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The Extension of Sarkovskii's Results and the Topological Entropy in Unimodal Transformations

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Introduction

The study of chaos is very important not only from a mathematical but also from a physical and biological point of view. One parameter family of continuous maps, from interval to itself, is an especially important example and recently the study of them has made great progress.

In this article we will examine unimodal transformations. Our aim is to extend Sarkovskii's result ([6], [7]) and to calculate the topological entropy of the transformations. A continuous map from interval $I=[a, b]$ $(-\infty<a<b<\infty)$ to itself will be called unimodal if there exists a unique point $c\in(a, b)$ such that

i) $f(c)>f(x)$ for any $x\in[a, b]$ $x\neq c$

ii) f is monotone increasing in $[a, c]$

and

iii) f is monotone decreasing in $[c, b]$.

Here we only treat those maps which satisfy

i) $I=[0, 1]$

ii) $f(1)=0$

iii) $f(c)=1$

and

iv) $0<f(0)<1$.

In general, all unimodal transformations except some trivial ones can be reduced to this case. If f is linear in both $[a, c]$ and $[c, b]$, f is called a unimodal linear transformation. Concerning unimodal linear transformations, see [1].

§1. Notations and preliminaries.

Let Θ be an aggregate of all the formal symbols

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$$(k_1, l_1)(k_2, l_2)\cdots(k_n, l_n)$$

where $n\geqq 1$, k_1 is a nonnegative integer and $k_2, \cdots, k_n, l_1, \cdots, l_n$ are positive integers. An element of Θ, which will play a major role in the following, will be called an orbit. We shall define some notations. Let

$$\theta^1=(k_1^1, l_1^1)\cdots(k_{n_1}^1, l_{n_1}^1)\ ,\qquad \theta^2=(k_1^2, l_1^2)\cdots(k_{n_2}^2, l_{n_2}^2)\in\Theta(k_1^2>0)\ .$$

Then we define

$$\theta^1\theta^2=(k_1^1, l_1^1)\cdots(k_{n_1}^1, l_{n_1}^1)(k_1^2, l_1^2)\cdots(k_{n_2}^2, l_{n_2}^2)\in\Theta\ .$$

Let $\theta=(k_1, l_1)\cdots(k_n, l_n)\in\Theta$. Then we define

i) $\#\theta=n$

ii) $|\theta|=\sum_{i=1}^n (k_i+l_i)$

iii) $\theta_i=(k_1, l_1)\cdots(k_i, l_i)\qquad(1\leqq i\leqq n)$

iv) $\sigma^i\theta=(k_{i+1}, l_{i+1})\cdots(k_n, l_n)\qquad(0\leqq i\leqq n)$

v) $(\theta)^1=\theta$

$(\theta)^m=(\theta)^{m-1}\theta\qquad(m\geqq 2$ and $k_1>0)$

and

vi) we call θ is even (odd), if $\sum_{i=1}^n l_i$ is even (odd), respectively.

Moreover we define ϕ for convenience; that is, for any $\theta\in\Theta$

i) $\theta_0=\phi$

ii) $\phi\theta=\theta\phi=\theta$

iii) $\#\phi=|\phi|=0$

iv) ϕ is even.

Now we can introduce a partial order in Θ. We begin with the orbits for which $\#\theta=1$. Define the relation $<$ as

i) $(k, l)<(k', l')$ whenever $k>k'$,

ii) $(k, 2l)<(k, 2l'+1)$ for any k, l and l',

iii) $(k, 2l)<(k, 2l+2)$, for any k, l,

iv) $(k, 2l+1)<(k, 2l-1)$, for any k, l.

Let $\theta, \theta'\in\Theta$. Suppose that there exists $i(i\geqq 0)$ such that $\theta_i=\theta'_i$ and $\theta_{i+1}\neq\theta'_{i+1}$. Then define $\theta>\theta'$ if θ and θ' satisfy one of the followings,

i) θ_i is even and $\sigma^i(\theta_{i+1})>\sigma^i(\theta'_{i+1})$,

ii) θ_i is odd and $\sigma^i(\theta_{i+1})<\sigma^i(\theta'_{i+1})$.

Up to now the concept of orbits has been defined formally. Related with a unimodal map f, which we shall fix until §3, the following meaning is given to this formal concept. Define $f_L=f|_{[0,c]}$ and $f_R=f|_{[c,1]}$. Then for $\theta=(k_1, l_1)\cdots(k_n, l_n)\in\Theta$, we identify θ with $f_R^{l_n}, f_L^{k_n}\cdots f_R^{l_1}, f_L^{k_1}$, moreover we identify ϕ with the identity. Let

i) $D[\theta]=$domain of $f_L\ \theta$

ii) $R[\theta]=\{\theta x; x\in D[\theta]\}$.
Notice that both $D[\theta]$ and $R[\theta]$ are intervals. Moreover if $D[\theta]=[a, b]$, then

$$R[\theta]=\begin{cases}[\theta a, \theta b] & \text{if } \theta \text{ is even}\\ [\theta b, \theta a] & \text{if } \theta \text{ is odd}\,.\end{cases}$$

If $R[\theta]=[0, c]$, θ is called complete.

We now define the special orbit θ^0 which is called the expansion of zero. This is one of the keys of this article. We say that θ^0 is Markov if there exists some $n>0$ such that

i) $f^n 0=0$

ii) $f^m 0\neq 0$ for $0<m<n$.

Then there exists two expressions of f^n. Let θ and θ' $(\theta<\theta', |\theta|=|\theta'|=n)$ be such expressions. In such cases we define θ^0 by

$$\theta^0=\theta\theta\cdots\,.$$

We call θ^0 periodic with period m(or period θ^0_m) if

$$\theta^0_j=(\theta^0_m)^{[j/m]}\theta^0_{j-m[j/m]}$$

for any j and there exists no $n<m$ such that

$$\theta^0_m=(\theta^0_n)^{[m/n]}\,.$$

If f is not Markov, let $\theta^0_i\in\Theta$ be the orbit which satisfies $0\in D[\theta^0_i]$ and let θ^0 be the inductive limit of θ^0_i. We denote

$$\theta^0=(k^0_1, l^0_1)(k^0_2, l^0_2)\cdots\,.$$

LEMMA 1.1. *Suppose $(k, l)>(k^0_1, l^0_1)$. Then (k, l) is complete.*

LEMMA 1.2.

$$R[(k^0_1, l^0_1)]=\begin{cases}[(k^0_1, l^0_1)0, c] & \textit{if } l^0_1 \textit{ is even}\\ [0, (k^0_1, l^0_1)0] & \textit{if } l^0_1 \textit{ is odd}\,.\end{cases}$$

The proofs of both lemmas easily follow from definition.

§2. The existence of the orbits.

In this section, we will show the necessary and sufficient condition for $D[\theta]\neq\phi(\theta\in\Theta)$. First we consider the orbits which are complete.

LEMMA 2.1. *Suppose θ is complete and $D[\theta']\neq\phi$, then $D[\theta\theta']\neq\phi$ and*

$R[\theta\theta']=R[\theta']$.

The proof easily follows from the intermediate value theorem.

COROLLARY 2.2. *Suppose that both θ and θ' are complete, then $\theta\theta'$ is also complete.*

COROLLARY 2.3. *Let $\theta=(k_1, l_1)\cdots(k_n, l_n)$. If $(k_i, l_i)>(k_1^0, l_1^0)$ for $1\leqq i\leqq n$, then θ is complete.*

LEMMA 2.4. *For $\theta\in\Theta$, if there exists i such that $\sigma^i\theta<\theta^0_{\#\theta-i}$, then $D[\theta]=\phi$.*

PROOF. It is sufficient to consider the orbit θ which satisfies $\theta<\theta^0_{\#\theta}$. Then there exists $j(1\leqq j\leqq\#\theta)$ such that

$$\theta=\theta^0_{j-1}(k_j, l_j)\cdots(k_{\#\theta}, l_{\#\theta})$$

and

$$\begin{aligned}(k_j, l_j)&<(k_j^0, l_j^0) \quad \text{if } \theta^0_{j-1} \text{ is even}\\ (k_j, l_j)&>(k_j^0, l_j^0) \quad \text{if } \theta^0_{j-1} \text{ is odd}\end{aligned}$$

where $\theta^0_0=\phi$.

i) If θ^0_{j-1} is even, then $R[\theta^0_{j-1}]\subset[\theta^0_{j-1}0, c]$ and $\theta^0_{j-1}0\in D[(k_j^0, l_j^0)]$. On the other hand

$$D[(k_j, l_j)\cdots(k_{\#\theta}, l_{\#\theta})]\subset D[(k_j, l_j)]\ .$$

Thus $D[(k_j, l_j)\cdots(k_{\#\theta}, l_{\#\theta})]\cap R[\theta^0_{j-1}]=\phi$ and hence $D[\theta]=\phi$.

ii) If θ^0_{j-1} is odd, then $R[\theta^0_{j-1}]\subset[0, \theta^0_{j-1}0]$. Thus it also follows that $D[\theta]=\phi$.

Let

$$\Theta(\theta^0)=\{\theta\in\Theta\colon \sigma^i\theta\geqq\theta^0_{\#\theta-i} \text{ for any } i(0\leqq i\leqq\#\theta-1)\}\ ,$$

$$\Theta^+(\theta^0)=\{\theta\in\Theta\colon \sigma^i\theta>\theta^0_{\#\theta-i} \text{ for any } i(0\leqq i\leqq\#\theta-1)\}\ ,$$

$$\Theta^0(\theta^0)=\{\theta=(k_1, l_1)\cdots(k_n, l_n)\in\Theta(\theta^0)\colon (k_1, l_1)=(k_1^0, l_1^0),\ (k_i, l_i)>(k_1^0, l_1^0) \text{ for } 2\leqq i\leqq n\},$$

and let

$$\Theta^c(\theta^0)=\{\theta=(k_1, l_1)\cdots(k_n, l_n)\in\Theta(\theta^0)\colon (k_i, l_i)>(k_1^0, l_1^0) \text{ for } 1\leqq i\leqq n\}\cup\{\phi\}\ .$$

We already know that any $\theta\in\Theta^c(\theta^0)$ is complete.

For $\theta\in\Theta(\theta^0)$ we define two integers n and m as follows.

i) Let n be the largest integer for which $\theta=\theta_{\#\theta-t}\theta^0_n$, where $t=\#\theta^0_n$.

If such n does not exist, we define $n=0$.

ii) If $n=0$, we define $m=0$. If $n\geqq 1$, let m be the largest integer which satisfies

1) $\theta_n^0=\theta_{n-m}^0\theta_m^0$

and

2) θ_{n-m}^0 is odd.

We also define $m=0$ if such m does not exist. We say that this θ is of (n, m)-type.

LEMMA 2.5. *If* $\theta\in\Theta^+(\theta^0)$, *then* θ *is* $(0, 0)$-*type.*

LEMMA 2.6. *Suppose* $\theta\in\Theta^0(\theta^0)$. *Let* j *be the largest integer such that* $\theta=\theta_j^0(k_{j+1}, l_{j+1})\cdots(k_{\#\theta}, l_{\#\theta})$. *Then we get*

i) *if* $j<\#\theta$, *then* θ *is complete.*

ii) *If* $j=\#\theta$, *then*

$$R[\theta]=\begin{cases}[\theta_{\#\theta}^0 0, c] & \text{if } \theta \text{ is even}\\ [0, \theta_{\#\theta}^0 0] & \text{if } \theta \text{ is odd}\ .\end{cases}$$

PROOF. ii) we show this assertion by induction. If $\#\theta=1$, we have already proved the assertion. Assume that the assertion holds for $\#\theta=n$. For $\theta\in\Theta^0(\theta^0)(\#\theta=n+1)$ we get

1) $R[\theta_n^0]=\begin{cases}[\theta_n^0 0, c] & \text{if } \theta_n^0 \text{ is even}\\ [0, \theta_n^0 0] & \text{if } \theta_n^0 \text{ is odd,}\end{cases}$

2) (k_{n+1}, l_{n+1}) is complete,

3) $\theta_n^0 0\in D[(k_{n+1}, l_{n+1})]$.

Thus the proof of ii) follows.

i) If $R[\theta_j^0]\cap D[(k_{j+1}, l_{j+1})]=\phi$, then there exist the following two cases.

a) If θ_j^0 is even, then we get from the assumption

$$0_j^0\theta>(k_{j+1}, l_{j+1})^{-1}0$$

and

$$\theta_j^0 0>(k_{j+1}, l_{j+1})^{-1}c\ .$$

This is a contradiction, because

$$\theta_j^0(k_{j+1}, l_{j+1})0>c \quad \text{if } (k_{j+1}, l_{j+1}) \text{ is even}\ ,$$

or

$$\theta_j^0(k_{j+1}, l_{j+1})0<0 \quad \text{if } (k_{j+1}, l_{j+1}) \text{ is odd}\ .$$

b) If θ_j^0 is odd, then we also get

$$\theta_j^0 0 > (k_{j+1}, l_{j+1})^{-1} 0$$

and

$$\theta_j^0 0 < (k_{j+1}, l_{j+1})^{-1} c .$$

Thus we can show the contradiction in the same way. As $(k_{j+1}, l_{j+1}) \neq (k_{j+1}^0, l_{j+1}^0)$ we can show $R[\theta_j^0] \supset D[(k_{j+1}, l_{j+1})]$. Moreover we already know that $(k_{j+1}, l_{j+1}) \cdots (k_{\#\theta}, l_{\#\theta})$ is complete. So this completes the proof.

THEOREM 2.7. *Let $\theta \in \Theta(\theta^0)$ be of (n, m)-type.*
 i) *If $n = m = 0$, then θ is complete and $\theta \in \Theta^+(\theta^0)$*
 ii) *If $n > 0$ and $m = 0$,*

$$R[\theta] = \begin{cases} [\theta_n^0 0, c] & \text{if } \theta_n^0 \text{ is even} \\ [0, \theta_n^0 0] & \text{if } \theta_n^0 \text{ is odd} . \end{cases}$$

 iii) *If both n and m are positive,*

$$R[\theta] = \begin{cases} [\theta_n^0 0, \theta_m^0 0] & \text{if } \theta_n^0 \text{ is even} \\ [\theta_m^0 0, \theta_n^0 0] & \text{if } \theta_n^0 \text{ is odd} . \end{cases}$$

PROOF. Let $\theta = \theta_{\#\theta - i_1 - \cdots - i_p} \theta^1 \cdots \theta^p$ such that

$$\theta_{\#\theta - i_1 - \cdots - i_p} \in \Theta^c(\theta^0) \quad \text{and} \quad \theta^k \in \Theta^0(\theta^0) (1 \leq k \leq p, \#\theta^k = i_k) .$$

Then we prove the assertion by the induction on p. If $p = 0$, it is trivial that θ is of $(0, 0)$-type and complete. If $p = 1$, then θ must be of $(n, 0)$-type $(n \geq 0)$; and we have already proved the assertion, in Lemma 2.6.

Now we assume that the assertion holds for any $r \leq p$. Let $\theta = \tilde{\theta}\theta^1 \cdots \theta^{p+1} (\#\theta^r = i_r, 1 \leq r \leq p+1)$.

i) If θ is $(0, 0)$-type, then θ^{p+1} must be complete. At the same time it is trivial that $R[\theta^1 \cdots \theta^p] \supset D[\theta^{p+1}]$ by the assumption of induction. Therefore θ is complete.

ii) Suppose that θ is of $(n, 0)$-type $(n > 0)$, $\theta^q \cdots \theta^{p+1} = \theta_n^0$ and that $\theta^q \cdots \theta^p$ is of $(n - i_{p+1}, m')$-type. In a similar way as in Lemma 2.6, we can show the following four assertions.

1) $\theta_{n-i_{p+1}}^0 0 \in D[\theta^{p+1}]$
2) $R[\tilde{\theta}\theta^1 \cdots \theta^{q-1}] \supset D[\theta^q \cdots \theta^{p+1}]$
3) $\theta_{m'}^0, 0 \notin D[\theta^{p+1}]$ if $m' > 0$
4) θ^{p+1} is complete if $\theta_{n-i_{p+1}}^0$ is odd

and

5) $R[\tilde{\theta}\theta^1\cdots\theta^p]=\begin{cases}[\theta^0_{n-i_{p+1}}0,\ \theta^0_{m'},\ 0] & \text{if } \theta^0_{n-i_{p+1}} \text{ is even}\\ [\theta^0_{m'},\ 0,\ \theta^0_{n-i_{p+1}}0] & \text{if } \theta^0_{n-i_{p+1}} \text{ is odd.}\end{cases}$

Thus it is trivial to see ii).

iii) Suppose that θ is (n, m)-type $(n, m>0)$, $\theta^q\cdots\theta^{p+1}=\theta^0_n$ and that $\theta^r\cdots\theta^{p+1}=\theta^0_m$. If $\theta^q\cdots\theta^{r+1}$ is of $(n-i_r-\cdots-i_{p+1}, m')$-type $(m'>0)$, then $\theta^0_{n-i_r\cdots-i_{p+1}}0\in D[\theta^r\cdots\theta^{p+1}]$ and $\theta^0_m, 0\notin D[\theta^r\cdots\theta^{p+1}]$. This contradicts the fact that $\theta^q\cdots\theta^{r-1}$ is odd and $D[\theta^r\cdots\theta^{p+1}]\ni 0$. Therefore we get, by the assumption of induction,

$$R[\theta^q\cdots\theta^{r-1}]=[0,\ \theta^0_{n-i_r-\cdots-i_{p+1}}0]\subset D[\theta^0_m]\ .$$

This proves the theorem.

COROLLARY 2.8. *$D[\theta]\neq\phi$ if and only if $\theta\in\Theta(\theta^0)$. Moreover, if $\theta\in\Theta^+(\theta^0)$, then θ is complete.*

PROOF. If $\theta\in\Theta(\theta^0)$, we have already proved $D[\theta]\neq\phi$ in Lemma 2.4. For $\theta\in\Theta(\theta^0)$ we proved in Theorem 2.7 that $R[\theta]\neq\phi$. Therefore we get $D[\theta]\neq\phi$, thus the first assertion is proved. On the other hand, if $\theta\in\Theta^+(\theta^0)$, then θ must be of (0, 0)-type. Hence we get the second assertion.

COROLLARY 2.9. *$\sigma^j\theta^0_i\geqq\theta^0_{i-j}$ for any $j(1\leqq j\leqq i-1)$.*

This corollary easily follows from Corollary 2.8.

COROLLARY 2.10. *If $\theta\in\Theta(\theta^0)\backslash\Theta^+(\theta^0)$ is complete, then θ^0 is periodic.*

PROOF. By the definition, we know $\theta=\tilde{\theta}\theta^0_i$ for some $i(1\leqq i\leqq \sharp\theta)$. If θ^0_i is even, then θ^0_i must be 0. This shows that θ^0_i is periodic.

On the other hand, if θ^0_i is odd, then θ^0_i must be c. Thus we get $\theta^0_i(1, 1)=\theta^0_{i+1}$ and $\theta^0_i(1, 1)0=0$. This completes the proof.

§3. The existence of the periodic orbits.

In this section we consider the periodic points and we will extend the result of Sarkovskii [6] [7]. We call

$$\{x, fx, \cdots, f^{k-1}x;\ f^ix\neq f^jx \text{ for } 0\leqq i<j\leqq k-1 \text{ and } f^kx=x\}$$

the periodic orbit of f with period k.

In Θ we introduce an equivalence relation. For $\theta^1, \theta^2\in\Theta$, we define $\theta^1\sim\theta^2$ if and only if there exists $\tilde{\theta}^1, \tilde{\theta}^2\in\Theta$ and positive integers n_1, n_2 such that $\theta^1=(\tilde{\theta}^1)^{n_1}$, $\theta^2=(\tilde{\theta}^2)^{n_2}$ and that $\tilde{\theta}^2=\sigma^k\tilde{\theta}^1\tilde{\theta}^1_{\sharp\theta^1-k}$ for some k. It is trivial that this relation is the equivalence relation. Let $[\Theta]$ be the quotient space defined by the equivalence relation and for $\theta\in\Theta$ let $[\theta]$

be the equivalence class to which θ belongs.

For $\theta=(k_1, l_1)\cdots(k_n, l_n)\in\Theta$, let

$$\tilde{\sigma}^i\theta=(k_{i+1}, l_{i+1})\cdots(k_n, l_n)(k_1, l_1)\cdots(k_i, l_i) \quad \text{for} \quad 1\leqq i\leqq n-1 .$$

Moreover for $[\theta]\in[\Theta]$, let

$$l([\theta])=\inf_{\theta\in[\theta]} \#\theta$$

and

$$[\bar{\theta}]=\inf\{\theta\in[\theta]: \#\theta=l([\theta])\} .$$

We also say that $[\theta]$ is odd, even or complete if $[\bar{\theta}]$ is odd, even or complete, respectively.

Now we can introduce the order on $[\Theta]$. We define $[\theta^1]\ll[\theta^2]$ as $([\bar{\theta}^1])^p<([\bar{\theta}^2])^q$, where $p\cdot l([\theta^1])=q\cdot l([\theta^2])$ is L.C.M. of $l([\theta^1])$ and $l([\theta^2])$.

Let

$$[\Theta](\theta^0)=\{[\theta]\in[\Theta]: ([\bar{\theta}])^n\geqq\theta^0_{nl([\theta])} \text{ for any } n\} .$$

DEFINITION. A periodic orbit of f,

$$\{p, fp, \cdots, f^{n-1}p: p=f^n p \text{ and } p<f^i p \text{ for } 1\leqq i\leqq n-1\}$$

is called of $[\theta]$-type if

$$p\in D[([\bar{\theta}])^m] \quad \text{for any } m\geqq 1 .$$

Let for $[\theta]\in[\Theta]$

$$A[[\theta]]=\bigcap_{n\geqq 1} D[([\bar{\theta}])^n]$$

and

$$I[[\theta]]=\bigcap_{n\geqq 1} ([\bar{\theta}])^n A[[\theta]] .$$

Then $A[[\theta]]$ is either empty, {one point} or an interval. On the other hand, if $A[[\theta]]$ is not empty, then $I[[\theta]]$ is {one point} or an interval. It is easy to see that the smallest point of any $[\theta]$-type periodic orbit belongs to $I[[\theta]]$. Thus, if $I[[\theta]]$ consists of a point, then this point belongs to a unique $[\theta]$-type periodic orbit and any point $x\in A[[\theta]]\backslash I[[\theta]]$ is attracted to this $[\theta]$-type periodic orbit with period $|[\bar{\theta}]|$. On the other hand, if $I[[\theta]]=[p, q](p<q)$, then the situations are different according

as $[\bar{\theta}]$ is even or odd. (See below.)

i) If $[\bar{\theta}]$ is even, then $x<y(x, y \in A[[\theta]])$ implies $[\bar{\theta}]x<[\bar{\theta}]y$. Thus both p and q belong to different $[\theta]$-type periodic orbits with period $|[\bar{\theta}]|$; and any $x \in A[[\theta]]$ such that $x<p$(resp. $x>q$) is attracted to the orbit which contains p (resp. q). Moreover, for any $x \in I[[\theta]]$ we have one of the following.

1) x belongs to some $[\theta]$-type periodic orbit with with period $|[\theta]|$.

2) $\{([\bar{\theta}])^n x\}_{n=-\infty}^{\infty}$ is monotone increasing (or decreasing) and $\lim_{n\to\infty}([\bar{\theta}])^n x$ and $\lim_{n\to-\infty}([\bar{\theta}])x$ belong to different $[\theta]$-type periodic orbits with period $|[\bar{\theta}]|$.

ii) If $[\bar{\theta}]$ is odd, then $x<y(x, y \in A[[\theta]])$ implies $[\bar{\theta}]x>[\bar{\theta}]y$. Therefore p and q belong to the same $[\theta]$-type periodic orbit with period $2|[\bar{\theta}]|$. Moreover there exists a unique point $x_0 \in I[[\theta]]$ which belongs to a unique $[\theta]$-type periodic orbit with period $|[\theta]|$; and any $x \in A[[\theta]]\backslash I[[\theta]]$ is attracted to the periodic orbit which contains both p and q. As in i), for any $x \in I[[\theta]]$ $(x<x_0)$ we have one of the following.

1) x belongs to some $[\theta]$-type periodic orbit with period $2|[\bar{\theta}]|$.

2) $\{([\bar{\theta}])^{2n}x\}_{n=-\infty}^{\infty}$ is monotone increasing (resp. decreasing) and $\{([\bar{\theta}])^{2n+1}x\}_{n=-\infty}^{\infty}$ is monotone decreasing (resp. increasing); and one has $([\bar{\theta}])^{2n}x<x_0<([\bar{\theta}])^{2n+1}x$ for any n. Moreover $\lim_{n\to\infty}([\bar{\theta}])^{2n}x$ and $\lim_{n\to\infty}([\bar{\theta}])^{2n+1}x$ ($\lim_{n\to-\infty}([\bar{\theta}])^{2n}x$ and $\lim_{n\to-\infty}([\bar{\theta}])^{2n+1}x$) belong to the same $[\theta]$-type periodic orbit with period $2|[\bar{\theta}]|$ or $|[\bar{\theta}]|$.

Now we consider when $I[[\theta]]$ is not empty, i.e., when $[\theta]$-type periodic orbit exists. The next two lemmas will give us the solution.

LEMMA 3.1. *If $[\theta] \in [\Theta](\theta^0)$, then $I[[\theta]]$ is not empty.*

PROOF. To show this it is sufficient to prove that $D[([\bar{\theta}])^n]$ is not empty for any n. On the other hand, by the assumption we get $\sigma^i[\bar{\theta}] \geqq [\bar{\theta}]$, therefore the rest of the proof easily follows.

LEMMA 3.2. *If $I[]\theta]]$ is not empty, then $[\theta] \in [\Theta](\theta^0)$.*

The proof of this lemma is trivial, so we omit it. Then we consider the property of $A[[\theta]]$ and $I[[\theta]]$.

LEMMA 3.3. *Suppose $[\theta]$ is complete, then $A[[\theta]]=I[[\theta]]$.*

PROOF. Let $x_n=([\bar{\theta}])^{-n}c$ and $y_n=([\bar{\theta}])^{-n}0$. If $[\theta]$ is even, then we have

$$y_1 \leqq y_2 \leqq \cdots \leqq y_\infty \leqq x_\infty \leqq \cdots \leqq x_2 \leqq x_1 ,$$

where $x_\infty=\lim_{n\to\infty} x_n$ and $y_\infty=\lim_{n\to\infty} y_n$. It is easy to prove that both y_∞ and x_∞ are the endpoints of $A[[\theta]]$ and at the same time they belong to

$[\theta]$-type periodic orbits.

If $[\theta]$ is odd, then

$$x_1 \leqq y_2 \leqq x_3 \leqq \cdots \leqq y_\infty \leqq x_\infty \leqq \cdots \leqq y_3 \leqq x_2 \leqq y_1 ,$$

where

$$y_\infty = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} y_{2n}$$

and

$$x_\infty = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n+1} .$$

The case where $[\theta]$ is even is analogous to the above.

LEMMA 3.4. *Suppose that $A[[\theta]]$ does not coincide with $I[[\theta]]$, then θ^0 is periodic and $[\theta]=[\theta^0]$.* (cf. §6).

PROOF. Let $[\bar{\theta}]$ be of (n, m)-type. Then by Lemma 3.3 it follows $n>0$. Assume that $[\bar{\theta}]>\theta^0_{l([\theta])}$. Then each of the following contradicts the minimality of $[\bar{\theta}]$ or the assumption on $[\bar{\theta}]$;

i) $\theta^0_n 0 \in D[[\theta]]$

ii) θ^0_n is odd

and

iii) $m>0$.

Therefore we get $R[[\theta]]=[\theta^0_n 0, c]$ and $\theta^0_n 0$ satisfies one of the following;

i) $\theta^0_n 0 > x$ for any $x \in D[[\theta]]$

ii) $\theta^0_n 0 < x$ for any $x \in D[[\theta]]$.

If the case i) holds, it is trivial that $A[[\theta]]=I[[\theta]]=\phi$. Otherwise, if we define $x_k=([\bar{\theta}])^k \theta^0_n 0$ and $y_k=([\bar{\theta}])^k c$, then x_k is monotone decreasing and y_k is monotone increasing. Thus we get $A[[\theta]]=[\lim_{k\to\infty} x_k, \lim_{k\to\infty} y_k]$ and therefore it is trivial that $I[[\theta]]=A[[\theta]]$. This contradicts the assumption. Hence we get $[\bar{\theta}]=\theta^0_{l([\theta])}$. The rest of the assertion easily follows.

To summarize the above results, we get:

THEOREM 3.5. *There exists at least one periodic orbit of $[\theta]$-type if and only if $[\theta] \in [\Theta](\theta^0)$.*

COROLLARY 3.6. *If there exists $[\theta]$-type periodic orbit, then there exists $[\theta']$-type periodic orbit for every $[\theta']$ which satisfies $[\theta'] \gg [\theta]$.*

This is the extension of Sarkovskii's result ([6], [7]). If we define

$$\{k\} = \inf \{[\theta] \in [\Theta] : \|[\bar{\theta}]\| = k\} .$$

Sarkovskii gives the order of $\{k\}$ $(k=1, 2, 3, \cdots)$. On the other hand we define all the orders on $[\Theta]$. Actually, it is not so easy to define $\{k\} \in [\Theta]$.

EXAMPLE 1.

$$\{2k+1\}=[(1, 2k)] .$$

This shows that the existence of a periodic orbit with period 3 implies the existence of periodic orbit with period 5 etc.

EXAMPLE 2. Define $\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots$ as

$$\alpha_0=(0. 1)$$
$$\alpha_1=(1. 1)$$
$$\alpha_2=(1. 3)$$

and

$$\alpha_n=\alpha_{n-1}\alpha_{n-2}\alpha_{n-2} \qquad (n\geqq 3) .$$

Then we get $[\bar{\alpha}_n]=\alpha_n$. And we can conclude that if $[\theta]\ll[\alpha_n]$ $(n\geqq 1)$, then $[\theta]=[\alpha_i]$ for some i $(0\leqq i\leqq n-1)$. This shows that $\{2^k\}=[\alpha_k]$ and that the existence of a periodic orbit with period 2^k implies the existence of periodic orbits with period 2^l $(0\leqq l\leqq k-1)$.

Let

$$[\bar{\Theta}]=\{\theta;\ \theta \text{ is an inductive limit of some } \{\theta^n\}_{n\geq 1} \text{ such that } \theta^n=[\bar{\theta}^n]\in\Theta \text{ and } \theta \text{ is admissible}\} ,$$

where $\theta\in[\bar{\Theta}]$ is called admissible if $\sigma^i\theta_j\geqq\theta_{j-i}$ for any $1\leqq i<j$. Hereafter we identify $[\theta]\in[\Theta]$ with $\theta=[\bar{\theta}][\bar{\theta}]\cdots$, then it is not difficult to see $\theta\in[\bar{\Theta}]$. Thus we can consider $[\Theta]\subset[\bar{\Theta}]$.

We have already shown that any expansion of zero belongs to $[\bar{\Theta}]$; moreover, for any $[\theta]\in[\Theta]$, it is not difficult to make a transformation whose expansion of zero is $[\theta]$.

§4. Topological entropy.

In this section we will treat the topological entropy. Let

$$\Xi_0=\left\{\xi=\{A_i\}_{i=1}^N;\ N<\infty,\ A_i \text{ is open and } \bigcup_{i=1}^N A_i\supset[0, 1]\right\} ,$$

and let

$$\Xi_1=\left\{\xi=\{A_i\}_{i=1}^N;\ N<\infty,\ A_i \text{ is an open interval and } \bigcup_{i=1}^N A_i\supset[0, 1]\right\} .$$

Moreover, let us define, for any $\xi \in \Xi_0$,

$$h(f, \xi) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathrm{Card} \left\{ A \in \bigvee_{i=0}^{n-1} f^{-i}\xi \right\} ,$$

where Card$\{\cdot\}$=the number of the elements of the set $\{\cdot\}$. Then we define the topological entropy by

$$h(f) = \sup_{\xi \in \Xi_0} h(f, \xi) .$$

But by the compactness of [0, 1] we get

$$h(f) = \sup_{\xi \in \Xi_1} h(f, \xi) .$$

On the other hand we get for $\xi \in \Xi_1$,

$$\begin{aligned} &\mathrm{Card}\, \{\theta \in \Theta(\theta^0);\ |\theta| \leqq n-1 \text{ and } \theta_1 \leqq (1.\,1)\} \\ &\leqq \mathrm{Card} \left\{ A \in \bigvee_{i=0}^{n-1} f^{-i}\xi \right\} \\ &\leqq \mathrm{Ex}\,(\xi)\ \mathrm{Card}\, \{\theta \in \Theta(\theta^0);\ |\theta| \leqq n-1 \text{ and } \theta_1 \leqq (1.\,1)\} , \end{aligned}$$

where Ex(ξ)=the number of the endpoints of the intervals which belongs to ξ, and θ^0 is the expansion of zero of the transformation f. Thus we get

$$h(f) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathrm{Card}\, \{\theta \in \Theta(\theta^0);\ |\theta| \leqq n-1 \text{ and } \theta_1 \leqq (1.\,1)\} .$$

Therefore we will write $h(\theta^0)$ instead of $h(f)$.

The type of θ_p^0 plays a major role in computing $h(\theta^0)$. We have the following lemma.

LEMMA 4.1. *If θ_p^0 is (p, q)-type, then θ_{p+1}^0 is either of $(p+1, q+1)$-type or of $(p+1, 0)$-type. Moreover if θ_{p+1}^0 is of $(p+1, 0)$-type and $q>0$, then θ_{q+1}^0 is of $(q+1, 0)$-type.*

PROOF. Suppose that θ_{p+1}^0 is neither of $(p+1, q+1)$-type nor of $(p+1, 0)$-type. Then there exists $1 \leqq r \leqq q$ such that θ_{p+1}^0 is $(p+1, r)$-type. Thus we have

a) $\sigma^{p-q}\theta_{p+1}^0 > \theta_{q+1}^0$

b) $\theta_{p+1-r}^0 \theta_r^0 = \theta_{p+1}^0$.

Noticing the fact that both θ_{p-q}^0 and θ_{p+1-r}^0 are odd, we get $\sigma^{q+1-r}\theta_{q+1}^0 < \theta_r^0$. This contradicts the admissibility of θ^0.

Suppose that θ_{p+1}^0 is of $(p+1, 0)$-type and $q>0$. Moreover, suppose that θ_q^0 is of (q, r)-type and that θ_{q+1}^0 is of $(q+1, r+1)$-type. Then we get by admissibility

c) $\theta_{q+1}^0 < \sigma^{p-q}\theta_{p+1}^0$

d) $\sigma^{q-r}\theta_{q+1}^0 = \theta_{r+1}^0$.

In a similar way, we can show

$$\theta_{r+1}^0 > \sigma^{p-r}\theta_{p+1}^0 .$$

This contradicts the admissibility of θ^0.

We now define a sequence of integer pairs $p = \{(s_i, q_i)\}_{i=1}^N$ as follows

i) $(s_1, q_1) = (0, 0)$

ii) $s_i = \min\{j > s_{i-1};\ \theta_j^0 \text{ is } (j, 0)\text{-type}\}$

iii) $q_i = \max\{j;\ \theta_{s_i+j}^0 \text{ is } (s_i+j, j)\text{-type}\}$.

If $q_i = \infty$, we define $N = i$.

Let

$$P(n) = \begin{cases} \text{Card}\,\{\theta \in \Theta(\theta^0);\ |\theta| = n\ \ \theta_1 \leqq (1, 1)\}, & \text{if } n \geqq 2 \\ 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$Q(p, n) = \begin{cases} \text{Card}\,\{\theta \in \Theta(\theta^0);\ |\theta| = n \text{ and } (1.1) \geqq \theta_1 \geqq \theta_1^0\}, & \text{if } p = 0, \\ \text{Card}\,\{\theta \in \Theta(\theta^0);\ |\theta| = n,\ \theta_p = \theta_p^0 \text{ and } \theta_{p+1} > \theta_{p+1}^0\}, & \text{if } p \geqq 1. \end{cases}$$

Then we get

$$\begin{aligned} P(n) &= \sum_p Q(p, n) \\ &= \sum_i Q(s_i + q_i, n) + \chi(n), \end{aligned}$$

where

$$\chi(n) = \begin{cases} 1, & \text{if } n = \theta_r^0 \text{ for some } r \text{ and if } r \text{ satisfies } s_i \leqq r < s_i + q_i \text{ for some } i \\ 0, & \text{otherwise}. \end{cases}$$

To compute $Q(p, n)$ we need some more notations.

LEMMA 4.2. *If there exists some i which satisfies $s_i < q_i$, then q_i equals ∞. Moreover $\theta_{s_i n}^0$ equals $(\theta_{s_i}^0)^n$ for any n.*

PROOF. Suppose that $s_i < q_i$ and that there exists $n \geqq 3$ such that

$$\begin{aligned} &\theta_{s_i n}^0 \neq (\theta_{s_i}^0)^n \\ &\theta_{s_i(m-1)}^0 = (\theta_{s_i}^0)^{m-1}, \quad \text{for } m \leqq n . \end{aligned}$$

Thus we get

$$\sigma^{s_i}\theta^0_{s_i n} > \theta^0_{s_i(n-1)} .$$

Since $\theta^0_{s_i}$ is odd, we also get

$$\sigma^{2s_i}\theta^0_{s_i n} < \sigma^{s_i}\theta^0_{s_i(n-1)} = \theta^0_{s_i(n-2)} .$$

This contradicts the assumption on θ^0.

Let

$$T(p, n) = \text{Card}\,\{(\alpha, \beta); \alpha+\beta=n \text{ and } (1, 1) \geqq (\alpha, \beta) > (k^0_{p+1}, l^0_{p+1})\}$$

$$\delta_{j,p} = \begin{cases} 1 & \text{if } k^0_{p+1}+l^0_{p+1}=j , \\ 0 & \text{otherwise} , \end{cases}$$

and

$$f(p, n) = \begin{cases} \sum_j T(p, j)P(n-|\theta^0_p|-j) , & \text{if } \theta^0_p \text{ is even} \\ P(n-|\theta^0_p|) - \sum_j T(p, j)P(n-|\theta^0_p|-j) - P(n-|\theta^0_{p+1}|) , & \text{if } \theta^0_p \text{ is odd} . \end{cases}$$

We can easily compute $T(p, n)$.

i) When $n \leqq k^0_{p+1}$, we have $T(p, n) = n-1$.

ii) When $n > k^0_{p+1}$, we have the following four cases.

1) When l^0_{p+1} is odd,

a) if $n < k^0_{p+1}+l^0_{p+1}$ and $n-k^0_{p+1}$ is odd, then we get $T(p, n) = k^0_{p+1}$,

b) otherwise, we get $T(p, n) = k^0_{p+1}-1$.

2) When l^0_{p+1} is even,

a) if $n < k^0_{p+1}+l^0_{p+1}$ and $n-k^0_{p+1}$ is even, then we get $T(p, n) = k^0_{p+1}-1$,

b) otherwise, we get $T(p, n) = k^0_{p+1}$.

We now consider $Q(s_i+q_i, n)$ $(0 \leqq i \leqq N-1)$. We have the following four cases.

i) If $q_i=0$ and $\theta^0_{s_i}$ is even, then we have

$$\begin{aligned} Q(s_i+q_i, n) &= \text{Card}\,\{\theta \in \Theta(\theta^0); |\theta|=n, \theta_{s_i}=\theta^0_{s_i} \text{ and } \theta_{s_i+1} > \theta^0_{s_i+1}\} \\ &= \sum_j T(s_i, j)P(n-|\theta^0_{s_i}|-j) \\ &= f(s_i+q_i, n) . \end{aligned}$$

ii) If $q_i=0$ and $\theta^0_{s_i}$ is odd, then we have

$$Q(s_i+q_i, n) = P(n-|\theta^0_{s_i}|) - \sum_j \text{Card}\,\{(\alpha, \beta); \alpha+\beta=j \text{ and } (\alpha, \beta) \geqq (k^0_{s_i+1}, l^0_{s_i+1})\}P(n-|\theta^0_{s_i}|-j)$$

$$=P(n-|\theta^0_{s_i}|)-\sum_j (T(s_i, j)+\delta_{j,s_i})P(n-|\theta^0_{s_i}|-j)$$
$$=f(s_i+q_i, n) .$$

iii) If $q_i>0$ and $\theta^0_{s_i+q_i}$ is even, then we have

$$Q(s_i+q_i, n)=\sum_j \text{Card}\,\{(\alpha, \beta); \alpha+\beta=j \text{ and } (k^0_{s_i+q_i+1}, l^0_{s_i+q_i+1})$$
$$<(\alpha, \beta)<(k^0_{q_i+1}, l^0_{q_i+1})\}P(n-|\theta^0_{s_i+q_i}|-j)+\sum_{p\geqq q_i+1} Q(p, n-|\theta^0_{s_i}|)$$
$$=\sum_j (T(s_i+q_i, j)-T(q_i, j)-\delta_{j,q_i})P(n-|\theta^0_{s_i+q_i}|-j)+P(n-|\theta^0_{s_i}|)$$
$$-\sum_{p\leqq q_i} Q(p, n-|\theta^0_{s_i}|)$$
$$=f(s_i+q_i, n)-\sum_j T(q_i, j)P(n-|\theta^0_{s_i+q_i}|-j)-P(n-|\theta^0_{s_i+q_i+1}|)$$
$$+P(n-|\theta^0_{s_i}|)-\sum_{p\leqq q_i} Q(p, n-|\theta^0_{s_i}|) .$$

iv) If $q_i>0$ and $\theta^0_{s_i+q_i}$ is odd, then we have

$$Q(s_i+q_i, n)=\sum_j \text{Card}\,\{(\alpha, \beta); \alpha+\beta=j \text{ and } (k^0_{s_i+q_i+1}, l^0_{s_i+q_i+1})$$
$$>(\alpha, \beta)>(k^0_{q_i+1}, l^0_{q_i+1})\}P(n-|\theta^0_{s_i+q_i}|-j)+\sum_{p\geqq q_i+1} Q(p, n-|\theta^0_{s_i}|)$$
$$=\sum_j (T(q_i, j)-T(s_i+q_i, j)-\delta_{j,s_i+q_i})P(n-|\theta^0_{s_i+q_i}|-j)+P(n-|\theta^0_{s_i}|)$$
$$-\sum_{p\geqq q_i} Q(p, n-|\theta^0_{s_i}|)$$
$$=f(s_i+q_i, n)-P(n-|\theta^0_{s_i+q_i}|)+\sum_j T(q_i, j)P(n-|\theta^0_{s_i+q_i}|-j)$$
$$+P(n-|\theta^0_{s_i}|)-\sum_{p\leqq q_i} Q(p, n-|\theta^0_{s_i}|) .$$

LEMMA 4.3. *For $k<N$, we have*

$$\sum_{i=0}^{k} Q(s_i+q_i, n)=\sum_{p=0}^{s_k+q_k} f(p, n) .$$

PROOF. We will prove by induction.

i) When k equals zero, both s_i an q_i also equal zero. And we already know that $Q(0, n)=f(0, n)$.

ii) We now assume the equation holds for $k\leqq m-1$. Then there are three cases.

1) If q_m equals zero, then $Q(s_m+q_m, n)$ equals $f(s_m+q_m, n)$.

2) If $q_m>0$ and $\theta^0_{s_m+q_m}$ is even, then we have

$$Q(s_m+j, n)=0 \quad \text{for} \quad 0\leqq j\leqq q_n-1$$

and

$$Q(s_m+q_m, n)=f(s_m+q_m, n)-\sum (T(q_m, j)+\delta_{j,q_m})P(n-|\theta^0_{s_m+q_m}|-j)$$
$$+P(n-|\theta^0_{s_m}|)-\sum_{p\leqq q_m} Q(p, n-|\theta^0_{s_m}|) .$$

On the other hand, by Lemma 4.1 we get

$$\sum_{p\leqq q_m} Q(p, n-|\theta^0_{s_m}|)-\sum_{p\leqq q_m} f(p, n-|\theta^0_{s_m}|) .$$

Therefore by assumption of induction, we get

$$\sum_{i=0}^{m} Q(s_m+q_m, n)=\sum_{p=0}^{s_{m-1}+q_{m-1}} f(p, n)+f(s_m+q_m, n)-\sum_j T(q_m, j)P(n-|\theta^0_{s_m+q_m}|-j)$$
$$-P(n-|\theta^0_{s_m+q_m+1}|)+P(n-|\theta^0_{s_m}|)-\sum_{p\leqq q_m} f(p, n-|\theta^0_{s_m}|)$$
$$=\sum_{p=0}^{s_{m-1}+q_{m-1}} f(p, n)+f(s_m+q_m, n)-\sum_j T(q_m, j)P(n-|\theta^0_{s_m+q_m}|-j)$$
$$-P(n-|\theta^0_{s_m+q_m+1}|)+P(n-|\theta^0_{s_m}|)$$
$$-\sum_{\substack{p\leqq q_m\\ \theta^0_p \text{ is even}}} \sum_j T(p, j)P(n-|\theta^0_{s_m}|-|\theta^0_p|-j)-\sum_{\substack{p\leqq q_m\\ \theta^0_p \text{ is odd}}} [P(n-|\theta^0_{s_m}|-|\theta^0_p|)$$
$$-\sum_j T(p, j)P(n-|\theta^0_{s_m}|-|\theta^0_p|-j)-P(n-|\theta^0_{s_m}|-|\theta^0_{p+1}|)] .$$

Notice that $\theta^0_{q_m}$ is odd, $T(p, j)=T(S_m+p, j)$ for $1\leqq p\leqq q_m-1$ and that $|\theta^0_{s_m}|+|\theta^0_p|=|\theta^0_{s_m+p}|$ for $0\leqq p\leqq q_m$. Thus the above equation can be written as

$$=\sum_{p=0}^{s_{m-1}+q_{m-1}} f(p, n)+f(s_m+q_m, n)-\sum_j T(q_m, j)P(n-|\theta^0_{s_m+q_m}|-j)$$
$$-P(n-|\theta^0_{s_m+q_m+1}|)+P(n-|\theta^0_{s_m}|)+\sum_{p=s_m}^{s_m+q_m-1} f(p, n)$$
$$+\sum_{\substack{p=s_m\\ \theta^0_p \text{ is odd}}}^{s_m+q_m-1} [-P(n-|\theta^0_p|)+P(n-|\theta^0_{p+1}|)]-\sum_{\substack{p=s_m\\ \theta^0_p \text{ is even}}}^{s_m+q_m-1} [P(n-|\theta^0_p|)-P(n-|\theta^0_{p+1}|)]$$
$$-P(n-|\theta^0_{s_m+q_m}|)+\sum_j T(q_m, j)P(n-|\theta^0_{s_m+q_m}|-j)+P(n-|\theta^0_{s_m+q_m+1}|)$$
$$=\sum_{p=0}^{s_m+q_m} f(p, n) .$$

3) If $q_m>0$ and $\theta^0_{s_m+q_m}$ is odd, then we can show the lemma in a same way.

COROLLARY 4.4. *For any n, we get*

$$P(n)=\sum_{p=0}^{s_{N-1}+q_{N-1}} f(p,\, n)+\chi(n)\ .$$

COROLLARY 4.5. *Suppose that θ^0 is periodic with period θ^0_m. Then we get*

$$P(n)=\begin{cases}\sum_{p=0}^{m-1} f(p,\, n)+P(n-|\theta^0_m|) & \text{if } \theta^0_m \text{ is even}\ ,\\ \sum_{p=0}^{m-1} f(p,\, n)+\chi(n) & \text{if } \theta^0_m \text{ is odd}\ .\end{cases}$$

Let

$$g(x,\, p)=\begin{cases}\dfrac{x^{-|\theta^0_{p+1}|}}{x-1}\left[\sum_{j=0}^{k^0_{p+1}-1} x^{j+l^0_{p+1}}+\sum_{j=0}^{l^0_{p+1}} (-1)^{l^0_{p+1}-j}x^j\right] \\ \qquad\qquad \text{if both } \theta^0_p \text{ and } \theta^0_{p+1} \text{ are even;}\\ \dfrac{x^{-|\theta^0_{p+1}|}}{x-1}\left[\sum_{j=0}^{k^0_{p+1}-1} x^{j+l^0_{p+1}}+\sum_{j=2}^{l^0_{p+1}-1} (-1)^{l^0_{p+1}-j}x^j\right], \\ \qquad\qquad \text{if } \theta^0_p \text{ is even and if } \theta^0_{p+1} \text{ is odd;}\\ \dfrac{x^{-|\theta^0_{p+1}|}}{x-1}\left[x^{k^0_{p+1}+l^0_{p+1}+1}-\sum_{j=1}^{k^0_{p+1}} x^{j+l^0_{p+1}}-\sum_{j=2}^{l^0_{p+1}} (-1)^{l^0_{p+1}-j}x^j\right], \\ \qquad\qquad \text{if } \theta^0_p \text{ is odd and if } \theta^0_{p+1} \text{ is even;}\\ \dfrac{x^{-|\theta^0_{p+1}|}}{x-1}\left[x^{k^0_{p+1}+l^0_{p+1}}-\sum_{j=1}^{k^0_{p+1}} x^{j+l^0_{p+1}}-\sum_{j=0}^{l^0_{p+1}} (-1)^{l^0_{p+1}-j}x^j\right], \\ \qquad\qquad \text{if both } \theta^0_p \text{ and } \theta^0_{p+1} \text{ are odd.}\end{cases}$$

This $g(x,\, p)$ is the limit of $f(p,\, n)/x^n$ if we put x^k instead of $P(k)$ for any k.

Now we first consider the periodic cases. Let θ^0_m be its period. Let

$$F(x,\, \theta^0)=\begin{cases}x^{|\theta^0_m|-1}-x^{|\theta^0_m|-2}-(x-1)x^{|\theta^0_m|-2}\left(\sum_{k=0}^{m-1} g(x,\, p)+x^{-|\theta^0_m|}\right), \\ \qquad\qquad \text{if } \theta^0_m \text{ is even;}\\ x^{|\theta^0_m|-1}-x^{|\theta^0_m|-2}-(x-1)x^{|\theta^0_m|-2}\sum_{k=0}^{m-1} g(x,\, p)\ , \\ \qquad\qquad \text{if } \theta^0_m \text{ is odd}\ ,\end{cases}$$

and

$$\alpha_i=\begin{cases}1 & \text{if there exists } j \text{ such that } \sum_{n=0}^{j} (k^0_n+l^0_n)\leqq i<\sum_{n=0}^{j} (k^0_n+l^0_n)+k_{j+1}\ ,\\ -1 & \text{otherwise}\ ,\end{cases}$$

where $k_0^0=l_0^0=0$. Then we get

$$F(x,\theta^0)=x^{|\theta_m^0|-1}-x^{|\theta_m^0|-2}-\sum_{k=0}^{|\theta_m^0|-3}\prod_{i=0}^{|\theta_m^0|-k-3}\alpha_i x^k .$$

THEOREM 4.6. *For $[\theta]\in[\Theta]$, we have $h([\theta])=\log\gamma([\theta])$, where $\gamma([\theta])$ is the maximal real solution of $F(x,[\theta])=0$.*

PROOF. Since $P(n)$ is defined by the recurrence formula of Corollary 5.4, we know that

$$P(n)=\sum_i a(\gamma_i([\theta]))(\gamma_i([\theta]))^n ,$$

where $\gamma_i([\theta])$ is a solution of $F(x,[\theta])=0$, and $\sum_i$ is a sum over all the solutions $F(x,[\theta])=0$, and $a(x)$ is a polynomial of degree with multiplicity of $\gamma_i([\theta])-1$. Noticing the fact that $\sum_{k=1}^{n-1}P(k)$ is monotone increasing, we get

$$h([\theta])=\log\max\{\gamma_i([\theta]);\ \gamma_i([\theta]) \text{ is real and } a(\gamma_i([\theta]))\neq 0\} .$$

Thus we only need to show that $a(\gamma([\theta]))\neq 0$.

i) We get $F(x,[(1.1)])=x-1$. Thus the assertion holds for this case.

ii) We now consider $[\theta]\in[\Theta]$ such that $|[\tilde\theta]|>2$.
But we need not consider the cases for which we have

$$[\tilde\theta]=(k_1,l_1)\cdots(k_{m-1},l_{m-1})(k_m,l) .$$

Because, if $k_m>1$, then we have

$$F(x,[(k_1,l_1)\cdots(k_{m-1},l_{m-1})(k_m,l)])=F(x,[(k_1,l_1)\cdots(k_{m-1},l_{m-1})(k_m-1,2)]) .$$

On the other hand, we have

$$F(x,[(k_1,l_1)\cdots(k_{m-1},l_{m-1})(1,1)])=F(x,[(k_1,l_1)\cdots(k_{m-1},l_{m-1}+2)]) .$$

Thus we can identify $(k_1,l_1)\cdots(k_m,1)$ with $(k_1,l_1)\cdots(k_m-1,2)$, if $k_m>1$ and we can also identify $(k_1,l_1)\cdots(k_{m-1},l_{m-1})$ $(1,1)$ with $(k_1,l_1)\cdots(k_{m-1},l_{m-1}+2)$. Then if θ even, we have

$$F(x,\theta)=xF(x,\theta')-1 .$$

Therefore if we assume that both $F(x,\theta')$ and $F'(x,\theta')$ are monotone increasing for $x>\gamma(\theta')$, then both $F(x, \theta)$ and $F'(x, \theta)$ are monotone increasing for $x>\gamma(\theta')$. On the other hand if θ is odd, we have

$$F(x, \theta)=xF(x, \theta')+1=x(F(x, \theta')+x^{-1}) .$$

Therefore if we assume that both $F(x, \theta')$ and $F'(x, \theta')$ are monotone increasing for $x>\gamma(\theta'')(\theta''=(\theta')')$, then we have the equation $F(x, \theta)=0$ has a unique solution for $x>\gamma(\theta'')$. Since

$$F'(x, \theta)=(F(x, \theta')+x^{-1})+(F'(x, \theta')-x^{-2})x ,$$

we find that both $F(x, \theta)$ and $F'(x, \theta)$ are monotone increasing for $x>\gamma(\theta)$. This completes the proof.

COROLLARY 4.7. *For any n, we have*

$$h(\{2^n\})=0 .$$

PROOF. We have

$$F(x, \{2^n\})=(x-1)(x^2-1)\cdots(x^{2^{n-1}}-1) .$$

The corollary is trivial.

COROLLARY 4.8. *For $[\theta]\in[\bar{\Theta}]\backslash[\Theta]$, we have*

$$h([\theta])=\lim_{n\to\infty}\log\gamma([[\theta]_n]) .$$

PROOF. Let $\{n_k\}_{k\geq 1}$ be a sequence for which $[\theta]_{n_k}$ is $(n_k, 0)$-type. Then $\{n_k\}_{k\geq 1}$ is an infinite sequence, otherwise it contradicts the assumption $[\theta]\in[\bar{\Theta}]\backslash[\Theta]$.

i) If $[\theta]_{n_k}$ is even, then $[[\theta]_{n_k}]\ll[\theta]\ll[[\theta]_{n_k}(1.1)]$

ii) If $[\theta]_{n_k}$ is odd, we can also show that $[[\theta]_{n_k}]\gg[\theta]\gg[[\theta]_{n_k}(1.1)]$.

Therefore the assertion easily follows.

§5. Concluding remarks.

We call f is of window type if its expansion of zero $\theta^0\in[\bar{\Theta}]$ satisfies

i) $[\theta^0]\in[\Theta]$ and $[\bar{\theta}^0]$ is odd, and

ii) $[\bar{\theta}^0]0>0$.

Suppose that f is of window type. Then any $x\in D([\bar{\theta}^0])$ belongs to $D([\bar{\theta}^0]^n)$ for any n.

On the other hand, we call f is of island type if the its expansion of zero $\theta^0\in[\bar{\Theta}]$ satisfies

i) We can represent $\theta^0=\theta^1\theta^2\theta^3\cdots$ such that

$\theta^1=\theta'(k, 1)$, $\quad \theta^2=\theta'(k-1, 2)$ or $\theta^1=\theta'(k, 2)$ $\quad \theta^2=\theta'(k+1, 1)$.

ii) θ^1 is odd, θ^2 is even and

$$\theta^i=\theta^1 \text{ or } \theta^2 \text{ for any } i=3, 4, \cdots .$$

iii) Suppose that $D[\theta^1]=[0, a]$. Then $\theta^1 a \in D[\theta^1]$ and $(\theta^1)^{-1}a>\theta^1 a$. Suppose that f is of island type. Then any $x \in D([\bar{\theta}^0])$ belongs to $D[\theta]$ such that

i) $\theta=\theta'^1\theta'^2\theta'^3\cdots$ such that

$$\theta'^1=\theta^1 \text{ and } \theta'^i=\theta^1 \text{ or } \theta^2 \text{ for any } i .$$

ii) $\max_j\{i;\ \theta'^{j+k}=\theta^2 \text{ for } 1\leqq i\leqq k\}\leqq\max_j\{i;\ \theta^{j+k}=\theta^2 \text{ for } 1\leqq i\leqq k\}$. Those are the slight extensions of [1].

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