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THE FABER-KRAHN TYPE ISOPERIMETRIC INEQUALITIES FOR A GRAPH

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Abstract. In this paper, a graph theoretic analog to the celebrated Faber-Krahn inequality for the first eigenvalue of the Dirichlet problem of the Laplacian for a bounded domain in the Euclidean space is shown. Namely, the optimal estimate of the first eigenvalue of the Dirichlet boundary problem of the combinatorial Laplacian for a graph with boundary is given.

1. Introduction.

The celebrated Faber-Krahn inequality is stated as follows (see [1], [2]):

FABER-KRAHN THEOREM. *Let $\lambda_1(\Omega)$ be the first eigenvalue of the Dirichlet Laplacian for a bounded domain Ω in $\boldsymbol{R}^n$. If $\mathrm{Vol}(\Omega)=\mathrm{Vol}(\Omega^*)$, where Ω^* is a ball in $\boldsymbol{R}^n$, then*

$$\lambda_1(\Omega)\geq\lambda_1(\Omega^*),$$

and the equality holds if and only if Ω is congruent to Ω^*.

In this paper, we show an analog of the Faber-Krahn theorem for a graph. A graph is a collection of vertices together with a collection of edges joining pairs of vertices. Let us take a connected graph with boundary, $G=(V\cup\partial V, E\cup\partial E)$ (see the definition in Section 2). We consider the Dirichlet boundary problem of the combinatorial Laplacian Δ on G:

$$\begin{cases}\Delta f(x)=\lambda f(x), & x\in V,\\ f(x)=0, & x\in\partial V.\end{cases}$$

Let us denote the eigenvalues for this problem by

$$0<\lambda_1(G)\leq\lambda_2(G)\leq\cdots\leq\lambda_k(G),$$

where k is the number of vertices in V. We call $\lambda_1(G)$ the *first eigenvalue* of G.

We give the following two examples (1), (2) of graphs with boundary: Here we denote by white (resp. black) circles, vertices in V (resp. ∂V) and by solid (resp. dotted) lines, edges in E (resp. ∂E).

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(1) (the graph of type L_m) The graph in Figure 1.1 will be denoted by L_m.

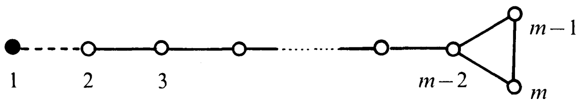

FIGURE 1.1.

(2) (the graph of type A_{m+1}) A_{m+1} will stand for the graph in Figure 1.2.

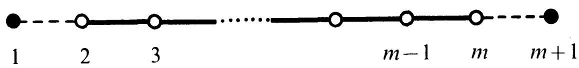

FIGURE 1.2.

Our main results are stated in Theorems A and B below.

THEOREM A. *Let $G=(V\cup\partial V, E\cup\partial E)$ be a connected graph with boundary. Assume that the cardinality of $E\cup\partial E$ satisfies $\sharp(E\cup\partial E)=m\geq 4$. Then*

$$\lambda_1(G)\geq\lambda_1(L_m),$$

and the equality holds if and only if G is isomorphic to L_m.

A graph with boundary $G=(V\cup\partial V, E\cup\partial E)$ is said (cf. [5]) to have the *non-separation property* if each connected component of the complement, $V-\{v\}$, of each vertex $v\in V$ contains at least one boundary vertex. A class of graphs having the separation property is also a large family. For instance, a tree with boundary has always the non-separation property. The following theorem singles out the graph of type A_{m+1}.

THEOREM B. *Let $G=(V\cup\partial V, E\cup\partial E)$ be a connected graph with boundary satisfying the non-separation property. Assume that $\sharp(E\cup\partial E)=m$. Then*

$$\lambda_1(G)\geq\lambda_1(A_{m+1}),$$

and the equality holds if and only if G is isomorphic to A_{m+1}.

We would like to express our gratitude to Professor Takashi Sakai for helpful discussions.

2. Preliminaries. In this section, we review basic notions about the Laplacian on a graph following [3] or [4].

Let $G=(V\cup\partial V, E\cup\partial E)$ be a graph with boundary (see for instance [4] or [5]), i.e., (i) each edge in E has both end points in V, (ii) each edge in ∂E has exactly one end point in V and one in ∂V and (iii) any vertex which has exactly one edge is in ∂V. We call vertices in V (resp. ∂V) the *interior* (resp. *boundary*) vertices, and similarly for the edges. We always consider a finite connected graph with boundary, and fix once and for all an orientation for each edge of G in this paper.

Let $C_0^0(G)$ be the set of all real-valued functions on $V \cup \partial V$ satisfying $f(x)=0$ for all $x \in \partial V$. Let $C^1(G)$ be the space of all functions φ defined on the set of all directed edges of G and satisfying

$$\varphi([x, y]) = -\varphi([y, x]),$$

where $[x, y]$, $x, y \in V \cup \partial V$, denotes a directed edge in $E \cup \partial E$ beginning at x and ending at y. We define the following inner products on these spaces by

$$\begin{cases} (f_1, f_2) := \sum_{x \in V} m(x) f_1(x) f_2(x), \\ (\varphi_1, \varphi_2) := \sum_{\sigma \in E \cup \partial E} \varphi_1(\sigma)\varphi_2(\sigma), \end{cases} \tag{2.1}$$

for $f_1, f_2 \in C_0^0(G)$ and $\varphi_1, \varphi_2 \in C^1(G)$. Here $m(x)$, $x \in V$ is the degree of x, which is by definition the number of edges in $E \cup \partial E$ incident to x. The coboundary operator

$$df([x, y]) := f(y) - f(x)$$

maps $C_0^0(G)$ into $C^1(G)$. The *combinatorial Laplacian* is defined as

$$\Delta f = d^* df, \quad f \in C_0^0(G),$$

where d^* is the adjoint of the coboundary operator d with respect to the above inner products. By definition,

$$(\Delta f_1, f_2) = (df_1, df_2), \quad f_1, f_2 \in C_0^0(G), \tag{2.2}$$

and

$$\Delta f(x) = f(x) - \frac{1}{m(x)} \sum_{y \sim x} f(y), \qquad x \in V, \quad f \in C_0^0(G), \tag{2.3}$$

where $y \sim x$ means that x and y are connected by an edge in $E \cup \partial E$. A real number λ is an eigenvalue of Δ on $C_0^0(G)$ if there exists a non-vanishing function $f \in C_0^0(G)$ such that $\Delta f(x) = \lambda f(x)$, $x \in V$. The function f is called the *eigenfunction* with eigenvalue λ. This means that f and λ satisfy the Dirichlet eigenvalue problem:

$$\begin{cases} \Delta f(x) = \lambda f(x), & x \in V, \\ f(x) = 0, & x \in \partial V. \end{cases}$$

The eigenvalues are labelled as follows:

$$0 < \lambda_1(G) \le \lambda_2(G) \le \cdots \le \lambda_k(G),$$

where $k := \sharp(V)$, the cardinality of V.

EXAMPLE 2.1. (1) The first eigenvalue $\lambda_1(A_{m+1})$ of the graph of type A_{m+1} is given (see, for instance, [5]) by

$$\lambda_1(A_{m+1}) = 1 - \cos\left(\frac{\pi}{m}\right).$$

For the graph of type A_{m+1}, we have $\sharp(E \cup \partial E) = m$ and $\sharp(V \cup \partial V) = m+1$.

(2) The first eigenvalue of the graph of type L_m, $m \geq 4$ is rather complicated. Let $H_m(t)$ be a polynomial of degree $m-1$ in t defined by

$$H_m(t) = \prod_{j=1}^{m-1} \left(t - 1 + \cos\left(\frac{j\pi}{m}\right)\right).$$

The eigenvalues of the Dirichlet problem for the graph of type L_m are 3/2 and the roots of the following equation of order $m-2$ in t:

$$(6t^2 - 9t + 1)H_{m-4}(t) - \left(t - \frac{1}{2}\right)H_{m-5}(t) = 0,$$

where we regard $H_{-1}(t) = 0$ and $H_0(t) = 1$.

For examples, $\lambda_1(L_4) = 0.24170$, $\lambda_1(L_5) = 0.12351$ and $\lambda_1(L_6) = 0.07809$.

For the graph of type L_m, we have $\sharp(E \cup \partial E) = \sharp(V \cup \partial V) = m$.

3. Surgery of a graph. Now let us describe our main tool—surgery of a graph. We consider the following cases:

(i) There exists $v_1 \in V$ such that the complement $G - \{v_1\}$ of v_1 has at least two connected components, say $G_1, G_2, \ldots$. Two cases occur:

(i-1) G_1 has an element $v_2 \in \partial V$.

(i-2) G_1 has no element of ∂V.

(ii) There exist $v_1, v_2 \in V$ such that the complement $G - \{v_1, v_2\}$ of $\{v_1, v_2\}$ also has at least two connected components, say $G_1, G_2, \ldots$.

We define *surgery* to obtain a new graph $G' = (V' \cup \partial V', E' \cup \partial E')$ by performing the following *operations* on $G = (V \cup \partial V, E \cup \partial E)$ in the above three cases:

DEFINITION 3.1. In the case (i-1), let us take an edge $e = [x, y] \in E$ such that $x, y \notin G_1$. The (G_1, e)-*operation of the first kind* consists of

(i) cutting G_1 at v_1 and e at x,

(ii) pasting the edges of G_1 to x, to have v_1 as an end point, and

(iii) pasting v_2 to e.

In this way, one gets a new graph $G' = (V' \cup \partial V', E' \cup \partial E')$ (see Figure 3.1).

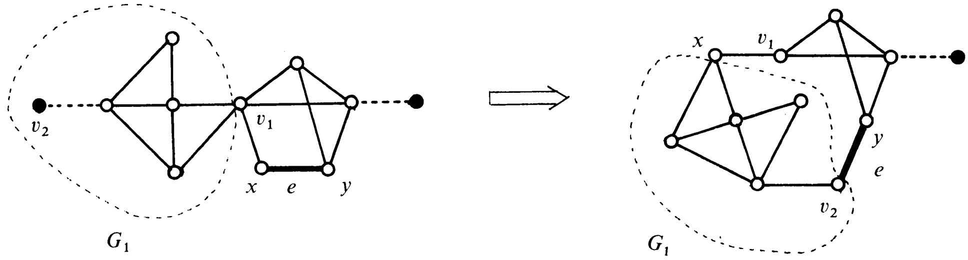

FIGURE 3.1.

REMARK 3.2. By the (G_1, e)-operation, the boundary vertex $v_2 \in \partial V$ is changed to an interior vertex of G', that is, $v_2 \in V'$.

DEFINITION 3.3. In the case (i-2), take $x \in V$ which does not belong to G_1 and is not equal to v_1. The (G_1, x)-*operation* on G is performed as follows.

(i) cutting G_1 at v_1, and

(ii) pasting the edges of G_1 to x, to have v_1 as an end point.

One gets a new graph $G' = (V' \cup \partial V', E \cup \partial E')$ (see Figure 3.2).

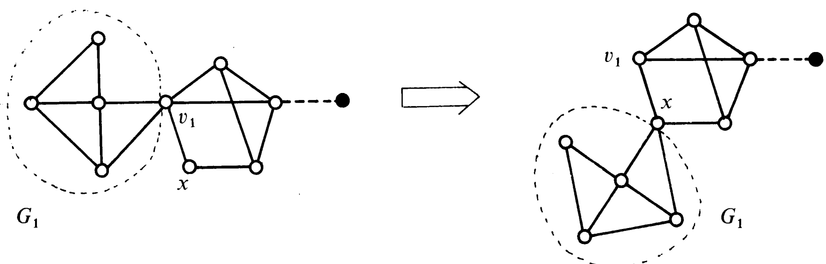

FIGURE 3.2.

DEFINITION 3.4. In the case (ii), we assume that both v_1 and v_2 are branch points. Recall that a *branch point* is $x \in V$ with $m(x) \geq 3$. Take an edge of G, $e = [x, y] \in E$ with $x, y \notin G_1$. The (G_1, e)-*operation of the second kind* on G is performed as follows.

(i) cutting G_1 at v_1 and v_2, and cutting e at x,

(ii) pasting edges of G_1 to x, to have v_1 as an end point,

(iii) adding a new vertex v_3, pasting it to e, and pasting the edges of G_1 to v_3, to have v_2 as an end point.

In this way one obtains a new graph $G' = (V' \cup \partial V', E' \cup \partial E')$ (see Figure 3.3). Note that both v_1 and v_2 remain interior points of G'.

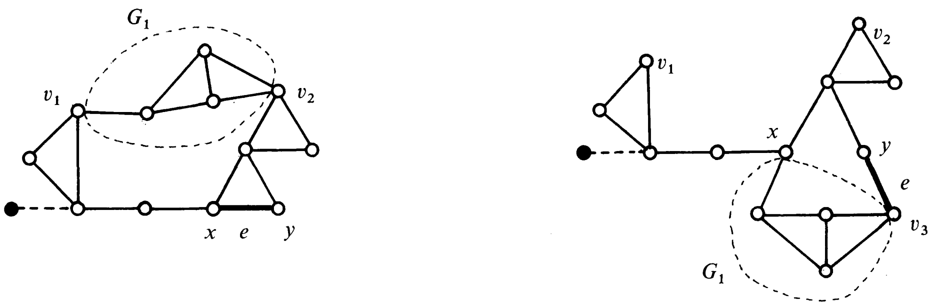

FIGURE 3.3.

Note that for a new graph $G'=(V'\cup\partial V', E'\cup\partial E')$ obtained by surgery, it holds that $\#(E\cup\partial E)=\#(E'\cup\partial E')$. Our key lemma is the following.

CRUCIAL LEMMA 3.5. *Assume that G_1 is one of the connected components of the complement of $v_1\in V$ or $v_2\in V$ in $G=(V\cup\partial V, E\cup\partial E)$. Let f be the first eigenfunction of G. Take $x\in V$ satisfying $f(x)=\max_{v\in V}f(v)$, and an edge $e=[x, y]\in E$ having x as an end point. Assume that G_1 and e have no vertices in common, and that $G'=(V'\cup\partial V', E'\cup\partial E')$ is obtained by surgeries on G. Then*

$$\lambda_1(G')\leq\lambda_1(G).$$

PROOF. Define a function $\tilde{f}$ on V' by

$$\tilde{f}(v)=\begin{cases} f(x) & \text{if } v \text{ is a vertex of } G_1, \text{ or } v_3 \text{ in the case (ii)}, \\ f(v) & \text{otherwise}, \end{cases}$$

for $v\in V'$. Since $\tilde{f}(v)=0$, $v\in\partial V'$, it suffices to show

$$(d\tilde{f}, d\tilde{f})_{G'}\leq(df, df)_G, \tag{3.1}$$

and

$$(\tilde{f}, \tilde{f})_{G'}\geq(f, f)_G, \tag{3.2}$$

whence we obtain

$$\lambda_1(G')\leq\frac{(d\tilde{f}, d\tilde{f})_{G'}}{(\tilde{f}, \tilde{f})_{G'}}\leq\frac{(df, df)_G}{(f, f)_G}=\lambda_1(G).$$

The inequality (3.1) follows as

$$\begin{aligned}(d\tilde{f}, d\tilde{f})_{G'}&=\sum_{e'\in E'\cup\partial E'} d\tilde{f}(e')^2=\sum_{\substack{e'\in E'\cup\partial E'\\ e'\notin G_1}} d\tilde{f}(e')^2\\ &=\sum_{\substack{e'\in E\cup\partial E\\ e'\notin G_1}} df(e')^2\leq(df, df)_G.\end{aligned}$$

For (3.2), let us consider the case where G' is obtained by the (G_1, e)-operation of the first kind. By definition, for some $a>0$,

$$\begin{aligned}(\tilde{f}, \tilde{f})_{G'} &= m_{G'}(v_1)\tilde{f}(v_1)^2 + m_{G'}(x)\tilde{f}(x)^2 + m_{G'}(v_2)\tilde{f}(v_2)^2 \\ &\quad + \sum_{\substack{v\in V', v\notin G_1 \\ v\neq v_1, x}} m_{G'}(v)\tilde{f}(v)^2 + \sum_{\substack{v\in V', v\in G_1 \\ v\neq v_2}} m_{G'}(v)\tilde{f}(v)^2 \\ &= (m_G(v_1)-a)f(v_1)^2 + (m_G(x)+a-1)f(x)^2 + 2f(x)^2 \\ &\quad + \sum_{\substack{v\in V, v\notin G_1 \\ v\neq v_1, x}} m_G(v)f(v)^2 + \sum_{\substack{v\in V, v\in G_1 \\ v\neq v_2}} m_G(v)f(x)^2 \\ &\geq (f, f)_G\,.\end{aligned}$$

In the case of the (G_1, x)-operation, for some $a>0$,

$$\begin{aligned}(\tilde{f}, \tilde{f})_{G'} &= m_{G'}(v_1)\tilde{f}(v_1)^2 + m_{G'}(x)\tilde{f}(x)^2 + \sum_{\substack{v\in V' \\ v\in G_1}} m_{G'}(v)\tilde{f}(v)^2 + \sum_{\substack{v\in V', v\notin G_1 \\ v\neq v_1, x}} m_{G'}(v)\tilde{f}(v)^2 \\ &= (m_G(v_1)-a)f(v_1)^2 + (m_G(x)+a)f(x)^2 \\ &\quad + \sum_{\substack{v\in V \\ v\in G_1}} m_G(v)f(x)^2 + \sum_{\substack{v\in V, v\notin G_1 \\ v\neq v_1, x}} m_G(v)f(v)^2 \\ &\geq (f, f)_G\,.\end{aligned}$$

In the case of the (G_1, e)-operation of the second kind, for some $a>0$ and $b>0$,

$$\begin{aligned}(\tilde{f}, \tilde{f})_{G'} &= m_{G'}(v_1)\tilde{f}(v_1)^2 + m_{G'}(v_2)\tilde{f}(v_2)^2 + m_{G'}(v_3)\tilde{f}(v_3)^2 + m_{G'}(x)\tilde{f}(x)^2 \\ &\quad + \sum_{\substack{v\in V', v\notin G_1 \\ v\neq v_1, v_2, v_3, x}} m_{G'}(v)\tilde{f}(v)^2 + \sum_{\substack{v\in V' \\ v\in G_1}} m_{G'}(v)\tilde{f}(v)^2 \\ &= (m_G(v_1)-a)f(v_1)^2 + (m_G(v_2)-b)f(v_2)^2 + (m_G(x)+a-1)f(x)^2 \\ &\quad + (b+1)f(x)^2 + \sum_{\substack{v\in V, v\notin G_1 \\ v\neq v_1, v_2, x}} m_G(v)f(v)^2 + \sum_{\substack{v\in V \\ v\in G_1}} m_G(v)f(x)^2 \\ &\geq (f, f)_G\,,\end{aligned}$$

hence we get Lemma 3.5. □

4. Proof of Theorem A. The main idea of our proof is to perform surgery on a given graph G, so as to decrease the numbers of cycles and boundary points and ultimately to obtain the graph of type L_m.

We use the following terminology: $c=(v_0, v_1, \ldots, v_s)$ is said to be a *path* emanating from a vertex $v\in V$ if $v_i\in V\cup\partial V$, $v_0=v$ and $[v_i, v_{i+1}]=e\in E\cup\partial E$. A *cycle* of G is a path $c=(v_0, v_1, \ldots, v_s)$ with $v_0=v_s$ with each $v_i\in V$ and $s\geq 3$. A *branch point* is a vertex $x\in V$

with $m(x) \geq 3$. A graph with one boundary point and one cycle as in Figure 4.1, where $\#(V \cup \partial V) = m$, is said to be of type $L_{m,i}$ with $m \geq 4$ and $i \geq 2$:

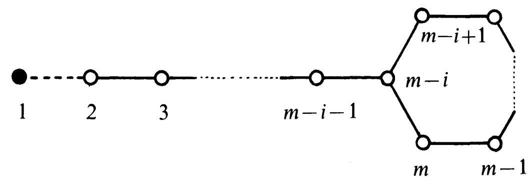

FIGURE 4.1.

Note that a graph of type $L_{m,2}$ is also of type L_m. We shall show

$$\lambda_1(G) \geq \lambda_1(L_{m,i}) \geq \lambda_1(L_m),$$

if $\#(V \cup \partial V) = m$.

LEMMA 4.1. *Let $G = (V \cup \partial V, E \cup \partial E)$ be a graph with boundary. Let us add boundary points to G so as to obtain a new graph G' of which each boundary point $v \in \partial V'$ has only one boundary edge* (*see Figure* 4.2). *Then*

$$\lambda_1(G) = \lambda_1(G').$$

PROOF. The set of interior points of G' is the same as that of G, so the eigenfunction of G can be regarded as a function on $V' \cup \partial V'$ by regarding it to vanish on the boundary, and the eigenfunction on G' vice versa. By definition, for all $v \in V$,

$$m_{G'}(v) = m_G(v),$$

which implies

$$\Delta_{G'} f(x) = \Delta_G f(x), \qquad x \in V.$$

□

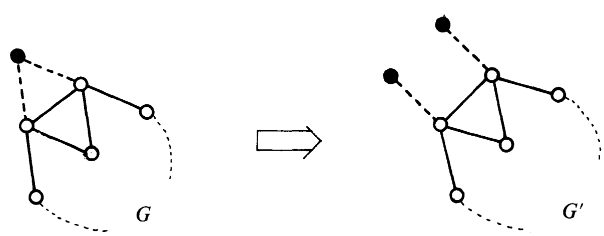

FIGURE 4.2.

In the rest of this paper, we choose an interior vertex $x_0 \in V$ satisfying

$$f(x_0) = \max_{v \in V} f(v).$$

The first step. For any boundary vertex $v \in \partial V$, let $e_v = (v, v_1, \ldots, v_{s-1}, v_s)$ be a path emanating from v and reaching the first branch point v_s of G. Let G_1 be the complement of v_s in e_v (see Figure 4.3). Then one of the following occurs:

Case (i) $x_0 \in e_v$;

Case (ii) $x_0 \notin e_v$.

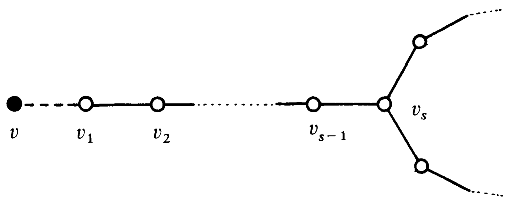

FIGURE 4.3.

In the case (ii), take an edge $e = [x_0, x_1] \in E$ which does not have v_s as a common end point. Perform a (G_1, e)-operation on G to obtain G'. Note that the number of the boundary vertices of G' is smaller than that of G and by Lemma 3.5,

$$\lambda_1(G) \geq \lambda_1(G') .$$

Carry out this process for each boundary vertex, until the case (i) occurs. The resulting graph, denoted by G', satisfies $\lambda_1(G) \geq \lambda_1(G')$, and it holds that either

(a) G' has only one boundary vertex v_1 and x_0 is a vertex in a path connecting v_1 to a branch vertex, or

(b) x_0 is a branch vertex to which all boundary vertices are connected.

The second step. Let G' be a graph which satisfies (a) or (b) in the first step. Here we use the following terminology: A cycle $c = (v_0, v_1, \ldots, v_s)$ with $v_s = v_0$ is *reducible* if there exist $1 \leq i < j \leq s-1$ and a path which connects v_i and v_j and is shorter than $(v_i, v_{i+1}, \ldots, v_j)$. Otherwise, a cycle is called *irreducible*.

In the second step, we perform surgery on the graph G' to obtain a graph G'' such that any cycle of G'' contains a unique branch point. Indeed, assume that G' admits a cycle c which has at least two branch points. We may assume that c is irreducible by taking first an irreducible cycle and considering cycles step by step. Recall that $x_0 \in V$ is a vertex satisfying $f(x_0) = \max_{v \in V} f(v)$. Let us take a path $\tilde{e}$ in c connecting two neighboring branch points, say v_1 and v_2, but not containing x_0. Let G_1 be the complement of v_1 and v_2 in $\tilde{e}$ which is the case (ii) in Section 3. Take an edge $e = [x_0, y]$ which does not have y as a common vertex to G_1. Now perform the (G_1, e)-operation of the second kind on G' to obtain a new graph G'' (see Figure 4.4).

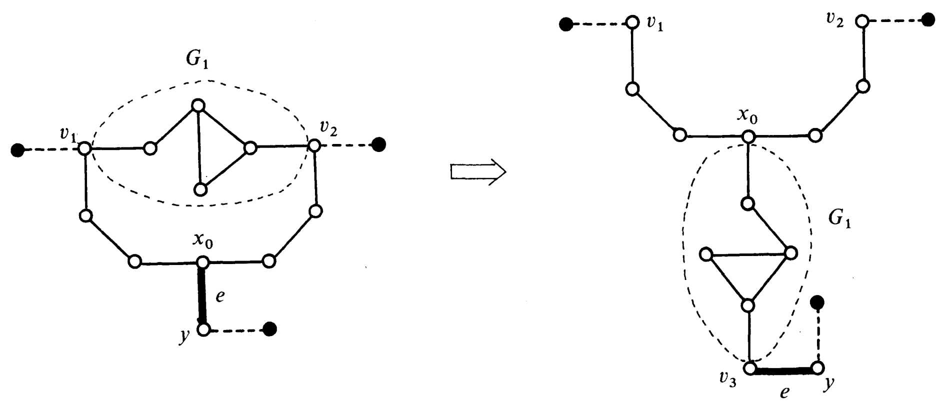

FIGURE 4.4.

The number of cycles of the graph G'' is smaller than that of G' and $\lambda_1(G') \geq \lambda_1(G'')$. Continue this process succesively. Then, finally we obtain the graph G'' all of whose cycles have only one branch point and $\lambda_1(G) \geq \lambda_1(G'')$.

The third step. If the graph G'' obtained in the second step admits at least two cycles, we shall perform surgery on such G'' to make a graph G''' whose number of cycles is smaller than that of G''. Finally we obtain a graph G''' which is of type $L_{m,i}$ or in general, a *star-shaped* graph, that is, a graph which has no cycle and one branch vertex (see Figure 4.5).

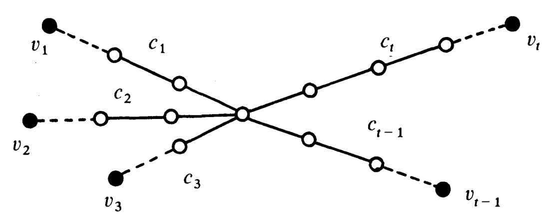

FIGURE 4.5.

Let G'' admit at least two cycles each of which has one branch point. Let c be any fixed cycle of G''. Let $e_c = (v, v_1, v_2, \ldots, v_j, \tilde{v})$ be a path emanating from a unique branch point v to a neighboring branch point $\tilde{v}$. Let $\tilde{e}_c$ be the union of e_c and c (see Figure 4.6).

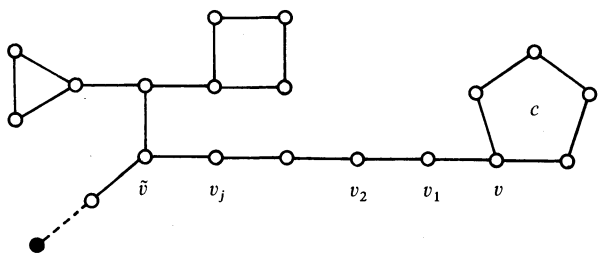

FIGURE 4.6.

Then we get:

LEMMA 4.2. *Let f be the first eigenfunction of G''. Then*

$$\max_{x\in\tilde{e}_c} f(x) = \max_{x\in c} f(x)\,.$$

PROOF. Assume that this is not the case. Then there exists $v_i \in e_c$ with $1 \le i \le j$ such that

$$f(v_i) = \max_{x\in\tilde{e}_c} f(x)\,.$$

Define $\tilde{f}$ on the set of vertices of G'' by

$$\tilde{f}(x) = \begin{cases} f(v_i)\,, & x \in c \cup \{v_1, \dots, v_i\}\,, \\ f(x)\,, & \text{otherwise}\,. \end{cases}$$

By the definition of $\tilde{f}$,

$$(\tilde{f}, \tilde{f})_{G''} > (f, f)_{G''}\,, \quad \text{and} \quad (d\tilde{f}, d\tilde{f})_{G''} \le (df, df)_{G''}\,,$$

which contradicts our assumption that f is the first eigenfunction of G''. □

Let us denote $f(c) = \max_{x\in c} f(x)$ for each cycle c of G''. Let us choose a cycle c_0 such that

$$f(c_0) = \max_{c} f(c)\,,$$

where c runs over all cycles of G''. By Lemma 4.2, we may assume that c_0 contains x_0, that is,

$$f(c_0) = \max_{v\in V_{G''}} f(v) = f(x_0)\,.$$

For each cycle c not equal to c_0, let v_c be its branch point, let G_1 be the complement of v_c in c. Now perform the (G_1, x_0)-operation on G'' to get a new graph G'''. Then

$\lambda_1(G'')\geq\lambda_1(G''')$ and the cycle of G''' containing x_0 has two branch points. Performing the process of the second step on G''' again, we get a new graph $G^{(4)}$ all of whose cycles have only one branch point and the number of cycles is smaller than that of G'''. Continue this process until the number of cycles is at most one. We obgain a graph of type $L_{m,i}$ or in general, a star-shaped graph.

The last step. We shall show:

LEMMA 4.3. *Let G_* be a star-shaped graph which is not of type A_{m+1}. For some $i>2$, we have*

$$\lambda_1(G_*)>\lambda_1(A_{m+1})\geq\lambda_1(L_{m,i}).$$

Moreover, for all $i>2$,

$$\lambda_1(L_{m,i})>\lambda_1(L_m).$$

PROOF. For the first inequality, let G_* be a star-shaped graph and f its first eigenfunction (see Figure 4.5). Let $\partial V_*=\{v_1, v_2, v_3, \dots, v_t\}$ be the set of all boundary vertices of G_*. Let c_i be the paths connecting x_0 and v_i $(1\leq i\leq t)$. Cut each c_i $(3\leq i\leq t)$ at x_0, paste c_i to v_{i-1} for all $3\leq i\leq t$ as to get a string, and change v_i $(2\leq i\leq t-1)$ to interior vertices and change boundary edge of c_i $(2\leq i\leq t-1)$ to interior edge. Then the resulting graph $\tilde{G}$ is of type A_{m+1}. Define a function $\tilde{f}$ on $\tilde{G}$ by

$$\tilde{f}(x)=\begin{cases} f(x_0), & x \text{ is a vertex of } c_i\ (2\leq i\leq t-1),\\ f(x), & \text{otherwise}.\end{cases}$$

Then

$$(d\tilde{f}, d\tilde{f})<(df, df), \quad \text{and} \quad (\tilde{f}, \tilde{f})>(f, f),$$

which implies that

$$\lambda_1(A_{m+1})\leq\frac{(d\tilde{f}, d\tilde{f})}{(\tilde{f}, \tilde{f})}<\frac{(df, df)}{(f, f)}=\lambda_1(G_*).$$

For the second inequality, let f be the first eigenfunction of a graph of type A_{m+1}. Let v_1 and v_2 be the two end points of the graph A_{m+1}, and let x_0 be the interior vertex attaining the maximum of f. Paste the end vertex v_2 to the vertex x_0 to get a cycle c and the graph $K_{m,i}$ for some i. Define a function $\tilde{f}$ on the graph $L_{m,i}$ by

$$\tilde{f}(x)=\begin{cases} f(x_0), & x\in c,\\ f(x), & \text{otherwise}.\end{cases}$$

Then $(\tilde{f}, \tilde{f})_{L_{m,i}}\geq(f, f)_{A_{m+1}}$ and $(d\tilde{f}, d\tilde{f})_{L_{m,i}}\leq(df, df)_{A_{m+1}}$, which implies that $\lambda_1(A_{m+1})\geq\lambda_1(L_{m,i})$ for some i.

It remains to show $\lambda_1(L_m)<\lambda_1(L_{m,i})$ for all $i>2$.

Let G be a graph of type $L_{m,i}$, x_0 its vertex attaining the maximum of the first eigenfunction f, and c its cycle. By Lemma 4.2, it follows that

(1) $x_0 \in c$.

To see the inequality, we want to show that:

(2) the function f is *monotone increasing* on the path $\tilde{e}=(v_1, v_2, \ldots, v_s, \tilde{v})$, where v_1 is the boundary vertex and $\tilde{v}$ is the branch point, that is, $f(v_i)<f(v_j)<f(\tilde{v})$ if $i<j$.

Indeed, otherwise, we replace f by $\tilde{f}$ on V in such a way that $\tilde{f}$ is linear on the part where f is lower convex. Then

$$\frac{(d\tilde{f}, d\tilde{f})}{(\tilde{f}, \tilde{f})}<\frac{(df, df)}{(f, f)}=\lambda_1(L_{m,i}),$$

which is a contradiction.

We also have:

(3) x_0 is a branch point of c.

Indeed, otherwise, for $G=L_{m,i}$, we cut G at $\tilde{v}$ one of the edges of c having $\tilde{v}$ as an end point, and paste the edge to x_0 to obtain G' (see Figure 4.7).

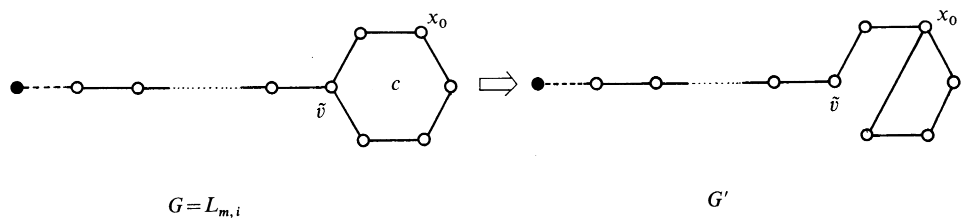

FIGURE 4.7.

Define $\tilde{f}$ on G' by

$$\tilde{f}(x)=\begin{cases} f(x_0), & x \text{ is in the cycle of } G', \\ f(x), & \text{otherwise}. \end{cases}$$

Then we get

$$\frac{(d\tilde{f}, d\tilde{f})}{(\tilde{f}, \tilde{f})}<\frac{(df, df)}{(f, f)}=\lambda_1(G)=\lambda_1(L_{m,i}),$$

which is a contradiction.

Let $c=(\tilde{v}, \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{j-2}, \tilde{v}_{j-1}, \tilde{v}_j)$ be a cycle of $G=L_{m,i}$. We have

(4) $$f(x)=f(\tilde{v}), \qquad \text{for all} \quad x \in c,$$

since, if there exists $\tilde{v}_s \in c$ such that $f(\tilde{v}_s)<f(\tilde{v})$, and we define $\tilde{f}$ on $L_{m,i}$ by

$$\tilde{f}(x)=\begin{cases} f(\tilde{v}), & x \in c, \\ f(x), & \text{otherwise}, \end{cases}$$

then we get

$$(d\tilde{f},\, d\tilde{f}) \leq (df,\, df)\,, \quad \text{and} \quad (\tilde{f}, \tilde{f}) > (f, f)\,,$$

which is a contradiction to our choice that f is the first eigenfunction.

Now cut the edge $[\tilde{v}, \tilde{v}_j]$ of c at $\tilde{v}$ and paste it to the vertex $\tilde{v}_{j-2}$. Then we get a graph $\tilde{G}$ of type L_m (see Figure 4.8).

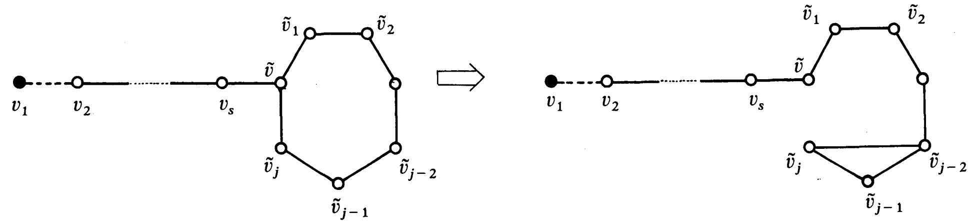

FIGURE 4.8.

Define $\tilde{f}$ on $\tilde{G}$ by the function corresponding to f. Then $(d\tilde{f}, d\tilde{f}) = (df, df)$ and $(\tilde{f}, \tilde{f}) = (f, f)$. However, $\tilde{f}$ cannot be the first eigenfunction on $\tilde{G}$, for otherwise, $\tilde{f}$ must be a strictly monotone function on the path emanating from the boundary vertex to the branch point by the fact (2). By definition, however, it is not the case, a contradiction.

Thus we obtain

$$\lambda_1(L_m) < \frac{(d\tilde{f},\, d\tilde{f})}{(\tilde{f}, \tilde{f})} = \frac{(df,\, df)}{(f, f)} = \lambda_1(L_{m,i})\,.$$

Therefore, we obtain Lemma 4.3, and hence Theorem A. □

5. Proof of Theorem B. To prove Theorem B, we first note that any cycle c of a graph $G = (V \cup \partial V, E \cup \partial E)$ with the non-separation property admits at least two branch points. Indeed, if c has only one branch point v, then $c - \{v\}$ is one of the connected components of the complement $G - \{v\}$. However, $c - \{v\}$ has no boundary vertex, a contradiction to the non-separation property of G.

Let G be a graph with the non-separation property. We first perform the (G_1, e)-operation on G as in the second step of the proof of Theorem A, and get a graph G', the number of whose cycles is smaller than that of G and which still has the non-separation property. We continue this process successively and finally obtain a graph, denoted by the same letter G', which has no cycle and the non-separation property.

Next, as in the third step of the proof of Theorem A, we perform the (G, x)-operation on G', and get a graph G'' whose number of boundary points is smaller than that of G'. Continuing this process successively until x_0 is the only one branch vertex, we obtain a star-shaped graph G''. By Lemma 4.3, we obtain Theorem B. □

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