The extreme points of a class of functions with positive real part

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Introduction

In spite of its elegance, extreme point theory plays a modest role in complex function theory. In a series of papers Brickman, Hallenbeck, Mac Gregor and Wilken determined the extreme points of some classical families of analytic functions. An excellent overview of their results is contained in [4]. Of fundamental importance is the availability of the extreme points of the set P of functions f analytic on the unit disc, with positive real part, normalized by $f(0) = 1$. These extreme points can be obtained from an integral representation formula given by Herglotz in 1911 [5]. A truly beautiful derivation of $\mathrm{Ext}P$ was given by Holland [6]. In this note we present yet another method, based on elementary functional analysis. As an application we determine the extreme points of the set F of functions f analytic on the unit disc, with imaginary part bounded by $\frac{\pi}{2}$ and normalized by $f(0) = 0$. They were originally determined by Milcetich [7] but our derivation is simpler. Finally we determine the extreme points of the set P_α of functions $f \in P$ for which $|\arg f| \le \alpha\frac{\pi}{2}$ for some constant $\alpha < 1$. These were earlier described by Abu-Muhanna and Mac Gregor [1].

Preliminaries

Let $H(\Delta)$ be the set of analytic functions on the unit disc Δ in $\mathbb{C}$. It is wellknown [9, page 1] that $H(\Delta)$ provided with the metric

$$d(f,g) = \sum_{n=2}^{\infty} \frac{1}{2^n} \max_{|z|\le\frac{n-1}{n}} \frac{|f-g|}{1+|f-g|}$$

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is a locally convex space. Convergence with respect to d is the same as locally uniform convergence. There is an explicit description of the dual space $H(\Delta)^*$.

Theorem: (Toeplitz [11]). There is a 1-1 correspondence between continuous linear functionals L on $H(\Delta)$ and sequences b_n with $\limsup \sqrt[n]{1b_n1} < 1$. If $f : z \to \sum_{n=0}^{\infty} a_n z^n$ belongs to $H(\Delta)$ then

$$L(f) = \sum_{n=0}^{\infty} a_n b_n.$$

The theorem can also be expressed as follows: There is a 1-1 correspondence between continuous linear functionals on $H(\Delta)$ and analytic functions b on some open neighbourhood Δ_r of $\overline{\Delta}$. If $f \in H(\Delta)$ then

$$L(f) = \frac{1}{2\pi i} \int\limits_{|z|=\rho} f(z) b(\frac{1}{z}) \frac{dz}{z} \quad \text{where} \quad \frac{1}{r} < \rho < 1.$$

Proof: It is evident that each such function b defines an element of $H(\Delta)^*$. Conversely if $L \in H(\Delta)^*$ we put

$$b_n = L(z^n).$$

If the sequence b_n had a subsequence b_{n_k} (with $b_{n_k} \neq 0$) for which $\lim\limits_{n_k \to \infty} \sqrt[n_k]{|b_{n_k}|} \geq 1$, then

$$\sum_{n_k} \frac{z^{n_k}}{b_{n_k}}$$

would determine an element f of $H(\Delta)$. Continuity of L would imply that

$$L(f) = \sum_{n_k} \frac{b_{n_k}}{b_{n_k}} = \infty$$

which is impossible. Therefore we conclude that $\limsup \sqrt[n]{|b_n|} < 1$.

Our main subject of interest is the set $P \subset H(\Delta)$ of functions

$$f : z \to 1 + \sum_{n=1}^{\infty} a_n z^n$$

for which $\operatorname{Re} f > 0$. Evidently P is convex. P is also a compact subset of $H(\Delta)$ [9, page 2]. We have the following result.

Lemma: (Schur [10]) Let $p : z \to 1 + 2 \sum_{n=1}^{\infty} p_n z^n$ and $q : z \to 1 + 2 \sum_{n=1}^{\infty} q_n z^n$ belong to P. Then $p * q \in P$ where

$$p * q(z) = 1 + 2 \sum_{n=1}^{\infty} p_n q_n z^n.$$

Proof: For $z \in \Delta$ we have

$$0 < \tfrac{1}{2\pi} \int\limits_{|w|=\rho>|z|} \operatorname{Re}\, p(\tfrac{z}{w}) \operatorname{Re}\, q(w) \tfrac{dw}{iw} = \tfrac{1}{2} \operatorname{Re}\, \tfrac{1}{2\pi i} \int\limits_{|w|=\rho} p(\tfrac{z}{w}) \{q(w) + \overline{q(w)}\} \tfrac{dw}{w} =$$

$$\tfrac{1}{2} \operatorname{Re}\, \{1 + 4 \sum_{n=1}^{\infty} p_n q_n z^n + 1\} = \operatorname{Re}\, p * q(z).$$

Extreme points of P

In order to determine the extreme points of P we shall apply the following result which is sometimes called the theorem of Milman and Rutman.

Lemma: Let X be a locally convex space, let Q be a compact subset of X and assume that its closed convex hull $\overline{co}(Q)$ is also compact. Then Q contains all the extreme points of $\overline{co}(Q)$.

For a proof of this (elementary) lemma we refer to [2, page 440].

For $\theta \in [0, 2\pi]$ we define

$$k_\theta(z) = \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} = 1 + 2\sum_{n=1}^{\infty} e^{in\theta} z^n.$$

Note that $k_\theta \in P$.

Theorem: The set of extreme points of P is

$$E = \{k_\theta : 0 \leq \theta < 2\pi\}.$$

Proof: It is easy to see that E is a compact subset of $H(\Delta)$.

We shall show that $\overline{co}(E) = P$. Assume that there exists a function $p \in P \backslash \overline{co}(E)$. Then, from Hahn-Banach's separation theorem [8, page 58] we deduce the existence of an $L \in H(\Delta)^*$ and a number λ such that for all $f \in \overline{co}(E)$

$$\text{Re } L(f) > \lambda > \text{ Re } L(p).$$

Since $\text{Re } L(f) = \text{ Re } b_0 + \text{ Re } \sum_{n=1}^{\infty} b_n f_n$ and $\text{Re } L(p) = \text{ Re } b_0 + \text{ Re } \sum_{n=1}^{\infty} b_n p_n$ we may assume that $b_0 \in \mathbb{R}$ and that

$$\text{Re } L(f) > 0 > \text{ Re } L(p).$$

In particular $\text{Re } L(k_\theta) = b_0 + 2 \text{ Re } \sum_{n=1}^{\infty} b_n e^{in\theta} > 0$.

From the maximum principle we see that for all $z \in \Delta$

$$b_0 + 2 \text{ Re } \sum_{n=1}^{\infty} b_n z^n > 0,$$

so in particular $b_0 > 0$, hence

$$\beta : z \to 1 + 2\sum_{n=1}^{\infty} \frac{b_n}{b_0} z^n$$

belongs to P. From Schur's Lemma we conclude that

$$\beta * p : z \to 1 + \sum_{n=1}^{\infty} p_n \frac{b_n}{b_0} z^n$$

is also an element of P and since $\limsup \sqrt[n]{|b_n|} < 1$, $\beta * p$ is continuous on $\overline{\Delta}$ and we even have Re $\beta * p(1) \geq 0$, i.e.

$$L(p) = b_0 \text{ Re } \beta * p(1) \geq 0,$$

a contradiction. Therefore we have $\overline{co}(E) = P$, and as a consequence of the theorem of Milman and Rutman we see that Ext $P \subset E$.
Since the group of rotations $z \to e^{i\theta} z$ acts transitively on E we conclude that Ext $P = E$.

Corollary: By Krein-Milman's theorem [8, p.71 th.3.22 and p.78 th.3.28] we obtain: For every $f \in P$ there exists a probability measure μ on $[0, 2\pi]$ such that

$$f = \int_0^{2\pi} k_\theta d\mu(\theta).$$

It is easy to see that there is a 1-1 correspondence between elements of P and probability measures on $[0, 2\pi]$. The integral representation is called Herglotz's integral representation.

The next theorem also follows from Hahn-Banach.

Theorem: Let A be an infinite subset of $\overline{\Delta}$ and let $h \in H(\Delta)$ be a function for which $h^{(n)}(0) \neq 0 (n = 0, 1, 2, \ldots)$.
Then the closed linear span

$$M = [\![z \to h(wz) : w \in A]\!]$$

is equal to $H(\Delta)$.

Proof: Again by Hahn-Banach's theorem [8, page 59] if M did not contain an element f of $H(\Delta)$ there would be an $L \in H(\Delta)^*$ such that L annihilates M, but $L(f) = 1$. From

$$h(wz) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} w^n z^n$$

we see that for $w \in A$

$$L\Big(h(wz)\Big) = \sum_{n=0}^{\infty} b_n \frac{h^{(n)}(0)}{n!} w^n = 0$$

hence the analytic function $z \to \sum_{n=0}^{\infty} b_n \frac{h^{(n)}(0)}{n!} z^n$ which is defined on Δ_r for some $r > 1$ has infinitely many zeros on $\overline{\Delta}$ and is therefore identically zero, so $b_n = 0$ for all n. Then $L(f) = 0$, a contradiction.

Extreme points of F

Let F be the subset of $H(\Delta)$ consisting of the functions

$$f : z \to \sum_{n=1}^{\infty} a_n z^n$$

for which $-\frac{\pi}{2} < \text{Im } f(z) < \frac{\pi}{2}$. A function f belongs to F if and only if $\exp \circ f \in P$. The exponential function however doesn't preserve linear relations. We shall employ the following criterion

$$f \in F \iff \begin{cases} 1 + \frac{2i}{\pi} f \in P \\ \\ 1 - \frac{2i}{\pi} f \in P. \end{cases} \tag{1}$$

Theorem: $f \in F \iff$ There is an integrable function φ on $[0, 2\pi]$ such that

$$-1 \le \varphi \le 1$$

$$\int_0^{2\pi} \varphi(\theta) d\theta = 0$$

and such that

$$f = \frac{\pi i}{2} \int_0^{2\pi} k_\theta \varphi(\theta) \frac{d\theta}{2\pi}.$$

Proof: From (1) and from Herglotz's representation we deduce the existence of probability measures μ and ν or $[0, 2\pi]$ such that

$$\begin{cases} 1 + \frac{2i}{\pi} f = \int_0^{2\pi} k_\theta d\mu(\theta) \\ \\ 1 - \frac{2i}{\pi} f = \int_0^{2\pi} k_\theta d\nu(\theta). \end{cases} \tag{2}$$

Addition leads to

$$\int_0^{2\pi} k_\theta d \frac{1}{2} \Big(\mu(\theta) + \nu(\theta) \Big) = 1$$

and from the uniqueness of Herglotz's representation we conclude that $\frac{1}{2}(\mu + \nu)$ is equal to normalized Lebesgue measure $\frac{d\theta}{2\pi}$. As a consequence, μ and ν are absolutely continuous. Thus there exist integrable functions u and v on $[0, 2\pi]$ such that $0 \le u$, $0 \le v$, $u + v = 2$, $\int_0^{2\pi} u d\theta = \int_0^{2\pi} v d\theta = 2\pi$

$$\mu = u \frac{d\theta}{2\pi}, \quad \nu = v \frac{d\theta}{2\pi}.$$

Substitution into (2) and subtraction leads to

$$f = \frac{\pi i}{4} \int_0^{2\pi} k_\theta \Big(v(\theta) - u(\theta) \Big) \frac{d\theta}{2\pi}.$$

This shows that $\varphi = \frac{1}{2}(v - u)$ satisfies the requirements of the theorem. Conversely, all functions

$$f = \frac{\pi i}{2} \int_0^{2\pi} k_\theta \varphi(\theta) \frac{d\theta}{2\pi}$$

evidently belong to F.

Corollary: $\text{Im } f = \frac{\pi}{2} \int_0^{2\pi} \text{Re } k_\theta \cdot \varphi(\theta) \frac{d\theta}{2\pi}$, and from well-known properties of the Poisson integral representation [3, page 5, Cor 2] we derive that

$$\lim_{r \uparrow 1} \text{Im } f(re^{-it}) = \frac{\pi}{2} \varphi(t)$$

From the last theorem, we obtain Ext F without any difficulty

Theorem: $f \in$ Ext $F \iff$ The corresponding function φ satisfies $|\varphi| = 1$ a.e.

Proof: $f \in$ Ext $F \iff \varphi \in$ Ext $\{\psi \in L^1[0, 2\pi] : -1 \leq \psi \leq 1, \int_0^{2\pi} \psi = 0\}$.

If $|\varphi| \neq 1$ on some set of positive measure, then there is also a set A of positive measure such that $0 \leq \varphi < 1$ (or such that $-1 < \varphi \leq 0$).
Split A into two subsets A_1 and A_2 such that

$$\int_{A_1} (1 - \varphi) = \int_{A_2} (1 - \varphi)$$

and define

$$\varphi_1 = \varphi \cdot 1_{A^c} + 1_{A_1} + (2\varphi - 1)1_{A_2}$$
$$\varphi_2 = \varphi \cdot 1_{A^c} + (2\varphi - 1)1_{A_1} + 1_{A_2}.$$

Then $\varphi = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$. Conversely if $|\varphi| = 1$ a.e. then evidently

$$\varphi \in \text{Ext}\{\psi \in L^1[0, 2\pi], -1 \leq \psi \leq 1, \int_0^{2\pi} \psi = 0\}.$$

Corollary: $f \in F$ is an extreme point of F if and only if

$$|\lim_{r \uparrow 1} \text{ Im } f(re^{\text{it}}), | = \frac{\pi}{2}$$

for almost all $t \in [0, 2\pi]$.

Of course, the extreme points of the set of functions $f \in H(\Delta)$ for which $f(0) = 0$ and $|\text{Im } f| < a$ are precisely those functions f for which

$$|\lim_{r \uparrow 1} \text{ Im } f(re^{\text{it}})| = a$$

for almost all $t \in [0, 2\pi]$.

Example: For $\varphi = -1_{[0,\pi]} + 1_{[\pi,2\pi)}$ the corresponding function

$$f : z \to \log \frac{1 + z}{1 - z}$$

maps Δ conformally onto the strip $\{z : |\text{Im } z| < \frac{\pi}{2}\}$. This f is an extreme point of F.

Remark: There is an analogue of Schur's Lemma for F. Let

$$f : z \to \sum_{n=1}^{\infty} f_n z^n \text{ and } g : z \to \sum_{n=1}^{\infty} g_n z^n$$

belong to F. Then

$$z \to \frac{1}{\pi i} \sum_{n=1}^{\infty} f_n g_n z^n$$

belongs to F.

Proof: From (1) we see that $1 \pm \frac{2i}{\pi} f \in P$ and $1 \pm \frac{2i}{\pi} g \in P$.
Thus, by Schur's lemma

$$(1 \pm \frac{2i}{\pi} f) * (1 \pm \frac{2i}{\pi} g) \in P$$

i.e.

$$z \to 1 \pm 2\Sigma \frac{f_n g_n}{\pi^2} z^n \in P.$$

Again from (1) we deduce that $z \to \frac{1}{\pi i} \sum_{n=1}^{\infty} f_n g_n z^n \in F$.

By similar arguments one can show that if $f : z \to \sum_{n=1}^{\infty} f_n z^n \in F$ and $p : z \to 1 + 2 \sum_{n=1}^{\infty} p_n z^n \in P$, then

$$z \to \sum_{n=1}^{\infty} p_n f_n z^n \in F.$$

Extreme points of P_α

Let $0 < \alpha < 1$. We focus our attention on the set

$$P_\alpha = \{f \in P : |\arg\, f| < \alpha \frac{\pi}{2}\}.$$

We have some characterizations of P_α.

$$f \in P_\alpha \iff f^{\frac{1}{\alpha}} \in P \iff \frac{1}{\alpha} \log f \in F,$$

but since neither exponentiation nor log preserve linearity we cannot derive Ext P_α directly from this correspondence. We start with two lemmas concerning the set

$$G = \{z \in \mathbb{C} : |\arg\, z| < \alpha \frac{\pi}{2}\}.$$

Lemma 1: Let $z, w \in \mathbb{C}$ have positive real part and let $z^2, w^2 \in G$.
If $\lambda \in \mathbb{R}$ and if

$$|\lambda| < \cos \frac{\alpha\pi}{2},$$

then

$$zw(1 + \lambda \frac{z - w}{z + w}) \in G.$$

Proof: We denote arg $z = t$, arg $w = s$; then $-\alpha\frac{\pi}{4} < s, t < \alpha\frac{\pi}{4}$, hence

$$\cos(t - s) > \cos \alpha \frac{\pi}{2} > |\lambda|.$$

By an elementary computation we obtain

$$\arg(1 + \lambda \frac{z - w}{z + w}) = \arctan \frac{2\lambda |z|\, |w| \sin(t - s)}{(1 + \lambda)|z|^2 + (1 - \lambda)|w|^2 + 2|z|\, |w| \cos(t - s)}.$$

Since

$$\frac{2|\lambda|\ |z|\ |w|}{(1+\lambda)|z|^2+(1-\lambda)|w|^2+2|z|\ |w|\cos(t-s)} <$$

$$< \frac{2|z|\ |w|\cos\alpha\frac{\pi}{2}}{(1+\lambda)|z|^2+(1-\lambda)|w|^2+2|z|\ |w|\cos\alpha\frac{\pi}{2}} < 1$$

we have

$$|\arg(1+\lambda\frac{z-w}{z+w})| \leq \arctan(\sin|t-s|) \leq |t-s|,$$

and therefore

$$2\min(|\arg z|, |\arg w|) \leq \arg zw(1+\lambda\frac{z-w}{z+w}) \leq 2\max(|\arg z|, |\arg w|),$$

i.e.

$$zw(1+\lambda\frac{z-w}{z+w}) \in G.$$

Lemma 2: Let $z \in G$ and let $w \in \mathbb{C}$. Suppose that $z+w \in G$ and $z-w \in G$. If $\lambda \in \mathbb{R}$ and if

$$|\lambda| < \frac{3}{16}\sin\alpha\pi$$

then

$$z\frac{z+\lambda w}{z-\lambda w} \in G.$$

Proof: It is sufficient to show that

$$|\arg z| + |\arg\frac{z+\lambda w}{z-\lambda w}| < \alpha\frac{\pi}{2}.$$

Since $z \pm w \in G$ we have $w \in (-z+G)\cap(z-G)$, i.e. w is an element of the parallellogram with vertices

$$\pm z, \text{ and } \pm\frac{2}{\sin\alpha\pi}(\text{Im } z\cos^2\alpha\frac{\pi}{2} + i\ \text{Re } z\sin^2\alpha\frac{\pi}{2}).$$

λw is an element of a homothetic parallellogram. Therefore

$$\left|\arg\frac{z+\lambda w}{z-\lambda w}\right|$$

is maximal if we choose

$$w = \frac{2}{\sin\alpha\pi}(\text{Im } z\cos^2\alpha\frac{\pi}{2} + i\ \text{Re } z\sin^2\alpha\frac{\pi}{2}).$$

For this choice of w we have (since $|\lambda| < \frac{1}{4}\sin\alpha\pi$)

$$\lambda^2|w|^2 \leq \frac{1}{4}\{(\text{Im } z)^2\cos^4\alpha\frac{\pi}{2} + (\text{Re } z)^2\sin^4\alpha\frac{\pi}{2}\} \leq \frac{1}{4}|z|^2.$$

By an elementary computation we obtain

$$\arg\frac{z+\lambda w}{z-\lambda w} = \arctan 4\lambda\frac{(\mathrm{Re}\ z)^2\sin^2\alpha\frac{\pi}{2} - (\mathrm{Im}\ z)^2\cos^2\alpha\frac{\pi}{2}}{(|z|^2-\lambda^2|w|^2)\sin\alpha\pi},$$

so we deduce that

$$\begin{aligned}\left|\arg\tfrac{z+\lambda w}{z-\lambda w}\right| &\le \arctan 4|\lambda|\tfrac{(\mathrm{Re}\ z)^2\sin^2\alpha\frac{\pi}{2}-(\mathrm{Im}\ z)^2\cos^2\alpha\frac{\pi}{2}}{\frac{3}{4}|z|^2\sin\alpha\pi}\\ &\le \arctan\tfrac{1}{|z|^2}\Big((\mathrm{Re}\ z)^2\sin^2\alpha\tfrac{\pi}{2} - (\mathrm{Im}\ z)^2\cos^2\alpha\tfrac{\pi}{2}\Big)\\ &= \arctan\sin(\alpha\tfrac{\pi}{2} - \arg z)\cdot\sin(\alpha\tfrac{\pi}{2} + \arg z)\\ &= \arctan\sin(\alpha\tfrac{\pi}{2} - |\arg z|)\cdot\sin(\alpha\tfrac{\pi}{2} + |\arg z|)\\ &\le \arctan\sin(\alpha\tfrac{\pi}{2} - |\arg z|) < \alpha\tfrac{\pi}{2} - |\arg z|,\end{aligned}$$

and the lemma is proved.

Now we are able to determine Ext P_α.

Theorem: Let $f \in P_\alpha$; then $f \in$ Ext P_α if and only if $\frac{1}{\alpha}\log f \in$ Ext F.

Proof: If $f \in P_\alpha$, then $\frac{1}{\alpha}\log f \in F$. Assume that $\frac{1}{\alpha}\log f \notin$ Ext F.

Then there are functions $f_1, f_2 \in F$, $f_1 \ne f_2$ such that $f = \frac{1}{2}(f_1 + f_2)$, or equivalently, there exist functions g, $h \in P_\alpha$, $g \ne h$ such that $f = \sqrt{gh}$. As a consequence of lemma 1 we have for all $|\lambda| < \cos\frac{\alpha\pi}{2}$

$$\sqrt{gh}(1+\lambda\frac{\sqrt{g}-\sqrt{h}}{\sqrt{g}+\sqrt{h}}) \in P_\alpha.$$

For such λ we have

$$f = \sqrt{gh} = \frac{1}{2}\sqrt{gh}(1+\lambda\frac{\sqrt{g}-\sqrt{h}}{\sqrt{g}+\sqrt{h}}) + \frac{1}{2}\sqrt{gh}(1-\lambda\frac{\sqrt{g}-\sqrt{h}}{\sqrt{g}+\sqrt{h}}),$$

hence $f \notin$ Ext P_α.

Conversely, if $f \in P_\alpha$, $f \notin$ Ext P_α, then there is a non-constant function $g \in H(\Delta)$ such that $f \pm g \in P_\alpha$. Now lemma 2 implies that for $|\lambda| < \frac{3}{16}\sin\alpha\pi$

$$f\frac{f+\lambda g}{f-\lambda g} \in P_\alpha.$$

For such λ we have

$$f = \sqrt{f\frac{f+\lambda g}{f-\lambda g}}\cdot\sqrt{f\frac{f-\lambda g}{f+\lambda g}}$$

i.e.

$$\frac{1}{\alpha}\log f = \frac{1}{2}\{\frac{1}{\alpha}\log\sqrt{f\frac{f+\lambda g}{f-\lambda g}} + \frac{1}{\alpha}\log\sqrt{f\frac{f-\lambda g}{f+\lambda g}}\},$$

hence $\frac{1}{\alpha}\log f \notin$ Ext F.

Corollary: Let $f \in P_\alpha$; then $f \in$ Ext P_α if and only if

$$|\lim_{r\uparrow 1}\arg f(re^{\mathrm{it}})| = \alpha\frac{\pi}{2}$$

for almost all $t \in [0, 2\pi]$.

There is an analogue of Schur's lemma for P_α. We make use of yet another characterization of P_α. The functions

$$\phi_1 : z \to \frac{i}{\sin \alpha\frac{\pi}{2}}(e^{-i\alpha\frac{\pi}{2}}z - \cos \alpha\frac{\pi}{2})$$

and

$$\phi_2 : z \to \frac{-i}{\sin \alpha\frac{\pi}{2}}(e^{i\alpha\frac{\pi}{2}}z - \cos \alpha\frac{\pi}{2})$$

map G into the right halfplane. Note that

$$f \in P_\alpha \iff \phi_j(f) \in P \qquad (j = 1, 2).$$

Theorem: If $f \in P_\alpha$ and $g \in P$, then $f * g \in P_\alpha$.

Proof: $\phi_j(f * g) = \phi_j(f) * g \in P$ $(j = 1, 2)$ by Schur's lemma, hence

$$f * g \in P_\alpha.$$

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