The extremal process of branching Brownian motion

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Abstract We prove that the extremal process of branching Brownian motion, in
the limit of large times, converges weakly to a cluster point process. The limiting
process is a (randomly shifted) Poisson cluster process, where the positions of the
clusters is a Poisson process with intensity measure with exponential density. The law
of the individual clusters is characterized as branching Brownian motions conditioned
to perform "unusually large displacements", and its existence is proved. The proof
combines three main ingredients. First, the results of Bramson on the convergence
of solutions of the Kolmogorov–Petrovsky–Piscounov equation with general initial
conditions to standing waves. Second, the integral representations of such waves as
first obtained by Lalley and Sellke in the case of Heaviside initial conditions. Third,
a proper identification of the tail of the extremal process with an auxiliary process

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(based on the work of Chauvin and Rouault), which fully captures the large time asymptotics of the extremal process. The analysis through the auxiliary process is a rigorous formulation of the *cavity method* developed in the study of mean field spin glasses.

Keywords Branching Brownian motion · Extreme value theory · Extremal process · Traveling waves

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1 Introduction

Branching Brownian motion (BBM) is a continuous-time Markov branching process which plays an important role in the theory of partial differential equations [9,10, 32,37], in the theory of disordered systems [13,15,26], and in biology [29]. It is constructed as follows.

Start with a single particle which performs standard Brownian motion $x(t)$ with $x(0) = 0$, which it continues for an exponential holding time T independent of x, with $\mathbb{P}[T > t] = e^{-t}$. At time T, the particle splits independently of x and T into k offspring with probability p_k, where $\sum_{k=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} k p_k = 2$, and $K \equiv \sum_k k(k-1)p_k < \infty$. These particles continue along independent Brownian paths starting at $x(T)$, and are subject to the same splitting rule, with the effect that the resulting tree $\mathfrak{X}$ contains, after an elapsed time $t > 0$, $n(t)$ particles located at $x_1(t), \ldots, x_{n(t)}(t)$, with $n(t)$ being the random number of particles generated up to that time. It holds that $\mathbb{E}n(t) = e^t$.

The link between BBM and partial differential equations is provided by the following observation due to McKean [37]: if one denotes by

$$v(t, x) \equiv \mathbb{P}\left[\max_{1 \le k \le n(t)} x_k(t) \le x\right] \tag{1.1}$$

the law of the maximal displacement, a renewal argument shows that $v(t, x)$ solves the Kolmogorov–Petrovsky–Piscounov or Fisher [F-KPP] equation,

$$v_t = \frac{1}{2}v_{xx} + \sum_{k=1}^{\infty} p_k v^k - v, \tag{1.2}$$

with Heaviside initial condition

$$v(0, x) = \begin{cases} 1, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{1.3}$$

The F-KPP equation admits traveling waves: there exists a unique solution satisfying

$$v(t, m(t) + x) \to \omega(x) \qquad \text{uniformly in } x \text{ as } t \to \infty, \tag{1.4}$$

with the centering term given by

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t, \tag{1.5}$$

and $\omega(x)$ the distribution function which solves the o.d.e.

$$\frac{1}{2}\omega'' + \sqrt{2}\omega' + \sum_{k=1}^{\infty} p_k \omega^k - \omega = 0. \tag{1.6}$$

If one excludes the trivial cases, solutions to (1.6) are unique up to translations: this will play a crucial role in our considerations.

Lalley and Sellke [33] provided a characterization of the limiting law of the maximal displacement in terms of a *random shift* of the Gumbel distribution. More precisely, denoting by

$$Z(t) \equiv \sum_{k=1}^{n(t)} (\sqrt{2}t - x_k(t)) \exp(-\sqrt{2}(\sqrt{2}t - x_k(t))), \tag{1.7}$$

the so-called *derivative martingale*, Lalley and Sellke proved that $Z(t)$ converges almost surely to a strictly positive random variable Z, and established the integral representation

$$\omega(x) = \mathbb{E}[\exp(-CZe^{-\sqrt{2}x})], \tag{1.8}$$

for some $C > 0$.

It is also known (see e.g. Bramson [16] and Harris [30]) that

$$\lim_{x \to \infty} \frac{1 - \omega(x)}{xe^{-\sqrt{2}x}} = C. \tag{1.9}$$

Despite the precise information on the maximal displacement of BBM, an understanding of the full statistics of the particles close to the maximal one has been lacking. The statistics of such particles are fully encoded in the *extremal process*, namely the random measure

$$\mathcal{E}_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t) - m(t)}. \tag{1.10}$$

The main result of this paper is an explicit construction of the extremal process in the limit of large times.

2 Main result

The description of the limiting extremal process has two parts: the *cluster-extrema*, and the *clusters*.

The cluster-extrema. Let Z be the limiting derivative martingale. Conditionally on Z, we consider the Poisson point process (PPP) on $\mathbb{R}$ of intensity $CZ\sqrt{2}\mathrm{e}^{-\sqrt{2}x}\mathrm{d}x$:

$$\mathcal{P}_Z \equiv \sum_{i\in\mathbb{N}} \delta_{p_i} \overset{\text{law}}{=} \mathrm{PPP}\left(CZ\sqrt{2}\mathrm{e}^{-\sqrt{2}x}\mathrm{d}x\right), \tag{2.1}$$

with C as in (1.8).

The clusters. Consider the extremal process shifted by $\sqrt{2}t$ instead of $m(t)$:

$$\overline{\mathcal{E}}_t = \sum_{k\leq n(t)} \delta_{x_k(t)-\sqrt{2}t}. \tag{2.2}$$

Obviously, the limit of such a process must be trivial, since in view of (1.5), the probability that the maximum of BBM shifted by $-\sqrt{2}t$ does not drift to $-\infty$ is vanishing in the large time limit. It however turns out, see Theorem 3.4 below, that *conditionally* on the event $\{\max_k x_k(t) - \sqrt{2}t \geq 0\}$, the process $\overline{\mathcal{E}}_t$ does converge to a well defined point process $\overline{\mathcal{E}} = \sum_j \delta_{\xi_j}$ in the limit of large times. We may then define the *point process of the gaps*

$$\mathcal{D} = \sum_j \delta_{\Delta_j}, \quad \Delta_j \equiv \xi_j - \max_i \xi_i. \tag{2.3}$$

Note that $\mathcal{D}$ is a point process on $(\infty, 0]$ with an atom at 0.

Theorem 2.1 (Main Theorem) *Let $\mathcal{P}_Z$ be as in (2.1) and let $\{\mathcal{D}^{(i)}, i \in \mathbb{N}\}$ be a family of independent copies of the gap-process (2.3). Then the point process $\mathcal{E}_t = \sum_{k\leq n(t)} \delta_{x_k(t)-m(t)}$ converges in law as $t \to \infty$ to a Poisson cluster point process $\mathcal{E}$ given by*

$$\mathcal{E} \equiv \lim_{t\to\infty} \mathcal{E}_t \overset{\text{law}}{=} \sum_{i,j} \delta_{p_i+\Delta_j^{(i)}}. \tag{2.4}$$

A graphical depiction is given in Fig. 1 below.

Given the inherent self-similarity of BBM, it can hardly come as a surprise that the extremal process of BBM in the limit of large times is fractal-like. As such, it has a number of interesting properties (e.g. the invariance under superpositions, see Corollary 3.3, or the representation of the clusters in terms of BBM with drift, see Remark 3.5).

Understanding the extremal process of BBM in the limit $t \to \infty$ has been a long-standing problem of fundamental interest. Most classical results for extremal processes of correlated random variables concern criteria to prove that the behavior is the same as for i.i.d. variables [34]. Bramson's result shows that this cannot be the case for BBM. A class of models where a more complex structure of *Poisson cascades* was shown to emerge are the *generalized random energy models* of Derrida [13,14,25]. These models, however, have a rather simple hierarchical structure involving only a finite number of hierarchies which greatly simplifies the analysis, which cannot be
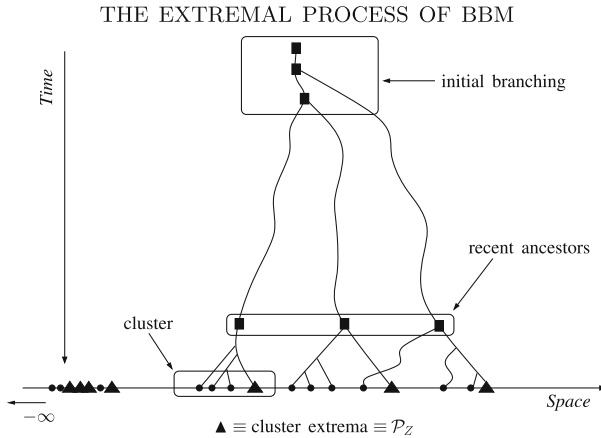
THE EXTREMAL PROCESS OF BBM

Fig. 1 Onset of the extremal process

carried over to models with infinite levels of branching such as BBM or the *continuous random energy models* studied in [15]. BBM is a case right at the borderline where correlations just start to effect the extremes and the structure of the extremal process. Results on the extremes of BBM allow one to peek into the world beyond the simple Poisson structures and hopefully open the gate towards the rigorous understanding of complex extremal structures.

Mathematically, BBM offers a spectacular interplay between probability and non-linear p.d.e's, as was noted already by McKean [37]. On one hand, the proof of Theorem 2.1 will rely on this dual aspect, in particular on precise estimates of the solutions of the F-KPP equations based on those of Bramson [16,17] and of Chauvin and Rouault [21], see Sect. 3.1. This aspect of the paper is in the spirit of Derrida and Spohn [26] who studied the free energy of the model by relying on results on the F-KPP equation. On the other hand, we propose a new method to study the extremal statistics of BBM based on the introduction of an auxiliary point process, whose correlation structure is much simpler than the one of BBM. As explained in Sect. 3.2, this approach finds its origins in the *cavity method* in spin glasses [41] and in the study of *competing particle systems* [3,39].

We believe this heuristic is a potentially powerful tool to understand other problems in extreme value statistics of correlated variables.

Finally, we remark that a structure similar to the one depicted in Theorem 2.1 is expected to emerge in all the models which are conjectured to fall into the universality class of branching Brownian motion, such as the *2-dim Gaussian Free Field* (2DGFF) [11,12,18], or the *cover time* for the simple random walk on the two dimensional discrete torus [23,24]. These are models whose correlations decay logarithmically with the distance. In particular, for log-correlated Gaussian field like the 2DGFF, we conjecture that the extremal process properly re-centered exists and is *exactly* of the form (2.4). Namely, the statistics of well-separated high points should be Poissonian with intensity measure with exponential density. In particular, the law of the maximum should be a mixture of a Gumbel as in (1.8) (after this paper has been submitted, partial progress towards this conjecture has been achieved in [27], and [28]). Finally, the law

of the clusters should be the one of a 2DGFF conditioned to be unusually high (that is, with maximal displacement of the order of $c \log N$, for the appropriate constant c, without a log log N correction, where N is the number of variables).

2.1 Relation to recent works

Many results have appeared in the past few years that shed light on the distribution of the particles of BBM close to the maximum in the limit of large times.

On the physics literature side, we mention the contributions of Brunet and Derrida [19,20], who reduce the problem of the statistical properties of particles at the edge of BBM to that of identifying the finer properties of the delay of travelling waves (here and below, "edge" stands for the set of particles which are at distances of order one from the maximum).

On the mathematics side, properties of the large time limit of the extremal process have been established in two papers of ours [5,6]. In the first paper we obtained a precise description of the *paths* of extremal particles which in turn imply a somewhat surprising restriction of the correlations of particles at the edge of BBM. These results were instrumental in the second paper where it is proved that a certain process obtained by a correlation-dependent thinning of the extremal particles converges to a random shift of a Poisson Point Process (PPP) with intensity measure with exponential density. The approach differs from the one used here to prove the Poissonian statistics of the cluster-extrema. There, the proof relied heavily on the description of the paths of extremal particles developed in [5], and did not prove that clusters are identically distributed, and describing their law. The approach based on the auxiliary point process fills this gap.

A description of the extremal process in terms of a Poisson cluster process has also been obtained independently by Aïdekon et al. [2] shortly after the first version of this paper appeared on arXiv. Their approach is based in part on a precise control of the localization of the paths of extremal particles that are essentially the ones of [5], cf. Proposition 2.5 in [2] and Theorems 2.2, 2.3 and 2.5 in [5]. It was pointed out in [5] that cluster-extrema perform an evolution which "resembles" that of a Bessel bridge. Aïdekon et al. put this on rigorous ground, using a spine decomposition developed in [1], showing that the law of such paths is in fact that of a Brownian motion in a potential, see [2, Theorem 2.4]. The clusters are then constructed by "attaching" (according to a certain random mechanism) to each such path independent BBMs, and then passing to the limit $t \to \infty$. Theorem 2.3 of [2] describes the resulting superposition of i.i.d. BBM's. Although the method of proof is different from ours (in particular, [2] does not rely on the convergence results established by Bramson [17]), the ensuing description of the extremal process is compatible with our main theorem. An additional bridge between the two descriptions goes through the work of Chauvin and Rouault [21,22] where a link between the spine decomposition and conditioned BBM is established, see Remark 3.5 below.

Very recently the ergodic properties at the edge of BBM have also been addressed. In [7], we proved a long-standing conjecture by Lalley and Sellke stating that the empirical (time-averaged) distribution function of the maximum of BBM converges almost surely to a double exponential, or Gumbel, distribution with a random shift.

The method of proof relies again on the localization of the paths established in [5]. Furthermore, combining the methods developed in the present paper and those from [7], we obtained in [8] a similar type of ergodic theorem for the whole extremal process.

Despite the steady progress over the last years, a number of interesting questions concerning the properties of the extremal process of BBM are still left open. In particular, a more quantitative understanding of the law of the clusters and its self-similar structure would be desirable. As an example, the average density of particles of the extremal process has been conjectured by Brunet and Derrida to behave asymptotically as $(-x)e^{-\sqrt{2}x}$ for $x \to -\infty$ [20]. This is the density of the intensity measure of the auxiliary process we introduce in this work, but the issues, although related, are not completely equivalent. As a matter of fact, none of the available descriptions of the extremal process and of the law of the clusters implies directly this result, though progress in this direction has been announced in [2].

Finally, we mention that Madaule [35] has recovered results on the extremal process of BBM for the discrete case of branching random walks.

3 The road to the Main Theorem

We first recall from [31] the standard setting for the study of point processes. Let $\mathcal{M}$ be the space of Radon measures on $\mathbb{R}$. Elements of $\mathcal{M}$ are in correspondence with the positive linear functionals on $\mathcal{C}_c(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ with compact support. In particular, any element of $\mathcal{M}$ is locally finite. The space $\mathcal{M}$ is endowed with the vague topology, that is, $\mu_n \to \mu$ in $\mathcal{M}$ if and only if for any $\phi \in \mathcal{C}_c(\mathbb{R})$, $\int \phi d\mu_n \to \int \phi d\mu$. The law of a random element Ξ of $\mathcal{M}$, or random measure, is determined by the collection of real random variables $\int \phi d\Xi, \phi \in \mathcal{C}_c(\mathbb{R})$. A sequence (Ξ_n) of random elements of $\mathcal{M}$ is said to converge to Ξ if and only if for each $\phi \in \mathcal{C}_c(\mathbb{R})$, the random variables $\int \phi d\Xi_n$ converges in the weak sense to $\int \phi d\Xi$. A point process is a random measure that is integer-valued almost surely. It is a standard fact that point processes are closed in the set of random elements of $\mathcal{M}$, and that a sufficient condition for their convergence is the convergence of Laplace functionals.

3.1 The Laplace transform of the extremal process of BBM

Our first step towards Theorem 2.1 consists in showing that the extremal process $\mathcal{E}_t$ is well defined in the limit of large times.

Theorem 3.1 (Existence of the limit) *The point process* $\mathcal{E}_t = \sum_{k \leq n(t)} \delta_{x_k(t)-m(t)}$ *converges in law to a point process* $\mathcal{E}$.

Brunet and Derrida [20] have obtained a similar existence result for the extremal process $\mathcal{E}_t$. To achieve this goal, they use the generating function that counts the number of the points to the right of fixed levels. Our method of proof is different, and relies on the convergence of Laplace functionals

$$\Psi_t(\phi) \equiv \mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{E}_t(\mathrm{d}y)\right)\right], \tag{3.1}$$

for $\phi \in C_c(\mathbb{R})$ non-negative. It is easy to see that the Laplace functional is a solution of the F-KPP equation following the observation of McKean, see Lemma 4.1 below. However, convergence is more subtle. It will follow from the convergence theorem of Bramson, see Theorem 4.2 below, but only after an appropriate truncation of the functional needed to satisfy the hypotheses of the theorem. The proof recovers a representation of the form (1.8) and, more importantly, it provides an expression for the constant C as a function of the initial condition. This observation is inspired by the work of Chauvin and Rouault [21]. It will be at the heart of the representation theorem of the extremal process as a cluster process.

Proposition 3.2 *Let $\mathcal{E}_t$ be the process* (1.10). *For $\phi \in C_c(\mathbb{R})$ non-negative and any $x \in \mathbb{R}$,*

$$\lim_{t \to \infty} \mathbb{E}\left[\exp\left(-\int \phi(y+x)\mathcal{E}_t(dy)\right)\right] = \mathbb{E}\left[\exp\left(-C(\phi)Ze^{-\sqrt{2}x}\right)\right] \quad (3.2)$$

where, for $v(t, y)$ solution of F-KPP (1.2) *with initial condition $v(0, y) = e^{-\phi(y)}$,*

$$C(\phi) = \lim_{t \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty \left(1 - v(t, y + \sqrt{2}t)\right) ye^{\sqrt{2}y} dy \quad (3.3)$$

is a strictly positive constant depending on ϕ only, and Z is the derivative martingale.

A straightforward consequence of Proposition 3.2 is the *Invariance under superpositions* of the random measure $\mathcal{E}$, conjectured by Brunet and Derrida [19,20].

Corollary 3.3 (Invariance under superposition) *The law of the extremal process of the superposition of n independent BBM started at $x_1, \ldots, x_n \in \mathbb{R}$ coincides in the limit of large time with that of a single BBM, up to a random shift.*

It is not hard to verify that a point process Ξ that is constructed from a Poisson point process with exponential density to which, at each atom, is attached an i.i.d. point process has the same law, up to a shift, as a superposition of n i.i.d. copies of Ξ for any $n \in \mathbb{N}$. Brunet and Derrida conjectured that any point process whose law is invariant, up to shift, under a superposition of i.i.d. copies of itself must in fact be of this form. Maillard has proved this conjecture [36], but this general theorem provides no information on the law of the clusters.

Theorem 3.4 (The law of the clusters) *Let $x \equiv a\sqrt{t} + b$ for some $a < 0, b \in \mathbb{R}$. The point process*

$$\sum_{k \leq n(t)} \delta_{x+x_k(t)-\sqrt{2}t} \text{ conditioned on the event } \left\{x + \max_k x_k(t) - \sqrt{2}t > 0\right\} \quad (3.4)$$

converges in law as $t \to \infty$ to a well-defined point process $\overline{\mathcal{E}}$. The limit does not depend on a and b, and the maximum of $\overline{\mathcal{E}}$ has the law of an exponential random variable.

The limiting process therefore inherits a "loss of memory" property from the large jump. This is the crucial point for the proof that the clusters are identically distributed.

Remark 3.5 BBMs conditioned on the atypical event $\{\max_k x_k(t) - \sqrt{2}t > 0\}$ in the limit of large times, such as the ones appearing in Theorem 3.4, have been studied by Chauvin and Rouault [21]. They show that the impact of the conditioning is twofold: first, on one path of the tree (and only one), the *spine*, the associated Brownian motion has a drift; second, the conditioning leads to an increase of both birth rate, and intensity. We refer the reader to [21, Theorem 3] for the precise statements.

3.2 An auxiliary point process

In this section, the method of proof of Theorem 2.1 using an auxiliary point process is explained. To make the distinction between the law of standard BBM, denoted by $\mathbb{P}$, and the auxiliary construction, we introduce for convenience a new probability space. Let $(\Omega', \mathcal{F}', P)$ be a probability space, and $Z : \Omega' \to \mathbb{R}_+$ with distribution as that of the limiting derivative martingale (1.7). Expectation with respect to P will be denoted by E. On $(\Omega', \mathcal{F}', P)$, let $(\eta_i; i \in \mathbb{N})$ be the atoms of a Poisson point process η on $(-\infty, 0)$ with intensity measure

$$\sqrt{\frac{2}{\pi}}(-x)e^{-\sqrt{2}x}dx. \tag{3.5}$$

For each $i \in \mathbb{N}$, consider independent BBMs on $(\Omega', \mathcal{F}', P)$ with drift $-\sqrt{2}$, i.e. $\{x_k^{(i)}(t) - \sqrt{2}t; k \leq n^{(i)}(t)\}$, $n^{(i)}(t)$ is the number of particles of the BBM i at time t. Remark that, by 1.4 and 1.5, for each $i \in \mathbb{N}$,

$$\max_{k \leq n^{(i)}(t)} x_k^{(i)}(t) - \sqrt{2}t \to -\infty, \quad P\text{-a.s.} \tag{3.6}$$

The auxiliary point process of interest is the superposition of the i.i.d. BBM's with drift and shifted by $\eta_i + \frac{1}{\sqrt{2}} \log Z$:

$$\Pi_t \equiv \sum_{i,k} \delta_{\frac{1}{\sqrt{2}} \log Z + \eta_i + x_k^{(i)}(t) - \sqrt{2}t} \tag{3.7}$$

The existence and non-triviality of the process in the limit $t \to \infty$ is not straightforward, especially in view of (3.6). It will be proved by recasting the problem into the frame of convergence of solutions of the F-KPP equations to travelling waves, as in the proof of Theorem 3.1.

It turns out that the density of the Poisson process points growing faster than exponentially as $x \to -\infty$ compensates for the fact that BBM's with drift wander off to $-\infty$.

The connection between the extremal process of BBM and the auxiliary process is the following result:

Theorem 3.6 (The auxiliary point process) *Let $\mathcal{E}_t$ be the extremal process (1.10) of BBM. Then*

$$\lim_{t\to\infty} \mathcal{E}_t \overset{law}{=} \lim_{t\to\infty} \Pi_t. \tag{3.8}$$

The above will follow from the fact that the Laplace functionals of $\lim_{t\to\infty} \Pi_t$ admits a representation of the form (3.2), and that the constants $C(\phi)$ in fact correspond. An elementary consequence of the above identification is that the extremal process $\mathcal{E}$ shifted back by $\frac{1}{\sqrt{2}} \log Z$ is an infinitely divisible point process. The reader is referred to [31] for definitions and properties of such processes.

We conjectured Theorem (3.6) in a recent paper [6], where it is pointed out that such a representation is a natural consequence of the results on the genealogies and the paths of the extremal particles in [5]. The proof of Theorem 3.6 provided here does not rely on such techniques. It is based on the analysis of Bramson [17] and the subsequent works of Chauvin and Rouault [21,22], and Lalley and Sellke [33]. However, the results on the genealogies of [5] provides a complementary intuition.

The approach to the problem in terms of the auxiliary point process is very close in spirit to the study of *competing particle systems* (CPS). These systems were introduced in the study of spin glasses [3,4,39], but are a good laboratory in general to study problems of extremal value statistics. The CPS approach can be seen as an attempt to formalize to so-called *cavity method* developed by Parisi and co-authors [38] for the study of spin glasses. The connection with Theorem 3.6 is as follows.

Let $\mathcal{E} = \sum_{i\in\mathbb{N}} \delta_{e_i}$ be the limiting extremal process of a BBM starting at zero. It is clear since $m(t) = m(t-s) + \sqrt{2}s + o(1)$ that the law of $\mathcal{E}$ satisfies the following invariance property: for any $s \geq 0$,

$$\mathcal{E} \overset{law}{=} \sum_{i,k} \delta_{e_i + x_k^{(i)}(s) - \sqrt{2}s} \tag{3.9}$$

where $\{x_k^{(i)}(s); k \leq n^{(i)}(s)\}_{i\in\mathbb{N}}$ are i.i.d. BBM's. On the other hand, by Theorem 3.6,

$$\mathcal{E} \overset{law}{=} \lim_{s\to\infty} \sum_{i,k} \delta_{\eta_i + x_k^{(i)}(s) - \sqrt{2}s} , \tag{3.10}$$

where now $(\eta_i; i \in \mathbb{N})$ are the atoms of a $\mathrm{PPP}(\sqrt{\frac{2}{\pi}}(-x)e^{-\sqrt{2}x}dx)$ on $(-\infty, 0]$ shifted by $\frac{1}{\sqrt{2}} \log Z$.

The main idea of the CPS-approach is to characterize extremal processes as invariant measures under suitable, model-dependent stochastic mappings. In the case of BBM, the mapping consists of adding to each ancestor e_i independent BBMs with drift $-\sqrt{2}$. This procedure randomly picks of the original ancestors only those with offspring in the lead at some future time. But the random thinning is performed through independent random variables: this suggests that one may indeed replace the process of ancestors $\{e_i\}$ by a Poisson process with suitable intensity measure.

Behind the random thinning, a crucial phenomenon of "energy vs. entropy" is at work. Under the light of (3.6), the probability that any such BBM with drift $-\sqrt{2}$ attached to the process of ancestors does not wander off to $-\infty$ vanishes in the limit of large times. On the other hand, the higher the position of the ancestors, the fewer one finds. A delicate balance must therefore be met, and only ancestors lying on a precise level below the lead can survive the random thinning. This is the content of Proposition 3.8 below, and a fundamental ingredient in the CPS-heuristics: particles at the edge come from a precise location of the tail in the past. A key step in identifying the equilibrium measure is thus to identify the tail, or at least a good approximation of it. In the example of BBM, we are able to show, perhaps indirectly, that a good approximation of the tail is a Poisson process with intensity measure $(-x)e^{-\sqrt{2}x}dx$ (up to constant) on $(-\infty, 0)$.

We now list some of the properties of the auxiliary process in the limit of large times, which by Theorem 3.6 coincide with those of the extremal process of BBM.

Proposition 3.7 (Poissonian statistics of the cluster-extrema) *Consider Π_t^{ext} the point process obtained by retaining from Π_t the maximal particles of the BBM's,*

$$\Pi_t^{ext} \equiv \sum_i \delta_{\frac{1}{\sqrt{2}} \log Z + \eta_i + \max_k \{x_k^{(i)}(t) - \sqrt{2}t\}}. \tag{3.11}$$

Then $\lim_{t\to\infty} \Pi_t^{ext} \overset{law}{=} \mathrm{PPP}(Z\sqrt{2}Ce^{-\sqrt{2}x}dx)$ as a point process on $\mathbb{R}$, where C is the same constant appearing in (1.8). In particular, the maximum of $\lim_{t\to\infty} \Pi_t^{ext}$ has the same law as the limit law of the maximum of BBM.

The fact that the laws of the maximum of the cluster-extrema and of BBM correspond is a consequence of (1.8) and the formula for the maximum of a Poisson process. The presence of Poissonian statistics in the extremes of BBM was proved in [6] using the results on the genealogies in [5]. The proof given here is conceptually very different and based on the auxiliary process. The last ingredient for the proof of Theorem 2.1 is a very precise control on the location of the atoms η for which the particles of the associated BBM may reach the leading edge. The proposition is central to the proof of Theorem 3.4.

Proposition 3.8 *Let $y \in \mathbb{R}$ and $\varepsilon > 0$ be given. There exist $0 < A_1 < A_2 < \infty$ and t_0 depending only on y and ε, such that*

$$\sup_{t \geq t_0} P[\exists_{i,k} : \eta_i + x_k^{(i)}(t) - \sqrt{2}t \geq y, \ \eta_i \notin [-A_1\sqrt{t}, -A_2\sqrt{t}]] < \varepsilon. \tag{3.12}$$

4 Proofs

In what follows, $\{x_k(t), \ k \leq n(t)\}$ will denote a branching Brownian motion on the interval of time $[0, t]$ with one particle at 0 at time 0. The law of BBM will be denoted by $\mathbb{P}$ and its expectation by $\mathbb{E}$. We recall that we will use P and E for the probability

and the expectation of the auxiliary process to distinguish with the original process. We will consider the F-KPP equation (1.2) with general initial condition $v(0, x) = f(x)$ for some function f that will depend on context. Initial condition (1.3) corresponds to $f(x) = 1_{\{-x \leq 0\}}$. It will be often convenient to consider $u(t, x) \equiv 1 - v(t, x)$ instead of $v(t, x)$. Note that $u(t, x)$ is solution of the equation

$$u_t = \frac{1}{2}u_{xx} + (1 - u) - \sum_{k=1}^{\infty} p_k (1 - u)^k, \tag{4.1}$$

with initial condition $u(0, x) = 1 - f(x)$. We will refer to this equation as the F-KPP Eq. (4.1).

4.1 Estimates of solutions of the F-KPP equation

We start by stating two fundamental results that will be used extensively. First, McKean's insightful observation:

Lemma 4.1 ([37]) *Let $f : \mathbb{R} \to [0, 1]$ and $\{x_k(t) : k \leq n(t)\}$ a branching Brownian motion starting at 0. The function*

$$v(t, x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} f(x + x_k(t))\right] \tag{4.2}$$

is solution of the F-KPP Eq. (1.2) with initial condition $v(0, x) = f(x)$.

The second fundamental result is the one of Bramson on the convergence of solutions of the F-KPP equation to travelling waves:

Theorem 4.2 (Theorem A, Theorem B and Example 2 in [17]) *Let u be a solution of the F-KPP Eq. (4.1) with $0 \leq u(0, x) \leq 1$. Then*

$$u(t, x + m(t)) \to 1 - \omega(x), \quad \text{uniformly in } x \text{ as } t \to \infty, \tag{4.3}$$

where ω is the unique solution (up to translation) of

$$\frac{1}{2}\omega'' + \sqrt{2}\omega' + \sum_{k=1}^{\infty} p_k \omega^k - \omega = 0, \tag{4.4}$$

if and only if

1. *for some $h > 0$, $\limsup_{t \to \infty} \frac{1}{t} \log \int_t^{t(1+h)} u(0, y)dy \leq -\sqrt{2}$;*
2. *and for some $v > 0$, $M > 0$, $N > 0$, $\int_x^{x+N} u(0, y)dy > v$ for all $x \leq -M$.*

Moreover, if $\lim_{x \to \infty} e^{bx} u(0, x) = 0$ for some $b > \sqrt{2}$, then one may choose

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t. \tag{4.5}$$

It is to be noted that the necessary and sufficient conditions hold for uniform convergence in x. Pointwise convergence could hold when, for example, condition 2 is not satisfied. This is the case in the proof of Theorem 3.1. The following Proposition provides sharp approximations to the solutions of the F-KPP equation.

Proposition 4.3 *Let u be a solution to the F-KPP equation* (4.1) *with initial data satisfying the assumptions of Theorem* 4.2 *and*

$$\int_0^\infty y e^{\sqrt{2}y} u(0, y) dy < \infty. \tag{4.6}$$

Define

$$\psi(r, t, X + \sqrt{2}t) \equiv \frac{e^{-\sqrt{2}X}}{\sqrt{2\pi(t-r)}} \int_0^\infty u(r, y' + \sqrt{2}r) e^{y'\sqrt{2}} e^{-\frac{(y'-X)^2}{2(t-r)}}$$

$$\left(1 - e^{-2y'\frac{(X+\frac{3}{2\sqrt{2}}\log t)}{t-r}}\right) dy'. \tag{4.7}$$

Then for r large enough, $t \geq 8r$, and $X \geq 8r - \frac{3}{2\sqrt{2}}\log t$,

$$\gamma^{-1}(r)\psi(r, t, X + \sqrt{2}t) \leq u(t, X + \sqrt{2}t) \leq \gamma(r)\psi(r, t, X + \sqrt{2}t), \tag{4.8}$$

for some $\gamma(r) \downarrow 1$ as $r \to \infty$.

The function ψ thus fully captures the large space-time behavior of the solution to the F-KPP equations. We will make extensive use of (4.8), mostly when both X and t are large in the positive, in which case the dependence on X becomes particularly easy to handle.

Proof of Proposition 4.3 For $T > 0$ and $0 < \alpha < \beta < \infty$, let $\{\mathfrak{z}_{\alpha,\beta}^T(s), 0 \leq s \leq T\}$ denote a Brownian bridge of length T starting at α and ending at β.

It has been proved by Bramson (see [17, Proposition 8.3]) that for u satisfying the assumptions in the Proposition 4.3, the following holds:

(1) for r large enough, $t \geq 8r$ and $x \geq m(t) + 8r$

$$u(t, x) \geq K_1(r)e^{t-r} \int_{-\infty}^\infty u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbb{P}[\mathfrak{z}_{x,y}^{t-r}(s) > \overline{\mathcal{M}}_{r,t}^x(t-s),$$

$$s \in [0, t-r]] \mathrm{d}y$$

and

$$u(t, x) \leq K_2(r)e^{t-r} \int\limits_{-\infty}^{\infty} u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbb{P}[\mathfrak{z}_{x,y}^{t-r}(s) > \underline{\mathcal{M}}_{r,t}'(t-s),$$

$$s \in [0, t-r]]dy$$

where the functions $\overline{\mathcal{M}}_{r,t}^x(t-s), \underline{\mathcal{M}}_{r,t}'(t-s)$ satisfy

$$\underline{\mathcal{M}}_{r,t}'(t-s) \leq n_{r,t}(t-s) \leq \overline{\mathcal{M}}_{r,t}^x(t-s),$$

for $s \mapsto n_{r,t}(s)$ being the linear interpolation between $\sqrt{2}r$ at time r and $m(t)$ at time t. Moreover, $K_1(r) \uparrow 1$, $K_2(r) \downarrow 1$ as $r \to \infty$.

(2) If $\psi_1(r, t, x)$ and $\psi_2(r, t, x)$ denote respectively the lower and upper bound to $u(t, x)$, we have

$$1 \leq \frac{\psi_2(r, t, x)}{\psi_1(r, t, x)} \leq \gamma(r)$$

where $\gamma(r) \downarrow 1$ as $r \to \infty$.

Hence, if we denote by

$$\widehat{\psi}(r, t, x) = e^{t-r} \int\limits_{-\infty}^{\infty} u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbb{P}(\mathfrak{z}_{x,y}^{t-r}(s) > n_{r,t}(t-s),$$

$$s \in [0, t-r])dy, \tag{4.9}$$

we have by domination $\psi_1 \leq \widehat{\psi} \leq \psi_2$. Therefore, for r, t and x large enough

$$\frac{u(t, x)}{\widehat{\psi}(r, t, x)} \leq \frac{\psi_2(r, t, x)}{\widehat{\psi}(r, t, x)} \leq \frac{\psi_2(r, t, x)}{\psi_1(r, t, x)} \leq \gamma(r), \tag{4.10}$$

and

$$\frac{u(t, x)}{\widehat{\psi}(r, t, x)} \geq \frac{1}{\gamma(r)}. \tag{4.11}$$

Combining (4.10) and (4.11) we thus get

$$\gamma^{-1}(r)\widehat{\psi}(r, t, x) \leq u(t, x) \leq \gamma(r)\widehat{\psi}(r, t, x). \tag{4.12}$$

We now consider $X \geq 8r - \frac{3}{2\sqrt{2}} \log t$, and obtain from (4.12) that

$$\gamma^{-1}(r)\widehat{\psi}(r, t, X + \sqrt{2}t) \leq u(t, X + \sqrt{2}t) \leq \gamma(r)\widehat{\psi}(r, t, X + \sqrt{2}t). \tag{4.13}$$

The probability involving the Brownian bridge in the definition of $\widehat{\psi}$ can be explicitly computed. The probability of a Brownian bridge of length t to remain below the interpolation of $A > 0$ at time 0 and $B > 0$ at time t is $1 - e^{-2AB/t}$, see e.g. [40]. In the above setting the length is $t - r$, $A = x - m(t) = x - \sqrt{2}t + \frac{3}{2\sqrt{2}}\log t > 0$ for t large enough and $B = y - \sqrt{2}r = y'$. Using this, together with the fact that $\mathbb{P}(\mathfrak{z}_{x,y}^{t-r}(s) > n_{r,t}(t-s), s \in [0, t-r])$ is 0 for $B = y' < 0$, and by change of variable $y' = y + \sqrt{2}t$ in the integral appearing in the definition of $\widehat{\psi}$, we get

$$\widehat{\psi}(r, t, X + \sqrt{2}t) = \frac{e^{-\sqrt{2}X}}{\sqrt{2\pi(t-r)}} \int\limits_{0}^{\infty} u(r, y' + \sqrt{2}r)e^{y'\sqrt{2}}e^{-\frac{(y'-X)^2}{2(t-r)}}$$

$$\times \left(1 - e^{-2y'\frac{(X+\frac{3}{2\sqrt{2}}\log t)}{t-r}}\right)dy'$$

$$= \psi(r, t, X + \sqrt{2}t). \tag{4.14}$$

This, together with (4.12), concludes the proof of the proposition. $\qquad\square$

The bounds in (4.8) have been used by Chauvin and Rouault to compute the probability of deviations of the maximum of BBM, see Lemma 2 [21]. Their reasoning applies to solutions of the F-KPP equation with other initial conditions than those corresponding to the maximum. We give the statement below, and reproduce Chauvin and Rouault's proof in a general setting for completeness.

Proposition 4.4 *Let u be a solution to the F-KPP equation* (4.1) *with initial data satisfying the assumptions of Theorem* 4.2 *and*

$$y_0 \equiv \sup\{y : u(0, y) > 0\} < \infty. \tag{4.15}$$

Then,

$$\lim_{t\to\infty} e^{x\sqrt{2}}\frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t}\,\psi(r, t, x + \sqrt{2}t) = \sqrt{\frac{2}{\pi}}\int\limits_{0}^{\infty} u(r, y + \sqrt{2}r)ye^{y\sqrt{2}}dy. \tag{4.16}$$

Moreover, the limit of the right-hand side exists as $r \to \infty$, and it is positive and finite.

The Proposition will often be used with the initial condition $u(0, x) = \mathbb{1}_{\{-x>\delta\}}$, $\delta \in \mathbb{R}$.

Proof Note that the condition (4.15) implies (4.6). The first claim is straightforward from Proposition 4.3 if the limit $t \to \infty$ can be taken inside the integral in the definition of ψ. This is justified by dominated convergence. For t large enough, since $e^{-x} \geq 1 - x$ for $x > 0$, it is easy to check that ψ times $e^{x\sqrt{2}}\frac{t^{3/2}}{\log t}$ is smaller than

$$\frac{4}{\sqrt{\pi}}\,y'e^{y'\sqrt{2}}u(r, y' + \sqrt{2}r). \tag{4.17}$$

It remains to show that (4.17) is integrable in $y' \geq 0$. To see this, let $u^{(2)}(t, x)$ be the solution to $\partial_t u^{(2)} = \frac{1}{2} u_{xx}^{(2)} - u^{(2)}$ (the *linearised* F-KPP-Equation (4.1) with initial conditions $u^{(2)}(0, x) = u(0, x)$. By the maximum principle for nonlinear, parabolic p.d.e.'s, see e.g. [17, Corollary 1, p. 29],

$$u(t, x) \leq u^{(2)}(t, x) \tag{4.18}$$

Moreover, by the Feynman–Kac representation and the definition of y_0,

$$u^{(2)}(t, x) \leq e^t \int_{-\infty}^{y^0} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy, \tag{4.19}$$

and for any $x > 0$ we thus have the bound

$$u^{(2)}(t, x) \leq e^t e^{-\frac{x^2}{2t}} e^{\frac{y_0 x}{t}}. \tag{4.20}$$

Hence,

$$u(r, y + \sqrt{2}r) \leq e^{-\frac{y^2}{2r}} e^{\frac{y_0 y}{r}} e^{-\sqrt{2}y} e^{\sqrt{2}y_0}. \tag{4.21}$$

The upper bound is integrable over the desired measure since

$$\int_0^\infty y e^{\sqrt{2}y} e^{-\frac{y^2}{2r}} e^{\frac{y_0 y}{r}} e^{-\sqrt{2}y} dy = \int_0^\infty y e^{-\frac{y^2}{2r}} e^{\frac{y_0 y}{r}} dy < \infty. \tag{4.22}$$

Therefore dominated convergence can be applied and the first part of the proposition follows.

It remains to show that

$$\lim_{r \to \infty} \int_0^\infty u(r, y + \sqrt{2}r) \sqrt{\frac{2}{\pi}} y e^{y\sqrt{2}} dy \text{ exists and is finite.} \tag{4.23}$$

Write $C(r)$ for the integral. By Proposition 4.3, for r large enough,

$$\limsup_{t \to \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} u(t, x + \sqrt{2}t)$$

$$\leq \gamma(r) \lim_{t \to \infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \psi(r, t, x + \sqrt{2}t) = C(r)\gamma(r),$$

and

$$\liminf_{t\to\infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} u(t, x + \sqrt{2t})$$

$$\geq \gamma(r)^{-1} \lim_{t\to\infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \psi(r, t, x + \sqrt{2t}) = C(r)\gamma(r)^{-1},$$

Therefore since $\gamma(r) \to 1$

$$\limsup_{t\to\infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} u(t, x + \sqrt{2t}) \leq \liminf_{r\to\infty} C(r)$$

and

$$\liminf_{t\to\infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} u(t, x + \sqrt{2t}) \geq \limsup_{r\to\infty} C(r).$$

It follows that $\lim_{r\to\infty} C(r) \equiv C$ exists and so does $\lim_{t\to\infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} u(t, x + \sqrt{2t})$. Moreover $C > 0$ otherwise

$$\lim_{t\to\infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} u(t, x + \sqrt{2t}) = 0$$

which is impossible since

$$\lim_{t\to\infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} u(t, x + \sqrt{2t}) \geq C(r)/\gamma(r)$$

for r large enough but finite. ($\gamma(r)$ and $C(r)$ are finite for r finite.) Moreover $C < \infty$, otherwise

$$\lim_{t\to\infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} u(t, x + \sqrt{2t}) = \infty,$$

which is impossible since $\lim_{t\to\infty} e^{x\sqrt{2}} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} u(t, x + \sqrt{2t}) \leq C(r)\gamma(r)$ for r large enough, but finite. □

For the proof of Theorem 3.4, the following refinement of Proposition 4.4 is needed.

Lemma 4.5 *Let u be a solution to the F-KPP equation* (4.1) *with initial data satisfying the assumptions of Theorem* 4.2 *and*

$$y_0 \equiv \sup\{y : u(0, y) > 0\} < \infty. \tag{4.24}$$

Then, for $x = a\sqrt{t}$, $a > 0$, and $Y \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{e^{\sqrt{2}x} t^{3/2}}{x} \psi(r, t, x + Y + \sqrt{2}t) = C(r)e^{-\sqrt{2}Y}e^{-a^2/2} \qquad (4.25)$$

where

$$C(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(r, y' + \sqrt{2}r)y'e^{y'\sqrt{2}}dy'. \qquad (4.26)$$

Moreover, the convergence is uniform for a in a compact set.

Proof The proof is a simple computation:

$$\lim_{t\to\infty} \frac{e^{\sqrt{2}x}}{x} t^{3/2}\psi(r, t, Y + x + \sqrt{2}t)$$

$$\stackrel{4.7}{=} e^{-\sqrt{2}Y} \lim_{t\to\infty} \frac{t^{3/2}}{x\sqrt{2\pi(t-r)}} \int_0^\infty u(r, y' + \sqrt{2}r)$$

$$e^{y'\sqrt{2}}e^{-\frac{(y'-x-Y)^2}{2(t-r)}} \left(1 - e^{-2y'\frac{(x+Y+\frac{3}{2\sqrt{2}}\log t)}{t-r}}\right)dy'$$

$$= e^{-\sqrt{2}Y} \int_0^\infty u(r, y' + \sqrt{2}r)e^{y'\sqrt{2}} \lim_{t\to\infty} \frac{t^{3/2}}{x\sqrt{2\pi(t-r)}}$$

$$\left[e^{-\frac{(y'-x-Y)^2}{2(t-r)}} \left(1 - e^{-2y'\frac{(x+Y+\frac{3}{2\sqrt{2}}\log t)}{t-r}}\right)\right]dy', \qquad (4.27)$$

the last step by dominated convergence (cfr. (4.18)–(4.22)). Using that $x = a\sqrt{t}$,

$$\lim_{t\to\infty} \frac{t^{3/2}}{x\sqrt{2\pi(t-r)}} \left[e^{-\frac{(y'-x-Y)^2}{2(t-r)}} \left(1 - e^{-2y'\frac{(x+Y+\frac{3}{2\sqrt{2}}\log t)}{t-r}}\right)\right] = \sqrt{\frac{2}{\pi}} y'e^{-a^2/2},$$

hence

$$\lim_{t\to\infty} \frac{e^{\sqrt{2}x}}{x} t^{3/2}\psi(r, t, Y + x + \sqrt{2}t) = C(r)e^{-\sqrt{2}Y}e^{-a^2/2}. \qquad (4.28)$$

$\square$

The contribution to $C = \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u(r, y + \sqrt{2}r) ye^{y\sqrt{2}}dy$ comes from the y's of the order of $\sqrt{r}$. This result will be used in the proof of Proposition 3.8.

Lemma 4.6 *Let u be a solution to the F-KPP equation* (4.1) *with initial data satisfying the assumptions of Theorem* 4.2 *and*

$$y_0 \equiv \sup\{y : u(0, y) > 0\} < \infty. \tag{4.29}$$

Then for any $x \in \mathbb{R}$

$$\lim_{A_1 \downarrow 0} \limsup_{r \to \infty} \int_0^{A_1 \sqrt{r}} u(r, x + y + \sqrt{2}r) \, y e^{y\sqrt{2}} dy = 0$$

$$\lim_{A_2 \uparrow \infty} \limsup_{r \to \infty} \int_{A_2 \sqrt{r}}^{\infty} u(r, x + y + \sqrt{2}r) \, y e^{y\sqrt{2}} dy = 0. \tag{4.30}$$

For the proof of the lemma, we use an estimate of the law of the maximum similar to the one of Proposition 4.3 that was used in [6].

Lemma 4.7 [6, Corollary 10] *For $X > 1$, and $t \geq t_o$ (for t_o a numerical constant),*

$$\mathbb{P}\left[\max_{k \leq n(t)} x_k(t) - m(t) \geq X\right] \leq \rho \cdot X \cdot \exp\left(-\sqrt{2}X - \frac{X^2}{2t} + \frac{3}{2\sqrt{2}}X\frac{\log t}{t}\right). \tag{4.31}$$

for some constant $\rho > 0$.

Proof of Lemma 4.6 We prove the first limit. The proof of the second is identical. By assumption, $u(0, y) \leq \mathbb{1}_{\{y < y_0 + 1\}}$. It follows from Lemma 4.1 that

$$u(r, x + y + \sqrt{2}r) \leq \mathbb{P}\left(\max x_k(r) - \sqrt{2}r > y_0 + 1 + x + y\right). \tag{4.32}$$

It thus suffices to prove the result for the right side. Write for simplicity $\delta = x + y_0 + 1$. We have by the change of variable $y \to y + \delta + \frac{3}{2\sqrt{2}} \log r$

$$\int_0^{A_1\sqrt{r}} \mathbb{P}\left(\max x_k(r) - \sqrt{2}r > \delta + y\right) y e^{y\sqrt{2}} dy$$

$$= e^{-\sqrt{2}\delta} r^{-3/2} \int_{\delta + \frac{3}{2\sqrt{2}} \log r}^{A_1\sqrt{r} + \delta + \frac{3}{2\sqrt{2}} \log r} \mathbb{P}\left(\max x_k(r) - m(r) > y\right) y e^{y\sqrt{2}} dy$$

$$- \left(\delta + \frac{3}{2\sqrt{2}} \log r\right) e^{-\sqrt{2}\delta} r^{-3/2} \int_{\delta + \frac{3}{2\sqrt{2}} \log r}^{A_1\sqrt{r} + \delta + \frac{3}{2\sqrt{2}} \log r} \mathbb{P}\left(\max x_k(r) - m(r) > y\right) e^{y\sqrt{2}} dy. \tag{4.33}$$

By Lemma 4.7, the first term of (4.33) is smaller than

$$\rho e^{-\sqrt{2}\delta} r^{-3/2} \int_{\delta+\frac{3}{2\sqrt{2}}\log r}^{A_1\sqrt{r}+\delta+\frac{3}{2\sqrt{2}}\log r} y^2 e^{-\frac{y^2}{2r}} e^{\frac{3}{2\sqrt{2}}y\frac{\log r}{r}} dy.$$

By the change of variable $y \to \frac{y}{\sqrt{r}}$, this becomes

$$\rho e^{-\sqrt{2}\delta} \int_{\frac{\delta}{\sqrt{r}}+\frac{3}{2\sqrt{2}}\frac{\log r}{\sqrt{r}}}^{A_1+\frac{\delta}{\sqrt{r}}+\frac{3}{2\sqrt{2}}\frac{\log r}{\sqrt{r}}} y^2 e^{-\frac{y^2}{2}} e^{\frac{3}{2\sqrt{2}}y\frac{\log r}{\sqrt{r}}} dy. \tag{4.34}$$

As $r \to \infty$, this integral is bounded by a Gaussian integral which goes to zero as $A_1 \downarrow 0$. The second term in 4.33 is bounded similarly. The resulting integral is the one of 4.34 with $y/\sqrt{r}$ instead of y^2 in the integrand. Therefore, it goes to zero as $r \to \infty$. This establishes (4.30). □

The following lemma tells us how $C = \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u(r, y + \sqrt{2}r) \, ye^{y\sqrt{2}}dy$ behaves when u is shifted.

Lemma 4.8 *Let u be a solution to the F-KPP equation* (4.1) *with initial data satisfying the assumptions of Theorem* 4.2 *and*

$$y_0 \equiv \sup\{y : u(0, y) > 0\} < \infty. \tag{4.35}$$

Let

$$C \equiv \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty ye^{y\sqrt{2}}u(r, y + \sqrt{2}r)dy, \tag{4.36}$$

whose existence is ensured by Proposition 4.4. *Then for any $x \in \mathbb{R}$:*

$$\lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty ye^{y\sqrt{2}}u(r, x + y + \sqrt{2}r)dy = Ce^{-\sqrt{2}x}. \tag{4.37}$$

Proof The claim will follow by Proposition 4.4, if it is shown that

$$\lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty ye^{y\sqrt{2}}u(r, x + y + \sqrt{2}r)dy = \lim_{r\to\infty} \lim_{t\to\infty} \frac{2\sqrt{2}}{3} \frac{t^{3/2}}{\log t} \psi(r, t, x + \sqrt{2}t).$$

$$\tag{4.38}$$

Note that the limit on the right-hand side exists by the proposition.

The definition of ψ in (4.7) and the change of variable $y' = y + x$ gives

$$\psi(r, t, x + \sqrt{2t}) = \frac{1}{\sqrt{2\pi(t-r)}} \int_{-x}^{\infty} u(r, x + y)$$
$$+ \sqrt{2}r)e^{y\sqrt{2}}e^{-\frac{y^2}{2(t-r)}} \left(1 - e^{-2(y+x)\frac{(x+\frac{3}{2\sqrt{2}}\log t)}{t-r}}\right) dy.$$

From this, a dominated convergence argument identical to the one in the proof of Proposition 4.4 implies

$$\lim_{t\to\infty} \frac{2\sqrt{2}}{3} \frac{t^{3/2}}{\log t} \psi(r, t, x + \sqrt{2t}) = \sqrt{\frac{2}{\pi}} \int_{-x}^{\infty} (y + x) \, u(r, x + y + \sqrt{2}r)e^{y\sqrt{2}} \, dy.$$

$$(4.39)$$

In the limit $r \to \infty$, the contribution for any finite interval is zero because $u(r, x + y + \sqrt{2}r) \to 0$ as $r \to \infty$. Therefore the integral can be replaced by $\int_0^\infty$. To establish 4.38, it remains to prove that

$$\lim_{r\to\infty} \int_0^{\infty} u(r, x + y + \sqrt{2}r)e^{y\sqrt{2}} \, dy = 0.$$
$$(4.40)$$

This is done exactly as in Lemma 4.6. In (4.33), take $A_1 = \infty$ and replace y^2 by y. Following the same estimate gives a Gaussian integral in (4.34) with $y/\sqrt{r}$ instead of y^2. Hence, the integral goes to zero as $r \to \infty$. □

4.2 Existence of a limiting process

Proof of Theorem 3.1 It suffices to show that, for $\phi \in C_c(\mathbb{R})$ positive, the Laplace transform $\Psi_t(\phi)$, defined in (3.1), of the extremal process of BBM converges. Remark first that this limit cannot be 0, since in the case of BBM it can be checked as in [5] that

$$\lim_{N\to\infty} \lim_{t\to\infty} \mathbb{P}[\mathcal{E}_t(B) > N] = 0, \text{ for any bounded measurable set } B \subset \mathbb{R}, \quad (4.41)$$

hence the limiting point process must be locally finite.

For convenience, we define $\max \mathcal{E}_t \equiv \max_{k \leq n(t)} x_k(t) - m(t)$. By Theorem 4.2 applied to the function

$$v(t, \delta + m(t)) = \mathbb{E}\left[\prod_{k=1}^{n(t)} \mathbb{1}_{\{x_k(t) - m(t) \leq \delta\}}\right] = \mathbb{P}[\max \mathcal{E}_t \leq \delta] \qquad (4.42)$$

it holds that

$$\lim_{\delta \to \infty} \lim_{t \to \infty} 1 - v(t, \delta + m(t)) = \lim_{\delta \to \infty} 1 - \omega(\delta) = 0. \qquad (4.43)$$

Now consider for $\delta > 0$

$$\mathbb{E}\left[\exp\left(-\int \phi \, d\mathcal{E}_t\right)\right] = \mathbb{E}\left[\exp\left(-\int \phi \, d\mathcal{E}_t\right) \mathbb{1}_{\{\max \mathcal{E}_t \leq \delta\}}\right]$$
$$+ \mathbb{E}\left[\exp\left(-\int \phi \, d\mathcal{E}_t\right) \mathbb{1}_{\{\max \mathcal{E}_t > \delta\}}\right] \qquad (4.44)$$

Note that by (4.43), the second term on the r.h.s of 4.44 satisfies

$$\limsup_{\delta \to \infty} \limsup_{t \to \infty} \mathbb{E}\left[\exp\left(-\int \phi d\mathcal{E}_t\right) \mathbb{1}_{\{\max \mathcal{E}_t > \delta\}}\right] \leq \limsup_{\delta \to \infty} \limsup_{t \to \infty} \mathbb{P}[\max \mathcal{E}_t > \delta] = 0.$$

It remains to address the first term on the r.h.s of (4.44). Write for convenience

$$\Psi_t^\delta(\phi) \equiv \mathbb{E}\left[\exp\left(-\int \phi \, d\mathcal{E}_t\right) \mathbb{1}_{\{\max \mathcal{E}_t \leq \delta\}}\right]. \qquad (4.45)$$

We claim that the limit

$$\lim_{\delta \to \infty} \lim_{t \to \infty} \Psi_t^\delta(\phi) \equiv \Psi(\phi) \qquad (4.46)$$

exists, and is strictly smaller than one. To see this, set $g_\delta(x) \equiv e^{-\phi(x)} \mathbb{1}_{\{x \leq \delta\}}$, and

$$u_\delta(t, x) \equiv \mathbb{E}\left[\prod_{k \leq n(t)} g_\delta(-x + x_k(t))\right]. \qquad (4.47)$$

By Lemma 4.1, u_δ is then solution to the F-KPP equation 1.2 with $u_\delta(0, x) = g_\delta(-x)$. Moreover, $g_\delta(-x) = 1$ for x large enough in the positive, and $g_\delta(-x) = 0$ for x large enough in the negative, so that conditions (1) and (2) of Theorem 4.2 are satisfied as well as (4.5) on the form of $m(t)$. Note that this would not be the case without the

presence of the cutoff. Therefore

$$u_\delta(t, x + m(t)) = \mathbb{E}\left[\prod_{k=1}^{n(t)} g_\delta(-x + x_k(t) - m(t))\right]$$

converges as $t \to \infty$ uniformly in x by Theorem 4.2. But

$$\Psi_t^\delta(\phi) = \mathbb{E}\left[\exp\left(-\int \phi \, d\mathcal{E}_t\right) \mathbb{1}_{\{\max \mathcal{E}_t \le \delta\}}\right]$$

$$= \mathbb{E}\left[\prod_{k \le n(t)} \exp(-\phi(x_k(t) - m(t))) \mathbb{1}_{\{x_k(t) - m(t) \le \delta\}}\right]$$

$$= \mathbb{E}\left[\prod_{k \le n(t)} g_\delta(x_k(t) - m(t))\right] = u_\delta(t, 0 + m(t)),$$

and therefore the limit $\lim_{t \to \infty} \Psi_t^\delta(\phi) \equiv \Psi^\delta(\phi)$ exists. But the function $\delta \mapsto \Psi^\delta(\phi)$ is increasing and bounded by construction. Therefore, $\lim_{\delta \to \infty} \Psi^\delta(\phi) = \Psi(\phi)$ exists. Moreover, there is the obvious bound

$$\mathbb{E}\left[\exp\left(-\int \phi \, d\mathcal{E}_t\right) \mathbb{1}_{\{\max \mathcal{E}_t \le \delta\}}\right] \le \mathbb{E}\left[\exp\left(-\phi(\max \mathcal{E}_t)\right) \mathbb{1}_{\{\max \mathcal{E}_t \le \delta\}}\right]$$

since the maximum is an atom of $\mathcal{E}_t$ and ϕ is non-negative. The limit $t \to \infty$ and $\delta \to \infty$ of the right side exists and is strictly smaller than 1 by the convergence in law of the re-centered maximum to $\omega(x)$ (note that the support of $\omega(x)$ is $\mathbb{R}$). Therefore $\Psi(\phi) = \lim_{\delta \to \infty} \lim_{t \to \infty} \Psi_t^\delta(\phi) < 1$ which proves (4.46). □

4.3 The Laplace functional and the F-KPP equation

Proof of Proposition 3.2 The proof of the proposition will be broken into proving two lemmas.

In the first lemma we establish an integral representation for the Laplace functionals of the extremal process of BBM which are truncated by a certain cutoff; in the second lemma we show that the results continues to hold when the cutoff is lifted. Throughout this section, $\phi : \mathbb{R} \to [0, \infty)$ is a non-negative continuous function with compact support.

Lemma 4.9 *Consider*

$$u_\delta(t, x) \equiv 1 - \mathbb{E}\left[\exp\left(-\sum_{k \le n(t)} \phi(-x + x_k(t))\right) \mathbb{1}_{\{\max_{k \le n(t)} -x + x_k(t) \le \delta\}}\right]. \quad (4.48)$$

Then $u_\delta(t, x)$ is the solution of the F-KPP equation (4.1) *with initial condition* $u_\delta(0, x) = 1 - e^{-\phi(-x)}1(-x \leq \delta)$. *Moreover, the following limit exists*

$$C(\phi, \delta) \equiv \lim_{t \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u_\delta(t, y + \sqrt{2}t) y e^{\sqrt{2}y} dy, \tag{4.49}$$

and

$$\lim_{t \to \infty} u_\delta(t, x + m(t)) = 1 - \mathbb{E}\left[\exp -C(\phi, \delta) Z e^{-\sqrt{2}x}\right]. \tag{4.50}$$

Proof The first part of the Lemma is proved in the proof of Theorem 3.1, whereas (4.50) follows from Theorem 4.2 and the representation (1.8). It remains to prove (4.49). The proof is a refinement of Proposition 4.4 that recovers the asymptotics (1.9).

For u_δ as above, let $\psi(r, t, x)$ be its approximation as in Proposition 4.3 and choose x, r so that $x \geq m(t) + 8r$. By Proposition 4.3 we then have the bounds

$$\frac{1}{\gamma(r)}\psi(r, t, x + m(t)) \leq u_\delta(t, x + m(t)) \leq \gamma(r)\psi(r, t, x + m(t)) \tag{4.51}$$

where

$$\psi(r, t, x + m(t)) = \frac{t^{3/2}e^{-\sqrt{2}x}}{\sqrt{2\pi(t-r)}} \int_0^\infty u_\delta(r, y' $$
$$+ \sqrt{2}r)e^{y'\sqrt{2}}e^{-\frac{(y'-x+\frac{3}{2\sqrt{2}\log t})^2}{2(t-r)}} \left(1 - e^{-2\frac{y'x}{t-r}}\right) dy'.$$

Using dominated convergence as in Proposition 4.4, one gets

$$\lim_{t \to \infty} \psi(r, t, x + m(t)) = \frac{2xe^{-\sqrt{2}x}}{\sqrt{2\pi}} \int_0^\infty u_\delta(r, y' + \sqrt{2}r)y'e^{y'\sqrt{2}}dy'.$$

Putting this back in (4.51),

$$\frac{1}{\gamma(r)}C(r) \leq \lim_{t \to \infty} \frac{u_\delta(t, x + m(t))}{xe^{-\sqrt{2}x}} \leq \gamma(r)C(r), \tag{4.52}$$

for $C(r) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty u_\delta(r, y' + \sqrt{2}r)y'e^{y'\sqrt{2}}dy'$, and $x > 8r$. We know that $\lim_{r \to \infty} C(r) \equiv C > 0$ exists by Proposition 4.4. Thus taking $x = 9r$, letting

$r \to \infty$ in (4.52), and using that $\gamma(r) \downarrow 1$, one has

$$\lim_{x\to\infty} \lim_{t\to\infty} \frac{u_\delta(t, x + m(t))}{xe^{-\sqrt{2}x}} = \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u_\delta(r, y' + \sqrt{2}r) y' e^{y'\sqrt{2}} dy'. \quad (4.53)$$

On the other hand, the representation (1.8) and the asymptotics (1.9) yield

$$\lim_{x\to\infty} \lim_{t\to\infty} \frac{u_\delta(t, x + m(t))}{xe^{-\sqrt{2}x}} = \lim_{x\to\infty} \frac{1 - \mathbb{E}\left[\exp\left(-C(\phi, \delta)Ze^{-\sqrt{2}x}\right)\right]}{xe^{-\sqrt{2}x}} = C(\phi, \delta).$$
$$(4.54)$$

The claim follows from the last two equations. □

The results of Lemma 4.9 also hold when the cutoff is removed. Essentially, one needs to prove that the limit $t \to \infty$ and $\delta \to \infty$ can be interchanged. The proof shows (non-uniform) convergence of the solution of the F-KPP equation when one condition of Theorem 4.2 is not fulfilled. With an appropriate continuity argument, a Lalley-Sellke type representation is also recovered.

Lemma 4.10 *Let $u(t, x)$, $u_\delta(t, x)$ be solutions of the F-KPP equation (4.1) with initial condition $u(0, x) = 1 - e^{-\phi(-x)}$ and $u_\delta(0, x) = 1 - e^{-\phi(-x)}\mathbb{1}_{\{-x\le\delta\}}$, respectively. Set $C(\delta, \phi) \equiv \lim_{t\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u_\delta(t, y+\sqrt{2}t) ye^{\sqrt{2}y} dy$. Then $\lim_{\delta\to\infty} C(\phi, \delta)$ exists, and*

$$C(\phi) \equiv \lim_{\delta\to\infty} C(\phi, \delta) = \lim_{t\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u(t, y + \sqrt{2}t) ye^{\sqrt{2}y} dy. \quad (4.55)$$

Moreover

$$\lim_{t\to\infty} u(t, x + m(t)) = 1 - \mathbb{E}\left[\exp{-C(\phi)Ze^{-\sqrt{2}x}}\right]. \quad (4.56)$$

Proof Since ϕ is non-negative, it is easy to see from Lemma 4.1 that

$$0 \le u_\delta(t, x) - u(t, x) \le \mathbb{P}(\max x_k(t) > \delta + x), \quad (4.57)$$

from which it follows that

$$\int_0^\infty u_\delta(t, x + \sqrt{2}t)xe^{\sqrt{2}x}\,dx - \int_0^\infty \mathbb{P}[\max x_k(t) - \sqrt{2}t > \delta + x]xe^{\sqrt{2}x}\,dx$$

$$\leq \int_0^\infty u(t, x + \sqrt{2}t)xe^{\sqrt{2}x}\,dx$$

$$\leq \int_0^\infty u_\delta(t, x + \sqrt{2}t)xe^{\sqrt{2}x}\,dx. \tag{4.58}$$

Define

$$F(t, \delta) \equiv \int_0^\infty u_\delta(t, x + \sqrt{2}t)xe^{\sqrt{2}x}\,dx, \tag{4.59}$$

$$U(t) \equiv \int_0^\infty u(t, x + \sqrt{2}t)xe^{\sqrt{2}x}\,dx, \tag{4.60}$$

and

$$M(t, \delta) \equiv \int_0^\infty \mathbb{P}[\max x_k(t) - \sqrt{2}t > \delta + x]xe^{\sqrt{2}x}\,dx. \tag{4.61}$$

The inequalities in (4.58) then read

$$F(t, \delta) - M(t, \delta) \leq U(t) \leq F(t, \delta). \tag{4.62}$$

By Lemma 4.8, it holds

$$\lim_{\delta \to \infty} \lim_{t \to \infty} M(t, \delta) = 0. \tag{4.63}$$

Remark also that Proposition 4.4 implies that for each δ, $\lim_{t \to \infty} F(t, \delta) \equiv F(\delta)$ exists and is strictly positive. We thus deduce from (4.62) that

$$\liminf_{\delta \to \infty} F(\delta) \leq \liminf_{t \to \infty} U(t) \leq \limsup_{t \to \infty} U(t) \leq \limsup_{\delta \to \infty} F(\delta). \tag{4.64}$$

We claim that $\lim_{\delta \to \infty} F(\delta)$ exists, is strictly positive and finite. To see this, we first observe that the function $\delta \to F(\delta)$ is by construction decreasing, and positive, therefore the limit $\lim_{\delta \to \infty} F(\delta)$ exists. Strict positivity is proved in a somewhat

indirect fashion: we proceed by contradiction, and rely on the convergence of the process of cluster extrema. Assume that

$$\lim_{\delta \to \infty} F(\delta) = 0, \tag{4.65}$$

and thus that

$$\lim_{t \to \infty} U(t) = 0. \tag{4.66}$$

Using the form of the Laplace functional of a Poisson process, we have that

$$
E\left[\exp\left(-\int \phi(x)\Pi_t(\mathrm{d}x)\right)\right]
$$
$$
= E\left[\exp\left(-Z\sqrt{\frac{2}{\pi}}\int_{-\infty}^{0}\left(1 - E\left[\exp - \sum_{k \leq n(t)} \phi(x + x_k(t) - \sqrt{2}t)\right]\right)\right.\right.
$$
$$
\left.\left. (-x)e^{-\sqrt{2}x}\mathrm{d}x\right)\right]
$$
$$
\stackrel{x \mapsto -x}{=} E\left[\exp\left(-Z\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\left(1 - E\left[\exp - \sum_{k \leq n(t)} \phi(-x + x_k(t) - \sqrt{2}t)\right]\right)\right.\right.
$$
$$
\left.\left. xe^{\sqrt{2}x}\mathrm{d}x\right)\right]
$$
$$
= E\left[\exp\left(-Z\sqrt{\frac{2}{\pi}}U(t)\right)\right]. \tag{4.67}
$$

Therefore, (4.66) would imply that

$$
\lim_{t \to \infty} E\left[\exp\left(-\int \phi(x)\Pi_t(\mathrm{d}x)\right)\right] = E\left[\exp\left(-Z\sqrt{\frac{2}{\pi}}\lim_{t \to \infty} U(t)\right)\right] = 1. \tag{4.68}
$$

This cannot hold. In fact, for Π_t^{ext} the process of the cluster-extrema defined earlier, one has the obvious bound

$$
E\left[\exp\left(-\int \phi(x)\Pi_t(\mathrm{d}x)\right)\right] \leq E\left[\exp\left(-\int \phi(x)\Pi_t^{\text{ext}}(\mathrm{d}x)\right)\right]. \tag{4.69}
$$

Since the process of cluster-extrema converges, by Proposition 3.7, to a PPP $(CZ\sqrt{2}e^{-\sqrt{2}x}dx)$,

$$\lim_{t\to\infty} E\left[\exp\left(-\int \phi(x)\Pi_t(dx)\right)\right] \le E\left[\exp\left(-CZ\int\left\{1-e^{-\phi(x)}\right\}\right.\right.$$
$$\left.\left.\sqrt{2}e^{-\sqrt{2}x}dx\right)\right] < 1. \tag{4.70}$$

This contradicts (4.68) and therefore also (4.66). □

Proposition 3.2 now follows directly from Lemma 4.10. □

Proof of Corollary 3.3 Consider n independent copies $\mathcal{E}^{(i)}$, $i = 1,\ldots,n$ of $\mathcal{E}$. By independence and by Proposition 3.2, the Laplace functional for $\phi \in C_c(\mathbb{R})$ of the superposition of these is

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^{n}\int \phi(y+x)\mathcal{E}_t^{(i)}(dy)\right)\right] = \mathbb{E}\left[\exp\left(-C(\phi)\sum_{i=1}^{n} Z^{(i)}e^{-\sqrt{2}x}\right)\right]$$

where $Z^{(i)}$, $i = 1,\ldots,n$ are i.i.d. copies of the derivative martingale. The log of the variable $\sum_{i=1}^{n} Z^{(i)}$ acts simply as a random shift in x. Therefore $\sum_{i=1}^{n} \mathcal{E}_t^{(i)}$ has the same law as $\mathcal{E}$ shifted by log of Z. □

4.4 The auxiliary process

Recall from Sect. 3.2 that $(\Omega', \mathcal{F}', P)$ stands for the probability space on which the auxiliary point process Π_t is defined. The expectation under P is denoted by E.

4.4.1 The process of cluster-extrema

In this section, we prove Proposition 3.7 on the convergence of the process of the cluster-extrema to a PPP$(CZ\sqrt{2}e^{-\sqrt{2}x}dx)$. We remark that the proof is independent of Lemma 4.10 which uses the result.

A consequence of Lemma 4.8 is the following vague convergence of the maximum when integrated over the appropriate density.

Lemma 4.11 *Let $b \in \mathbb{R}$ and $h(x) = \mathbb{1}_{[b,\infty)}(x)$. Then*

$$\lim_{t\to\infty} \int_{-\infty}^{0} \left\{\int_{\mathbb{R}} h(x)\, P(y+\max_i x_i(t) - \sqrt{2}t \in dx)\right\} \sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}dy = Ce^{-\sqrt{2}b}. \tag{4.71}$$

Moreover, the convergence also holds for a continuous function $h(x)$ that is bounded and is zero for x small enough:

$$\lim_{t \to \infty} \int_{-\infty}^{0} \left\{ \int_{\mathbb{R}} h(x) \, P(y + \max_i x_i(t) - \sqrt{2}t \in dx) \right\} \sqrt{\frac{2}{\pi}} (-y) e^{-\sqrt{2}y} dy$$

$$= \int_{\mathbb{R}} h(z) \sqrt{2} C e^{-\sqrt{2}z} dz \qquad (4.72)$$

Proof The first assertion is direct from Lemma 4.8 with initial condition $u(0, x) = \mathbb{1}_{\{x < 0\}}$. For the second claim, note that the statement holds for any linear combination of functions of the form $\mathbb{1}_{[b, \infty)}(x)$. Suppose for simplicity that $|h(x)| \le 1$. By assumption, it is possible to find $\delta > 0$ such that $h(x) = 0$ for $x < -\delta$. Moreover, for any $\varepsilon > 0$, δ can be taken large enough so that $Ce^{-\sqrt{2}\delta} < \varepsilon$. On $[-\delta, \delta]$, h can be approximated uniformly by linear combinations of indicator functions of the form $\mathbb{1}_{[b, \infty)}(x)$. Therefore, (4.72) holds for h restricted to this interval. On (δ, ∞), the term

$$\lim_{t \to \infty} \int_{-\infty}^{0} \left\{ \int_{\delta}^{\infty} h(x) \, P(y + \max_i x_i(t) - \sqrt{2}t \in dx) \right\} \sqrt{\frac{2}{\pi}} (-y) e^{-\sqrt{2}y} dy \le Ce^{-\sqrt{2}\delta}$$

is smaller than ε by the choice of δ and the bound $|h(x)| \le 1$. Since ε is arbitrary, the conclusion holds. $\qquad \square$

Proof of Proposition 3.7 Consider

$$E\left[\exp\left(-\sum_i \phi(\eta_i + M^{(i)}(t) - \sqrt{2}t) \right) \right] \qquad (4.73)$$

where $\eta = (\eta_i)$ is a Poisson process on $(-\infty, 0)$ with intensity measure $\sqrt{\frac{2}{\pi}} (-y) e^{-\sqrt{2}y} dy$, and $M^{(i)}(t) \equiv \max_k x_k^{(i)}(t)$. We show that

$$\lim_{t \to \infty} E\left[\exp\left(-\sum_i \phi(\eta_i + M^{(i)}(t) - \sqrt{2}t) \right) \right]$$

$$= \exp\left(-C \int_{\mathbb{R}} (1 - e^{-\phi(a)}) \sqrt{2} e^{-\sqrt{2}a} da \right). \qquad (4.74)$$

Since the underlying process is Poisson and the $M^{(i)}$'s are i.i.d. ,

$$E\left[\exp\left(-\sum_i \phi(\eta_i + M^{(i)}(t) - \sqrt{2}t)\right)\right]$$

$$= \exp\left(-\int_{-\infty}^{0} E\left[1 - e^{-\phi(y+M(t)-\sqrt{2}t)}\right] \sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}dy\right),$$

where $M(t)$ has the same distribution as one copy of the variables $M^{(i)}(t)$. The result then follows from Lemma 4.11 with $h(x) = 1 - e^{-\phi(x)}$ after taking the limit. □

4.4.2 Identification with the extremal process

Proposition 4.10 yields a short proof of Theorem 3.6.

Proof of Theorem 3.6 The Laplace functional of Π_t using the form of the Laplace functional of a Poisson process reads

$$E\left[\exp\left(-\int \phi(x)\Pi_t(dx)\right)\right]$$

$$= E\left[\exp\left(-\int_{-\infty}^{0}\left\{1-E\left[\exp\left(-\sum_{k\leq n(t)} \phi(x+x_k(t) - \sqrt{2}t + \frac{1}{\sqrt{2}}\log Z)\right)\right]\right\}\right.\right.$$
$$\left.\left.\sqrt{\frac{2}{\pi}}(-x)e^{-\sqrt{2}x}dx\right)\right]$$

$$= E\left[\exp\left(-\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} u(t, x + \sqrt{2}t - \frac{1}{\sqrt{2}}\log Z)xe^{\sqrt{2}x}dx\right)\right], \qquad (4.75)$$

with

$$u(t, x) = 1 - E\left[\exp\left(-\sum_{k=1}^{n(t)} \phi(-x + x_k(t))\right)\right].$$

By (4.37),

$$\lim_{t\to\infty} \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} u(t, x + \sqrt{2}t - \frac{1}{\sqrt{2}}\log Z)xe^{\sqrt{2}x}dx$$

$$= Z\sqrt{\frac{2}{\pi}} \lim_{t\to\infty} \int_{0}^{\infty} u(t, x + \sqrt{2}t)xe^{\sqrt{2}x}dx,$$

and the limit exists and is strictly positive by Lemma 4.10. This implies that the Laplace functionals of $\lim_{t\to\infty} \Pi_t$ and of the extremal process of BBM are equal. $\qquad\square$

4.4.3 Properties of the clusters

In this section we prove Proposition 3.8 and Theorem 3.4.

Proof of Proposition 3.8 Throughout the proof, the probabilities are considered conditional on Z. We show that for $\varepsilon > 0$ there exist A_1, A_2 such that

$$\sup_{t \ge t_0} P\left[\exists_{i,k} : \eta_i + \frac{1}{\sqrt{2}} \log Z + x_k^{(i)}(t) - \sqrt{2}t \ge Y, \text{ but } \eta_i \notin [-A_2\sqrt{t}, -A_1\sqrt{t}]\right] \le \varepsilon. \tag{4.76}$$

We claim that, for t large enough, there exists $A_1 > 0$ small enough such that

$$P\left[\exists_{i,k} : \eta_i + \frac{1}{\sqrt{2}} \log Z + x_k^{(i)}(t) - \sqrt{2}t \ge Y, \text{ but } \eta_i \ge -A_1\sqrt{t}\right] \le \varepsilon/2, \tag{4.77}$$

and $A_2 > 0$ large enough such that

$$P\left[\exists_{i,k} : \eta_i + \frac{1}{\sqrt{2}} \log Z + x_k^{(i)}(t) - \sqrt{2}t \ge Y, \text{ but } \eta_i \le -A_2\sqrt{t}\right] \le \varepsilon/2. \tag{4.78}$$

By Markov's inequality and the fact that $\sqrt{\frac{2}{\pi}} < 1$, the left-hand side of (4.77) is less than

$$\int_{-A_1\sqrt{t}}^{0} P\left[\max_k x_k(t) \ge \sqrt{2}t - x + Y - \frac{1}{\sqrt{2}} \log Z\right] (-x)e^{-\sqrt{2}x} dx$$

$$= \int_{0}^{A_1\sqrt{t}} P\left[\max_k x_k(t) \ge \sqrt{2}t + x + Y - \frac{1}{\sqrt{2}} \log Z\right] xe^{\sqrt{2}x} dx \tag{4.79}$$

This can be smaller than $\varepsilon/2$ by Lemma 4.6 by taking A_1 small enough. The same reasoning can be applied to (4.78) thereby establishing the proposition. $\qquad\square$

Proof of Theorem 3.4 Let $(x_i(t), i \le n(t))$ be a standard BBM of law $\mathbb{P}$. Let $a \in I$, a compact interval in $(-\infty, 0)$ and $b \in \mathbb{R}$. Set $x = a\sqrt{t} + b$. Consider

$$\overline{\mathcal{E}}_t \equiv \sum_i \delta_{x_i(t)-\sqrt{2}t}, \tag{4.80}$$

and $\max \overline{\mathcal{E}}_t \equiv \max_i x_i(t) - \sqrt{2}t$. We first claim that $x + \max \overline{\mathcal{E}}_t$ conditionally on $\{x + \max \overline{\mathcal{E}}_t > 0\}$ converges weakly to an exponential random variable,

$$\lim_{t \to \infty} \mathbb{P}\left[x + \max \overline{\mathcal{E}}_t > X \mid x + \max \overline{\mathcal{E}}_t > 0\right] = e^{-\sqrt{2}X}, \qquad (4.81)$$

for $X > 0$ and 0 otherwise. Remark, in particular, that the limit does not depend on x.

To see (4.81), we write the conditional probability as

$$\frac{\mathbb{P}\left[x + \max \overline{\mathcal{E}}_t > X\right]}{\mathbb{P}\left[x + \max \overline{\mathcal{E}}_t > 0\right]}. \qquad (4.82)$$

For t large enough (and hence $-x$ large enough in the positive) we may apply the uniform bounds from Proposition 4.3 in the form

$$\mathbb{P}\left[\max_{k \leq n(t)} x_k(t) \geq X - x + \sqrt{2}t\right] \leq \gamma(r)\psi(r, t, X - x + \sqrt{2}t) \qquad (4.83)$$

and

$$\mathbb{P}\left[\max_{k \leq n(t)} x_k(t) \geq -x + \sqrt{2}t\right] \geq \gamma^{-1}(r)\psi(r, t, -x + \sqrt{2}t) \qquad (4.84)$$

where ψ is as in (4.7) and the u entering into its definition is solution to F-KPP with Heaviside initial conditions, and r is large enough. Therefore,

$$\gamma^{-2}(r)\frac{\psi(r, t, X - x + \sqrt{2}t)}{\psi(r, t, -x + \sqrt{2}t)} \leq \frac{\mathbb{P}[x + \max \overline{\mathcal{E}}_t > X]}{\mathbb{P}[x + \max \overline{\mathcal{E}}_t > 0]}$$
$$\leq \gamma^2(r)\frac{\psi(r, t, X - x + \sqrt{2}t)}{\psi(r, t, -x + \sqrt{2}t)}. \qquad (4.85)$$

By Lemma 4.5,

$$\lim_{t \to \infty} \frac{\psi(r, t, X - x + \sqrt{2}t)}{\psi(r, t, -x + \sqrt{2}t)} = e^{-\sqrt{2}X}. \qquad (4.86)$$

Taking the limit $t \to \infty$ first, and then $r \to \infty$ (and using that $\gamma(r) \downarrow 1$) we thus see that (4.85) implies (4.81).

Second, we show that for any function ϕ that is continuous with compact support, the limit of

$$\mathbb{E}\left[\exp - \int \phi(x + z)\overline{\mathcal{E}}_t(\mathrm{d}z)\Big| x + \max \overline{\mathcal{E}}_t > 0\right] \qquad (4.87)$$

exists and is independent of x. It follows from the first part of the proof that the conditional process has a maximum almost surely. It is thus sufficient to consider the truncated Laplace functional, that is for $\delta > 0$,

$$\mathbb{E}\left[\exp\left(-\int \phi(x+z)\overline{\mathcal{E}}_t(\mathrm{d}z)\right)\mathbb{1}_{\{x+\max\overline{\mathcal{E}}_t\leq\delta\}}\Big| x+\max\overline{\mathcal{E}}_t > 0\right] \qquad (4.88)$$

The above conditional expectation can be written as

$$= \frac{\mathbb{E}\left[\prod_{k=1}^{n(t)}\exp\left(-\phi(x+x_k(t)-\sqrt{2}t)\right)\mathbb{1}_{\{x+x_k(t)-\sqrt{2}t\leq\delta\}}\right]}{\mathbb{P}\left[x+\max\overline{\mathcal{E}}_t > 0\right]}$$

$$- \frac{\mathbb{E}\left[\prod_{k=1}^{n(t)}\exp\left(-\phi(x+x_k(t)-\sqrt{2}t)\right)\mathbb{1}_{\{x+x_k(t)-\sqrt{2}t\leq 0\}}\right]}{\mathbb{P}\left[x+\max\overline{\mathcal{E}}_t > 0\right]}. \qquad (4.89)$$

Define

$$u_1(t, y) \equiv 1 - \mathbb{E}\left[\prod_{k=1}^{n(t)} e^{-\phi(-y+x_k(t))}\mathbb{1}_{\{-y+x_k(t)\leq 0\}}\right]$$

$$u_2(t, y) \equiv 1 - \mathbb{E}\left[\prod_{k=1}^{n(t)} e^{-\phi(-y+x_k(t))}\mathbb{1}_{\{-y+x_k(t)\leq\delta\}}\right] \qquad (4.90)$$

$$u_3(t, y) \equiv \mathbb{P}\left[-y + \max_k x_k(t) \leq 0\right]$$

so that

$$(4.89) = \frac{u_2(t, -x+\sqrt{2}t)}{u_3(t, -x+\sqrt{2}t)} - \frac{u_1(t, -x+\sqrt{2}t)}{u_3(t, -x+\sqrt{2}t)}. \qquad (4.91)$$

Remark that the functions u_1, u_2 and u_3, all solve the F-KPP equation 4.1 with initial conditions

$$\begin{aligned} u_1(0, y) &= 1 - e^{-\phi(-y)}\mathbb{1}_{\{-y\leq 0\}}, \\ u_2(0, y) &= 1 - e^{-\phi(-y)}\mathbb{1}_{\{-y\leq\delta\}}, \\ u_3(0, y) &= 1 - \mathbb{1}_{\{-y\leq 0\}}. \end{aligned} \qquad (4.92)$$

They also satisfy the assumptions of Theorem 4.2 and Proposition 4.3.

Let ψ_i be as in (4.7) with u replaced by the appropriate u_i, $i = 1, 2, 3$. By Proposition 4.3,

$$\lim_{t\to\infty} \frac{u_2(t, -x+\sqrt{2}t)}{u_3(t, -x+\sqrt{2}t)} - \frac{u_1(t, -x+\sqrt{2}t)}{u_3(t, -x+\sqrt{2}t)}$$

$$= \lim_{r\to\infty}\lim_{t\to\infty}\left\{\frac{\psi_2(r, t, -x+\sqrt{2}t)}{\psi_3(r, t, -x+\sqrt{2}t)}\right\} - \lim_{r\to\infty}\lim_{t\to\infty}\left\{\frac{\psi_1(r, t, -x+\sqrt{2}t)}{\psi_3(r, t, -x+\sqrt{2}t)}\right\}. \qquad (4.93)$$

By Lemma 4.5, the above limits exist and do not depend on x. This shows the existence of (4.88). It remains to prove that this limit is non-zero for a non-trivial ϕ, thereby showing the existence and local finiteness of the conditional point process. To see this, note that, by Proposition 4.4, the limit of (4.88) equals

$$\frac{C_2 - C_1}{C_3}$$

where $C_i = \lim_{t \to \infty} \int_{-\infty}^{0} u_i(t, y' + \sqrt{2}t)(-y')e^{-\sqrt{2}y'} \, dy'$ for u_i as above. Note that $0 < C_3 < \infty$, since by the representation of Proposition 3.2

$$\lim_{t \to \infty} \mathbb{P}\left(\max_{i \leq n(t)} x_i(t) - m(t) \leq z\right) = \mathbb{E}\left[\exp -C_3 Z e^{-\sqrt{2}z}\right],$$

and this probability is non-trivial. Now suppose $C_1 = C_2$. Then by Theorem 3.1 again, this would entail

$$\lim_{t \to \infty} \mathbb{E}\left[\exp\left(-\int \phi(x)\mathcal{E}_t(dx)\right)\mathbb{1}_{\{\max \mathcal{E}_t < \delta\}}\right]$$

$$= \lim_{t \to \infty} \mathbb{E}\left[\exp\left(-\int \phi(x)\mathcal{E}_t(dx)\right)\mathbb{1}_{\{\max \mathcal{E}_t < 0\}}\right],$$

where $\mathcal{E}_t = \sum_i \delta_{x_i(t) - m(t)}$ and $\max \mathcal{E}_t = \max_i x_i(t) - m(t)$. Thus,

$$\lim_{t \to \infty} \mathbb{E}\left[\exp\left(\int \phi(x)\mathcal{E}_t(dx)\right)\mathbb{1}_{\{0 < \max \mathcal{E}_t < \delta\}}\right] = 0.$$

But this is impossible since the maximum has positive probability of occurrence in $[0, \delta]$ for any δ and the process $\lim_{t \to \infty} \mathcal{E}_t$ is locally finite. This concludes the proof of the Proposition. □

Define the gap process at time t

$$\mathcal{D}_t \equiv \sum_i \delta_{x_i(t) - \max_j x_j(t)}. \tag{4.94}$$

Let us write $\overline{\mathcal{E}}$ for the point process obtained in Theorem 3.4 from the limit of the conditional law of $\overline{\mathcal{E}}_t$ given $\max \overline{\mathcal{E}}_t > 0$. We denote by $\max \overline{\mathcal{E}}$ the maximum of $\overline{\mathcal{E}}$, and by $\mathcal{D}$ the process of the gaps of $\overline{\mathcal{E}}$, that is the process $\overline{\mathcal{E}}$ shifted back by $\max \overline{\mathcal{E}}$. The following corollary is the fundamental result showing that $\mathcal{D}$ is the limit of the conditioned process $\mathcal{D}_t$, and, perhaps surprisingly, the process of the gaps in the limit is independent of the location of the maximum.

Corollary 4.12 *Let $x = a\sqrt{t}$, $a < 0$. In the limit $t \to \infty$, the random variables $\mathcal{D}_t$ and $x + \max \overline{\mathcal{E}}$ are conditionally independent on the event $x + \max \overline{\mathcal{E}} > b$ for any $b \in \mathbb{R}$. More precisely, for any bounded continuous function f, h and $\phi \in C_c(\mathbb{R})$,*

$$\lim_{t \to \infty} \mathbb{E}\left[f\left(\int \phi(z)\mathcal{D}_t(dz) \right) h(x + \max \overline{\mathcal{E}}_t) \Big| x + \max \overline{\mathcal{E}}_t > b \right]$$

$$= \mathbb{E}\left[f\left(\int \phi(z)\mathcal{D}(dz) \right) \right] \int_b^\infty h(y) \frac{\sqrt{2}e^{-\sqrt{2}y}dy}{e^{-\sqrt{2}b}}. \qquad (4.95)$$

Moreover, convergence is uniform in $x = a\sqrt{t}$ for a in a compact set.

Proof By standard approximation, it suffices to establish the result for $h(y) = \mathbb{1}_{\{y > b'\}}$ for $b' > b$. By the property of conditioning and since $b' > b$

$$\mathbb{E}\left[f\left(\int \phi(z)\mathcal{D}_t(dz) \right) \mathbb{1}_{\{x + \max \mathcal{E}_t > b'\}} \Big| x + \max \overline{\mathcal{E}}_t > b \right]$$

$$= \mathbb{E}\left[f\left(\int \phi(z)\mathcal{D}_t(dz) \right) \Big| x - b' + \max \overline{\mathcal{E}}_t > 0 \right] \frac{\mathbb{P}[x - b + \max \overline{\mathcal{E}}_t > b' - b]}{\mathbb{P}[x - b + \max \overline{\mathcal{E}}_t > 0]}.$$

The conclusion will follow from Theorem 3.4 by taking the limit $t \to \infty$, once it is shown that convergence of $(\overline{\mathcal{E}}_t, y + \max \overline{\mathcal{E}}_t)$ under the conditional law implies convergence of the gap process $\mathcal{D}_t$. This is a general continuity result which is done in the next lemma. $\qquad \square$

Lemma 4.13 *Let (μ_t, X_t) be a sequence of random variables on $(\mathcal{M} \times \mathbb{R}, \mathbb{P})$ that converges to (μ, X) in the sense that for any bounded continuous function f, h on $\mathbb{R}$ and any $\phi \in \mathcal{C}_c(\mathbb{R})$*

$$\mathbb{E}\left[f\left(\int \phi(y)\mu_t(dy) \right) h(X_t) \right] \to \mathbb{E}\left[f\left(\int \phi(y)\mu(dy) \right) h(X) \right]. \qquad (4.96)$$

Then for any $\phi \in \mathcal{C}_c(\mathbb{R})$ and $g : \mathbb{R} \to \mathbb{R}$, bounded continuous,

$$\mathbb{E}\left[g\left(\int \phi(y + X_t)\mu_t(dy) \right) \right] \to \mathbb{E}\left[g\left(\int \phi(y + X)\mu(dy) \right) \right]. \qquad (4.97)$$

Proof Let $\varepsilon > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Introduce the notation

$$T_x \mu(\phi) \equiv \int \phi(y + x)\mu(dx). \qquad (4.98)$$

We need to show that for t large enough

$$\left| \mathbb{E}\left[f\left(T_{X_t} \mu_t(\phi) \right) \right] - \mathbb{E}\left[f\left(T_X \mu(\phi) \right) \right] \right| < \varepsilon. \qquad (4.99)$$

By standard approximations, it is enough to suppose f is Lipschitz, whose constant we assume to be 1 for simplicity.

Since the random variables (X_t) are tight by assumption, there exists a compact interval K_ε of $\mathbb{R}$ such that for t large enough

$$(4.99) \le \left| \mathbb{E}\left[f\left(T_{X_t} \mu_t(\phi)\right) ; X_t \in K_\varepsilon \right] - \mathbb{E}\left[f\left(T_X \mu(\phi)\right) ; X \in K_\varepsilon \right] \right| + \varepsilon. \quad (4.100)$$

Now divide K_ε into N intervals I_j of equal length. Write $\bar{x}_j$ for the midpoint of I_j. For each of these intervals, one has

$$\mathbb{E}\left[f(T_{X_t} \mu_t(\phi)); X_t \in I_j \right] = \mathbb{E}\left[f(T_{\bar{x}_j} \mu_t(\phi)); X_t \in I_j \right] + \mathcal{R}(t, j) \quad (4.101)$$

for

$$\mathcal{R}(t, j) \le \mathbb{E}\left[| f(T_{X_t} \mu_t(\phi)) - f(T_{\bar{x}_j} \mu_t(\phi)) |; X_t \in I_j \right]$$

Since f is Lipschitz, the right-hand side is smaller than

$$\mathbb{E}[| T_{X_t} \mu_t(\phi) - T_{\bar{x}_j} \mu_t(\phi) |; X_t \in I_j]. \quad (4.102)$$

Moreover

$$| T_{X_t} \mu_t(\phi) - T_{\bar{x}_j} \mu_t(\phi) | \le \int | \phi(y + X_t) - \phi(y + \bar{x}_j) | \mu_t(dy).$$

Note that there exists a compact C, independently of t and j so that $| \phi(y + X_t) - \phi(y + \bar{x}_j) | = 0$ for $y \notin C$ (it suffices to take C so that it contains all the translates $\mathrm{supp}\, \phi + k, k \in K_\varepsilon$). By taking N large enough, $|y + X_t - (y + \bar{x}_j)| = |\bar{x}_j - X_t| < \delta_\phi$ for the appropriate δ_ϕ making $| \phi(y + X_t) - \phi(y + \bar{x}_j) | < \varepsilon$, uniformly on $y \in C$. Hence, 4.102 is smaller than

$$\varepsilon \mathbb{E}\left[\mu_t(C); X_t \in I_j \right].$$

The summation over j is thus smaller than $\varepsilon \mathbb{E}[\mu_t(C)]$. By the convergence of (μ_t), this can be made smaller for t large enough.

The same approximation scheme for (μ, X) yields

$$\mathbb{E}\left[f\left(T_X \mu(\phi)\right); X_t \in I_j \right] = \mathbb{E}\left[f\left(T_{\bar{x}_j} \mu(\phi)\right); X \in I_j \right] + \mathcal{R}(j) \quad (4.103)$$

where $\sum_j \mathcal{R}(j) \le \varepsilon \mathbb{E}[\mu(C)]$. Therefore (4.99) will hold provided that the difference of the first terms of the right-hand side of (4.101) and of (4.103) is small for t large enough and N fixed. But this is guaranteed by the hypotheses on the convergence of (μ_t, X_t). $\qquad \square$

4.5 Characterisation of the extremal process

Proof of Theorem 2.1 It suffices to show that for $\phi : \mathbb{R} \to \mathbb{R}_+$ continuous with compact support, the Laplace functional $\Psi_t(\phi)$ of the extremal process $\mathcal{E}_t$ of BBM satisfies

$$\lim_{t \to \infty} \Psi_t(\phi) = E\left[\exp\left(-CZ \int_{\mathbb{R}} E[1 - e^{-\int \phi(y+z)\mathcal{D}(\mathrm{d}z)}]\sqrt{2}e^{-\sqrt{2}y}\mathrm{d}y\right)\right]$$

(4.104)

for the point process $\mathcal{D}$ of Corollary 4.12.

Now, by Theorem 3.6,

$$\lim_{t \to \infty} \Psi_t(\phi) = \lim_{t \to \infty} E\left[\exp\left(-\sum_{i,k}\phi(\eta_i + \frac{1}{\sqrt{2}}\log Z + x_k^{(i)}(t) - \sqrt{2}t)\right)\right].$$

(4.105)

Using the form for the Laplace transform of a Poisson process we have for the right side

$$\lim_{t \to \infty} E\left[\exp\left(-\sum_{i,k}\phi(\eta_i + \frac{1}{\sqrt{2}}\log Z + x_k^{(i)}(t) - \sqrt{2}t)\right)\right]$$

$$= E\left[\exp\left(-Z\lim_{t \to \infty}\int_{-\infty}^{0}E\left[1 - \exp\left(-\int\phi(x+y)\overline{\mathcal{E}}_t(\mathrm{d}x)\right)\right]\right.\right.$$

$$\left.\left.\sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}\mathrm{d}y\right)\right].$$

(4.106)

Let $\mathcal{D}_t$ as in (4.94). The integral on the right-hand side above can be written as

$$\lim_{t \to \infty}\int_{-\infty}^{0}E\left[f\left(\int\{T_{y+\max\overline{\mathcal{E}}_t}\phi(z)\}\mathcal{D}_t(\mathrm{d}z)\right)\right]\sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}\mathrm{d}y$$

for the bounded (on $[0, \infty)$) continuous function $f(x) = 1-e^{-x}$, and where $T_x\phi(y) = \phi(y+x)$. Note that $f(0) = 0$. By Proposition 3.8, there exist A_1 and A_2 suchthat

$$\int_{-\infty}^{0} E\left[f\left(\int \left\{T_{y+\max \overline{\mathcal{E}}_t}\phi(z)\right\} \mathcal{D}_t(\mathrm{d}z)\right)\right]\sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}\,\mathrm{d}y$$

$$= \Omega_t(A_1, A_2) + \int_{-A_2\sqrt{t}}^{-A_1\sqrt{t}} E\left[f\left(\int \left\{T_{y+\max \overline{\mathcal{E}}_t}\phi(z)\right\} \mathcal{D}_t(\mathrm{d}z)\right)\right]\sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}\,\mathrm{d}y,$$

$$(4.107)$$

where the error term satisfies $\lim_{A_1\downarrow 0, A_2\uparrow\infty} \sup_{t\geq t_0} \Omega_t(A_1, A_2) = 0$. Let m_ϕ be the minimum of the support of ϕ. Note that

$$f\left(\int \left\{T_{y+\max \overline{\mathcal{E}}_t}\phi(z)\right\} \mathcal{D}_t(\mathrm{d}z)\right)$$

is zero when $y + \max \overline{\mathcal{E}}_t < m_\phi$, and that the event $\{y + \max \overline{\mathcal{E}}_t = m_\phi\}$ has probability zero. Therefore,

$$E\left[f\left(\int \left\{T_{y+\max \overline{\mathcal{E}}_t}\phi(z)\right\} \mathcal{D}_t(\mathrm{d}z)\right)\right]$$

$$= E\left[f\left(\int \left\{T_{y+\max \overline{\mathcal{E}}_t}\phi(z)\right\} \mathcal{D}_t(\mathrm{d}z)\right) \mathbb{1}_{\{y+\max \overline{\mathcal{E}}_t>m_\phi\}}\right]$$

$$= E\left[f\left(\int \left\{T_{y+\max \overline{\mathcal{E}}_t}\phi(z)\right\} \mathcal{D}_t(\mathrm{d}z)\right) \Big| y + \max \overline{\mathcal{E}}_t > m_\phi\right] P\left[y+\max \overline{\mathcal{E}}_t > m_\phi\right].$$

$$(4.108)$$

By Corollary 4.12, the conditional law of the pair $(\mathcal{D}_t, y + \max \overline{\mathcal{E}}_t)$ *given* $\{y + \max \overline{\mathcal{E}}_t > m_\phi\}$ exists in the limit. Moreover the convergence is uniform in $y \in [-A_1\sqrt{t}, -A_2\sqrt{t}]$. By Lemma 4.13, the convergence applies to the random variable $\int \left\{T_{y+\max \overline{\mathcal{E}}_t}\phi(z)\right\} \mathcal{D}_t(\mathrm{d}z)$. Therefore

$$\lim_{t\to\infty} E\left[f\left(\int \left\{T_{y+\max \overline{\mathcal{E}}_t}\phi(z)\right\} \mathcal{D}_t(\mathrm{d}z)\right) \Big| y + \max \overline{\mathcal{E}}_t > m_\phi\right]$$

$$= \int_{m_\phi}^{\infty} E\left[f\left(\int (T_y\phi(z)\mathcal{D}(\mathrm{d}z)\right)\right]\frac{\sqrt{2}e^{-\sqrt{2}y}\,\mathrm{d}y}{e^{-\sqrt{2}m_\phi}} \qquad (4.109)$$

On the other hand,

$$\int_{-A_2\sqrt{t}}^{-A_1\sqrt{t}} P\left[y + \max \overline{\mathcal{E}}_t > m_\phi\right]\sqrt{\frac{2}{\pi}}(-y)e^{-\sqrt{2}y}\,\mathrm{d}y = Ce^{-\sqrt{2}m_\phi} + \Omega_t(A_1, A_2)$$

$$(4.110)$$

by Lemma 4.11 and by the same approximation as in (4.107).

Combining (4.110), (4.109) and (4.108) gives that (4.106) converges to

$$E\left[\exp\left(-CZ\int_{\mathbb{R}}E[1-e^{-\int\phi(y+z)\mathcal{D}(\mathrm{d}z)}]\sqrt{2}e^{-\sqrt{2}y}dy\right)\right],\qquad(4.111)$$

which is by (4.105) also the limiting Laplace transform of the extremal process of BBM: this shows (4.104) and thus concludes the proof of Theorem 2.1. □

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