THE EXTENSION PROPERTY OF COMPLEX BANACH SPACES

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The structure of real normed linear spaces with the extension property was clarified by Nachbin [10], Goodner [6] and Kelley [8]. Such a space is known to be equivalent to the Banach space of all real-valued continuous functions on a suitable stonean space with the topology of uniform norm. In connection with this, it has been conjectured that an analogous theorem will hold for complex normed linear spaces (cf. Grothendieck [7]). The object of the present note is to give an affirmative solution to this problem by utilizing the device of Kelley [8]. In our discussion, a theorem on continuous selections, proved in § 1, enables us to apply well the device of Kelley to our case.

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1. A continuous selection theorem. Let X, Y be any topological spaces and ψ a mapping which assigns to each $x \in X$ a non-void subset $\psi(x)$ of Y. ψ is called upper (lower) semi-continuous if $\{x \in X : \psi(x) \subset U\}$ ($\{x \in X: \psi(x) \cap U \neq \phi\}$) is open in X for any open set U in Y. We denote by $\mathfrak{F}(Y)$ the totality of non-void closed subsets of Y.

THEOREM 1. *Let X be a stonean space, Y a compact space and ψ a mapping of X into $\mathfrak{F}(Y)$. If ψ is upper semi-continuous, then there exists a continuous mapping f of X into Y such that $f(x) \in \psi(x)$ for every $x \in X$.*

Before proceeding to the proof of the theorem, we state several lemmas.

LEMMA 1. *Let X be a topological space, Y a compact space and ψ_1, ψ_2 two upper semi-continuous mappings of X into $\mathfrak{F}(Y)$. If W is a closed entourage of Y such that $\theta(x) = \psi_1(x) \cap W(\psi_2(x))$ is non-void for every $x \in X$, then θ is an upper semi-continuous mapping of X into $\mathfrak{F}(Y)$.*

PROOF. Since the closedness of $\theta(x)$ follows from the compactness of $\psi_1(x)$, $\psi_2(x)$ and W, it suffices to show that $\{x \in X : \theta(x) \cap F \neq \phi\}$ is closed for any closed set F in Y. As $X(x) = \psi_1(x) \times \psi_2(x)$ is an upper semi-continuous mapping of X into $\mathfrak{F}(Y \times Y)$, the lemma follows from the equality

$$\{x \in X : \theta(x) \cap F \neq \phi\} = \{x \in X : X(x) \cap \{W \cap (F \times Y\} \neq \phi\}.$$

LEMMA 2. *Let X be a topological space, Y a compact space and $\{\psi_\lambda\}$ a family of upper semi-continuous mappings of X into $\mathfrak{F}(Y)$. If $\{\psi_\lambda\}$ is decreasingly directed in the sense that, for any ψ_λ, $\psi_{\lambda'}$, there exists a $\psi_{\lambda''}$ satisfying $\psi_\lambda(x) \supset \psi_{\lambda''}(x)$ and $\psi_{\lambda'}(x) \supset \psi_{\lambda''}(x)$ simultaneously, then $\psi(x) = \bigcap_\lambda \psi_\lambda(x)$ is an upper semi-continuous mapping of X into $\mathfrak{F}(Y)$.*

PROOF. $\psi(x)$ is clearly non-void and closed. For any open set U in Y, we have

$$\{x \in X : \psi(x) \subset U\} = \bigcup_\lambda \{x \in X : \psi_\lambda(x) \subset U\}.$$

Since ψ_λ is upper semi-continuous, $\{x \in X : \psi_\lambda(x) \subset U\}$ is open for each λ and consequently $\{x \in X : \psi(x) \subset U\}$ is open. Hence ψ is upper semi-continuous.

LEMMA 3. *Let X be a stonean space, Y a uniform space and ψ a lower semi-continuous mapping of X into a family of non-void subsets of Y. Then, for any open symmetric entourage W of Y, there exists a continuous mapping f of X into Y such that $f(x) \in W(\psi(x))$ for every $x \in X$.*

PROOF. For any $y \in Y$, define

$$G_y = \{x \in X : y \in W(\psi(x))\}.$$

Since $G_y = \{x \in X : \psi(x) \cap W(y) \neq \phi\}$ and ψ is lower semi-continuous, G_y is open for any y. The totality of non-void G_y forms an open covering of X. Since X is stonean, there exists a refinement $\{G'_i : i = 1, 2, \dots, n\}$ of this covering which is a partition of X into a finite number of open-closed sets. Take, for any i $(1 \leqq i \leqq n)$, a set G_{y_i} satisfying $G'_i \subset G_{y_i}$ and define a mapping f of X into Y by setting

$$f(x) = y_i \qquad \text{for } x \in G'_i, \ i = 1, 2, \dots, n.$$

Then f satisfies the required conditions in the lemma. q. e. d.

Let, once for all, X be a stonean space and Y a compact space. Let ψ be an upper semi-continuous mapping of X into $\mathfrak{F}(Y)$ and set

$$M_U = \{x \in X : \psi(x) \subset U\}$$

for any open set U in Y. Since ψ is upper semi-continuous and X is stonean, both M_U and $\overline{M_U}$ are open in X. Now define

$$\tilde{\psi}(x) = \bigcap (\overline{U} : x \in \overline{M_U}),$$

where the intersection is taken over all open subsets U of Y satisfying $x \in \overline{M_U}$. Then, as we shall see from the following lemma, $\tilde{\psi}(x)$ is non-void for any $x \in X$ and we obtain a mapping of X into $\mathfrak{F}(Y)$. We shall call $\tilde{\psi}$ the regularization of ψ.

LEMMA 4.

$$\bigcap_{i=1}^{n} \overline{M_{U_i}} = \overline{M_V}$$

where $\{U_i\}$ is any finite number of open sets in Y and $V = \bigcap_{i=1}^{n} U_i$.

PROOF. We denote by N the first member of the equality. Since $\overline{M_V} \subset N$ is clear, we may suppose that N is non-void. To prove $N \subset \overline{M_V}$, it is sufficient to show that, for any $x \in N$ and any open neighborhood G of x, $G \cap M_V \neq \phi$. Since N is open, we may assume $G \subset N$. As $G \subset \overline{M_{U_1}}$ and M_{U_1} is open, $G_1 = G \cap M_{U_1}$ is a non-void open set. Since $G_1 \subset \overline{M_{U_2}}$, $G_2 = G_1 \cap M_{U_2}$ is also non-void and open. Repeating the same argument a finite number of times, we find that $G_n = G \cap \left(\bigcap_{i=1}^{n} M_{U_i}\right)$ is non-void. Since we can easily verify that $M_V = \bigcap_{i=1}^{n} M_{U_i}$, G contains a point of M_V.

LEMMA 5. *The regularization $\tilde{\psi}$ of an upper semi-continuous mapping ψ of X into $\mathfrak{F}(Y)$ is semi-continuous in both senses and satisfies $\psi(x) \supset \tilde{\psi}(x)$ for every $x \in X$.*

PROOF. Let J be any open set in Y such that $G = \{x \in X : \tilde{\psi}(x) \subset J\}$ is non-void. If x_0 is any point in G, then $\tilde{\psi}(x_0) \subset J$. Since $\tilde{\psi}(x_0)$ is the intersection of compact sets $\bar{U}$ such that $x_0 \in \overline{M_U}$, there exist a finite number of open sets $U_i (i = 1, 2, \ldots, n)$ in Y such that $x_0 \in \overline{M_{U_i}}$ $(i = 1, 2, \ldots\ldots, n)$ and $\tilde{\psi}(x_0) \supset \bigcap_{i=1}^{n} \bar{U}_i \subset J$. By Lemma 4, $x_0 \in \bigcap_{i=1}^{n} \overline{M_{U_i}} = \overline{M_V}$ where $V = \bigcap_{i=1}^{n} U_i$. If $x \in \overline{M_V}$, then $\tilde{\psi}(x) \subset \bar{V} \subset \bigcap_{i=1}^{n} \bar{U}_i \subset J$. Thus $x_0 \in \overline{M_V} \subset G$. Since $\overline{M_V}$ is open and x_0 is arbitrary in G, G is open. Hence $\tilde{\psi}$ is upper semi-continuous.

To show that $\tilde{\psi}$ is lower semi-continuous, it suffices to verify that $A = \{x \in X : \tilde{\psi}(x) \subset F\}$ is closed for any closed set F in Y. Suppose that $x_\lambda \in A$ and $x_\lambda \to x$. Let W be any open entourage of Y. Since $W(F)$ is open and $\tilde{\psi}(x_\lambda) \subset F \subset W(F)$, there exist, for each λ, a finite number of open sets $U_{\lambda,i}$ $(i = 1, 2, \ldots, n_\lambda)$ in Y such that $x_\lambda \in \overline{M_{U_{\lambda;i}}} (i = 1, 2, \ldots\ldots, n_\lambda)$ and $\tilde{\psi}(x_\lambda) \subset \bigcap_{i=1}^{n_\lambda} \bar{U}_{\lambda;i} \subset W(F)$. It follows that

$$x_\lambda \in \bigcap_{i=1}^{n_\lambda} \overline{M_{U_{\lambda;i}}} = \overline{M_{V_\lambda}} \subset \overline{M_{W(F)}} \qquad \text{for each } \lambda,$$

where $V_\lambda = \bigcap_{i=1}^{n_\lambda} U_{\lambda;i}$. Since $\overline{M_{W(F)}}$ is closed, $x \in \overline{M_{W(F)}}$. Thus $\tilde{\psi}(x) \subset \overline{W(F)}$. As W is arbitrary, $\tilde{\psi}(x) \subset \bigcap_W \overline{W(F)} = F$. Hence $x \in A$ and A is closed.

The latter part is clear. q.e.d.

PROOF OF THEOREM 1. Let $\mathfrak{U}$ be the set of all upper semi-continuous mappings of X into $\mathfrak{F}(Y)$. If we define an ordering relation in $\mathfrak{U}$ by setting $\psi_1 \geqq \psi_2$ when and only when $\psi_1(x) \supset \psi_2(x)$ for every $x \in X$, then Lemma 2 implies that $\mathfrak{U}$ is inductively ordered with respect to $\geqq$. Now, let ψ be any upper semi-continuous mapping of X into $\mathfrak{F}(Y)$, i.e. any element in $\mathfrak{U}$. Then there exists, by the Zorn lemma, a minimal element $\theta \in \mathfrak{U}$ satisfying $\psi \geqq \theta$. Let $\tilde{\theta}$ be the regularization of θ. Since $\tilde{\theta}$ is upper semi-continuous and $\theta \geqq \tilde{\theta}$, we have $\tilde{\theta} = \theta$ by the minimality of θ. Hence θ is also lower semi-continuous. We assert that $\theta(x)$ consists of a single point for any $x \in X$. Suppose, on the contrary, that $\theta(x_0)$ contains two distinct points y_1 and y_2 for some $x_0 \in X$. Then we can find symmetric entourages W_1, W_2 of Y such that W_1 is closed, W_2 is open, $W_1 \supset W_2$ and $y_1 \notin W_1^2(y_2)$. By Lemma 3, there exists a continuous mapping g of X into Y such that $g(x) \in W_2(\theta(x))$ for any $x \in X$. Then $\theta_1(x) = \theta(x) \cap W_1(g(x))$ is non-void for any $x \in X$ and, by Lemma 1, θ_1 is upper semi-continuous. Since $\theta \geqq \theta_1$, we have $\theta_1 = \theta$ by the minimality of θ. Thus

$\theta_1(x_0)$ must contain y_1 and y_2. Therefore, we have $y_1,\ y_2 \in W_1(g(x_0))$, which implies $y_1 \in W_1^2(y_2)$. This contradiction shows that $\theta(x)$ consists of a single point for every $x \in X$. If f denotes a mapping of X into Y which assings to each $x \in X$ the point contained in $\theta(x)$, then f is clearly continuous and $f(x) \in \theta(x) \subset \psi(x)$ for any $x \in X$. This completes the proof.

2. The extension property. Let K be a compact space and $C(K)$ the Banach space of all complex-valued continuous functions on K with the uniform norm. The dual of $C(K)$ is denoted by $C^*(K)$, whose elements are measures on K. For each $p \in K$, an elements $\varepsilon_p \in C^*(K)$, defined by $\varepsilon_p(f) = f(p)$ for $f \in C(K)$, is called an evaluation at p.

LEMMA 6. *Each measure μ on a compact space K is weakly* adherent to the set Γ of linear combinations $\sum \alpha_j \varepsilon_{p_j}$ of evaluations ε_{p_j} where $\{p_j\}$ varies over all finite subsets of the carrier of μ and $\{\alpha_j\}$ varies over all finite systems of complex numbers such that $\sum |\alpha_j| \leqq \|\mu\|$* (cf. Bourbaki [4], p. 75).

LEMMA 7. *A measure μ on a compact space K is an extreme point (in the sense of the real vector space theory) of the unit sphere Σ^* of $C^*(K)$ if and only if there exists a point $p \in K$ and a complex number α with $|\alpha| = 1$ such that $\mu = \alpha\varepsilon_p$.*

PROOF. Suppose μ is an extreme point of Σ^* and the carrier S of μ contains more than one point. Let S_1 be a proper closed subset of S such that $\overline{S - S_1} \neq S$ and let Γ be the same as in the preceding lemma. Then any $\nu \in \Gamma$ can be written uniquely in the form $\nu = \varphi_1(\nu) + \varphi_2(\nu)$ where the carriers of $\varphi_1(\nu)$ and $\varphi_2(\nu)$ are contained in S_1 and in $S - S_1$, respectively. φ_1 and φ_2 are mappings of Γ into itself. By Lemma 6, there is a filter-base $\mathfrak{F}$ on Γ which is convergent weakly* to μ. Then $\mathfrak{F}_1 = \varphi_1(\mathfrak{F})$ is a filter-base on Γ. By the weak* compactness of Σ^*, there is a filter-base $\mathfrak{F}_1'$ on Γ which is finer than $\mathfrak{F}_1$ and convergent weakly* to a $\mu_1 \in \Sigma^*$. Let $\mathfrak{F}_2$ be the family of sets of the form $\varphi_2(M \cap \varphi_1^{-1}(M_1))$ where $M \in \mathfrak{F}$ and $M_1 \in \mathfrak{F}_1'$. Then $\mathfrak{F}_2$ is also a filter-base on Γ and there is a filter-base $\mathfrak{F}_2'$ on Γ which is finer than $\mathfrak{F}_2$ and convergent weakly* to a $\mu_2 \in \Sigma^*$. It is easy to see that $\mu = \mu_1 + \mu_2$ and $\|\mu\| = \|\mu_1\| + \|\mu_2\| = 1$. Since the carriers of μ_1 and μ_2 are included in S_1 and in $\overline{S - S_1}$, respectively, we have $\mu_1 \neq 0$, $\mu_2 \neq 0$ and $\mu_1 \neq \mu_2$. Putting $\|\mu_1\| = \alpha_1$ and $\|\mu_2\| = \alpha_2$, we get $\mu = \mu_1 + \mu_2 = \alpha_1(\alpha_1^{-1} \cdot \mu_1) + \alpha_2(\alpha_2^{-1} \cdot \mu_2)$, which is clearly a contradiction. Thus S consists of a single point $p \in K$ and we have $\mu = \alpha\varepsilon_p$ where $|\alpha| = 1$.

To show that ε_p is an extreme point of Σ^*, suppose that $\varepsilon_p = \alpha\mu_1 + (1-\alpha)\mu_2$ where $\mu_1, \mu_2 \in \Sigma^*$ and $0 < \alpha < 1$. Of course, $\varepsilon_p(f) = f(p) = \alpha\mu_1(f) + (1-\alpha)\mu_2(f)$ for any $f \in C_r(K)$ where $C_r(K)$ denotes the Banach space of all real-valued continuous functions on K. We may write $\mu_j = \mu_j^{(1)} + i\mu_j^{(2)}$ for $j = 1, 2$, where $\mu_j^{(k)}$ are real measures on K. Then a theorem of Arens-Kelley [1] implies that $\mu_1^{(1)}(f) = \mu_2^{(1)}(f) = f(p)$ and $\mu_1^{(2)}(f) = \mu_2^{(2)}(f) = 0$ for any $f \in C_r(K)$. Thus

$\mu_1(f) = \mu_2(f) = \varepsilon_p(f)$ for any $f \in C_r(K)$. Hence, by linearity, $\mu_1(f) = \mu_2(f) = \varepsilon_p(f)$ for any $f \in C(K)$. This shows that ε_p is an extreme point. It is then clear that $\alpha\varepsilon_p$ with $|\alpha| = 1$ are extreme points of Σ^*. q. e. d.

We say that a complex normed linear space B has the extension property if, for any complex normed linear space D and for any linear subspace D_1 of D, every bounded linear mapping φ of D_1 into B has a linear extension Φ of D into B such that $\|\Phi\| = \|\varphi\|$.

THEOREM 2. *A complex normed linear space B has the extension property if and only if B is isomorphic in an algebraic and norm preserving fashion to $C(X)$ where X is a stonean space.*

PROOF. It is easy to show the "if"-part of the theorem. Let D be a complex normed linear space, D_1 any linear subspace of D and φ a bounded linear mapping of D_1 into $C(X)$ where X is a stonean space. If we denote by $C_r(X)$ the totality of real-valued functions in $C(X)$ and define

$$[\varphi_1(a)](x) = \text{real part of } [\varphi(a)](x)$$

for any $a \in D_1$ and any $x \in X$, then φ_1 is a mapping of D_1 into $C_r(X)$ which is linear with respect to the real scalars and satisfies $\|\varphi_1\| \leqq \|\varphi\|$ and $\varphi(a) = \varphi_1(a) - i\varphi_1(ia)$ for any $a \in D_1$. Since $C_r(X)$ has the extension property as a real Banach space by a theorem of Nachbin [10], there exists a real-linear mapping Φ_1 of D into $C_r(X)$ which extends φ_1 and satisfies $\|\Phi_1\| = \|\varphi_1\|$. Put $\Phi(a) = \Phi_1(a) - i\Phi_1(ia)$. Then Φ is a bounded linear mapping of D into $C(X)$ which extends φ. We assert that $\|\Phi\| \leqq \|\Phi_1\|$. By definition,

$$\|\Phi\| = \sup_{\|a\| \leqq 1} \|\Phi(a)\|_{C(X)} = \sup_{\|a\| \leqq 1, x \in X} |[\Phi(a)](x)|$$

and

$$\|\Phi_1\| = \sup_{\|a\| \leqq 1} \|\Phi_1(a)\|_{C_r(X)} = \sup_{\|a\| \leqq 1, x \in X} |[\Phi_1(a)](x)|.$$

For any $\varepsilon > 0$, there exists an $a_0 \in D$ with $\|a_0\| \leqq 1$ and an $x_0 \in X$ such that $\|\Phi\| < |[\Phi(a_0)](x_0)| + \varepsilon$. There exists a real number $\theta = \theta(x_0, a_0)$ such that $[\Phi(e^{i\theta}a_0)](x_0) = e^{i\theta}([\Phi(a_0)](x_0))$ is real and therefore $[\Phi_1(e^{i\theta}a_0)](x_0) = [\Phi(e^{i\theta}a_0)](x_0)$. Thus we have

$$\begin{aligned}\|\Phi\| &< |[\Phi(a_0)](x_0)| + \varepsilon = |[\Phi(e^{i\theta}a_0)](x_0)| + \varepsilon \\ &= |[\Phi_1(e^{i\theta}a_0)](x_0)| + \varepsilon < \|\Phi_1\| + \varepsilon.\end{aligned}$$

As ε is arbitrary, $\|\Phi\| \leqq \|\Phi_1\|$. Consequently, $\|\Phi\| \leqq \|\Phi_1\| = \|\varphi_1\| \leqq \|\varphi\|$. Since $\|\varphi\| \leqq \|\Phi\|$ is clear, we conclude that $\|\Phi\| = \|\varphi\|$. Hence the space $C(X)$ has the extension property.

Now we have to show the "only if" part of the theorem. Suppose B has the extension property. Let E be the set of extreme points of the unit sphere Σ^* of B^*, the dual of B, and Y the weak* closure of E. Y is clearly weakly* compact. If we set $y_1 \equiv y_2$ for $y_1, y_2 \in Y$ when and only when there exists a complex number α with $|\alpha| = 1$ such that $y_1 = \alpha y_2$, then we obtain an equivalence relation in Y which we denote by R_0. We say that an equivalence relation R defined in a topological space M is closed if the saturation of any closed subset of M with respect to R is closed in M. Then the relation R_0

is closed with respect to the relative weak* topology for Y. Let F be any weakly* closed subset of Y. If we denote by h the mapping of $C_0\times Y$ into Y defined by $h(\alpha,y)=\alpha y$ where C_0 is the unit circle($\{\alpha:|\alpha|=1\}$) in the complex plane, then the saturation of F with respect to R_0 is obviously $h(C_0\times F)$. Since C_0 and F are compact and h is continuous, $h(C_0\times F)$ is weakly* compact and, consequently, weakly* closed in Y. Hence R_0 is closed.

Next, we shall prove

LEMMA 8. *The quotient space $X=Y/R_0$ is a stonean space where the topology of X is the quotient of the relative weak* topology for Y.*

We recall that a non-void subset L of a convex set K in any (real or complex) linear space is called a support of K if each line segment contained in K which has an interior point in L is contained in L, and that, if a point is an extreme point of a support of K, it is also an extreme point of K.

PROOF OF LEMMA 8. X being clearly compact, we shall show that $\overline{G}$ is open for any open set G in X. Let h be the natural mapping of Y onto X and put $U=h^{-1}(G)$. Then U is a saturated open set in Y. Since R_0 is a closed equivalence relation, $h(\overline{U})$ is closed and we have $\overline{G}=\overline{h(U)}=h(\overline{U})$. As $\overline{U}$ is saturated with respect to R_0, we have only to prove that $\overline{U}\cap\overline{V}=\phi$ where V is the complement of $\overline{U}$ in Y. For this end, we argue as follows. Set $Z=(\{0\}\times\overline{U})\cup(\{1\}\times\overline{V})$, the topology of which is defined such that a set in Z is open if and only if it is of the form $(\{0\}\times U_1)\cup(\{1\}\times V_1)$ where U_1 and V_1 are relatively open in $\overline{U}$ and in V, respectively. We notice that $\overline{V}$ is also saturated. Now, Z being the union of disjoint compact spaces, $C(Z)$ is the direct sum of $C(\{0\}\times\overline{U})$ and $C(\{1\}\times\overline{V})$. Accordingly, the dual $C^*(Z)$ is the direct sum of $C^*(\{0\}\times\overline{U})$ and $C^*(\{1\}\times\overline{V})$, each of which is weakly* closed in $C^*(Z)$.

Define a mapping φ of B into $C(Z)$ by putting $[\varphi(b)](0,u)=\langle b,u\rangle$ and $[\varphi(b)](1,v)=\langle b,v\rangle$, where $b\in B$, $u\in\overline{U}$ and $v\in V$. It is clear that φ is a linear isometric mapping of B into $C(Z)$. A simple calculation shows that, for any $u\in\overline{U}$, any $v\in\overline{V}$ and any complex number α,

$$\varphi^*(\alpha\varepsilon_{(0,u)})=\alpha u \quad\text{and}\quad \varphi^*(\alpha\varepsilon_{(1,v)})=\alpha v, \tag{1}$$

where φ^* is the adjoint of φ. For any $w\in U\cup V$, we set $K(w)=\varphi^{*-1}(w)\cap\Sigma_1^*$ where Σ_1^* is the unit sphere of $C^*(Z)$. If $u\in U$ is an extreme point of the unit sphere Σ^* of B^*, then $K(u)$ is a support of Σ_1^* which is weakly* compact. By the Krein-Milman theorem (cf. Bourbaki [3], p. 84), $K(u)$ is the closed convex envelope of the extreme points of $K(u)$. Since every extreme point of $K(u)$ is an extreme point of Σ_1^*, it follows from Lemma 7 and the first equality in (1) that the extreme points of $K(u)$ are of the form $\alpha^{-1}\varepsilon_{(0,\alpha u)}$ with $|\alpha|=1$. Thus $K(u)\subset C^*(\{0\}\times\overline{U})$. Similarly, if $v\in V$ is an extreme point of Σ^*, then $K(v)\subset C^*(\{1\}\times\overline{V})$.

Since B has the extension property, there exists a linear mappnig Φ of

$C(Z)$ onto B such that $\|\Phi\| = 1$ and $\Phi\varphi$ is the identity mapping on B. It is obvious that Φ^* carries Σ^* into Σ_1^* and $(\Phi\varphi)^* = \varphi^*\Phi^*$ is the idnetity mapping on B^*. Thus $\varphi^*\Phi^*(w) = w$ implies $\Phi^*(w) \in K(w)$ for any $w \in U \cup V$. Set $U_1 = U \cap E$ and $V_1 = V \cap E$. Then $\Phi^*(u) \in K(u) \subset C^*(\{0\} \times \overline{U})$ for any $u \in U_1$ and $\Phi^*(v) \in K(v) \subset C^*(\{1\} \times \overline{V})$ for any $v \in V_1$. Thus we have $\overline{\Phi^*(U_1)} \cap \overline{\Phi^*(V_1)} = \phi$, where the bars denote the weak* closure in $C^*(Z)$. Since U_1 and V_1 are dense in $\overline{U}$ and in $\overline{V}$, respectively, we have

$$\Phi^*(\overline{U}) \cap \Phi^*(\overline{V}) = \Phi^*(\overline{U_1}) \cap \Phi^*(\overline{V_1}) \subset \overline{\Phi^*(U_1)} \cap \overline{\Phi^*(V_1)} = \emptyset.$$

Hence $\overline{U} \cap \overline{V} = \emptyset$, which proves Lemma 8.

Since $X = Y/R_0$, each element $x \in X$ is regarded as a subset of Y which is denoted by $\psi(x)$. $\psi(x)$ is clearly a closed subset of Y for every $x \in X$. From the weak* closedness of the equivalence relation R_0 follows that $\psi(x)$ is upper semi-continuous. Hence, by Theorem 1, there exists a continuous mapping π of X into Y such that $\pi(x) \in \psi(x)$ for every $x \in X$.

Let φ be a linear mapping of B into $C(X)$ defined by $[\varphi(b)](x) = \langle b, \pi(x)\rangle$ for any $b \in B$ and any $x \in X$. Then φ is clearly isometric. Since B has the extension property, there exists a linear mapping Φ of $C(X)$ onto B such that $\|\Phi\| = 1$ and $\Phi\varphi$ is the identity mapping on B. Let Σ^* and Σ_1^* be the unit spheres of B^* and $C^*(X)$, respectively. If u is an extreme point of Σ^*, then $K(u) = \varphi^{*-1}(u) \cap \Sigma_1^*$ is a support of Σ_1^* which is weakly* compact. If μ is any extreme point of $K(u)$, then μ is an extreme point of Σ_1^* and, by Lemma 7, there exists a point $x \in X$ and a complex number α with $|\alpha| = 1$ such that $\mu = \alpha\varepsilon_x$. Since $\varphi^*(\mu) = u$, we have, for any $b \in B$,

$$\begin{aligned}\langle b, u\rangle &= \langle b, \varphi^*(\mu)\rangle = \langle\varphi(b), \mu\rangle = \langle\varphi(b), \alpha\varepsilon_x\rangle \\ &= \alpha[\varphi(b)](x) = \alpha\langle b, \pi(x)\rangle = \langle b, \alpha\pi(x)\rangle.\end{aligned}$$

Hence $u = \alpha\pi(x)$ and, since $|\alpha| = 1$, $u \equiv \pi(x)$ (mod. R_0). It follows from the hypothesis on π that x and α are determined uniquely by u. Thus $K(u)$ consists of a single point $\alpha\varepsilon_x$ and we have $\Phi^*(u) = \alpha\varepsilon_x$. Putting $\Omega_1 = \{\alpha\varepsilon_x : \pi(x) \in E, |\alpha| = 1\}$, we have shown that Φ^* maps E into Ω_1. Conversely, let $\alpha\varepsilon_x$ be any element in Ω_1. Then $\alpha\pi(x)$ is an element in E and $\Phi^*(\alpha\pi(x)) = \alpha\varepsilon_x$. Hence Φ^* maps E onto Ω_1. Denote by E_1 the set of extreme points of Σ_1^*. Lemma 7 implies that $E_1 = \{\alpha\varepsilon_x : x \in X, |\alpha| = 1\}$. Since E is weakly* dense in Y, Ω_1 is weakly* dense in E_1. Thus we conclude, by the weak* compactness of Y, that $\Phi^*(Y) \supset E_1$. Hence, by the Krein-Milman theorem, $\Phi^*(\Sigma^*) \supset \Sigma_1^*$ and therefore Φ^* maps B^* onto $C^*(X)$. On the other hand, since $\Phi\varphi$ is the identity mapping on B, $(\Phi\varphi)^* = \varphi^*\Phi^*$ is the identity mapping on B^* and, consequently, φ^* must be a one-to-one mapping of $C^*(X)$ onto B^*. It is an easy matter to see that any normed linear space with the extension property is necessarily complete, i.e., a Banach space. Hence $\varphi(B)$ is closed in $C(X)$ and therefore φ maps B onto $C(X)$. Thus B and $C(X)$ are isometrically isomorphic and Theorem 2 is established.

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