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THE EXTENSION OF G-FOLIATIONS TO TANGENT BUNDLES OF HIGHER ORDER

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Introduction

In this paper we describe a canonical procedure for constructing the extension of a G-foliation on a differentiable** manifold X to its tangent bundles of higher order and by applying the Bott-Haefliger's construction of characteristic classes of G-foliations ([2], [3]) we obtain an infinite sequence $\{\overset{0}{\varphi}, \overset{1}{\varphi}, \cdots, \overset{r}{\varphi}, \cdots\}$ of characteristic classes for those foliations (Theorem 4.8).

By the way, a new equivalence relation between G-foliations weaker than the homotopy is defined (Definition 3.7) which we call r-homotopy and show that the set of characteristic classes of a G-foliation is an invariant of its r-homotopy class; some new results in the theory of tangent bundles of higher order are shown (Theorem 1.1 and Lemma 3.10) and the concept of tangent pseudogroup of higher order of a transitive Lie pseudogroup is introduced (Theorem 2.1 and Definition 2.1).

§ 1. Tangent bundles of higher order ([5])

Let $r \geqslant 0$ be an integer.

Let M be a differentiable C^∞ manifold, $\dim M = n$, and let $C^\infty(M)$ be the algebra of all differentiable functions on M. We denote by $S(M)$ the set of all differentiable maps $\varphi: \boldsymbol{R} \to M$; we define an equivalence relation on $S(M)$ in the following way: if $\varphi, \psi \in S(M)$ we say $\varphi \underset{r}{\sim} \psi$ if and only if $\varphi(0) = \psi(0)$ and, for every $f \in C^\infty(M), f \circ \varphi$ and $f \circ \psi$ have the same r-jet in 0, the origin of $\boldsymbol{R}$; if $\varphi \in S(M), [\varphi]_r$ will denote its class of equivalence and if $\varphi(0) = \mathrm{p} \in M, [\varphi]_r$ is called the r-tangent vector

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** Always differentiable will mean differentiable of class C^∞.

defined by φ at the point p of M.

Let $\overset{r}{T}M$ be the set of all r-tangent vectors at all points of M; there is a canonical projection

$$\overset{r}{\Pi}_M : \overset{r}{T}M \to M$$

given by $\overset{r}{\Pi}_M([\varphi]_r) = \varphi(0)$.

In order to define a structure of differentiable manifold on $\overset{r}{T}M$, consider a differentiable atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ of M and let $(x_1^\alpha, \cdots, x_n^\alpha)$ be the coordinate functions on U_α. On the set $(\overset{r}{\Pi}_M)^{-1}(U_\alpha)$ a coordinate system $(\overset{(\nu)\alpha}{x_i})$, $i = 1, 2, \cdots, n, \nu = 0, 1, \cdots, r$, is defined by

$$\overset{(\nu)\alpha}{x_i}([\varphi]_r) = \frac{1}{\nu!}\left[\frac{d^\nu(x_i^\alpha(\varphi(t)))}{dt^\nu}\right]_{t=0}$$

for every $[\varphi]_r \in (\overset{r}{\Pi}_M)^{-1}(U_\alpha)$.

Therefore, $\overset{r}{T}M$ is an $n(r+1)$-dimensional differentiable manifold and $\overset{r}{\Pi}_M$ is a submersion. Besides, for every $p \in M, \overset{r}{T}_pM = (\overset{r}{\Pi}_M)^{-1}(p)$ is canonically diffeomorphic to $\boldsymbol{R}^{rn}$.

M can be canonically imbedded in $\overset{r}{T}M$ by taking

$$i_M : M \to \overset{r}{T}M$$

defined by $i_M(x) = \tilde{x}, x \in M$, being $\tilde{x} = [\gamma_x]_r$, with $\gamma_x \in S(M)$ given by $\gamma_x(t) = x$, for every $t \in \boldsymbol{R}$.

Let N be another differentiable manifold and $\phi : M \to N$ a differentiable map; then, a differentiable map

$$\overset{r}{T}\phi : \overset{r}{T}M \to \overset{r}{T}N$$

is canonically defined by

$$(\overset{r}{T}\phi)([\varphi]_r) = [\phi \circ \varphi]_r , \qquad \text{for every } \varphi \in S(M) .$$

Let M_0, M_1, M_2 and M_3 be differentiable manifolds and let

$$\phi : M_0 \to M_1, \phi_1 : \mathrm{M}_1 \to M_2, \phi' : M_0 \to M_2 \quad \text{and} \quad \psi : M_2 \to M_3$$

be differentiable maps. Then, it is verified that

$$\overset{r}{T}(\phi_1 \circ \phi) = \overset{r}{T}\phi_1 \circ \overset{r}{T}\phi\,, \qquad \overset{r}{T}(\phi, \phi') = (\overset{r}{T}\phi, \overset{r}{T}\phi')$$
$$\overset{r}{T}(\phi \times \psi) = \overset{r}{T}\phi \times \overset{r}{T}\psi\,, \qquad \overset{r}{T}1_M = 1_{\overset{r}{T}M}$$

where 1_M is the identity diffeomorphism and $\overset{r}{T}(M \times M_2)$ is canonically identified to $\overset{r}{T}M \times \overset{r}{T}M_2$.

Likewise, if $\phi: M \to N$ is a submersion (respect. an immersion) $\overset{r}{T}\phi$ is also a submersion (respect. an immersion); if ϕ is a diffeomorphism, $\overset{r}{T}\phi$ is also a diffeomorphism.

If $\phi: M \to N$ is a differentiable map and $\psi: \overset{r}{T}M \to \overset{r}{T}N$ is a differentiable map in such a form that

$$\begin{array}{ccc} \overset{r}{T}M & \xrightarrow{\psi} & \overset{r}{T}N \\ {\scriptstyle \overset{r}{\Pi}_M}\downarrow & & \downarrow{\scriptstyle \overset{r}{\Pi}_N} \\ M & \xrightarrow{\phi} & N \end{array}$$

is commutative we shall say "ψ is over ϕ"; note that, for each ϕ, the set of differentiable maps which are over ϕ is not empty and let us denote this set S_ϕ.

The following theorem will be important for our purposes and gives a topological relation between a differentiable manifold and its tangent bundles of higher order.

THEOREM 1.1. *For every integer $r \geqslant 0, M$ and $\overset{r}{T}M$ have the same homotopy type.*

Proof. Let i_M and $\overset{r}{\Pi}_M$ as above; it is clear that $\overset{r}{\Pi}_M \circ i_M = 1_M$.

Now, define a continuous map

$$F: \overset{r}{T}M \times \boldsymbol{R} \to \overset{r}{T}M$$

by $F([\varphi]_r, t) = [\varphi_t]_r$, for $[\varphi]_r \in \overset{r}{T}M$ and $t \in \boldsymbol{R}$, where $[\varphi_t]_r \in \overset{r}{T}M$ is defined in the following way: if $\varphi: \boldsymbol{R} \to M$ defines $[\varphi]_r$, we take, for each $t \in \boldsymbol{R}, \varphi_t: \boldsymbol{R} \to M$ given by $\varphi_t(s) = \varphi(s(1-t)), \forall s \in \boldsymbol{R}$; it is clear that $[\varphi_t]_r$ is well-defined and

$$F|_{\overset{r}{T}M \times \{0\}} = 1_{\overset{r}{T}M}$$
$$F|_{\overset{r}{T}M \times \{1\}} = i_M \circ \overset{r}{\Pi}_M$$

Q.E.D.

COROLLARY 1.1. *For every integer $r \geqslant 0$, the de Rham complex $H^*(M)$ and $H^*(\overset{r}{T}M)$ are canonically isomorphic.*

§ 2. Tangent pseudogroups of higher order.

Let M an n-dimensional differentiable manifold and let $\overset{r}{T}M$ be its tangent bundle of order $r, r \geqslant 0$. Let G be a pseudogroup of local diffeomorphisms of M and consider, for every $g \in G$, the set S_g of all local diffeomorphisms of $\overset{r}{T}M$ which are over g. Then, ${}^rG = \bigcup_{g \in G} S_g$ is a pseudogroup of local diffeomorphisms of $\overset{r}{T}M$.

DEFINITION 2.1. We shall call ${}^rG$ *the tangent pseudogroup of G of order r.*

Now, consider the euclidean space $\boldsymbol{R}^n$ and its tangent bundle of order $r, \overset{r}{T}\boldsymbol{R}^n$; for each coordinate open neighborhood U in $\boldsymbol{R}^n$ with coordinate functions $(x_1, \cdots, x_n)$, consider the coordinate open neighborhood $\overset{r}{T}U$ in $\overset{r}{T}\boldsymbol{R}^n$ and its coordinate functions $(\overset{(\nu)}{x_i}), i = 1, 2, \cdots, n, \nu = 0, 1, \cdots, r$, and denote $\varphi^r : \overset{r}{T}U \to$ open set $\subset \boldsymbol{R}^{n(r+1)}$ the diffeomorphism defined by the coordinate functions $\overset{(\nu)}{x_i}$. Let

$$p_1 : \boldsymbol{R}^n \times \overset{r+1}{\cdots} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$$

be the canonical projection onto the first factor; then, every diffeomorphism

$$\lambda : \varphi^r(\overset{r}{T}U) \to \varphi^r(\overset{r}{T}M)$$

such that $p_1 \circ \lambda = p_1$ defines canonically a differentiable transformation of $\overset{r}{T}U$ which is over 1_U.

Now, take $G = \Gamma_n$, the Lie pseudogroup of diffeomorphisms on $\boldsymbol{R}^n$ (for definition of Lie pseudogroup see [4], p. 36).

THEOREM 2.1. *${}^r\Gamma_n$ is a transitive Lie pseudogroup.*

Proof. Let $A, B \in \overset{r}{T}\boldsymbol{R}^n, A \neq B$. We have to show there is ${}^rf \in {}^r\Gamma_n$ in such a form that ${}^rf(A) = B$. It may be $\overset{r}{\Pi}_{\boldsymbol{R}^n}(A) = \overset{r}{\Pi}_{\boldsymbol{R}^n}(B)$ or $\overset{r}{\Pi}_{\boldsymbol{R}^n}(A) \neq \overset{r}{\Pi}_{\boldsymbol{R}^n}(B)$; suppose we are in the second case and put $a = \overset{r}{\Pi}_{\boldsymbol{R}^n}(A)$, b

$= \overset{r}{\Pi}_{R^n}(B)$; then, there exists $f \in \Gamma_n$ such that $f(a) = b$ and by using $\overset{r}{T}f \in {}^r\Gamma_n$ we obtain $\overset{r}{\Pi}_{R^n}((\overset{r}{T}f)(A)) = \overset{r}{\Pi}_{R^n}(B)$. Therefore we can restrain us to consider $a = b$.

Thus, let $U \subset \boldsymbol{R}^n$ be an open set and $a \in U$; then, $A, B \in \overset{r}{T}U$ and put $a' = \varphi^r(A), b' = \varphi^r(B), \varphi^r$ being the diffeomorphism of $\overset{r}{T}U$ on an open set in $\boldsymbol{R}^{n(r+1)}$; clearly, there is a diffeomorphism

$$\lambda : \varphi^r(\overset{r}{T}U) \to \varphi^r(\overset{r}{T}U)$$

in such a form that $\lambda(a') = b'$ and satisfying $p_1 \circ \lambda = p_1$. The differentiable transformation η of $\overset{r}{T}U$ on itself defined through λ is over 1_U and, therefore, $\eta \in {}^r\Gamma_n$; besides, $\eta(A) = B$ and this shows ${}^r\Gamma_n$ is transitive.

Now, let $J^k_{\tilde{0}}({}^r\Gamma_n)$ be the space of k-jets at $\tilde{0}$ of elements of ${}^r\Gamma_n$, with $\tilde{0} = i_{R^n}(0)$ and 0 being the origin of $\boldsymbol{R}^n$. Our purpose is to show that $J^k_{\tilde{0}}({}^r\Gamma_n)$ is canonically a differentiable principal bundle over $\overset{r}{T}\boldsymbol{R}^n$ with group $({}^r\Gamma_n)^k_{\tilde{0}}$, the Lie group of k-jets of elements of ${}^r\Gamma_n$ which keep $\tilde{0}$ fixed.

$({}^r\Gamma_n)^k_{\tilde{0}}$ acts freely on $J^k_{\tilde{0}}({}^r\Gamma_n)$ on the right in the natural way: if $j^k_{\tilde{0}}({}^rf) \in ({}^r\Gamma_n)^k_{\tilde{0}}$ and $j^k_{\tilde{0}}({}^rg) \in J^k_{\tilde{0}}({}^r\Gamma_n)$, then

$$j^k_{\tilde{0}}({}^rg) \circ j^k_{\tilde{0}}({}^rf) = j^k_{\tilde{0}}({}^rg \circ {}^rf)$$

is well-defined and if ${}^rg \in S_g, {}^rf \in S_f$, then $({}^rg \circ {}^rf) \in S_{(g \circ f)}$ and, therefore $j^k_{\tilde{0}}({}^rg \circ {}^rf) \in J^k_{\tilde{0}}({}^r\Gamma_n)$. In order to obtain the local trivialization of $J^k_{\tilde{0}}({}^r\Gamma_n)$, consider the open covering of $\overset{r}{T}\boldsymbol{R}^n$ given by $\{\overset{r}{T}U\}$, $\{U\}$ being the open sets of $\boldsymbol{R}^n$; then, if $p : J^k_{\tilde{0}}({}^r\Gamma_n) \to \overset{r}{T}\boldsymbol{R}^n$ is the canonical projection, for every $U \subset \boldsymbol{R}^n$ we define

$$\phi_{\overset{r}{T}U} : p^{-1}(\overset{r}{T}U) \to \overset{r}{T}U \times ({}^r\Gamma_n)^k_{\tilde{0}}$$

as follows: for every $j^k_{\tilde{0}}({}^rf) \in p^{-1}(\overset{r}{T}U)$ with $p(j^k_{\tilde{0}}({}^rf)) = \tilde{x}$, let ${}^rg_U \in {}^r\Gamma_n$ such that ${}^rg_U(\tilde{0}) = \tilde{x}$; then

$$\phi_{\overset{r}{T}U}(j^k_{\tilde{0}}({}^rf)) = (\tilde{x}, j^k_{\tilde{0}}(({}^rg)^{-1} \circ {}^rf))$$

Q.E.D.

§ 3. r-extension and r-homotopy of foliations.

Let M be a differentiable manifold and G a pseudogroup of local diffeomorphisms acting transitively on M; consider the manifold $\overset{r}{T}M$ and the tangent pseudogroup ${}^{r}G$ of order r, for every $r \in \{0, 1, 2, \cdots\}$. We shall suppose from now on that ${}^{r}G$ is a transitive Lie pseudogroup (that is the case when $M = \boldsymbol{R}^n$ and $G = \Gamma_n$ as we have shown in theorem 2.1).

Let X be a differentiable manifold, $\dim X \geqslant \dim M$.

DEFINITION 3.1. *A G-foliation on X* is a maximal family F of submersions

$$f_U : U \to M$$

of open sets U in X, $\{U\}$ being an open covering of X and the family $\{f_U\}$ satisfying the following condition: for every $x \in U \cap V$ there exists an element $g_{UV} \in G$ with $f_U = g_{UV} \circ f_V$ in some vicinity of x.

Given a smooth map $f : X' \to X, f$ is transverse to F if the composed maps $f_U \circ f$ are submersions; in this case, the maps $f_U \circ f$ are the local projections of a G-foliation on X' called the inverse image $f^{-1}F$ of F via f. With this concept, f is called a morphism from $f^{-1}F$ to F and, thus, the G-foliations form a category denoted $\mathscr{F}(G)$.

Let $\mathscr{F}({}^{r}G)$ be the category of ${}^{r}G$-foliations.

THEOREM 3.2. *Let F be a G-foliation on X. There exists, canonically defined, a ${}^{r}G$-foliation $\overset{r}{F}$ on $\overset{r}{T}X$ in such a form that the correspondence $F \to \overset{r}{F}$ defines a contravariant functor $\mathscr{R}$ from $\mathscr{F}(G)$ to $\mathscr{F}({}^{r}G)$.*

Proof. Let $\{U\}$ be the open covering of X and let $\{f_U\}$ be the family of submersions which define the foliation F. The ${}^{r}G$-foliation $\overset{r}{F}$ on $\overset{r}{T}X$ is defined taking the open covering $\{\overset{r}{T}U = (\overset{r}{\Pi}_X)^{-1}(U)\}$ and the family of submersions $\{\overset{r}{T}f_U\}$; since this family satisfies the compatibility condition, there exists a maximal family containing it and defining $\overset{r}{F}$. Now, let $f : X' \to X$ be a differentiable map which is transverse to F. Then, it is clear that $\overset{r}{T}f : \overset{r}{T}X' \to \overset{r}{T}X$ is transverse to $\overset{r}{F}$ and it follows $(f^{-1}\overset{r}{F}) = (\overset{r}{T}f)^{-1}\overset{r}{F}$. The functoriality of the correspondence $F \to \overset{r}{F}$ is shown by a direct computation.

Q.E.D.

DEFINITION 3.3. Let F be a G-foliation on X. The ${}^rG$-foliation $\overset{r}{F}$ on $\overset{r}{T}X$ defined in theorem 3.2 will be called *the r-extension of F.*

Remark. The construction of Theorem 3.2 is true for every finite positive integer r, and, therefore, to each G-foliation F on X, a sequence $\{\overset{0}{F}, \overset{1}{F}, \overset{2}{F}, \cdots\}$, with $\overset{0}{F} = F$, is associated. If $\dim M = m$, that is $\operatorname{codim} F = m$, then $\operatorname{codim} \overset{r}{F} = m(r+1)$, for each $r \geqslant 0$.

Let F_0 and F_1 be two G-foliations on X. For each $t \in \boldsymbol{R}$

$$i_t : X \rightarrow X \times \boldsymbol{R}$$

denotes the canonical injection $x \rightarrow (x, t)$.

DEFINITION 3.4. The G-foliations F_0 and F_1 are said *homotopic*, $F_0 \sim F_1$, if there exists a G-foliation F on $X \times \boldsymbol{R}$ in such a way that i_0 and i_1 are transverse to F and $i_0^{-1}F = F_0, i_1^{-1}F = F_1$.

As it is well known, the homotopy of G-foliations is an equivalence relation. Denote $\mathscr{H}_G(X)$ the set of homotopy class of G-foliations on X; if $f : X' \rightarrow X$ is a morphism of F, G-foliation on X, to $f^{-1}F$, G-foliation on X', it is clear that f defines a map

$$\mathscr{H}(f) : \mathscr{H}_G(X) \rightarrow \mathscr{H}_G(X')$$

and the following theorem is easily proved:

THEOREM 3.5. *$\mathscr{H}_G(\cdot)$ is a homotopy invariant contravariant functor.*

Now, we return to our r-extensions.

THEOREM 3.6. *Let F_0 and F_1 be two homotopic G-foliations on X. Then, for every $r \geqslant 0$, their r-extensions $\overset{r}{F}_0$ and $\overset{r}{F}_1$ are homotopic ${}^rG$-foliations on $\overset{r}{T}X$.*

Proof. Let F be the G-foliation on $X \times \boldsymbol{R}$ defining the homotopy between F_0 and F_1. Consider

$$\overset{r}{T}X \times \boldsymbol{R} \xrightarrow{1_{\overset{r}{T}X} \times i_{\boldsymbol{R}}} \overset{r}{T}X \times \overset{r}{T}\boldsymbol{R} \xrightarrow{\simeq} \overset{r}{T}(X \times \boldsymbol{R}) \xrightarrow{\overset{r}{\Pi}_{X \times \boldsymbol{R}}} X \times \boldsymbol{R}$$

and denote $\lambda = \simeq \circ (1_{\overset{r}{T}X} \times i_{\boldsymbol{R}})$; then, $\lambda^{-1}\overset{r}{F}$ is a ${}^rG$-foliation on $\overset{r}{T}X \times \boldsymbol{R}$ which defines a homotopy between $\overset{r}{F}_0$ and $\overset{r}{F}_1$; this fact follows from the commutativity of the following diagram, for every $t \in \boldsymbol{R}$,

$$\begin{array}{ccc} & & \overset{r}{T}X \times \mathbf{R} \\ & \overset{i_t}{\nearrow} & \downarrow \lambda \\ \overset{r}{T}X & \xrightarrow{\overset{r}{T}i_t} & \overset{r}{T}(X \times \mathbf{R}) \\ \downarrow \overset{r}{\Pi}_X & & \downarrow \overset{r}{\Pi}_{X \times \mathbf{R}} \\ X & \xrightarrow{i_t} & X \times \mathbf{R} \end{array}$$

Q.E.D.

Observe that if F_0 and F_1 are not homotopic G-foliations on X, their r-extensions could be homotopic, but the converse is an open problem, the answer of which we think to be negative. That leads us to the following definition.

DEFINITION 3.7. Let $r \geqslant 0$ be an integer. Two G-foliations F_0 and F_1 on X will be said *r-homotopic*, $F_0 \underset{r}{\sim} F_1$, if their r-extensions $\overset{r}{F}_0$ and $\overset{r}{F}_1$ are homotopic, $\overset{r}{F}_0 \sim \overset{r}{F}_1$.

PROPOSITION 3.8. $\underset{r}{\sim}$ *is an equivalence relation.*

Remark. The 0-homotopy is the usual homotopy of G-foliations and if F_0 and F_1 are 0-homotopic then they are r-homotopic for every $r > 0$.

Denote, for each $r \geqslant 0$, $\mathscr{H}^r_G(X)$ the set of r-homotopy classes of G-foliations on X. Then, we have

THEOREM 3.9. $\mathscr{H}^r_G(\cdot)$ *is a homotopy invariant contravariant functor.*

This theorem is a direct consequence of Theorems 3.5 and 3.6 and of the following Lemma.

LEMMA 3.10. *Let* $f_0, f_1 : X' \to X$ *two differentiable (differentiably) homotopic maps. Then, for each* $r \geqslant 0$, $\overset{r}{T}f_0, \overset{r}{T}f_1 : \overset{r}{T}X' \to \overset{r}{T}X$ *are (differentiably) homotopic.*

Proof. Let $g : X' \times \mathbf{R} \to X$ be the differentiable map defining the homotopy between f_0 and f_1. We define a differentiable map

$$^r g : \overset{r}{T}X' \times \mathbf{R} \to \overset{r}{T}X$$

by $^r g = \overset{r}{T}g \circ \simeq \circ (1_{\overset{r}{T}X'} \times i_{\mathbf{R}})$, where $\simeq : \overset{r}{T}X' \times \overset{r}{T}\mathbf{R} \to \overset{r}{T}(X' \times \mathbf{R})$ is the canonical diffeomorphism; ${}^{r}g$ defines actually a homotopy between $\overset{r}{T}f_0$ and $\overset{r}{T}f_1$, because for each $t \in \boldsymbol{R}$ the following diagram is commutative:

$$
\begin{array}{ccccc}
 & & \overset{r}{T}X' \times \boldsymbol{R} & & \\
 & {}^{r}i_t \nearrow & \big\downarrow 1_{\overset{r}{T}X'} \times i_{\boldsymbol{R}} & \searrow {}^{r}g & \\
 & & \overset{r}{T}X' \times \overset{r}{T}\boldsymbol{R} & & \\
 & & \big\downarrow \simeq & & \\
\overset{r}{T}X' & \xrightarrow{\overset{r}{T}i_t} & \overset{r}{T}(X' \times \boldsymbol{R}) & \xrightarrow{\overset{r}{T}g} & \overset{r}{T}X
\end{array}
$$

Q.E.D.

§ 4. Characteristic classes of G-foliations.

Recall briefly the construction of the Bott-Haefliger's characteristic homomorphism for G-foliations, following Haefliger ([3]).

Let G be a Lie pseudogroup acting transitively on a differentiable manifold M; a vector field defined on an open set of M is called a G-vector field if the local one parameter group which it generates is in G.

Fix a point $0 \in M$; the set of k-jets at 0 of G-vector fields is a vector space $\underline{G}^k$ which is not a Lie algebra. Then, consider the inverse limit

$$\underline{G} = \varprojlim \underline{G}^k$$

which is a Lie algebra called "the Lie algebra of formal G-vector fields". Denote by $A(\underline{G})$ the direct limit of the algebras $A(\underline{G}^k)$ of multilinear alternate forms on $\underline{G}^k$; the bracket on $\underline{G}$ induces a differential on $A(\underline{G})$, and we write $H^*(\underline{G})$ for the resulting cohomology group.

Denote $J_0^k(G)$ the space of k-jets at 0 of the elements of G; this is the total space of a fiber space on M; besides, if G_0^k denotes the Lie group of elements of $J_0^k(G)$ keeping 0 fixed, G_0^k acts on $J_0^k(G)$ on the right and makes it a differentiable principal bundle. Besides, G acts on $J_0^k(G)$ on the left as a pseudogroup of transformations. Denote $J_0^\infty(G)$ the inverse limit of the $J_0^k(G)$; $J_0^\infty(G)$ is endowed with a differentiable structure as follows: a map of a differentiable manifold X on $J_0^\infty(G)$ is differentiable if its projection on each $J_0^k(G)$ is differentiable; in this way $J_0^\infty(G)$ can be looked as a differentiable principal bundle over M

with group G_0^∞, the inverse limit of the G_0^k; besides, G acts on $J_0^\infty(G)$ on the left. We define the algebra $A(J_0^\infty(G))$ of differential forms on $J_0^\infty(G)$ as the direct limit of the algebras $A(J_0^k(G))$ of differential forms on $J_0^k(G)$.

THEOREM 4.1 ([3]). *$A(\underline{G})$ is canonically isomorphic to the algebra of differential forms on $J_0^\infty(G)$ which are invariant under the action of G and this isomorphism commutes with the differential operators.*

A compact subgroup K of G_0^∞, playing the role of maximal compact subgroup, is defined being isomorphic to (up to conjugation) the inverse limit of the maximal compact subgroups K^s of G_0^s, for each positive integer s; the complex $A(\underline{G}, K)$ is the subcomplex of K-basic elements of $A(\underline{G})$ and its cohomology algebra will be denoted $H^*(\underline{G}, K)$.

THEOREM 4.2 ([3]). *Let F be a G-foliation on X. There is an algebra homomorphism*

$$\varphi(F) : H^*(\underline{G}, K) \to H^*(X)$$

in such a form that if $f : X' \to X$ is transverse to F, then

$$f^* \circ \varphi(F) = \varphi(f^{-1}F)$$

DEFINITION 4.3. $\operatorname{Im} \varphi(F)$ is called *the set of characteristic classes of F.*

PROPOSITION 4.4. *If F_0 and F_1 are homotopic G-foliations on X, then*

$$\operatorname{Im} \varphi(F_0) = \operatorname{Im} \varphi(F_1) .$$

This means that the characteristic classes of a G-foliation are invariants of its homotopy class; the following theorem gives a finer characterization.

THEOREM 4.5. *Let F_0 and F_1 G-foliations on X. If there is some integer $r \geqslant 0$ such that F_0 and F_1 are r-homotopic, then*

$$\operatorname{Im} \varphi(F_0) = \operatorname{Im} \varphi(F_1) .$$

This theorem follows from Proposition 4.4 and the following theorem.

THEOREM 4.6. *Let F be a G-foliation on X and let $r \geqslant 0$ an integer; if $\varphi(\overset{r}{F})$ denotes the Bott-Haefliger's characteristic homomorphism, we have*

$$\operatorname{Im}\varphi(F) = i_X^*(\operatorname{Im}\varphi(\overset{r}{F}))$$

where i_X^* *is the isomorphism induced in cohomology by* $i_X : X \to \overset{r}{T}X$.

To show this theorem, we need a preparatory Lemma. For that, denote $H^*({}^r\underline{G}, {}^rK)$ the cohomology of ${}^rK$-basic differential forms on ${}^r\underline{G}$, the Lie algebra of formal ${}^rG$-vector fields; we keep the notations above, only adding the index r in each case.

LEMMA 4.7. *Let* $r \geqslant 0$ *be an arbitrary fixed integer. Let* F *be a* G-*foliation on* X *and let* $\overset{r}{F}$ *be its* r-*extension. Then:*

a) *There exists a canonical homomorphism*

$$\sigma : H^*(\underline{G}, K) \to H^*({}^r\underline{G}, {}^rK)$$

such that

$$\begin{array}{ccc} H^*({}^r\underline{G}, {}^rK) & \xrightarrow{\varphi(\overset{r}{F})} & H^*(\overset{r}{T}X) \\ \sigma\uparrow & & \downarrow i_X^* \\ H^*(\underline{G}, K) & \xrightarrow{\varphi(F)} & H^*(X) \end{array} \tag{4.1}$$

commutes.

b) *There exists a canonical homomorphism*

$$\tau : H^*({}^r\underline{G}, {}^rK) \to H^*(\underline{G}, K)$$

such that

$$\begin{array}{ccc} H^*({}^r\underline{G}, {}^rK) & \xrightarrow{\varphi(\overset{r}{F})} & H^*(\overset{r}{T}X) \\ \tau\downarrow & & \uparrow (\overset{r}{\Pi}_X)^* \\ H^*(\underline{G}, K) & \xrightarrow{\varphi(F)} & H^*(X) \end{array} \tag{4.2}$$

commutes.

c) $\tau \circ \sigma = 1_{H^*(\underline{G}, K)}$ *and, hence,* τ *is onto.*

Proof. 1. Construction of σ.

Fix the point $\tilde{0} \in \overset{r}{T}M, \tilde{0} = i_M(0)$. Now, consider the map, for each $k \geqslant 0$,

$$\sigma_k : J_{\tilde{0}}^k({}^rG) \to J_0^k(G)$$

defined as follows: let $j_0^k({}^rf) \in J_0^k({}^rG)$ and let ${}^rf \in {}^rG$ a representative of this jet; then, there is a unique $f \in G$ such that ${}^rf$ is over f; we define

$$\sigma_k(j_0^k({}^rf)) = j_0^k(f)$$

and σ_k is, clearly, a well-defined map. Actually, σ_k induces a homomorphism of Lie groups

$$\sigma_k : {}^rG_0^k \to G_0^k$$

and, in fact, we get a homomorphism of differentiable principal bundles making commutative the following diagram

$$\begin{array}{ccc} J_0^k({}^rG) & \xrightarrow{\sigma_k} & J_0^k(G) \\ \downarrow & & \downarrow \\ \overset{r}{T}M & \xrightarrow{\overset{r}{\Pi}_M} & M \end{array}$$

Moreover, if for every ${}^rf \in {}^rG$ with ${}^rf \in S_f, f \in G$, we denote λ_{r_f} (respect. λ_f) the differentiable transformation of $J_0^k({}^rG)$ (respect. $J_0^k(G)$) defined by the action on the left of ${}^rf$(respect. f), a direct computation shows

$$\lambda_f \circ \sigma_k = \sigma_k \circ \lambda_{r_f}$$

If σ_k denotes, still, the induced homomorphism between the algebras of differential forms

$$\sigma_k : A(J_0^k(G)) \to A(J_0^k({}^rG))$$

the differential forms invariant under the action of G are sent on the differential forms invariant under the action of ${}^rG$. As a consequence, we have canonically a homomorphism

$$\sigma : A(J_0^\infty(G)) \to A(J_0^\infty({}^rG))$$

which induces a new one

$$\sigma : A(\underline{G}) \to A({}^r\underline{G})$$

Actually, σ induces a homomorphism

$$\sigma : A(\underline{G}, K) \to A({}^r\underline{G}, {}^rK)$$

which induces a homomorphism in cohomology

$$\sigma : H^*(\underline{G}, K) \to H^*({}^r\underline{G}, {}^rK)$$

In order to prove the commutativity of (4.1) it is sufficient to show the commutativity of

$$\begin{array}{ccccc} A(J_0^k({}^rG)) & \xrightarrow{{}^r\eta} & A(P^k(\overset{r}{F})|_{\overset{r}{T}U}) & \xrightarrow{{}^rp} & A(\overset{r}{T}U) \\ \uparrow{\scriptstyle \sigma_k} & & & & \downarrow{\scriptstyle i_U^*} \\ A(J_0^k(G)) & \xrightarrow{\eta} & A(P^k(F)|_U) & \xrightarrow{p} & A(U) \end{array} \tag{4.3}$$

where U is a distinguished open set on $X, P^k(F)|_U$ (respect. $P^k(\overset{r}{F})|_{\overset{r}{T}U}$) is the restriction to U (respect. to $\overset{r}{T}U$) of the principal bundles of k-jets of the local projections of F (respect. of $\overset{r}{F}$); p (respect. ${}^rp$) is the homomorphism canonically induced by the local embedding j_U (respect. $j_{\overset{r}{T}U}$) in $P^k(F)|_U$ (respect. $P^k(\overset{r}{F})|_{\overset{r}{T}U}$) and η (respect. ${}^r\eta$) is induced by the identification of $J_0^k(G)$ (respect. $J_0^k({}^rG)$) to $P^k(F)|_U$ (respect. $P^k(\overset{r}{F})|_{\overset{r}{T}U}$) via f_U (respect. $\overset{r}{T}f_U$). This diagram, in the limit, and for the K-basic G-invariant differential forms, induces (4.1).

The embedding $j_U: U \to P^k(F)|_U$ is defined as follows: if $f_U: U \to M$ is the local submersion, for each point $x \in U, j_U(x) = j_0^k(g^{-1}f_U)$, where $g \in G$ verifies $g(0) = f_U(x)$, that is, j_U is defined through the local trivialization of $P^k(F)$; $j_{\overset{r}{T}U}$ is defined in the same way.

Then, if $\omega \in A(J_0^k(G))$, we have

$$p(\eta(\omega))|_x = \eta(\omega)|_{j_0^k(g^{-1}f_U)} = \omega|_{j_0^k(g)}$$

and, if $\tilde{x} = i_U(x)$

$$\begin{aligned} i_U^*({}^rp({}^r\eta(\sigma_k(\omega))))|_x &= {}^rp({}^r\eta(\sigma_k(\omega)))|_{\tilde{x}} \\ &= {}^r\eta(\sigma_k(\omega))|_{j_0^k((\overset{r}{T}g)^{-1}\overset{r}{T}f_U)} = \sigma_k(\omega)|_{j_0^k(\overset{r}{T}g)} = \omega|_{j_0^k(g)} \end{aligned}$$

Hence, (4.3) commutes.

2. Construction of τ.

For each $k \geqslant 0$, we define a differentiable map

$$\tau_k : J_0^k(G) \to J_0^{k-r}({}^rG)$$

by $\tau_k(j_0^k(f)) = j_0^{k-r}(\overset{r}{T}f)$ for $f \in G$, if $k > r$, and $\tau_k(j_0^k(f)) = j_0^0(\overset{r}{T}f)$ if $k \leqslant r$. It is clear that τ_k is a well-defined differentiable map and it induces a homomorphism

$$\tau_k : A(J_0^{k-r}({}^rG)) \to A(J_0^k(G))$$

and, in the limit, we have the homomorphism

$$\tau : A(J_0^\infty({}^rG)) \to A(J_0^\infty(G)) .$$

As above, τ sends the differential forms invariant under the action of ${}^rG$ on differential forms invariant under the action of G, because

$$\lambda_{\overset{r}{T}f} \circ \tau_k = \tau_k \circ \lambda_f$$

for every $k \geqslant 0$. Hence, τ defines a homomorphism

$$\tau : A({}^r\underline{G}) \to A(\underline{G}) .$$

Obviously, for each $k \geqslant 0, \tau_k$ defines a homomorphism of differentiable principal bundles, making commutative the following diagram

$$\begin{array}{ccc} J_0^k(G) & \xrightarrow{\tau_k} & J_0^{k-r}({}^rG) \\ \downarrow & & \downarrow \\ M & \xrightarrow{i_M} & \overset{r}{T}M \end{array}$$

and, in fact, τ induces a homomorphism in cohomology

$$\tau : H^*({}^r\underline{G}, {}^rK) \to H^*(\underline{G}, K) .$$

The commutativity of (4.2) follows from the commutativity of

$$\begin{array}{ccccc} A(J_0^{k-r}({}^rG)) & \xrightarrow{{}^r\eta} & A(P^{k-r}(\overset{r}{F})|_{\overset{r}{T}U}) & \xrightarrow{{}^rp} & A(\overset{r}{T}U) \\ {\scriptstyle\tau_k}\downarrow & & & & \uparrow{\scriptstyle(\overset{r}{\Pi}_U)^*} \\ A(J_0^k(G)) & \xrightarrow{\eta} & A(P^k(F)|_U) & \xrightarrow{p} & A(U) \end{array} \tag{4.4}$$

because if $\omega \in A(J_0^{k-r}({}^rG))$ and $\tilde{x} \in \overset{r}{T}U$ with $\overset{r}{\Pi}_U(\tilde{x}) = x$, we have

$${}^rp({}^r\eta(\omega))|_{\tilde{x}} = {}^r\eta(\omega)|_{j_{\tilde{0}}^{k-r}(({}^rg)^{-1}\overset{r}{T}f_U)} = \omega|_{j_{\tilde{0}}^{k-r}({}^rg)}$$

and

$$\begin{aligned} (\overset{r}{\Pi}_U) * (p(\eta(\tau_k(\omega))))|_{\tilde{x}} &= p(\eta(\tau_k(\omega)))|_x \\ &= \eta(\tau_k(\omega))|_{j_0^k(g^{-1}f_U)} = \tau_k(\omega)|_{j_0^k(g)} = \omega|_{j_{\tilde{0}}^{k-r}(\overset{r}{T}g)} \end{aligned}$$

but ${}^rg \in S_g$ and, by definition of $j_{\overset{r}{T}U}$ it is ${}^rg = \overset{r}{T}g$ and we have the commutativity of (4.4).

3. $\tau \circ \sigma = 1_{H^*(\underline{G},K)}$

For that, it is sufficient to show that

$$A(J_0^k(G)) \xrightarrow{\sigma_k} A(J_0^k({}^rG)) \xrightarrow{\tau_{k+r}} A(J_0^{k+r}(G))$$
$$\tau_{k+r}\circ\sigma_k = \mu_k$$

induces the identity in the limit. Then, consider, for each $k > 0$,

$$\begin{array}{ccc} A(J_0^k(G)) & \xrightarrow{\mu_k} & A(J_0^{k+r}(G)) \\ & {}_{1}\searrow \quad \nearrow_{(p_k^{k+r})^*} & \\ & A(J_0^k(G)) & \end{array} \tag{4.5}$$

where $1 = 1_{A(J_0^k(G))}$ and

$$p_k^{k+r} : J_0^{k+r}(G) \to J_0^k(G)$$

is the canonical projection. But (4.5) commutes because the following diagram

$$\begin{array}{ccc} J_0^{k+r}(G) \xrightarrow{\tau_{k+r}} & J_0^k({}^rG) & \xrightarrow{\sigma_k} J_0^k(G) \\ {}_{p_k^{k+r}}\searrow & & \swarrow_{1} \\ & J_0^k(G) & \end{array}$$

commutes trivially.

The assertion, now, follows from the commutativity of (4.5).

Proof of Theorem 4.6. (4.1) implies

$$i_X^*(\operatorname{Im}\varphi(\overset{r}{F})) \supseteq \operatorname{Im}\varphi(F)$$

and (4.2) implies

$$(\overset{r}{\Pi}_X)^*(\operatorname{Im}\varphi(F)) \supseteq \operatorname{Im}\varphi(\overset{r}{F})$$

because τ is onto. Then, as $i_X^* \circ (\overset{r}{\Pi}_X)^* = 1_{H^*(X)}$, we obtain

$$\operatorname{Im}\varphi(F) = i_X^*(\operatorname{Im}\varphi(\overset{r}{F})) .$$

Q.E.D.

Finally, combining the Bott-Haefliger's result (theorem 4.2), their definition of characteristic class of a G-foliation and our results, we can assert:

THEOREM 4.8. *Let $\mathscr{F}(G)$ the category of G-foliations; there exists* *an infinite sequence* $\{\overset{0}{\varphi}, \overset{1}{\varphi}, \cdots, \overset{r}{\varphi}, \cdots\}$ *of characteristic classes of G-foliations, that is, natural transformations*

$$\overset{r}{\varphi} : \mathscr{F}(G) \to H^*(\ \ ; \boldsymbol{R})$$

satisfying

$$\overset{r}{\varphi}(f^{-1}F) = f^* \circ \overset{r}{\varphi}(F)$$

and $\overset{0}{\varphi}$ *being the Bott-Haefliger's characteristic class.*

Proof. Define, for a G-foliation F, $\overset{r}{\varphi}(F) = \overset{r}{\varphi}(\bar{F})$, and apply the above theorem.

Q.E.D.

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