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The Extended Loop Group: An Infinite Dimensional Manifold Associated With the Loop Space

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Abstract. A set of coordinates in the non-parametric loop-space is introduced. We show that these coordinates transform under infinite dimensional linear representations of the diffeomorphism group. An extension of the group of loops in terms of these objects is proposed. The enlarged group behaves locally as an infinite dimensional Lie group. Ordinary loops form a subgroup of this group. The algebraic properties of this new mathematical structure are analyzed in detail. Applications of the formalism to field theory, quantum gravity and knot theory are considered.

1. Introduction

Loop space has been used in several non-perturbative approaches to gauge theories and gravitation [1–7]. In the eighties, the loop representation of gauge theories was accomplished [8–11]. This representation has proved to be a suitable framework where one can develop a complete canonical scheme for the quantization of gauge theories. The two remarkable features of this formulation are the manifest gauge invariance of the quantization method and the solution of the constraints through geometrical requirements. Other approaches to the loop space have also been developed, based fundamentally on Polyakov's and Makeenko–Migdal's [4, 5, 12] treatment of the dynamics of loop dependent objects in Yang–Mills theories. At present, the results obtained in the loop representation are in agreement with those obtained in the usual approach to gauge theories. The loop representation also gives new insights into the non-perturbative aspects of gauge theories.

Recently, the loop representation has been used in quantum general relativity [13, 14]. This representation emerges naturally from Ashtekar's new formulation of general relativity [15]. Any quantum field theory whose configuration variable is a connection can be realized in loop space language. But in the quantum gravity case, the use of the loop space turns out to be an essential tool in order to develop a complete non-perturbative quantization program [16]. Non-perturbative solutions to all the constraints of quantum gravity have been found [17, 18], including those associated with nondegenerate metrics [19]. In the latter case, knot invariants of intersecting loops have proved to play an essential role in the construction of the solutions.

Knot and quantum field theories can be related in different ways. For instance, by making use of the deep connection between the three dimensional Chern–Simons models and certain two dimensional conformal field theories, Witten [20] was able to derive a skein relation for the Wilson line expectation value, $\langle W(L) \rangle$, and show that there exists a correspondence between $\langle W(L) \rangle$ and Jones [21] and HOMFLY [22] polynomials. Also the construction of link polynomials from exactly solvable models in statistical mechanics was developed [23]. On the other hand, as we just mentioned, knot invariants are intimately related to the non-perturbative quantum states of general relativity. Actually, at this stage one can speak about a loop space formalism. This formalism seems to be an appropriate framework to develop a realistic quantum theory of space time [16].

In the loop representation, states are given as functionals of loops. Loops can be precisely defined in a non-parametric form [9] as an equivalence class of closed oriented paths in a manifold. We will declare the loop L equivalent to L' if the product $L' \circ \bar{L}$, the contour obtained by following first L' and then L with the opposite orientation, is a closed path which is contractible within itself to the null path. The basic mathematical property that makes the loop space on appropriate scenario to construct representations of gauge theories and gravitation is the group property. Loops form a group and the holonomy of a particular gauge theory may be obtained by considering a representation of the group of loops into the particular group being gauged.

The group of loops does not contain one parameter subgroups and consequently, it is not a Lie group. However, by analyzing its infinitesimal elements one can appropriately introduce generators (through the loop derivative operator) and construct with them any element of the group. The group composition law is the fundamental mathematical property that enables us to operate in loop space and to use it as the domain of functionals where the gauge theory is realized.

The lack of a Lie structure for the loop group restricts our capability to work with the loop space. We are not dealing with a manifold and then we cannot use all the functional techniques available in those cases. From a mathematical point of view, it would be highly desirable to develop an extension of the group of loops in order to include it in a larger manifold structure.

From a physical point of view, the fact that loops are non-local distributional objects with a one dimensional support introduces certain typical difficulties in the hamiltonian formulation of any diffeomorphism invariant gauge theory in the loop representation. It is a well known fact that many of the diffeomorphism invariant quantities up to now considered (i.e. the Gauss number) are ill defined when evaluated on loops and need to be framed. Moreover, the framing destroys invariance under diffeomorphism of the theory. Another typical difficulty is related with our inability to define an integration measure in loop space and consequently, an inner product in the Hilbert space of quantum gravity. We expect that the extension of the loop space would allow to transform loop integrals into functional integrals.

In this paper we develop a new approach to the loop space formalism. The basic idea underlying this approach is the possibility to define a *coordinate representation* on the loop space. The coordinates on loop space were first introduced by Gambini and Leal [24] and were used by Brügmann, Gambini and Pullin in the determination of the first nondegenerate quantum state of general relativity [19].

We start by introducing a set of coordinates which allows to describe any element of the non-parametric loop space. We show that these coordinates transform under linear representations of the diffeomorphism group. We find that it is possible to generate an infinite set of gauge invariant objects generalizing the coordinate expansion of the holonomy. The identification of a set of invariant tensors under diffeomorphism transformations allows to give rules for the construction of an, in principle infinite, set of knot invariants. We also show how to generate new knot invariants from others and how to relate link and knot invariants in the SU(2) case. Within this approach, the basic conditions necessary to define an affine geometry in loop space seem to be fulfilled. What is more important, loop coordinates allow to show that there exist an infinite dimensional manifold with a local Lie group structure associated with the loop space, the Special-extended Loop Group. This group provides the basis of a new formulation of the loop representation. A complete description of the algebraic properties of this enlarged structure is accomplished. Finally we would like to stress that we are not going to discuss analytical details of regularization, and in general, this presentation pretends a physicist's rather than a mathematician's level of rigor.

We organize the article as follows: in Sect. 2 we introduce a set of coordinates on loop space and show that they are connected by infinite dimensional linear transformations under diffeomorphisms. We also show that these coordinates are not independent, but satisfy a set of algebraic and differential constraints. Section 3 is devoted to the differential constraint. In Sect. 3.1 we solve the differential constraint and in Sect. 3.2 non-trivial representations of the diffeomorphism group (with a fixed point) are introduced. The free coordinates associated with loops are defined in Sect. 4. In Sect. 5 invariant tensors in the space of free coordinates are introduced and the relationship between the invariant tensors and knot invariants is analyzed. The first five sections lay the groundwork for the further study of the extended loop group. In Sect. 6 we show how the loop coordinates can be incorporated in a more general algebraic structure, the SeL group. The algebra of the SeL group is studied in Sect. 7. The generators are introduced in Sect. 7.1, in Sect. 7.2 a basis for the algebra is constructed and Sect. 7.3 is devoted to the study of invariant forms and its relationship with the automorphism transformations of the algebra. Final comments and conclusions are made in Sect. 8.

1. Coordinates on the Loop Space

The loop representation [8, 13] of a quantum field theory can be constructed in terms of an algebra of linear operators defined on a state space of loop functionals. These states may be expressed as follows:

$$\Psi(L_1, \ldots, L_n) = \int d\mu[A]\,\Psi[A]\,W_A(L_1)\cdots W_A(L_n)\,, \tag{1}$$

where

$$W_A(L) = \mathrm{Tr}\left[P\exp\oint_L A_a dy^a\right] = \mathrm{Tr}[U_A(L)] \tag{2}$$

is the Wilson line functional corresponding to the holonomy $U_A(L)$. Thus, holonomies are the building blocks of any loop dependent object. The holonomy of a particular gauge theory [9] may be obtained by considering a representation of the group of loops into the particular group G being gauged. For Quantum Gravity $U_A(L)$ belongs to SU(2) [25]. As it has been emphasized by a number of authors [9, 13, 26–29], loops are equivalence classes of closed oriented paths in a manifold M. As it was already emphasized in the introduction, the set of loops forms a group. It is important to note that whereas closed curves are essentially parametrization dependent objects, loops may be described in nonparametric form [10, 30]. This means that the space of loops on which the loop representation is defined is not the standard parametrized space of closed curves.

In this section we are going to introduce a set of coordinates which allows to describe any element of the nonparametric loop space. Then, we are going to show that the corresponding coordinates of two diffeomorphic loops are connected by an infinite dimensional linear transformation. In other words, the loop coordinates transform under linear representations of the diffeomorphism group.

Let us start by introducing a set of coordinates in loop space. They are related with the holonomies as follows

$$U_A(L) = 1 + \sum_{n=1}^{\infty} \int dx_1^3 \cdots dx_n^3 A_{a_1}(x_1) \cdots A_{a_n}(x_n) X^{a_1 \cdots a_n}, x_1, \ldots, x_n, L) \,, \quad (3)$$

where L are loops with a base point O taken as their origin and the loop dependent objects X are given by:

$$\begin{aligned} X^{a_1 \cdots a_n}(x_1, \ldots, x_n, L) &= \int_L dy_n^{a_n} \int_0^{y_n} dy_{n-1}^{a_{n-1}} \cdots \int_0^{y_2} dy_1^{a_1} \delta(x_n - y_n) \cdots \delta(x_1 - y_1) \\ &= \oint_L dy_n^{a_n} \cdots \oint_L dy_1^{a_1} \delta(x_n - y_n) \cdots \delta(x_1 - y_1) \\ &\quad \times \Theta_L(0, y_1, \ldots, y_n) \end{aligned} \quad (4)$$

and $\Theta_L(0, y_1, \ldots, y_n)$ orders the points along the contour starting at the origin of the loop. These relations define the X objects of "rank" n. We shall call them the multitangents of the loop L.

In what follows, it will be convenient to introduce the notation

$$X^{\mu_1 \cdots \mu_n}(L) = X^{a_1 x_1 \cdots a_n x_n}(L) = X^{a_1 \cdots a_n}(x_1, \ldots, x_n, L) \,, \quad (5)$$

with $\mu_i \equiv (a_i x_i)$, which is more suggestive of the role played by the x variables under diffeomorphic transformations.

The distributional objects X allow to determine the Wilson line element (2) for any connection. As it was first pointed out by Makeenko and Migdal [31], the wave functions in the loop representation depend on the cyclic permutations of the multitangent fields, given by

$$X_c^{\mu_1 \cdots \mu_n}(L) = \frac{1}{n} \{X^{\mu_1 \cdots \mu_n}(L) + X^{\mu_2 \cdots \mu_n \mu_1}(L) + \text{c.p.}\} \,. \quad (6)$$

The X objects transform as multivector densities under the subgroup of coordinate transformations that leaves the base point O fixed. In other words if

$$x^a \to x'^a = D^a(x) \,, \quad (7)$$

then

$$X^{a_1 x'_1 \cdots a_n x'_n}(DL) = \frac{\partial x'^{a_1}_1}{\partial x^{b_1}_1} \cdots \frac{\partial x'^{a_n}_n}{\partial x^{b_n}_n} \frac{1}{J(x_1)} \cdots \frac{1}{J(x_n)} X^{b_1 x_1 \cdots b_n x_n}(L) , \tag{8}$$

where J is the jacobian of the transformation.

Introducing a matrix notation this relation may be rewritten in a different way which is more suggestive of the role of the multitangents in loop space. Let us define the vector-like object $\vec{X}(L) \equiv (X^{\mu_1}(L), \ldots, X^{\mu_1 \cdots \mu_i}(L), \ldots)$. The components of this object are multivector density fields of any rank (being the rank, the number of indexes). Now we introduce the matrix Λ_D with components

$$\Lambda_D{}^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} \equiv \delta_{n,m} \Lambda_D{}^{\mu_1}{}_{\nu_1} \cdots \Lambda_D{}^{\mu_n}{}_{\nu_n} , \tag{9}$$

where

$$\Lambda_D{}^{ay}{}_{bx} = \frac{1}{J(x)} \frac{\partial D^a(x)}{\partial x^b} \delta(x - D^{-1}(y)) = \frac{\partial D^a(x)}{\partial x^b} \delta(D(x) - y)) . \tag{10}$$

Then (8) can be written in the form

$$X^{\mu_1 \cdots \mu_n}(DL) = \sum_{m=1}^{\infty} \Lambda_D{}^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} X^{\nu_1 \cdots \nu_m}(L) , \tag{11}$$

or in shorthand,

$$\vec{X}(DL) = \Lambda_D \cdot \vec{X}(L) , \tag{12}$$

where use has been made of a generalized Einstein convention for the repeated indexes, given by

$$A_{bx} B^{bx} \equiv \sum_{b=1}^{3} \int d^3x A_{bx} B^{bx} . \tag{13}$$

Hence, the multitangents associated with two diffeomorphic loops are related by linear transformations and therefore they transform as generalized tensors. The constant (loop independent) linear transformations are the elements of a linear representation of the diffeomorphism group. In a certain sense it is a trivial linear representation of the diffeomorphism group because it is a direct consequence of the multivector character of the multitangent field. Later on, we shall introduce a new non-trivial representation which is relevant in the determination of the knot invariant quantities.

The multitangents contain all the relevant information necessary to uniquely determine a loop and therefore they may be considered as good prospects for coordinates of a loop. However, they are not independent variables. In fact, they obey a set of simple algebraic and differential constraints.

The algebraic constraints are a direct consequence of the properties of the Θ_L functions,

$$\Theta_L(0, y_1) = 1, \quad \Theta_L(0, y_1, y_2) + \Theta_L(0, y_2, y_1) = 1 ,$$
$$\Theta_L(0, y_1, y_2, y_3) + \Theta_L(0, y_2, y_1, y_3) + \Theta_L(0, y_2, y_3, y_1) = \Theta_L(0, y_2, y_3) , \tag{14}$$

and so on. The corresponding constraints for the X objects are

$$X^{\mu_1} = X^{\mu_1}, \quad X^{\mu_1 \mu_2} + X^{\mu_2 \mu_1} = X^{\mu_1} X^{\mu_2} ,$$
$$X^{\mu_1 \mu_2 \mu_3} + X^{\mu_2 \mu_1 \mu_3} + X^{\mu_2 \mu_3 \mu_1} = X^{\mu_1} X^{\mu_2 \mu_3} . \tag{15}$$

Their general form is

$$X^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1} \cdots \mu_n} \equiv \sum_{P_k} X^{P_k(\mu_1 \cdots \mu_n)} = X^{\mu_1 \cdots \mu_k} X^{\mu_{k+1} \cdots \mu_n} , \tag{16}$$

and the sum goes over all the permutations of the μ variables which preserve the ordering of the $\mu_1, \ldots, \mu_k$ and the $\mu_{k+1}, \ldots, \mu_n$ between themselves.

The differential constraint may be easily obtained from Eq. (4) and takes the form:

$$\frac{\partial}{\partial x_i^{a_i}} X^{a_1 x_1 \cdots a_i x_i \cdots a_n x_n}$$
$$= (\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1})) X^{a_1 x_1 \cdots a_{i-1} x_{i-1} a_{i+1} x_{i+1} \cdots a_n x_n} \tag{17}$$

with x_0 and x_{n+1} equal to the origin of the loop.

The differential constraint is related to the gauge transformation properties of the holonomy. Any object $\vec{E} = (E^{\mu_1}, \ldots, E^{\mu_1 \cdots \mu_n}, \ldots)$ satisfying the differential constraints allows one to define a gauge covariant quantity

$$U_A(E) = 1 + \sum_{n=1}^{\infty} \int dx_1 \cdots dx_n A_{\mu_1} \cdots A_{\mu_n} E^{\mu_1 \cdots \mu_n} . \tag{18}$$

Even though for any loop there corresponds a distributional object $\vec{X}(L)$ such that (18) generates their Wilson functional, the converse is not true. The space of functions that satisfy both algebraic and differential constraints is more general and includes the multitangents $\vec{X}(L)$ and smooth functions. For simplicity, one can consider the abelian case. For U(1), loops are completely described through the multitangent fields of rank one. We know that $X^{\mu_1}(L)$ is given by integration of a distribution along the loop L. This function is transverse (divergence free) and trivially satisfy the algebraic constraint. But this set of functions is only a sector of the entire space of transverse functions.

3. The Differential Constraint

3.1. The Solution of the Differential Constraint. We consider now the solution of the differential constraint. Let us start by introducing transverse and longitudinal projectors in the multivector density space. We shall first consider a covariant metric in the space of transverse vector densities of rank one. Given two transverse fields V^{ax} and W^{ax} one can define [32]

$$g(V, W) = \int d^3x V^a A_a^W ,$$
$$\partial_a V^a = \partial_a W^a = 0 , \tag{19}$$

where A_a^W is the potential associated with the curl free tensor $W_{ab} = \varepsilon_{abc} W^c = \partial_{[a} A_{b]}^W$.

In the transverse (non-covariant) gauge

$$\partial^a A_a^W = 0 , \tag{20}$$

it takes the form

$$g(V, W) = g_{o\,axby} V^{ax} W^{by} \tag{21}$$

with

$$g_{o\,axby} = -\,\varepsilon_{abc}\frac{\partial^c}{\Delta}\,\delta(x-y) = -\,\frac{1}{4\pi}\,\varepsilon_{abc}\,\frac{x^c-y^c}{|x-y|^3}\,. \tag{22}$$

This well known object is the kernel of the Gauss knot invariant. It is the expression in a particular gauge of the covariant metric in the space of transverse vector densities defined by (19). In general, the covariant metric is defined up to gradients

$$g_{axby} = g_{o\,axby} + \rho_{ax\,y,b} + \rho_{by\,x,a}\,. \tag{23}$$

Now transverse and longitudinal projectors may be easily written in terms of g and its inverse in the transverse space

$$g^{axby} = \varepsilon^{abc}\partial_c\delta(x-y)\,. \tag{24}$$

In fact

$$\delta_T{}^{ax}{}_{by} \equiv g^{ax\,cz}g_{cz\,by} \tag{25}$$

and

$$\delta_L{}^{ax}{}_{by} \equiv \delta^{ax}{}_{by} - \delta_T{}^{ax}{}_{by} \equiv \delta_{a,b}\,\delta(x-y) - \delta_T{}^{ax}{}_{by} \tag{26}$$

are orthogonal projectors

$$\delta_T{}^{\mu}{}_{\rho}\delta_T{}^{\rho}{}_{\nu} = \delta_T{}^{\mu}{}_{\nu}\,,$$

$$\delta_L{}^{\mu}{}_{\rho}\delta_L{}^{\rho}{}_{\nu} = \delta_L{}^{\mu}{}_{\nu}\,,$$

$$\delta_L{}^{\mu}{}_{\rho}\delta_T{}^{\rho}{}_{\nu} = \delta_T{}^{\mu}{}_{\rho}\delta_L{}^{\rho}{}_{\nu} = 0\,.$$

By using the explicit form of the covariant metric, one can prove that

$$\delta_L{}^{ax}{}_{by} = \phi^{ax}{}_{y,b}\,, \tag{27}$$

where

$$\frac{\partial}{\partial x^a}\,\phi^{ax}{}_{y} = -\,\delta(x-y)\,. \tag{28}$$

The ambiguity in the definition of the metric induces an ambiguity in the decomposition in transverse and longitudinal parts. Each function ϕ satisfying (28) determines a particular prescription of the decomposition. It is important to note that the transverse density fields and in particular the contravariant metric (24) are prescription independent. When the transverse metric g_o is chosen we have

$$\phi_o^{ax}{}_{y} = \frac{1}{4\pi}\frac{\partial}{\partial x^a}\frac{1}{|x-y|}\,, \tag{29}$$

$$\delta_{oT}{}^{ax}{}_{by} = \delta^{ax}{}_{by} + \frac{\partial^a\partial_b}{4\pi}\,\frac{1}{|x-y|}\,. \tag{30}$$

Now we are ready to solve the differential constraint. Consider the set $\mathscr{E}^*$ of all quantities $\vec{E}$ that satisfy the differential constraint and whose components $E^{\mu_1\cdots\mu_n}$ are multivector density fields. This set forms a linear vector space. A transverse projector acting on the vector space $\mathscr{E}^*$ can be introduced through the matrix δ_T, defined as

$$\delta_T{}^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} \equiv \delta_{n,m}\,\delta_T{}^{\mu_1}{}_{\nu_1}\cdots\delta_T{}^{\mu_n}{}_{\nu_n}\,. \tag{31}$$

The transverse part of any element of $\mathscr{E}^*$ is given by

$$\vec{Z} = \delta_T \cdot \vec{E} \,. \tag{32}$$

The vector $\vec{Z}$ satisfies the homogeneous differential constraints. The set of all $\vec{Z}$'s forms a linear vector space $\mathscr{Z}$. Equation (32) may be inverted allowing to write $\vec{E}$ in terms of $\vec{Z}$. To do that, we start evaluating

$$E^{\mu_1 \cdots \mu_n} = \delta^{\mu_1}{}_{\nu_1} \cdots \delta^{\mu_n}{}_{\nu_n} E^{\nu_1 \cdots \nu_n} \tag{33}$$

and making use of the decomposition of the identity (26), the differential constraint and recalling that $E^{\mu_1} = Z^{\mu_1}$, we get

$$\vec{E} = \sigma \cdot \vec{Z} \,, \tag{34}$$

where the *soldering quantities* σ only depend on the function ϕ which characterize the choice of decomposition in transverse and longitudinal parts,

$$\sigma^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} = \begin{cases} \delta_T{}^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n}, & \text{if } m = n \\ Q^{\mu_1 \cdots \mu_n}_{\rho_1 \cdots \rho_{n-1}} \sigma^{\rho_1 \cdots \rho_{n-1}}{}_{\nu_1 \cdots \nu_m}, & \text{if } m < n \\ 0, & \text{if } m > n \end{cases} \tag{35}$$

with

$$Q^{a_1 x_1 \cdots a_n x_n}_{c_1 y_1 \cdots c_{n-1} y_{n-1}} \equiv \sum_{j=1}^{n} \delta^{a_1 x_1 \cdots a_{j-1} x_{j-1}}_{c_1 y_1 \cdots c_{j-1} y_{j-1}} (\phi^{a_j x_j}_{y_j} - \phi^{a_j x_j}_{y_{j-1}}) \delta_T{}^{a_{j+1} x_{j+1} \cdots a_n x_n}_{c_j y_j \cdots c_{n-1} y_{n-1}} \,. \tag{36}$$

The quantities σ have definite transversal properties

$$\delta_T \cdot \sigma = \delta_T \,, \tag{37}$$

$$\sigma \cdot \delta_T = \sigma \,, \tag{38}$$

and under a change of the prescription $\phi^{ax}_{1y} \to \phi^{ax}_{2y}$ we have

$$\sigma[\phi_1] = \sigma[\phi_2] \cdot \sigma[\phi_1] \,. \tag{39}$$

Let us note that (34) connects the component $E^{\mu_1 \cdots \mu_n}$ of rank n with the components of $\vec{Z}$ of rank one to n. That is

$$E^{\mu_1 \cdots \mu_n} = \sum_{m=1}^{n} \sigma^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} Z^{\nu_1 \cdots \nu_m} \,. \tag{40}$$

It is useful to introduce a vector product on $\mathscr{E}^*$. In general, given two vectors $\vec{E}_1$ and $\vec{E}_2$ we define its $\times$-product as the vector $(\vec{E}_1 \times \vec{E}_2)$ with components

$$(\vec{E}_1 \times \vec{E}_2)^{\mu_1 \cdots \mu_n} = \sum_{i=1}^{n-1} E_1^{\mu_1 \cdots \mu_i} E_2^{\mu_{i+1} \cdots \mu_n} \,. \tag{41}$$

The $\times$-product is associative and have the important property that satisfies the differential constraint if any $\vec{E}_j$ does.

Equations (32) and (34) define an isomorphism between the vector spaces $\mathscr{E}^*$ and $\mathscr{Z}$. The vector product can be introduced in the vector space $\mathscr{Z}$ and, due to the isomorphism we have

$$\vec{E}_1 \times \vec{E}_2 = \sigma \cdot (\vec{Z}_1 \times \vec{Z}_2) \,. \tag{42}$$

It would be important for our purpose to consider the subset $\mathscr{F} \subset \mathscr{E}^*$ of elements that satisfy the homogeneous algebraic constraint. Any element $\vec{F} \in \mathscr{F}$ identically vanish under the action of the permutation operator associated with the algebraic constraint

$$F^{\underline{\mu_1 \cdots \mu_k \mu_{k+1}} \cdots \mu_n} = 0 \, . \tag{43}$$

This set is a subspace of $\mathscr{E}^*$. The isomorphism between the spaces $\mathscr{E}^*$ and $\mathscr{Z}$ induces a one to one correspondence between the vectors of $\mathscr{F}$ and a subspace $\mathscr{Y} \subset \mathscr{Z}$. In Appendix B we demonstrate that $\mathscr{Y}$ is the set of *all* vectors of $\mathscr{Z}$ that satisfy the homogeneous algebraic constraint.

3.2. Non-Trivial Representations of the Diffeomorphism Group. Here, we study the transformation laws of the elements of the vector space $\mathscr{Z}$ under diffeomorphism transformations with a fixed point. The vectors $\vec{E} \in \mathscr{E}^*$ behave as multivector densities, that is

$$\vec{E}' = \Lambda_D \cdot \vec{E} \, . \tag{44}$$

Thus

$$\vec{Z}' = \delta_T \cdot \Lambda_D \cdot \vec{E} = \mathscr{L}_D \cdot \vec{Z} \tag{45}$$

with

$$\mathscr{L}_D \equiv \delta_T \cdot \Lambda_D \cdot \sigma \, . \tag{46}$$

The isomorphism between the vector spaces $\mathscr{E}^*$ and $\mathscr{Z}$ makes $\mathscr{L}_D$ a representation of the diffeomorphism group. This representation emerges as the push-forward of the natural action of diffeomorphisms on the space of solutions of the differential constraints by the isomorphism of that space with the space of transverse vectors $\mathscr{Z}$.

The presence of the nondiagonal matrix σ in $\mathscr{L}_D$ makes this representation highly non-trivial. This is an important result, due to the possibility to introduce "spinor-like" objects in a theory invariant under diffeomorphisms. In fact, the isomorphism guarantees the following property of the σ's:

$$\sigma = \Lambda_D \cdot \sigma \cdot \mathscr{L}_{D^{-1}} \, . \tag{47}$$

This relationship clearly shows the role played by the σ's as the soldering quantities between the "vector" representation Λ_D and the "spinor-like" representation $\mathscr{L}_D$.

It is straightforward to see that the subspaces $\mathscr{F}$ and $\mathscr{Y}$ are invariants under diffeomorphisms. The vector product of $\vec{Y}$'s belongs to $\mathscr{Z}$ and in consequence has the same transformation law under diffeomorphisms that a single $\vec{Y}$.

$$(\vec{Y}_1' \times \cdots \times \vec{Y}_n') = \mathscr{L}_D \cdot (\vec{Y}_1 \times \cdots \times \vec{Y}_n) \, . \tag{48}$$

This fact will be important for the construction of knot invariants.

4. Free Coordinates on the Loop Space

We proceed now to determinate the free coordinates associated with an arbitrary element of the loop space. This means that we have to identify all those elements of the subspace $\mathscr{Y}$ of the space of solutions of the homogeneous differential constraint that can be put in correspondence with loops.

Let us start defining the algebraic-free coordinates $\vec{F}(L) \in \mathscr{F}$ of the loop L (an alternative expression will be given in Sect. 7). A complete solution of the full set of algebraic equations can be obtained in terms of the combinations

$$\begin{aligned}\vec{R}(L) &= \vec{X}(L) + \vec{X}(\bar{L}) ,\\ \vec{P}(L) &= \vec{X}(L) - \vec{X}(\bar{L}) .\end{aligned} \tag{49}$$

We notice that $R^{\mu_1 \cdots \mu_n}$ is always reducible to a product of X's of smaller rank. From the identities

$$\vec{X}(L_1 \circ L_2) = \vec{X}(L_1) + \vec{X}(L_2) + \vec{X}(L_1) \times \vec{X}(L_2) , \tag{50}$$

$$X^{\mu_1 \cdots \mu_n}(\bar{L}) = (-1)^n X^{\mu_n \cdots \mu_1}(L) , \tag{51}$$

and using the previous definitions one immediately obtains

$$\vec{R}(L) = \frac{1}{4}(\vec{P}(L) \times \vec{P}(L)) - \frac{1}{4}(\vec{R}(L) \times \vec{R}(L)) . \tag{52}$$

The last equation enables to express $\vec{R}(L)$ in terms of even products of $\vec{P}$'s. We conclude that the algebraic-free coordinates are contained in the combination $\vec{P}(L)$. Applying the first constraint on $\vec{P}(L)$ one obtains

$$P^{\underline{\mu_1}\mu_2 \cdots \mu_n}(L) = X^{\mu_1}(L) R^{\mu_2 \cdots \mu_n}(L) . \tag{53}$$

Both $\vec{P}(L)$ and $\vec{R}(L)$ satisfy the differential constraint and have a definite behavior under the inversion of the orientation of the loop,

$$\vec{R}(\bar{L}) = +\vec{R}(L), \quad \vec{P}(\bar{L}) = -\vec{P}(L) . \tag{54}$$

We observe that the $\times$-product $(\vec{P} \times \vec{R}^m) \equiv (\vec{P} \times \vec{R} \times \overset{m \text{ times}}{\cdots} \times \vec{R})$ has the symmetry of the $\vec{P}$'s under inversion of the loop and

$$(P \times R^m)^{\underline{\mu_1}\mu_2 \cdots \mu_n} = X^{\mu_1}[2(1 + 2m)(R^m)^{\mu_2 \cdots \mu_n} + (m + 1)(R^{m+1})^{\mu_2 \cdots \mu_n}] . \tag{55}$$

Then the following combination

$$\vec{F}(L) = \vec{P}(L) + \sum_{m=1}^{\infty} \alpha_m (\vec{P}(L) \times \vec{R}^m(L)) \tag{56}$$

satisfies the constraint $F^{\underline{\mu_1}\mu_2 \cdots \mu_n} = 0$ if the coefficients take the values

$$\alpha_m = \frac{(-1)^m m!}{2^m (2m + 1)!!} . \tag{57}$$

The action of a higher order algebraic constraint on $(P \times R^m)$ can be computed in a similar way. It is found that the combination (56) is annihilated by the constraint for the same values of the coefficients α_m in all cases. The algebraic-free coordinate $\vec{F}$ given by (56) automatically satisfy the differential constraint. This result can be inverted to express $\vec{X}$ in terms of the coordinates $\vec{F}$ leading to an expansion containing the $\times$-product of any number of $\vec{F}$'s.

All the $\vec{F}(L)$'s and its vector products belong to the space $\mathscr{E}^*$. In consequence, one can generate new gauge covariant quantities, other than (3), replacing in (18) $\vec{E}$ by $\vec{F}$ or the $\times$-product of any number of F's.

The free coordinates $\vec{Y}(L)$ are defined through the isomorphism between the vector spaces $\mathscr{F}$ and $\mathscr{Y}$. We get

$$\vec{Y}(L) = \delta_T \cdot \vec{F}(L) \tag{58}$$

and

$$\vec{F}(L) = \sigma \cdot \vec{Y}(L) . \tag{59}$$

Due to the isomorphism, it is clear that we are using the term "coordinate" of a loop in a wide sense. The one to one correspondence is established between objects that belong to more general spaces that contain, among others, all the elements associated with loops. In spite of this fact, we choose to call $\vec{Y}(L)$ the coordinates of the loop L since this terminology seems to us the best way to describe the meaning of these objects.

To conclude, the loop coordinates $\vec{Y}(L)$ have been identified as the transverse part of the algebraic independent quantities $\vec{F}(L)$. Both are related by a linear transformation and the coordinates $\vec{Y}$ of certain loop L depend on the projection prescription.

5. Knot Invariants

In this section we analyze the relationship between knot invariants and the invariant forms defined on the space $\mathscr{Y}$. Covariant vectors are forms in the linear space $\mathscr{Y}$ such that

$$\mathbf{f}(\alpha \vec{Y}_1 + \beta \vec{Y}_2) = \alpha \mathbf{f}(\vec{Y}_1) + \beta \mathbf{f}(\vec{Y}_2) \tag{60}$$

where $\mathbf{f}$ is the row matrix $\mathbf{f} = (f_{\mu_1}, \ldots, f_{\mu_1 \cdots \mu_n}, \ldots)$. Covariant vectors are defined up to gradients. However a particular transverse prescription can be fixed using

$$\mathbf{f} \equiv \mathbf{f} \cdot \delta_T . \tag{61}$$

When a diffeomorphism transformation is performed, the transformed covariant vectors will be in the same transverse prescription and are given by

$$\mathbf{f}' = \mathbf{f} \cdot \mathscr{L}_{D^{-1}} . \tag{62}$$

Consider now the following quantity:

$$\mathscr{I}(L) = \mathbf{g} \cdot (\vec{Y}(L) \times \overset{n \text{ times}}{\cdots} \times \vec{Y}(L)) , \tag{63}$$

where $\mathbf{g}$ is a covariant vector and $\vec{Y}(L)$ gives the coordinates of the loop L. If $\mathbf{g}$ is a covariant invariant vector,

$$\mathbf{g}' = \mathbf{g} \cdot \mathscr{L}_{D^{-1}} = \mathbf{g} , \tag{64}$$

from Eq. (48) it is immediate to conclude that $\mathscr{I}(L) = \mathscr{I}(L')$, L' being the diffeomorphism transform of L. In other words, $\mathscr{I}(L)$ is a knot invariant. It is important to remark that we are not going to discuss analytical details of regularization and consequently this statement is formal.

The first non-trivial invariant tensor $\mathbf{g}_G$ has only one non-vanishing component,

$$g_{G\,\widehat{\mu_1 \cdots \mu_n}} = \delta_{n,2}\, g_{\mu_1 \mu_2} , \tag{65}$$

where $g_{\mu_1\mu_2}$ is the previously obtained metric (22), and leads to the Gauss knot invariant

$$I_G = \mathbf{g}_G \cdot (\vec{Y} \times \vec{Y}) = g_{\mu_1\mu_2} Y^{\mu_1} Y^{\mu_2} . \tag{66}$$

The second non-trivial invariant tensor is

$$g_{A\mu_1 \cdots \mu_n} = \delta_{n,3} h_{\mu_1\mu_2\mu_3} + \delta_{n,4} g_{\mu_1\mu_3} g_{\mu_2\mu_4} \tag{67}$$

with

$$h_{a_1x_1a_2x_2a_3x_3} = \int d^3y \varepsilon^{\text{def}} g_{dya_1x_1} g_{eya_2x_2} g_{fya_3x_3} , \tag{68}$$

and leads to the Alexander Conway 2^{ND} coefficient [33]

$$\begin{aligned} I_A &= \mathbf{g}_A \cdot (\vec{Y} \times \vec{Y}) \\ &= 2h_{\mu_1\mu_2\mu_3} Y^{\mu_1} Y^{\mu_2\mu_3} \\ &\quad + g_{\mu_1\mu_3} g_{\mu_2\mu_4} (Y^{\mu_1\mu_2} Y^{\mu_3\mu_4} + 2Y^{\mu_1\mu_2\mu_3} Y^{\mu_4}) . \end{aligned} \tag{69}$$

Notice that one can directly express knot invariants in terms of the the algebraic-free coordinates $\vec{F}$'s. In fact, $\mathbf{g} \cdot (\vec{Y}^n) = \mathbf{g} \cdot (\vec{F}^n)$ due to the transverse character of the invariant tensor $\mathbf{g}$ and the definition of $\vec{Y}$.

Knot invariants are prescription independent. Let us fix some prescription for $\mathbf{g}$, $\mathbf{g}_1 = \mathbf{g}_1 \cdot \delta_{T1}$. Then

$$\mathbf{g}_1 \cdot \vec{Y}_1 = \mathbf{g}_1 \cdot \delta_{T1} \cdot \vec{F} = \mathbf{g}_1 \cdot \vec{F} . \tag{70}$$

But $\vec{F} = \sigma_2 \cdot \vec{Y}_2$, then

$$\mathbf{g}_1 \cdot \vec{Y}_1 = \mathbf{g}_1 \cdot \sigma_2 \cdot \vec{Y}_2 = \mathbf{g}_2 \cdot \vec{Y}_2 , \tag{71}$$

where

$$\mathbf{g}_2 = \mathbf{g}_1 \cdot \sigma_2 \tag{72}$$

is the invariant tensor in prescription 2. Using the algebraic-free coordinates we have

$$\mathbf{g}_1 \cdot \vec{F} = \mathbf{g}_1 \cdot \vec{Y}_1 = \mathbf{g}_2 \cdot \vec{Y}_2 = \mathbf{g}_2 \cdot \vec{F} . \tag{73}$$

In Quantum Gravity one can use the Mandelstam identities for SU(2) [14],

$$W_A(L_1) W_A(L_2) = W_A(L_1 L_2) + W_A(L_1 \bar{L}_2) , \tag{74}$$

in order to give a general expression of the knot invariants involved in the physical state space. Taking L_1 as the identity loop we find that the Wilson loop functionals have the symmetry of the R's under inversion of the loop orientation. So in this case these functionals would depend only on the cyclic permutation of $R^{\mu_1 \cdots \mu_n}$. Making use of Eq. (1) the one loop states take the form

$$\Psi(L) = \sum_{n=1}^{\infty} \psi_{\mu_1 \cdots \mu_n} R_c^{\mu_1 \cdots \mu_n}(L) . \tag{75}$$

The cyclic permutations $R_c^{\mu_1 \cdots \mu_n}$ can be written in terms of even vector products of $\vec{F}$'s or $\vec{Y}$'s coordinates, so

$$\Psi(L) = \sum_{m=1}^{\infty} \beta_m \psi \cdot (\vec{F}^{2m})_c = \sum_{m=1}^{\infty} \beta_m \mathbf{g} \cdot (\vec{Y}^{2m})_c , \tag{76}$$

where

$$\mathbf{g} \equiv \psi \cdot \sigma \,. \tag{77}$$

The coefficients β_i that appear in the expansion of R_c in terms of F can be directly evaluated using the results of Sect. 3.1.

The wavefunction Ψ would represent a quantum state of the gravitational field if it is annihilated by all the constraints of quantum gravity [14]. One of these constraints is the generator of diffeomorphisms on a three dimensional manifold. Then, a *cyclic* invariant tensor in the $\mathscr{Y}$ space and a quantum state of general relativity turn out to be closely related. The cyclic property of the invariant tensor means independence with respect to the base point taken as the origin of the loops (the diffeomorphism transformations considered here do not contain general translations). Equation (76) shows the general structure of such a state: it is a multilinear invariant form constructed by even vector products of the vector $\vec{Y}$ with itself. The restriction to even products is a direct consequence of the Mandelstam identities for SU(2).

Let us make some remarks about this result. By (48), each term of the sum is a knot invariant in its own right. But the invariant tensor involved in the whole sum is the same, so a *family* of knot invariants is associated with each invariant tensor $\mathbf{g}$. In the above examples, $\mathbf{g}_G$ gives only the Gauss invariant, but $\mathbf{g}_A$ produces also other knot invariants when is contracted with $(\vec{Y}^4)$. In this case it reduces simply to $(I_G)^2$. In general, an invariant tensor involving at most rank $2p$ components allows to construct p in principle independent knot invariants. This fact is a direct consequence of the transformation properties of the $\times$-product of contravariant transverse objects.

To conclude this section, we show another interesting results related to the construction of link invariants from knot invariants in the case of Quantum Gravity. Let $\Psi(L)$ be a knot invariant of Quantum Gravity and let $L_1 L_2$ be two arbitrary loops. Then consider any open path P connecting the origins of L_1 and L_2. From (1) and the Mandelstam identity (74) we know that the combination $\Psi(L_1 P L_2 \bar{P}) + \Psi(L_1 P \bar{L}_2 \bar{P})$ is independent of P and it only depends on the original loops L_1 and L_2. This means that

$$\Psi(L_1, L_2) = \Psi(L_1 P L_2 \bar{P}) + \Psi(L_1 P \bar{L}_2 \bar{P}) \tag{78}$$

is a link invariant.

6. The Extended Loop Group

In the previous sections it was shown that any element of the loop space can be described by a set of multivector density fields constrained by a set of algebraic and differential identities. We develop now the basis of a more complete and rigorous mathematical description of these objects.

We begin by considering the set of all objects of the type

$$\mathbf{E} = (E, E^{\mu_1}, \ldots, E^{\mu_1 \cdots \mu_n}, \ldots) \equiv (E, \vec{E}) \,, \tag{79}$$

where E is a real number and $E^{\mu_1 \cdots \mu_n}$ (for any $n \neq 0$) is an arbitrary multivector density field. This set has the structure of a vector space (denoted as $\mathscr{E}$) with the usual composition laws of addition and multiplication between functions.

A product law on $\mathscr{E}$ can be introduced as follows: given two vectors $\mathbf{E}_1$ and $\mathbf{E}_2$, we define $\mathbf{E}_1 \times \mathbf{E}_2$ as the vector with components

$$\mathbf{E}_1 \times \mathbf{E}_2 = (E_1 E_2, E_1 \vec{E}_2 + \vec{E}_1 E_2 + \vec{E}_1 E_2 + \vec{E}_1 \times \vec{E}_2)\,, \tag{80}$$

where $\vec{E}_1 \times \vec{E}_2$ is given by Eq. (41). The composition law (80) is an extension of the vector product among loop coordinates introduced in Sect. 3.1. For any value of n, the rank n component of the $\times$-product can be expressed as

$$(\mathbf{E}_1 \times \mathbf{E}_2)^{\mu_1 \cdots \mu_n} = \sum_{i=0}^{n} E_1^{\mu_1 \cdots \mu_i} E_2^{\mu_{i+1} \cdots \mu_n} \tag{81}$$

with the convention

$$E^{\mu_1 \cdots \mu_0} = E^{\mu_{n+1} \cdots \mu_n} = E\,. \tag{82}$$

The product law is associative and distributive with respect to the addition of vectors. It has a null element (the null vector) and an identity element, given by

$$\mathbf{I} = (1, 0, \ldots, 0, \ldots)\,. \tag{83}$$

An inverse element exists for all vectors with nonvanishing zero rank component. It is given by

$$\mathbf{E}^{-1} = E^{-1}\mathbf{I} + \sum_{i=1}^{\infty} (-1)^i E^{-i-1} (\mathbf{E} - E\mathbf{I})^i \tag{84}$$

such that

$$\mathbf{E} \times \mathbf{E}^{-1} = \mathbf{E}^{-1} \times \mathbf{E} = \mathbf{I}\,. \tag{85}$$

It should be noticed that, when evaluating the components of $\mathbf{E}^{-1}$, the sum involved in (84) is actually finite due to the fact that $(\mathbf{E} - E\mathbf{I})$ is a vector with its zero rank component equal to zero. In consequence

$$[(\mathbf{E} - E\mathbf{I})^i]^{\mu_1 \cdots \mu_n} = 0 \quad \text{if } n < i\,. \tag{86}$$

The set of all vectors with nonvanishing zero rank component forms a group with the $\times$-product law.

Now we concentrate on the set $\mathscr{X}$ of vectors of $\mathscr{E}$ that have its zero rank component equal to one, $\mathbf{X} = (1, \vec{X})$, with the components of $\vec{X}$ multivector density fields that obey the algebraic constraint (16) and differential constraint (17).

The set $\mathscr{X}$ is closed under the $\times$-product law. If $\mathbf{X}_1 \in \mathscr{X}$ and $\mathbf{X}_2 \in \mathscr{X}$, it is clear from the definition of the group product that $\mathbf{X}_1 \times \mathbf{X}_2$ satisfy the differential constraint. One can also demonstrate that $\mathbf{X}_1 \times \mathbf{X}_2$ satisfies the algebraic constraint. In a similar way one can show that the inverse $\mathbf{X}^{-1}$ given by (84) satisfies the constraints if $\mathbf{X}$ does. A detailed proof of these properties is given in Appendix C. These results show that the algebraic and differential constraint preserves the group structure under the $\times$-composition law. We call $\mathscr{X}$ the Special-extended Loop group (SeL group) [35].

The group of loops is a subgroup of the SeL group since $\mathbf{X}(L) \in \mathscr{X}$ and

$$\mathbf{X}(L_1 \circ L_2) = \mathbf{X}(L_1) \times \mathbf{X}(L_2)\,, \tag{87}$$

where $\circ$ indicates the group composition law in loop space.

We now show that this group is in fact larger than the group of loops. For that, consider the group element $\mathbf{X}^m \equiv \mathbf{X} \times \overset{m}{\cdots} \times \mathbf{X}$. Note that if $\mathbf{X}$ gives the multitangent field of a certain loop L, $\mathbf{X}^m$ would be the multitangent field of the loop L swept itself m times. We get by the binomial expansion

$$\mathbf{X}^m \equiv [\mathbf{I} + (\mathbf{X} - \mathbf{I})]^m = \mathbf{I} + \sum_{i=1}^{m} \binom{m}{i} (\mathbf{X} - \mathbf{I})^i . \tag{88}$$

The extension of (88) to real values of m is straightforward, being defined as

$$\mathbf{X}^\lambda = \mathbf{I} + \sum_{i=1}^{\infty} \binom{\lambda}{i} (\mathbf{X} - \mathbf{I})^i \tag{89}$$

with λ real. We usually call this object the analytic extension of $\mathbf{X}$. Note that for $\lambda = -1$ we recover the expression of the inverse of $\mathbf{X}$. Also in this case, due to (86) the analytic extension is well defined for all elements of $\mathscr{X}$. One can prove that if $\mathbf{X}$ is constrained by the differential and algebraic identities, its analytic extension also verifies the constraints (see Appendix C). So, the analytic extension of any $\mathbf{X}$ is in $\mathscr{X}$. Moreover, we have

$$\mathbf{X}^\lambda \times \mathbf{X}^\mu = \mathbf{X}^{\lambda+\mu} . \tag{90}$$

We conclude that the set $\{\mathbf{X}^\lambda / \lambda \in R \text{ and } \mathbf{X} \in \mathscr{X}\}$ defines an abelian one-parameter subgroup of the $\mathscr{X}$ group.

For non-integer values of λ, the analytic extension of a loop coordinate is not a loop coordinate. This fact explicitly shows that there exists in $\mathscr{X}$ other elements besides the loop coordinates.

Matrix representations of the SeL group can be generated through a natural extension of the holonomy. The extended holonomy associated with a nonabelian connection A_{ax} is defined as $U_A(\mathbf{X}) = \mathbf{A} \cdot \mathbf{X}$, where $\mathbf{A} \equiv (1, A_{a_1 x_1}, \ldots, A_{a_1 x_1}, \ldots, A_{a_n x_n}, \ldots)$, $\mathbf{X} \equiv (1, X^{a_1 x_1}, \ldots, X^{a_1 x_1 \cdots a_n x_n}, \ldots)$ and the dot acts like a generalized Einstein convention including contractions of the discrete indices a_i and the continuous indices x_i. We have

$$\begin{aligned} U_A(\mathbf{X}_1) U_A(\mathbf{X}_2) &= \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} A_{\mu_1 \cdots \mu_i} A_{\mu_{i+1} \cdots \mu_j} X_1^{\mu_1 \cdots \mu_i} X_2^{\mu_{i+1} \cdots \mu_j} \\ &= \sum_{j=0}^{\infty} A_{\mu_1 \cdots \mu_j} \left(\sum_{i=0}^{j} X_1^{\mu_1 \cdots \mu_i} X_2^{\mu_{i+1} \cdots \mu_j} \right) = U_A(\mathbf{X}_1 \times \mathbf{X}_2) , \end{aligned} \tag{91}$$

where the convention (82) was applied over all the indices. The correspondence $\mathbf{X} \to U_A(\mathbf{X})$ gives a representation of the SeL group into a particular gauge group. The differential constraint imposed on $\mathbf{X}$ assures $U_A(\mathbf{X})$ to be a gauge covariant quantity.

We have shown that the analytic extension of any element of the SeL group defines a one-parameter subgroup. By studying its properties one can find the algebra associated with the SeL group.

7. The Algebra of the SeL Group

7.1. Generators. Consider the one-parameter subgroup $\{\mathbf{X}^\lambda\}$ and suppose that λ varies in an infinitesimal amount. We can write

$$\mathbf{X}^{\lambda + d\lambda} = \mathbf{X}^\lambda \times \mathbf{X}^{d\lambda} = \mathbf{X}^\lambda + \frac{d\mathbf{X}^\lambda}{d\lambda} d\lambda . \tag{92}$$

Taking $\lambda = 0$,

$$\mathbf{X}^{d\lambda} = \mathbf{I} + F d\lambda \,, \tag{93}$$

where

$$F \stackrel{\text{def}}{=} \left.\frac{d\mathbf{X}^\lambda}{d\lambda}\right|_{\lambda=0} = \left(0, \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \vec{X}^i\right) = (0, \vec{F}) \,. \tag{94}$$

Introducing (93) in (92) we obtain the following differential equation for the elements of $\{\mathbf{X}^\lambda\}$:

$$\frac{d\mathbf{X}^\lambda}{d\lambda} = \mathbf{X}^\lambda \times F = F \times \mathbf{X}^\lambda \,. \tag{95}$$

This equation can be integrated, obtaining

$$\mathbf{X}^\lambda = \mathbf{I} + \sum_{k=1}^{n} \frac{\lambda^k}{k!} F^k + F^{n+1} \times \int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \cdots \int_0^{\lambda_n} d\lambda_{n+1} \mathbf{X}^{\lambda_{n+1}} \,. \tag{96}$$

The iterative integration actually stops for any finite rank n component ($F^{n+1} = F \times \overset{n+1}{\cdots} \times F = 0$ in this case). So

$$\mathbf{X}^\lambda = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} F^k = \exp(\lambda F) \,. \tag{97}$$

We conclude that the vector F given by (94) is the generator of the one-parameter subgroup $\{\mathbf{X}^\lambda\}$. It is evident that the generator satisfies the differential constraint. We prove now the following fundamental property: F satisfies the homogeneous algebraic constraint. In other words, the generator of the one-parameter subgroup $\{\mathbf{X}^\lambda\}$ is the algebraic free part of $\mathbf{X}$.

We know that

$$(\mathbf{X}^\lambda)^{\underline{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n}} = (\mathbf{X}^\lambda)^{\mu_1 \cdots \mu_k} (\mathbf{X}^\lambda)^{\mu_{k+1} \cdots \mu_n} \,. \tag{98}$$

Differentiating with respect to λ and evaluating for $\lambda = 0$ we get

$$\frac{d}{d\lambda} (\mathbf{X}^\lambda)^{\underline{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n}}_{\lambda=0} = \left(\frac{d\mathbf{X}^\lambda}{d\lambda}\right)^{\mu_1 \cdots \mu_k}_{\lambda=0} \mathbf{I}^{\mu_{k+1} \cdots \mu_n} + \mathbf{I}^{\mu_1 \cdots \mu_k} \left(\frac{d\mathbf{X}^\lambda}{d\lambda}\right)^{\mu_{k+1} \cdots \mu_n}_{\lambda=0} \,. \tag{99}$$

As $1 \leqq k < n$, we conclude

$$F^{\underline{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n}} = 0, \quad 1 \leqq k < n \,. \tag{100}$$

Reciprocally, one can demonstrate that the exponential of any algebraic-free quantity produces an object that satisfies the algebraic constraint. It is important to stress that these results permit to obtain a general solution for the algebraic constraint (Eq. (97) with $\lambda = 1$ and its inverse (94) gives the relationship between an object that satisfies the algebraic constraint and its algebraic-free part).

7.2. The Algebra. The set of all F's that satisfies the differential constraint and are annihilated by the algebraic constraint forms a vector space $\mathscr{F}$ [36]. One can define a bilinear operation on $\mathscr{F}$ in the following way:

$$[F_1, F_2] = F_1 \times F_2 - F_2 \times F_1 \quad \text{for any } F_1, F_2 \in \mathscr{F} \,. \tag{101}$$

This operation is closed on $\mathscr{F}$. The vector space $\mathscr{F}$ together with the bracket operation (101) defines the algebra associated to the SeL group.

A basis of this algebra can be found by means of the transverse part of the algebraic free quantities. The longitudinal and transverse orthogonal projectors acting on the space of transverse vector densities were defined in Sect. 3.1.

The transverse part of any vector $F \in \mathscr{F}$ is then given by

$$\mathbf{Y} = \delta_T \cdot F \,. \tag{102}$$

When $F = F(L)$, Eq. (102) gives the free coordinates of the loop L defined in Sect. 3.2. The inverse relation is

$$F = \sigma \cdot \mathbf{Y} \,, \tag{103}$$

where σ are the soldering quantities.

The homogeneous algebraic constraint can be imposed by defining a projector Ω over the algebraic free part of an arbitrary vector. Consider the matrix

$$\delta^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} = \delta_{n,m} \delta^{\mu_1}_{\nu_1} \cdots \delta^{\mu_n}_{\nu_n} \tag{104}$$

and the vector

$$\delta_{\nu_1 \cdots \nu_i} = (0, \delta^{\mu_1}{}_{\nu_1 \cdots \nu_i}, \ldots, \delta^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_i}, \ldots) \,. \tag{105}$$

We define then a new matrix, given by

$$\begin{aligned} \Omega^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} &\equiv \frac{\delta_{n,m}}{m} [[\ldots [\delta_{\nu_1}, \delta_{\nu_2}], \ldots], \delta_{\nu_n}]^{\mu_1 \cdots \mu_n} \theta(m-1) + \delta_{m,1} \delta^{\mu_1}_{\nu_1} \\ &= \delta^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} + \sum_{k=1}^{n-1} \frac{(n-k)}{n} (-1)^k \delta^{\underline{\mu_k \cdots \mu_1} \mu_{k+1} \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} \,. \end{aligned} \tag{106}$$

The matrix Ω has the following important property: it satisfies the homogeneous algebraic constraint in the upper indexes. This fact immediately shows that Ω is a projector. Given an arbitrary vector $\mathbf{E}$, $\Omega \cdot \mathbf{E}$ is an algebraic-free object. In particular we have $F = \Omega \cdot F$.

Let us introduce now the following set of vectors

$$\mathscr{S}_{\nu_1 \cdots \nu_m} = (0, \vec{\mathscr{S}}_{\nu_1 \cdots \nu_m}) \tag{107}$$

with

$$\mathscr{S}^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m} = (\sigma \cdot \Omega)^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} \,. \tag{108}$$

The vectors $\mathscr{S}_{\nu_1 \cdots \nu_m}$ belong to $\mathscr{F}$. They form a linear independent set of vectors that generates all the vector space $\mathscr{F}$. From (103) one can write

$$F = \mathscr{S} \cdot \mathbf{Y} \,. \tag{109}$$

The set $\{\mathscr{S}_{\nu_1 \cdots \nu_m}\}$ defines then a base of the algebra. The components of any element of the algebra are simply given in this base by the transverse part of the vector.

In order to obtain the structure constants associated with $\{\mathscr{S}_{\nu_1 \ldots \nu_m}\}$ one can start evaluating the commutator of two Ω's. One finds

$$[\Omega_{\nu_1 \ldots \nu_n}, \Omega_{\mu_1 \ldots \mu_m}] = \frac{n+m}{2} \Omega \cdot \left(\frac{1}{n} \delta_{\nu_1 \ldots \nu_n} \times \Omega_{\mu_1 \ldots \mu_m} - \frac{1}{m} \delta_{\mu_1 \ldots \mu_m} \times \Omega_{\nu_1 \ldots \nu_n} \right). \tag{110}$$

This relation can be obtained by first evaluating $[\Omega_{\nu_1 \ldots \nu_{n-1}}, \delta_{\nu_n}]$ and then proceeding by induction. Now it is straightforward to calculate the commutator of two elements of the base. We obtain

$$[\mathscr{S}_{\nu_1 \cdots \nu_n}, \mathscr{S}_{\mu_1 \cdots \mu_m}] = \mathscr{S} \cdot f_{\nu_1 \cdots \nu_n, \mu_1 \cdots \mu_m}, \tag{111}$$

where the structure constants $f^{\rho_1 \cdots \rho_k}_{\nu_1 \cdots \nu_n, \mu_1 \cdots \mu_m}$ are given by

$$f^{\rho_1 \cdots \rho_k}_{\nu_1 \cdots \nu_n, \mu_1 \cdots \mu_m} = [\delta_T \cdot \Omega_{\nu_1 \cdots \nu_n}, \delta_T \cdot \Omega_{\mu_1 \cdots \mu_m}]^{\rho_1 \cdots \rho_k}. \tag{112}$$

7.3. Diffeomorphism Invariants. Any vector F belonging to the SeL algebra is a multivector density field under a general diffeomorphism transformation. In matrix form the transformation law is

$$F' = \Lambda_D \cdot F \tag{113}$$

while the transformation law of the transverse algebraic-free vectors $\mathbf{Y}$ was derived in Sect. 3.2

$$\mathbf{Y}' = \delta_T \cdot F' = \mathscr{L}_D \cdot \mathbf{Y}, \tag{114}$$

where

$$\mathscr{L}_D \equiv \delta_T \cdot \Lambda_D \cdot \sigma. \tag{115}$$

The general diffeomorphism transformation considered in (113) is in fact a particular example of an automorphism of the algebra. Another automorphism transformation can be considered, for example those induced by the conjugation classes $\mathbf{X} \times F \times \mathbf{X}^{-1}$.

Diffeomorphism invariants can be related to some kind of invariant forms defined in the vector space $\mathscr{F}$. Consider a covariant vector $\mathbf{g} = (0, g_{\mu_1\mu_2}, \ldots, g_{\mu_1 \cdots \mu_n}, \ldots)$ that satisfies the following properties:

$$\mathbf{g} = \mathbf{g} \cdot \mathscr{L}_D, \tag{116}$$

$$g_{\mu_1 \cdots \mu_n} = g_{(\mu_1 \cdots \mu_n)\text{cyclic}}. \tag{117}$$

This tensor allows to define the following multilinear form

$$(F_1, \ldots, F_n)_g = \mathbf{g} \cdot (\mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n), \tag{118}$$

that is invariant with respect to both automorphism transformations just considered. The invariance property (116) together with Eq. (48) makes (118) invariant with respect to diffeomorphism transformations with a fixed point. The cyclicity property (117) ensures invariance with respect to the conjugation classes. Knot invariants are formally generated evaluating on loops the multilinear forms associated with the invariant tensors.

Each invariant tensor $\mathbf{g}$ is related with a *metric* function in the algebra in the following way:

$$(\mathscr{S}_{\nu_1\cdots\nu_n}, \mathscr{S}_{\mu_1\cdots\mu_m})_g = \tilde{g}_{\nu_1\cdots\nu_n,\mu_1\cdots\mu_m} = \mathbf{g}\cdot(\Omega_{\nu_1\cdots\nu_n} \times \Omega_{\mu_1\cdots\mu_m}) . \qquad (119)$$

The metrics associated with the Gauss and Alexander–Conway invariants are singular in the sense that there exist non-null vectors with zero norm. Several open questions about the general structure of the invariant forms and its relationship with the irreducible representations of the algebra remain to be studied. A complete study of the SeL group algebra is a non-trivial task since we are dealing with an infinite dimensional (Lie) algebra.

8. Conclusions

A coordinate representation on loop space has been introduced. In this representation any loop is described by a set of contravariant tensors of any rank defining a vector in the space of the transverse objects $\mathscr{Y}$. The loop coordinates depend on a prescription due to the non-uniqueness of the transverse part of a tensor. However knot invariants turn out to be independent of the prescription, as it should be. Each invariant form defined on $\mathscr{Y}$ generates a family of knot invariants. The loop coordinates provide the basis to define an affine geometry supported on the loop space.

Within this approach we have shown how the group of loops can be embedded in a natural way in a more general algebraic structure, the Special extended Loop group. The existence in the SeL group of other elements besides the multitangent fields associated with loops has been explicitly demonstrated. The algebra of the SeL group has been studied and a primary approach to the formulation of invariant forms in the vector space of the algebraic free coordinates has been done.

It is important to remark the connection between the solution of the constraints and the new emerging group structure. The elements of the algebra are algebraic-free vectors that satisfy the differential constraint. The free-coordinates $\mathbf{Y}$ may be considered as free parameters of the group. The soldering quantities σ connects two representations of the diffeomorphism group. One can prove in a direct way that the following property holds, $\Lambda_D \cdot \sigma \cdot \mathscr{L}_{D-1} = \sigma$. So, the matrix σ connects the "vector" representation Λ_D with the "spinor-like" representation $\mathscr{L}_D$. They act then like the Pauli matrices in an infinite dimensional Lie algebra.

The quantum formulation of General Relativity in the Extended Loop Group becomes closer to the familiar configuration space of any quantum field theory. The extension of the Hilbert space of quantum gravity including smooth functions gives a new perspective on the framing problem associated with knot invariants and in the search of a measure allowing to define an inner product on it. In the cases where an inner product is known, the U(1) case [34, 37] and linearized gravitation [38], it has been defined by means of the loop coordinates. This fact strongly suggests that the extended loop group could be the natural framework of a "loop representation."

Finally, we would like to remember that it was the study of simultaneous partial differential equations that led Lie to investigate continuous transformation groups. From this point of view, it is not surprising that many of the differential equations of mathematical physics can be solved using the Lie group techniques. It becomes natural to think then that the existence of the local infinite dimensional Lie group associated with loop space would allow to treat functional problems related with gauge theories and gravitation in a simpler form.

Appendix A

In this appendix we list several useful formulae concerning the σ introduced in Sect. 3.1. These quantities satisfy several useful relations: their transversal properties are given by

$$\delta_T \cdot \sigma = \delta_T \,, \tag{120}$$

$$\sigma \cdot \delta_T = \sigma \,, \tag{121}$$

and they obey the differential constraint

$$\frac{\partial}{\partial x_i^{a_i}} \sigma^{a_1 x_1 \cdots a_i x_i \cdots a_n x_n}{}_{v_1 \cdots v_m}$$
$$= (\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1}))\sigma^{a_1 x_1 \cdots a_{i-1} x_{i-1} a_{i+1} x_{i+1} \cdots a_n x_n}{}_{v_1 \cdots v_m} \,. \tag{122}$$

We show now that the vector product of two σ's reproduce another σ. From (122) we see that the $\times$-product of two σ's constructed with its contravariant indices automatically satisfy the differential constraint, so we can write

$$(\sigma_{v_1 \cdots v_j} \times \sigma_{\rho_1 \cdots \rho_m})^{\mu_1 \cdots \mu_n} = \sum_{k=1}^{\infty} \sigma^{\mu_1 \cdots \mu_n}{}_{\alpha_1 \cdots \alpha_k} (\sigma_{v_1 \cdots v_j} \times \sigma_{\rho_1 \cdots \rho_m})^{\alpha_1 \cdots \alpha_k} \,. \tag{123}$$

Then using (120) and (121) we obtain

$$\begin{aligned}
(\sigma_{v_1 \cdots v_j} \times \sigma_{\rho_1 \cdots \rho_m})^{\mu_1 \cdots \mu_n} &= \sum_{k=1}^{\infty} (\sigma \cdot \delta_T)^{\mu_1 \cdots \mu_n}{}_{\alpha_1 \cdots \alpha_k} (\sigma_{v_1 \cdots v_j} \times \sigma_{\rho_1 \cdots \rho_m})^{\alpha_1 \cdots \alpha_k} \\
&= \sum_{k=1}^{\infty} \sigma^{\mu_1 \cdots \mu_n}{}_{\alpha_1 \cdots \alpha_k} ((\delta_T \cdot \sigma)_{v_1 \cdots v_j} \times (\delta_T \cdot \sigma)_{\rho_1 \cdots \rho_m})^{\alpha_1 \cdots \alpha_k} \\
&= \sum_{k=1}^{\infty} \sigma^{\mu_1 \cdots \mu_n}{}_{\alpha_1 \cdots \alpha_k} (\delta_{T\, v_1 \cdots v_j} \times \delta_{T\, \rho_1 \cdots \rho_m})^{\alpha_1 \cdots \alpha_k} \\
&= \sum_{k=1}^{\infty} \sigma^{\mu_1 \cdots \mu_n}{}_{\alpha_1 \cdots \alpha_k} \delta_T{}^{\alpha_1 \cdots \alpha_k}{}_{v_1 \cdots v_j \rho_1 \cdots \rho_m} \,,
\end{aligned} \tag{124}$$

that is to say

$$\sigma^{\mu_1 \cdots \mu_n}{}_{v_1 \cdots v_j \rho_1 \cdots \rho_m} = (\sigma_{v_1 \cdots v_j} \times \sigma_{\rho_1 \cdots \rho_m})^{\mu_1 \cdots \mu_n} \,. \tag{125}$$

Under a change of the projection prescription we have

$$\phi_{1z}^{ax} \to \phi_{2z}^{ax}, \quad \delta_{1T} \to \delta_{2T}, \quad \delta_{1L} \to \delta_{2L}, \quad \sigma_1 \to \sigma_2 \,; \tag{126}$$

then the following relationships can be easily proved:

$$\delta_{1T} = \delta_{2T} \cdot \delta_{1T} \,, \tag{127}$$

$$\delta_{2L} = \delta_{2L} \cdot \delta_{1L} \,, \tag{128}$$

$$\sigma_1 = \sigma_2 \cdot \sigma_1 \,. \tag{129}$$

Now we give the transformation laws of σ under a general diffeomorphism transformation $x^a \to x'^a = D^a(x)$. We start by introducing the quantities δ_{DT} and δ_{DL} defined by

$$\delta_{DT} \equiv \Lambda_{D^{-1}} \cdot \delta_T \cdot \Lambda_D, \quad \delta_{DL} \equiv \delta - \delta_{DT} = \Lambda_{D^{-1}} \cdot \delta_L \cdot \Lambda_D\,, \tag{130}$$

where Λ_D is given by (9). Using the identity

$$\frac{\partial}{\partial x^a} \Lambda_{D^{-1}}{}^{ax}{}_{by} = -\frac{\partial}{\partial y^b} \delta(x - D^{-1}(y)) \tag{131}$$

it may be immediately proved that δ_{DT} and δ_{DL} are transverse and longitudinal projectors. The function ϕ_D, which characterizes the projection prescription, is related to ϕ by

$$\phi_D{}^{ax}{}_y = J(x) \frac{\partial x^a}{\partial D^b(x)} \phi^{bD(x)}{}_{D(y)}\,. \tag{132}$$

Making use of the identity

$$\Lambda_D{}^{ax}{}_{cz} \delta(t - z) \Lambda_{D^{-1}}{}^{cz}{}_{by} = \delta(t - D^{-1}(x)) \delta^{ax}{}_{by}\,, \tag{133}$$

we get

$$\sigma_D \equiv \sigma[\phi_D{}^{ax}{}_y] = \Lambda_{D^{-1}} \cdot \sigma \cdot \Lambda_D\,, \tag{134}$$

where σ_D is the soldering quantity constructed with ϕ_D.

Appendix B

We shall demonstrate in this appendix that there exists an isomorphism between the vector space $\mathscr{F}$ of vectors that satisfy the differential constraint and the homogeneous algebraic constraint, and the space $\mathscr{Y}$ of vectors that satisfy both homogeneous constraints.

Consider a vector $\vec{F} \in \mathscr{F}$. It is straightforward to see that $\delta_T \cdot \vec{F}$ satisfies the homogeneous differential and algebraic constraint. In consequence, $\delta_T \cdot \vec{F} \in \mathscr{Y}$.

The reciprocal states that any algebraic-free transverse vector must generate when contracting with a σ an object that satisfies the homogeneous algebraic identities.

Let us first consider an arbitrary vector $\vec{E}$ that satisfies the differential constraint, i.e. $\vec{E} = \sigma \cdot \vec{Z}$, where $\vec{Z}$ is the transverse part of the vector. It is straightforward to show that

$$\frac{\partial}{\partial x_i^{a_i}} E^{\underline{\mu_1 \cdots \mu_k} \mu_{k+1} \cdots \mu_n} = [\delta(x_i - b_{ik}) - \delta(x_i - c_{ik})]$$

$$\times \{ E^{\underline{\mu_1 \cdots \not{\mu}_i \cdots \mu_k} \mu_{k+1} \cdots \mu_n} \theta(k + 1 - i) + E^{\underline{\mu_1 \cdots \mu_k} \mu_{k+1} \cdots \not{\mu}_i \cdots \mu_n} \theta(i - k) \}\,, \tag{135}$$

where

$$b_{ik} = \begin{cases} x_0 & \text{if } i = 1 \text{ or } i = k + 1 \\ x_{i-1} & \text{elsewhere} \end{cases}, \quad c_{ik} = \begin{cases} x_0 & \text{if } i = n \text{ or } i = k \\ x_{i+1} & \text{elsewhere} \end{cases}. \tag{136}$$

For $n = 2$ we obtain from the last equation

$$\frac{\partial}{\partial x_i^{a_i}} E^{\underline{\mu_1}\mu_2} = 0 , \tag{137}$$

so $E^{\underline{\mu_1}\mu_2}$ is a transverse vector. Suppose now that $\vec{Z} \in \mathscr{Y}$, that is

$$\vec{Z}^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_n} = 0 \quad 1 \leq k < n . \tag{138}$$

For $n = 2$ we can conclude that $E^{\underline{\mu_1}\mu_2} = Z^{\underline{\mu_1}\mu_2} = 0$. We now proceed by induction. Suppose that

$$\vec{E}^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_n} = 0 \quad \forall n \in [2, h], \forall k \in [1, n-1] . \tag{139}$$

For $n = h + 1$, (135) shows that $E^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_{h+1}}$ is a transverse vector. Then

$$E^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_{h+1}} = Z^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_{h+1}} = 0 . \tag{140}$$

We conclude then that $\vec{E} \in \mathscr{F}$.

Appendix C

In this appendix we shall demonstrate: 1) the ×-product is closed with respect to the algebraic constraint and 2) the analytic extension $\mathbf{X}^\lambda$ of any element of the SeL group satisfies the constraints. In particular, taking $\lambda = -1$ we conclude that the inverse element also satisfies the constraints.

We start considering the algebraic constraint over the element $X_1 \times X_2$. From (80) we have

$$(X_1 \times X_2)^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_n} = [\vec{X}_1 + \vec{X}_2 + \vec{X}_1 \times \vec{X}_2]^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_n} . \tag{141}$$

The last term of (141) can be written in the form

$$(\vec{X}_1 \times \vec{X}_2)^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_n} = \sum_{i=1}^{n-1} (X_1, X_2)_i^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_n} , \tag{142}$$

where $(X_1, X_2)_i^{\mu_1 \cdots \mu_n} \stackrel{\text{def}}{=} X_1^{\mu_1 \cdots \mu_i} X_2^{\mu_{i+1} \cdots \mu_n}$. One can see that

$$(X_1, X_2)_i^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_n} = \sum_{h=\mathrm{Max}(0,k+i-n)}^{\mathrm{Min}(i,k)} X_1^{\underline{\mu_1 \cdots \mu_h}\mu_{k+1}\cdots\mu_{k+i-h}} X_2^{\underline{\mu_{h+1} \cdots \mu_k}\mu_{k+i-h+1}\cdots\mu_n} , \tag{143}$$

where we use the following notation, $X^{\mu_{i+1} \cdots \mu_i \mu_j \cdots \mu_m} = X^{\mu_j \cdots \mu_m \mu_{i+1} \cdots \mu_i} = X^{\mu_j \cdots \mu_m}$. Summing now on i, rearranging the sums and operating, we obtain

$$\begin{aligned}(\vec{X}_1 \times \vec{X}_2)^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_n} = {} & X_1^{\mu_1 \cdots \mu_k} [\vec{X}_1 \times \vec{X}_2 + \vec{X}_2]^{\mu_{k+1} \cdots \mu_n} \\ & + X_2^{\mu_1 \cdots \mu_k} [\vec{X}_1 \times \vec{X}_2 + \vec{X}_1]^{\mu_{k+1} \cdots \mu_n} \\ & + (\vec{X}_1 \times \vec{X}_2)^{\mu_1 \cdots \mu_k} [\vec{X}_1 \times \vec{X}_2 + \vec{X}_1 + \vec{X}_2]^{\mu_{k+1} \cdots \mu_n} .\end{aligned} \tag{144}$$

Introducing (144) in (141) we obtain after a few direct manipulations

$$(X_1 \times X_2)^{\underline{\mu_1 \cdots \mu_k}\mu_{k+1}\cdots\mu_n} = (X_1 \times X_2)^{\mu_1 \cdots \mu_k} (X_1 \times X_2)^{\mu_{k+1} \cdots \mu_n} . \tag{145}$$

Consider now the analytical extension of an **X**. We study first the differential constraint. One can immediately recognize that $\vec{X}_1 \times \vec{X}_2$ satisfies the differential constraint. The linearity of the differential constraint assures then that $\mathbf{X}^\lambda$ obey the differential identities.

Let us compute the action of the algebraic constraint over $\mathbf{X}^\lambda$. We have

$$(\vec{\mathbf{X}}^\lambda)^{\underline{\mu_1 \cdots \mu_k} \mu_{k+1} \cdots \mu_n} = \sum_{i=1}^{\infty} \binom{\lambda}{i} [(\vec{X})^i]^{\underline{\mu_1 \cdots \mu_k} \mu_{k+1} \cdots \mu_n} . \tag{146}$$

The algebraic constraint acting over a $\times$-product of i elements $\vec{X}$ produces the following result:

$$(\vec{X}^i)^{\underline{\mu_1 \cdots \mu_k} \mu_{k+1} \cdots \mu_n} = \sum_{m=1}^{i} \sum_{j=1}^{i} \binom{i}{m} \binom{m}{i-j} (\vec{X}^m)^{\mu_1 \cdots \mu_k} (\vec{X}^j)^{\mu_{k+1} \cdots \mu_n} . \tag{147}$$

This relation can be obtained directly by induction. Substituting (147) in (146), rearranging the sums, and operating we obtain

$$(\vec{\mathbf{X}}^\lambda)^{\underline{\mu_1 \cdots \mu_k} \mu_{k+1} \cdots \mu_n} = (\vec{\mathbf{X}}^\lambda)^{\mu_1 \cdots \mu_k} (\vec{\mathbf{X}}^\lambda)^{\mu_{k+1} \cdots \mu_n} . \tag{148}$$

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