter and De Moor, 1995a) we have proved that in order to check whether a set  $\mathcal{E}_s$  is cross-complementary it suffices to test only one strictly positive combination of the generators in  $\mathcal{E}_s$ , e.g., the combination with  $\beta_k = 1$  for

all  $k$ . Now we can determine  $\Gamma$ , the set of maximal cross-complementary sets of extreme generators:  $\Gamma = \{ \mathcal{E}_s \mid \mathcal{E}_s \text{ is maximal and cross-complementary} \}$ .

**Algorithm 2 : Determination of the cross-complementary sets of extreme generators.**

**Initialization:**

- $\Gamma := \emptyset$
- Construct the cross-complementarity graph  $\mathcal{G}$  with a node  $e_i$  for each generator  $e_i \in \mathcal{E}$  and an edge between nodes  $e_k$  and  $e_l$  if the set  $\{e_k, e_l\}$  is cross-complementary.
- $\mathcal{S} := \{e_1\}$

**Depth-first search in  $\mathcal{G}$ :**

- Select a new node  $e^{\text{new}}$  that is connected by an edge to all nodes of the set  $\mathcal{S}$  and add the corresponding generator to the test set:  $\mathcal{S}^{\text{new}} := \mathcal{S} \cup \{e^{\text{new}}\}$ .
- **if**  $\mathcal{S}^{\text{new}}$  is cross-complementary  
**then** Select a new node and add it to the test set.  
**else** Add  $\mathcal{S}$  to  $\Gamma$ :  $\Gamma := \Gamma \cup \{\mathcal{S}\}$ , and go back to the last point where a choice was made.  
 Continue until all possible choices have been considered.

**Result:**  $\Gamma$

Now  $u$  is a solution of the homogeneous ELCP if and only if there exists a set  $\mathcal{E}_s \in \Gamma$  such that  $u$  can be written as

$$u = \sum_{c_k \in \mathcal{C}} \alpha_k c_k + \sum_{e_k \in \mathcal{E}_s} \beta_k e_k, \quad (1.24)$$

with  $\alpha_k \in \mathbb{R}$  and  $\beta_k \geq 0$ .

Finally, we have to extract the solution set  $\mathcal{S}$  of the original ELCP (cf. equation (1.7)), i.e., we have to retain solutions of the form (1.24) that have an  $\alpha$  component equal to 1 ( $u_\alpha = 1$ ). So we transform the sets  $\mathcal{C}$ ,  $\mathcal{E}$ , and  $\Gamma$  as follows:

- If  $c \in \mathcal{C}$  then  $c_\alpha = 0$ . We drop the  $\alpha$  component and put the result in  $\mathcal{X}^{\text{cen}}$  (i.e., the basis the linear subspace associated with the maximal affine subspace of  $\mathcal{S}$ ).
- If  $e \in \mathcal{E}$  then there are two possibilities:
  - If  $e_\alpha = 0$  then we drop the  $\alpha$  component and put the result in  $\mathcal{X}^{\text{ext}}$  (i.e., the set of generators for the extreme rays of  $\mathcal{S}$ ).
  - If  $e_\alpha > 0$  then we normalize  $e$  such that  $e_\alpha = 1$ . Next, we drop the  $\alpha$  component and put the result in  $\mathcal{X}^{\text{fin}}$  (i.e., the set of finite vertices of  $\mathcal{S}$ ).

- For each set  $\mathcal{E}_s \in \Gamma$  we construct the set of corresponding extreme generators  $\mathcal{X}_s^{\text{ext}}$  and the set of corresponding finite vertices  $\mathcal{X}_s^{\text{fin}}$ . If  $\mathcal{X}_s^{\text{fin}} \neq \emptyset$  then we add the pair  $(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}})$  to  $\Lambda$ , the set of pairs of maximal cross-complementary sets of finite vertices and extreme generators (where each pair corresponds to a face of  $\mathcal{S}$ ).

For a more detailed and precise description of these algorithms and a worked example the interested reader is referred to (De Schutter and De Moor, 1995a). Also note that the running time and memory requirements of the algorithms presented above increase exponentially with the size of the ELCP (see (De Schutter, 1996) for more details). This implies that the above ELCP algorithm, which determines the entire solution set of the ELCP, is not well suited for large ELCPs with a large number of variables and (in)equalities, or a complex solution set. Therefore, we will now present some method to compute only one solution of an ELCP.

### 3.2 Algorithms to compute one solution

Some of the methods that could be used to compute one solution of an ELCP are:

- via global minimization (Mangasarian and Solodov, 1993):  
We could minimize the left-hand side of the complementarity condition (1.6) subject to the linear equality and inequality constraints (1.4)–(1.5). This results in a nonlinear non-convex optimization problem with linear constraints, that could, e.g., be solved using multi-start local optimization (SQP), simulated annealing, tabu search, etc. (Pardalos and Resende, 2002).
- as a system multi-variate polynomial equations:  
if we introduce a dummy variable  $s_i$  then the  $i$ th inequality of the system  $Ax \geq c$  can be transformed into an equality:  $A_{i,\cdot}x - s_i^2 = c_i$ . Note that  $s_i = 0$  if and only if  $A_{i,\cdot}x = c_i$ . If we repeat this reasoning for each inequality, then we find that the complementarity condition (1.6) results in  $\sum_{j=1}^m \prod_{i \in \phi_j} s_i = 0$ . The resulting system of multi-variate polynomial equations could then be solved using, e.g., a homotopy method (Li, 2003).
- using a combinatorial approach:  
We could select one index  $i_j$  out of each set  $\phi_j$  for  $j = 1, \dots, m$ . Each index  $i_j$  then corresponds to an inequality of  $Ax \geq c$  that should hold with equality. So in that case we just get a system of linear equalities and inequalities. If this system has a solution, we have obtained a solution of the ELCP; if not, we have to select another combination of indices, and repeat the process.
- using a mixed-integer linear programming approach:  
This approach is based on Theorem 1.6 and applies if the surplus variables of the inequalities of the ELCP are bounded over the feasible set.

Note that a sufficient condition for this is that the feasible set of the ELCP is bounded. For engineering problems such bounds are often available, e.g., as a consequence of physical or other constraints, operating ranges, etc. If we add a dummy linear objective function to the MILFP (1.10)–(1.12) we obtain a mixed-integer linear programming problem. This problem can then be solved using, e.g., a branch-and-bound method (Fletcher and Leyffer, 1998; Taha, 1987) or a branch-and-cut method (Cordier et al., 1999). Moreover, there exist good commercial and free solvers for mixed-integer linear programming problems (such as, e.g., CPLEX, Xpress-MP, GLPK, lp\_solve, etc.; see (Atamtürk and Savelsbergh, 2005; Linderoth and Ralphs, 2004) for an overview)

Note that all these approaches are essentially of combinatorial nature. However, based on our own experiences the best results are usually obtained using the mixed-integer linear programming approach.

## 4. Link with max-plus equations

In this section we consider max-plus equations as they arise in various applications in the max-plus algebra and in the analysis and control of max-plus-linear systems. But first we give a short introduction to the basic concepts of the max-plus algebra.

### 4.1 Max-plus algebra

The basic operations of the max-plus algebra (Cuninghame-Green, 1979; Baccelli et al., 1992) are maximization and addition, which are represented by  $\oplus$  and  $\otimes$  respectively:

$$x \oplus y = \max(x, y) \quad \text{and} \quad x \otimes y = x + y$$

for  $x, y \in \mathbb{R}_\varepsilon \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$ . The structure  $(\mathbb{R}_\varepsilon, \oplus, \otimes)$  is called the max-plus algebra. The operations  $\oplus$  and  $\otimes$  are called the max-plus-algebraic addition and max-plus-algebraic multiplication respectively since many properties and concepts from linear algebra can be translated to the max-plus algebra by replacing  $+$  by  $\oplus$  and  $\times$  by  $\otimes$ . Note that  $0$  is the identity element for  $\otimes$  and that  $-\infty$  is absorbing for  $\otimes$ .

The matrix  $E_n$  is the  $n \times n$  max-plus-algebraic identity matrix:  $(E_n)_{ii} = 0$  for all  $i$  and  $(E_n)_{ij} = -\infty$  for all  $i, j$  with  $i \neq j$ . The basic max-plus-algebraic operations are extended to matrices as follows. If  $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ ,  $C \in \mathbb{R}_\varepsilon^{n \times p}$  then

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$(A \otimes C)_{ij} = \bigoplus_{k=1}^n a_{ik} \otimes c_{kj} = \max_k(a_{ik} + c_{kj})$$

for all  $i, j$ . Note the analogy with the definitions of matrix sum and product in conventional linear algebra.



The max-plus-algebraic matrix power of  $A \in \mathbb{R}_\varepsilon^{n \times n}$  is defined as follows:  $A^{\otimes 0} = E_n$  and  $A^{\otimes k} = A \otimes A^{\otimes k-1}$  for  $k = 1, 2, \dots$ . For scalar numbers  $x, r \in \mathbb{R}$  we have  $x^{\otimes r} = r \cdot x$ .

## 4.2 Systems of max-plus-polynomial equations

In the next section we shall see that many max-plus-algebraic problems can be written in the following form:

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^n x_j^{\otimes c_{kij}} = b_k \quad \text{for } k = 1, \dots, p_1 \quad (1.25)$$

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^n x_j^{\otimes c_{kij}} \leq b_k \quad \text{for } k = p_1 + 1, \dots, p_1 + p_2, \quad (1.26)$$

i.e., the max-plus-algebraic equivalent of a system of polynomial equations. Therefore, we call (1.25)–(1.26) a system of multivariate polynomial equalities and inequalities in the max-plus algebra, or a system of max-plus-polynomial equations for short. Note that the exponents can be negative or real. Using the notations introduced in Section 1.4.1 it is easy to verify that in conventional algebra this problem can be rewritten as follows:

Given a set of integers  $\{m_k\}$  and three sets of coefficients  $\{a_{ki}\}$ ,  $\{b_k\}$  and  $\{c_{kij}\}$  with  $i \in \{1, \dots, m_k\}$ ,  $j \in \{1, \dots, n\}$  and  $k \in \{1, \dots, p_1, p_1 + 1, \dots, p_1 + p_2\}$ , find  $x \in \mathbb{R}^n$  such that

$$\max_{i=1, \dots, m_k} \left( a_{ki} + \sum_{j=1}^n c_{kij} x_j \right) = b_k \quad \text{for } k = 1, \dots, p_1 \quad (1.27)$$

$$\max_{i=1, \dots, m_k} \left( a_{ki} + \sum_{j=1}^n c_{kij} x_j \right) \leq b_k \quad \text{for } k = p_1 + 1, \dots, p_1 + p_2. \quad (1.28)$$

Let us now we show that (1.27)–(1.28) can be recast as an ELCP.

## 4.3 Translation into an ELCP

Clearly, the  $k$ th equation of (1.27) is equivalent to the system of linear inequalities

$$a_{ki} + c_{ki1}x_1 + c_{ki2}x_2 + \dots + c_{kin}x_n \leq b_k \quad \text{for } i = 1, \dots, m_k,$$

where at least one inequality should hold with equality. So equation (1.27) will lead to  $p_1$  groups of linear inequalities, where in each group at least one inequality should hold with equality.

Using the same reasoning equations of the form (1.28) can also be transformed into a system of linear inequalities, but without an extra condition.

If we define  $p_1 + p_2$  matrices  $C_k$  and  $p_1 + p_2$  column vectors  $d_k$  such that  $(C_k)_{ij} = c_{kij}$  and  $(d_k)_i = b_k - a_{ki}$ , then our original problem is equivalent to

$p_1 + p_2$  groups of linear inequalities  $C_k x \leq d_k$ , where there has to be at least one inequality that holds with equality in each group  $C_k x \leq d_k$  for  $k = 1, \dots, p_1$ . Now we define

$$\tilde{A} = \begin{bmatrix} -C_1 \\ -C_2 \\ \vdots \\ -C_{p_1+p_2} \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} -d_1 \\ -d_2 \\ \vdots \\ -d_{p_1+p_2} \end{bmatrix},$$

and  $p_1$  sets  $\phi_j$  such that  $\phi_j = \{s_j + 1, \dots, s_j + m_j\}$  for  $j = 1, \dots, p_1$ , where  $s_1 = 0$  and  $s_{j+1} = s_j + m_j$  for  $j = 1, \dots, p_1 - 1$ . Our original problem (1.27)–(1.28) is then equivalent to the following ELCP:

Find  $x \in \mathbb{R}^n$  such that

$$\begin{aligned} \tilde{A}x &\geq \tilde{c} \\ \sum_{j=1}^{p_1} \prod_{i \in \phi_j} (\tilde{A}x - \tilde{c})_i &= 0. \end{aligned}$$

Conversely, we can also show that any ELCP can be written as a system of max-plus equations of the form (1.27)–(1.28), which yields the following theorem (De Schutter and De Moor, 1996):

**THEOREM 1.8** *A system of multivariate polynomial equalities and inequalities in the max-plus algebra is equivalent to an ELCP.*

**Proof:** As we have already shown that (1.27)–(1.28) can be recast as an ELCP. To show that the ELCP (1.4)–(1.6) can also be recast as a system of the form (1.27)–(1.28), we consider the equivalent ELCP of the form (1.8)–(1.9), and we rewrite the ELCP inequalities into the form  $c - Ax \leq 0$  and we note that if in a group of several homogeneous inequalities of this form at least one inequality should hold with equality, then the maximum of the left-hand sides of the inequalities in this group should be equal to 0. Hence, we get on equation of the form (1.27) for the ELCP inequalities that belong to some subset  $\phi_j$ , and an equation of the form (1.28) for the other ELCP inequalities.

## 5. Applications: Analysis and control of max-plus-linear systems

### 5.1 Max-plus-linear discrete event systems

Typical examples of discrete-event systems are flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems and logistic systems. The class of discrete-event systems essentially consists of man-made systems that contain a finite number of resources (e.g., machines, communications channels, or processors) that are shared by several users (e.g., product types, information packets, or jobs) all of which contribute to the achievement of some common goal (e.g., the assembly of products, the

end-to-end transmission of a set of information packets, or a parallel computation) (Baccelli et al., 1992).

In general, models that describe the behavior of a discrete-event system are nonlinear in conventional algebra. However, there is a class of discrete-event systems – the max-plus-linear discrete-event systems – that can be described by a model that is “linear” in the max-plus algebra (Baccelli et al., 1992). The max-plus-linear discrete-event systems can be characterized as the class of discrete-event systems in which only synchronization and no concurrency or choice occurs. More specifically, these systems can be described by a model of the form

$$x_i(k) = \max \left( \max_{j=1, \dots, n} (a_{ij} + x_j(k-1)), \right. \\ \left. \max_{j=1, \dots, m} (b_{ij} + u_j(k)) \right) \quad \text{for } i = 1, \dots, n \quad (1.29)$$

$$y_i(k) = \max_{j=1, \dots, n} (c_{ij} + x_j(k)) \quad \text{for } i = 1, \dots, l, \quad (1.30)$$

where  $x(k)$  represents the time instants at which the internal processes of the system start for the  $k$ th time (i.e., the state of the system),  $u(k)$  represents the time instants at which the system is fed with new data or products for the  $k$ th (i.e., the input of the system), and  $y(k)$  represents the time instants at which the  $k$ th batch of final data or finished products leave the system (i.e., the output of the system). The additions with  $a_{ij}$ ,  $b_{ij}$ , and  $c_{ij}$  in (1.29)–(1.30) correspond to the time delays like processing times, production times, traveling times, etc. The maximizations correspond to synchronization: a new activity can only start as soon as all predecessor activities are finished.

In a manufacturing context  $x(k)$  contains the time instants at which the processing units start working for the  $k$ th time,  $u(k)$  the time instants at which the  $k$ th batch of raw material is fed to the system, and  $y(k)$  the time instants at which the  $k$ th batch of finished product leaves the system.

Using the notations from max-plus algebra introduced in Section 1.4.1 the model (1.29)–(1.30) can be written as

$$x_i(k) = \bigoplus_{j=1}^n a_{ij} \otimes x_j(k-1) \oplus \bigoplus_{j=1}^m b_{ij} \otimes u_j(k) \quad \text{for } i = 1, \dots, n$$

$$y_i(k) = \bigoplus_{j=1}^n c_{ij} \otimes x_j(k) \quad \text{for } i = 1, \dots, l,$$

or in a more compact matrix-vector format as

$$x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \quad (1.31)$$

$$y(k) = C \otimes x(k) \quad (1.32)$$

This latter form also illustrates where the name “max-plus-linear” systems comes from: for these systems the state and the output are a linear combination (in the max-plus sense) of the previous state and the input.

Using the model (1.31)–(1.32) we can compute the output sequence  $y(1), \dots, y(N)$  of the system for a given input sequence  $u(1), \dots, u(N)$  and initial state  $x(0)$  as follows:

$$y(k) = C \otimes A^{\otimes k} \otimes x(0) \oplus C \otimes A^{\otimes k-1} \otimes B \otimes u(1) \oplus C \otimes A^{\otimes k-2} \otimes B \otimes u(2) \oplus \dots \oplus C \otimes B \otimes u(k) \quad (1.33)$$

for  $k = 1, \dots, N$ .

To illustrate the definition presented above we now consider a simple (max-plus-linear) manufacturing system, determine its evolution equations, and write them in the forms (1.29)–(1.30) and (1.31)–(1.32).

EXAMPLE 1.9 Consider the production system of Figure 1.1.

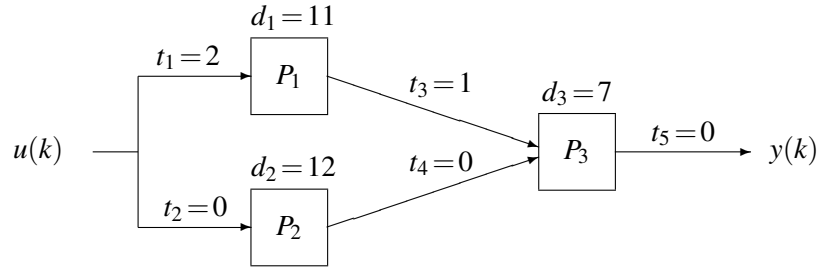


Figure 1.1. A simple manufacturing system.

This manufacturing system consists of three processing units:  $P_1$ ,  $P_2$  and  $P_3$ , and works in batches (one batch for each finished product). Raw material is fed to  $P_1$  and  $P_2$ , processed and sent to  $P_3$  where assembly takes place. The processing times for  $P_1$ ,  $P_2$  and  $P_3$  are respectively  $d_1 = 11$ ,  $d_2 = 12$  and  $d_3 = 7$  time units. It takes  $t_1 = 2$  time units for the raw material to get from the input source to  $P_1$ , and  $t_3 = 1$  time unit for a finished product of  $P_1$  to get to  $P_3$ . The other transportation times and the set-up times are assumed to be negligible. A processing unit can only start working on a new product if it has finished processing the previous product. Each processing unit starts working as soon as all parts are available.

Let us now we determine the time instant at which processing unit  $P_1$  starts working for the  $k$ th time. If we feed raw material to the system for the  $k$ th time, then this raw material is available at the input of processing unit  $P_1$  at time  $t = u(k) + 2$ . However,  $P_1$  can only start working on the new batch of raw material as soon as it has finished processing the previous, i.e., the  $(k-1)$ th batch. Since the processing time on  $P_1$  is  $d_1 = 11$  time units, the  $(k-1)$ th intermediate product will leave  $P_1$  at time  $t = x_1(k-1) + 11$ . Since  $P_1$  starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

$$x_1(k) = \max(x_1(k-1) + 11, u(k) + 2) . \quad (1.34)$$

Using a similar reasoning we find the following expressions for the time instants at which  $P_2$  and  $P_3$  start working for the  $k$ th time and for the time instant at which the  $k$ th finished product leaves the system:

$$x_2(k) = \max(x_2(k-1) + 12, u(k) + 0) \quad (1.35)$$

$$x_3(k) = \max(x_1(k) + 11 + 1, x_2(k) + 12 + 0, x_3(k-1) + 7) \quad (1.36)$$

$$= \max(x_1(k-1) + 23, x_2(k-1) + 24, x_3(k-1) + 7, u(k) + 14) \quad (1.37)$$

$$y(k) = x_3(k) + 7 + 0 \quad (1.38)$$

Let us now rewrite the evolution equations of the production system using the symbols  $\oplus$  and  $\otimes$ . It is easy to verify that (1.34) can be rewritten as

$$x_1(k) = 11 \otimes x_1(k-1) \oplus 2 \otimes u(k) \quad .$$

Equations (1.35)–(1.38) result in

$$x_2(k) = 12 \otimes x_2(k-1) \oplus u(k)$$

$$x_3(k) = 23 \otimes x_1(k-1) \oplus 24 \otimes x_2(k-1) \oplus 7 \otimes x_3(k-1) \oplus 14 \otimes u(k)$$

$$y(k) = 7 \otimes x_3(k) \quad .$$

If we rewrite these evolution equations in max-algebraic matrix notation, we obtain the description

$$x(k) = \begin{bmatrix} 11 & -\infty & -\infty \\ -\infty & 12 & -\infty \\ 23 & 24 & 7 \end{bmatrix} \otimes x(k-1) \oplus \begin{bmatrix} 2 \\ 0 \\ 14 \end{bmatrix} \otimes u(k)$$

$$y(k) = \begin{bmatrix} -\infty & -\infty & 7 \end{bmatrix} \otimes x(k) \quad .$$

## 5.2 Max-plus-algebraic problems and analysis of max-plus systems

It is easy to verify that the following max-plus-algebraic problems can be recast as a system of max-plus-polynomial equations and inequalities and thus also as an ELCP (De Schutter and De Moor, 1996; De Schutter, 1996):

- solving two-sided max-plus-linear equations:

Given  $A, B \in \mathbb{R}_{\varepsilon}^{m \times n}$ , and  $c, d \in \mathbb{R}_{\varepsilon}^m$ , find  $x \in \mathbb{R}_{\varepsilon}^n$  such that

$$A \otimes x \oplus c = B \otimes x \oplus d \quad .$$

- max-plus-algebraic matrix decomposition:

Given a matrix  $A \in \mathbb{R}_{\varepsilon}^{m \times n}$  and an integer  $p > 0$ , find  $B \in \mathbb{R}_{\varepsilon}^{m \times p}$  and  $C \in \mathbb{R}_{\varepsilon}^{p \times n}$  such that

$$A = B \otimes C \quad .$$

- determining state space realizations of max-plus-linear systems:

Given a partial impulse response  $\{G_k\}_{k=1}^N$  of a max-plus-linear system with unknown system matrices  $A$ ,  $B$  and  $C$ , and a system order  $n$ , determine the system matrices of the system.

For a single-input system the impulse response is the output of the system for the input sequence given by  $u(1) = 0$  and  $u(k) = -\infty$  for all  $k > 0$  (i.e., an impulse signal), and for the initial state  $x(0) = [-\infty \ -\infty \ \dots \ -\infty]^T$ . In general, for a multi-input system, the sequence of the  $i$ th columns of the  $G_k$ 's corresponds to the output sequence obtained when an impulse signal is applied to the  $i$ th input and the other inputs are kept at  $-\infty$ . Using (1.33) it is then easy to verify that the impulse response satisfies

$$G_k = C \otimes A^{\otimes k-1} \otimes B \quad \text{for all } k .$$

If  $\{G_k\}_{k=1}^N$  is known this results in a system of max-plus-polynomial equations in  $A$ ,  $B$ , and  $C$ .

- transformation of state space models:

Given system matrices  $A$ ,  $B$ ,  $C$ , find  $L$ ,  $\hat{A}$ , and  $\hat{C}$  such that

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \otimes L .$$

If we can find such a decomposition, and if we define

$$\tilde{A} = L \otimes \hat{A}, \quad \tilde{B} = L \otimes B, \quad \tilde{C} = \hat{C} ,$$

then it is easy to verify that the state space models corresponding to the triplets  $(A, B, C)$  and  $(\tilde{A}, \tilde{B}, \tilde{C})$  of systems matrices have the same impulse response, i.e.,

$$C \otimes A^{\otimes k} \otimes B = \tilde{C} \otimes \tilde{A}^{\otimes k} \otimes \tilde{B} \quad \text{for all } k .$$

In that case we say that  $(A, B, C)$  and  $(\tilde{A}, \tilde{B}, \tilde{C})$  are equivalent realizations of the same max-plus-linear system.

An alternative transformation is the following

Given system matrices  $A$ ,  $B$ ,  $C$ , find  $M$ ,  $\hat{A}$ , and  $\hat{B}$  such that

$$\begin{bmatrix} A & B \end{bmatrix} = M \otimes \begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix} .$$

In this case we should consider

$$\tilde{A} = \hat{A} \otimes M, \quad \tilde{B} = \hat{B}, \quad \tilde{C} = C \otimes M .$$

Other applications related to the max-plus algebra that result in an ELCP include computing singular value decompositions, QR decompositions, and other matrix factorizations in the extended max-plus-algebra, and systems of max-min-plus equations (De Schutter and De Moor, 1998c)

### 5.3 Model-based predictive control of max-plus-linear systems

**5.3.1 Framework.** As a final application we consider model predictive control (MPC) of max-plus-linear systems. MPC (Maciejowski, 2002) was pioneered simultaneously by Richalet *et al.* (Richalet et al., 1978), and Cutler and Ramaker (Cutler and Ramaker, 1979). Since then, MPC has probably become the most applied advanced control technique in the process industry. A key advantage of MPC is that it can accommodate constraints on the inputs and outputs. Usually MPC uses linear or nonlinear discrete-time models. However, we now consider the extension of MPC to max-plus-linear discrete-event systems (De Schutter and van den Boom, 2001a).

In MPC we determine at each event step  $k$  the optimal input sequence  $u(k), u(k+1), \dots, u(k+N_p-1)$  over a given prediction horizon  $N_p$ . We assume that at event step  $k$ , the previous value  $x(k-1)$  of the state can be measured or estimated using previous measurements. We can then use (1.33) to estimate the evolution of the output of the system for the input sequence  $u(k), \dots, u(k+N_p-1)$ :

$$\hat{y}(k+j|k) = C \otimes A^{\otimes j} \otimes x(k-1) \oplus \bigoplus_{i=0}^j C \otimes A^{\otimes j-i} \otimes B \otimes u(k+i) , \quad (1.39)$$

where  $\hat{y}(k+j|k)$  is the estimate of the output at event step  $k+j$  based on the information available at event step  $k$ . If the due dates  $r$  for the finished products are known and if we have to pay a penalty for every delay, a well-suited output cost criterion is the tardiness:

$$J_{\text{out}}(k) = \sum_{j=0}^{N_p-1} \sum_{i=1}^I \max(\hat{y}_i(k+j|k) - r_i(k+j), 0) . \quad (1.40)$$

On the other hand we also want to keep the throughput time and the internal buffer levels as low as possible. Therefore, we will *maximize* the input time instants. For a manufacturing system, this would correspond to a scheme in which raw material is fed to the system as late as possible. This results in the following input cost criterion

$$J_{\text{in}}(k) = \sum_{j=0}^{N_p-1} \sum_{i=1}^m u(k+j) . \quad (1.41)$$

The input and output cost criteria are combined as follows in the overall performance function  $J$ :

$$J(k) = J_{\text{out}}(k) + \lambda J_{\text{in}}(k) ,$$

with  $\lambda > 0$ .

Since for discrete-event systems the inputs  $u(k)$  correspond to consecutive feeding times, this sequence should be nondecreasing, resulting in the constraint

$$u(k+j) \geq u(k+j-1) \quad \text{for } j = 0, \dots, N_p-1 .$$

Furthermore, we sometimes also have constraints such as minimum or maximum separation between input and output events:

$$\begin{aligned} a_1(k+j) &\leq u(k+j) - u(k+j-1) \leq b_1(k+j) && \text{for } j = 0, \dots, N_p - 1 \\ a_2(k+j) &\leq \hat{y}(k+j|k) - \hat{y}(k+j-1|k) \leq b_2(k+j) && \text{for } j = 0, \dots, N_p - 1, \end{aligned}$$

maximum due dates for the output events:

$$\hat{y}(k+j|k) \leq r(k+j) \quad \text{for } j = 0, \dots, N_p - 1,$$

or maximum deviations from the due dates:

$$\begin{aligned} r(k+j) - \delta^-(k+j) &\leq \hat{y}(k+j|k) \\ &\leq r(k+j) + \delta^+(k+j) \quad \text{for } j = 0, \dots, N_p - 1, \end{aligned}$$

If we define

$$\tilde{u}(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N_p-1) \end{bmatrix}, \quad \tilde{y}(k) = \begin{bmatrix} \hat{y}(k|k) \\ \hat{y}(k+1|k) \\ \vdots \\ \hat{y}(k+N_p-1|k) \end{bmatrix},$$

we can collect all the above constraints into one system of linear equations of the form

$$A_c(k)\tilde{u}(k) + B_c(k)\tilde{y}(k) \leq c_c(k). \quad (1.42)$$

### 5.3.2 The MPL-MPC problem and its link with the ELCP.

If we combine the material of previous subsection, we finally obtain the following problem:

At event step  $k$ , find the input sequence vector  $\tilde{u}(k)$  that minimizes  $J(k) = J_{\text{out}}(k) + \lambda J_{\text{in}}(k)$  subject to the evolution equations (1.39) and the constraints (1.42).

This problem will be called the max-plus-linear MPC (MPL-MPC) problem for event step  $k$ . MPL-MPC also uses a receding horizon principle, which means that at event step  $k$  the future control sequence  $u(k), \dots, u(k+N_p-1)$  is determined such that the cost criterion is minimized subject to the constraints. At event step  $k$  the first element of the optimal sequence (i.e.,  $u(k)$ ) is then applied to the system. At the next event step, the horizon is shifted, the model is updated with new information of the measurements, and a new optimization at event step  $k+1$  is performed, and so on.

Let us now have a closer look at the MPL-MPC problem. We could consider both  $\tilde{u}(k)$  and  $\tilde{y}(k)$  as optimization variables. Clearly, as the constraints of the MPL-MPC problem are a combination of max-plus-polynomial constraints and linear constraints, they can be recast as an ELCP. This implies that the optimal sequence  $\tilde{u}(k)$  can be determined by optimizing  $J(k)$  over the solution set of this ELCP.



**5.3.3 Algorithms for the MPL-MPC problem.** Now we discuss some methods to solve the MPL-MPC problem. The material in this section is inspired by (De Schutter and van den Boom, 2001a), but due to the fact that we focus on one particular performance function (i.e., (1.40)–(1.41)) we can make some significant simplifications in our explanation and in our approach with respect to (De Schutter and van den Boom, 2001a).

As indicated above we can solve the MPL-MPC problem by first determining the entire solution set of the ELCP that corresponds to the constraints of the MPL-MPC problem in a parameterized way using the algorithms of Section 1.3.1, and then optimizing  $J(k)$  over this solution set. However, as the MPL-MPC problem has to be solved at each event step, this approach is not feasible in practice.

Alternatively, we could consider the MPL-MPC problem as a nonlinear non-convex optimization problem and use standard multi-start nonlinear non-convex local optimization methods to compute the optimal control policy. However, in practice this approach is also often not feasible.

We could also apply the mixed-integer programming approach as follows: note that since  $A$ ,  $B$  and  $C$  are known, the evolution equations (1.39) can be rewritten as

$$\tilde{y}_i(k) = \max_{j=1, \dots, mN_p} (h_{ij} + \tilde{u}_j(k), g_j(k)) \quad \text{for } i = 1, \dots, lN_p, \quad (1.43)$$

for some matrix  $H$  and a vector  $g(k)$  that depends on  $x(k-1)$  (see (De Schutter and van den Boom, 2001a) for the exact expressions). We can now eliminate  $\tilde{y}(k)$  from the objective function  $J(k)$ , resulting in an expression of the form

$$\begin{aligned} J(k) &= \sum_{i=1}^{lN_p} \left( \max_{j=1, \dots, mN_p} (h_{ij} + \tilde{u}_j(k), g_j(k)) - \tilde{r}_i(k) \right) + \lambda \sum_{j=1}^{mN_p} \tilde{u}_j(k) \\ &= \max_{i=1, \dots, K} \max_{j=1, \dots, mN_p} (p_{ij} \tilde{u}_j(k) + q_j(k)) \\ &= \max_{i=1, \dots, K} (P\tilde{u}(k) + q(k))_i \end{aligned}$$

for an appropriately defined matrix  $P$ , vector  $q$ , and constant  $K$  where  $\tilde{r}(k)$  is defined in a similar way as  $\tilde{y}(k)$  and where we have made recursive use of the following basic property: for  $\alpha, \beta, \gamma \in \mathbb{R}$  we have  $\max(\alpha, \beta) + \gamma = \max(\alpha + \gamma, \beta + \gamma)$ . If we now introduce a scalar dummy variable  $t$  such that

$$t = \max_{i=1, \dots, K} (P\tilde{u}(k) + q(k))_i, \quad (1.44)$$

then the MPL-MPC problem reduces to minimizing a linear objective function ( $J(k) = t$ ) subject to the constraints (1.42), (1.43), and (1.44). Note that these constraints are a combination of max-plus and linear constraints, i.e., they correspond to an ELCP. As shown in the proof of Theorem 1.6, these constraints thus can be rewritten as a system of mixed-integer linear equations (in fact the detour via the ELCP is not necessary, and the equations can directly be transformed into mixed-integer linear constraints). Hence, the MPL-MPC problem can be recast as a mixed-integer linear programming problem.

If in addition the matrix  $B_c(k)$  in (1.42) only has nonnegative entries we can make a further simplification, which will ultimately result in a linear programming problem. In fact, if all entries of  $B_c(k)$  are nonnegative (this occurs, e.g., when there are no constraints on  $\tilde{y}(k)$ , or if there are only upper bound constraints on  $\tilde{y}(k)$ ), then we can also easily eliminate  $\tilde{y}(k)$  from the linear constraints (1.42), resulting in

$$(A_c(k)\tilde{u}(k))_\ell + \sum_{i=1}^{IN_p} (B_c)_{\ell i} \max_{j=1, \dots, mN_p} (h_{ij} + \tilde{u}_j(k), g_j(k)) \leq (c_c(k))_\ell \quad \text{for all } \ell,$$

or equivalently an expression of the form

$$\max_{i=1, \dots, L} (S_{(\ell)}(k)\tilde{u}(k) + s_{(\ell)}(k))_\ell \leq (c_c(k))_\ell \quad \text{for all } \ell,$$

for an appropriately defined matrix  $S_{(\ell)}(k)$  and vector  $s_{(\ell)}(k)$ , or even more simply

$$S_\ell(k)\tilde{u}(k) + s_\ell(k) \leq c_{(\ell)}(k) \quad \text{for all } \ell, \quad (1.45)$$

for an appropriately defined vector  $c_{(\ell)}(k)$ . As now we have eliminated  $\tilde{y}(k)$  completely, we have to minimize

$$J(k) = \max_{i=1, \dots, K} (P\tilde{u}(k) + q(k))_i$$

over the linear constraint (1.45). If we again introduce a dummy variable  $t$  and solve the following linear optimization problem

$$\begin{aligned} & \min_{t, \tilde{u}(k)} t \\ & \text{subject to (1.45) and } t \geq (P\tilde{u}(k) + q(k))_i \text{ for } i = 1, \dots, K, \end{aligned}$$

then it is easy to verify that in the optimal solution, at least one of the bounds on  $t$  is tight, i.e., (1.44) holds. So in this case we can find the optimal solution of the MPL-MPC problem via linear programming, for which efficient algorithms exist such as (variants of) the simplex method or interior point methods (Nesterov and Nemirovskii, 1994; Pardalos and Resende, 2002).

For a worked example and a comparison of several of these alternative MPL-MPC algorithms, we refer the interested reader to (De Schutter and van den Boom, 2001a).

In (De Schutter and van den Boom, 2000; De Schutter and van den Boom, 2001b) we have extended the above results to max-min-plus-scaling systems, a class of discrete-event systems that can be modeled using the operations maximization, minimization, addition and scalar multiplication. Related work involving the determination of optimal switching times for traffic signals and for first-order linear hybrid systems with saturation is described in (De Schutter, 2000; De Schutter, 2002).

## 6. Summary

In this chapter we have presented the extended linear complementarity problem (ELCP) and its relation to the regular linear complementarity problem (LCP) and to various linear generalizations of the LCP. We have shown that the ELCP can in a way be considered to be the most general linear extension of the LCP. We have also discussed some properties of the solution set of an ELCP and presented some algorithms to solve an ELCP: we have considered an algorithm for determining the complete solution set of an ELCP, and also several algorithms to determine only one solution. Next, we have shown that a system of max-plus-polynomial equations is equivalent to an ELCP, which allows us to solve several problems that arise in the max-plus algebra, and in the analysis and control of max-plus-linear systems. In particular, for the model-based predictive control of max-plus-linear systems the original ELCP-based problem can be reduced to a linear programming problem, which can be solved very efficiently.

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## Notes

1. We only need boundedness from above since the surplus variables are always nonnegative due to the condition  $Ax \geq c$ .
2. Note, however, that if we want to solve an ELCP using, e.g., the algorithm of Section 1.3.1, then the formulation (1.5)–(1.6) leads to a more efficient solution than the reformulation (1.8)–(1.9).
3. Regarding the Extended LCP of Mangasarian and Pang, note that we may assume without loss of generality that  $\mathcal{P}$  can be represented as  $\mathcal{P} = \{u \in \mathbb{R}^m \mid Su \geq t\}$  for some matrix  $S \in \mathbb{R}^{l \times m}$  and vector  $t \in \mathbb{R}^l$ .

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