THE FACTORIZATION PROBLEM FOR NONNEGATIVE OPERATOR VALUED FUNCTIONS

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Introduction. Let F be a function defined on the circle Γ $= \{e^{i\theta} : 0 \leq \theta < 2\pi\}$ or line $R = (-\infty, \infty)$ whose values are nonnegative operators on a separable complex Hilbert space. We are concerned with the problem of finding conditions that $F = G^*G$ a.e. where G is the strong boundary value function of a suitable operator valued analytic function defined in the disk $|z| < 1$ or half-plane $y > 0$. Mainly we are interested in special classes of functions in which such a factorization is always possible.

Our study is motivated by the Fejér-Riesz theorem on the factorization of nonnegative trigonometric polynomials, and Ahiezer's version [1] of its generalization to entire functions of exponential type which are nonnegative on the real axis. Both results generalize to operator valued functions, and, in fact, both appear as special cases of a very general result (Theorem 3.1).

More generally we present a unified treatment of the factorization problem, and thus much of §1 is expository. There we develop the theory of a corresponding abstract factorization problem for nonnegative Hilbert space operators. Both the results and methods of §1 are purely operator theoretic. In §2 we show how the abstract theory relates to the theory of operator valued functions defined on the circle Γ or line R. The main applications to the factorization problem for nonnegative operator valued functions are deferred to §3.

The factorization problem arises in the prediction theory of stationary stochastic processes. For this connection see Helson and Lowdenslager [11], Rozanov [27], and Wiener and Masani [28].

We wish to thank Professor Loren Pitt for calling our attention to the paper by E. Robinson [23]. We have extended Robinson's results in §2.

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1. **Shift analysis and factorization.** By a *shift operator* we mean an isometry S on a Hilbert space such that $S^{*n} \to 0$ strongly as $n \to \infty$. In this section we fix a Hilbert space $\mathcal{3C}$ and a shift operator S on $\mathcal{3C}$. We write $\mathcal{C} = \ker S^*$. The dimension of $\mathcal{C}$ is called the *multiplicity* of S. The projection mapping $\mathcal{3C}$ onto $\mathcal{C}$ is denoted P_0. For any $j = 0, 1, 2, \cdots$, the projection mapping $\mathcal{3C}$ onto $S^j \mathcal{C}$ is given by $P_j = S^j(I - SS^*)S^{*j}$. Since the projections $P_0, P_1, P_2, \cdots$ are pairwise orthogonal and sum strongly to the identity operator on $\mathcal{3C}$, we have

$$\mathcal{3C} = \sum_0^\infty \oplus\, S^j \mathcal{C}.$$

It is easy to exhibit the reducing subspaces of S in terms of this decomposition. If $\mathcal{3C}_0$ is a subspace of $\mathcal{3C}$ which reduces S, then $S_0 = S \mid \mathcal{3C}_0$ is a shift operator on $\mathcal{3C}_0$. Hence

$$\mathcal{3C}_0 = \sum_0^\infty \oplus\, S^j \mathcal{C}_0$$

for some subspace $\mathcal{C}_0$ of $\mathcal{C}$. Conversely, any subspace $\mathcal{3C}_0$ of $\mathcal{3C}$ of this form reduces S.

We write $\mathcal{B}(\mathcal{3C})$, $\mathcal{B}(\mathcal{3C}, \mathcal{C})$, for example, for the spaces of bounded operators mapping $\mathcal{3C}$ into $\mathcal{3C}$, respectively $\mathcal{3C}$ into $\mathcal{C}$.

We associate with each $T \in \mathcal{B}(\mathcal{3C})$ a semi-infinite matrix, $T \sim [A_{jk}]$, with entries $A_{jk} \in \mathcal{B}(\mathcal{C})$, $j, k = 0, 1, 2, \cdots$. The action of T is determined by the matrix according to this rule: $Tf = g$ where $f = \sum_0^\infty S^j a_j$, $g = \sum_0^\infty S^j b_j$ and

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} \cdots \\ A_{10} & A_{11} & A_{12} \cdots \\ A_{20} & A_{21} & A_{22} \cdots \\ & \cdots & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix}.$$

Explicitly,

(1) $A_{jk} = P_0 S^{*j} T S^k P_0 \mid \mathcal{C}, \qquad j, k = 0, 1, 2, \cdots.$

If $T_1 \sim [A_{jk}]$, $T_2 \sim [B_{jk}]$, then $T_1 T_2 \sim [C_{jk}]$ where, for each $j, k = 0, 1, 2, \cdots$, $C_{jk} = \sum_{p=0}^\infty A_{jp} B_{pk}$ with convergence in the strong operator topology. This follows from (1) and the fact, already noted, that the projections $P_p = S^p P_0 S^{*p}$, $p = 0, 1, 2, \cdots$, sum strongly to the identity operator on $\mathcal{3C}$.

1.1. DEFINITION. Let $T \in \mathcal{B}(\mathcal{3C})$. We say that T is

(i) *S-Toeplitz* if $S^*TS = T$,

(ii) *S-analytic* if $TS = ST$,

(iii) *S-inner* if T is S-analytic and partially isometric,

(iv) *S-outer* if T is S-analytic and $CL(T\mathfrak{IC})$ reduces S.

It is easy to see that T is S-Toeplitz or S-analytic if and only if its matrix has the form

$$\begin{bmatrix} A_0 & A_{-1} & A_{-2} \cdots \\ A_1 & A_0 & A_{-1} \cdots \\ A_2 & A_1 & A_0 \cdots \\ & \cdots \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} A_0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & \cdots \\ A_2 & A_1 & A_0 & \cdots \\ & \cdots \end{bmatrix}$$

respectively. Then

$$A_j = P_0 S^{*j} T P_0 \big| \mathfrak{C}, \qquad j = 0, 1, 2, \cdots,$$

and

$$A_{-j} = P_0 T S^j P_0 \big| \mathfrak{C}, \qquad j = 1, 2, 3, \cdots,$$

by (1).

We will usually denote S-analytic operators by letters $A, B, \cdots$. Then without further explanation we will write $(A_j)_0^\infty$, $(B_j)_0^\infty$, $\cdots$ for the corresponding operators in $\mathfrak{B}(\mathfrak{C})$ which appear in the matrix representations as above.

Notice that the initial space of any S-inner operator B reduces S. In fact $\ker B$ is invariant under S since $BS = SB$, while for any $f \in H$, $\|Bf\| = \|f\|$ implies $\|BSf\| = \|SBf\| = \|Bf\| = \|f\| = \|Sf\|$. On the other hand, the final space of an S-inner operator B reduces S if and only if B^* is also S-inner. In this degenerate case, B_0 is a partially isometric operator on $\mathfrak{C}$, and B is completely determined by B_0 according to the formula

$$B\left(\sum_0^\infty S^j c_j\right) = \sum_0^\infty S^j(B_0 c_j),$$

where $(c_j)_0^\infty$ is any sequence in $\mathfrak{C}$ such that $\sum_0^\infty |c_j|^2 < \infty$.

If A is an S-analytic operator on $\mathfrak{IC}$, then $T = A^*A$ is S-Toeplitz and nonnegative. We are mainly concerned with the problem of finding conditions on a nonnegative S-Toeplitz operator such that it will be of this form.

It is convenient to begin with a characterization of products AA^* where A is S-analytic. This characterization is implicit in de Branges and Rovnyak [4]. See also Sz.-Nagy and Foias [21].

1.2. THEOREM. *Let T be a nonnegative operator on $\mathfrak{IC}$. Then $T = AA^*$ for some S-analytic operator A if and only if $T - STS^* = J^*J$ for some $J \in \mathfrak{B}(\mathfrak{IC}, \mathfrak{C})$. In this case, A is determined by T up to a trivial right factor. More precisely, if A and C are two S-analytic operators such that $T = AA^*$ and $T = CC^*$, then $A = CV$, $C = AV^*$ for some operator V such that both V and V^* are S-inner, with initial spaces $\sum_0^\infty \oplus S^j \mathfrak{C}_A$ and $\sum_0^\infty \oplus S^j \mathfrak{C}_C$ respectively, where $\mathfrak{C}_A = \mathrm{CL}(P_0 A^* \mathfrak{IC})$ and $\mathfrak{C}_C = \mathrm{CL}(P_0 C^* \mathfrak{IC})$.*

PROOF. If $T = AA^*$ where A is S-analytic, then $T - STS^* = A(I - SS^*)A^* = AP_0 A^* = J^*J$ where $J = P_0 A^* \in \mathfrak{B}(\mathfrak{IC}, \mathfrak{C})$.

Suppose $T - STS^* = J^*J$ for some $J \in \mathfrak{B}(\mathfrak{IC}, \mathfrak{C})$. On iterating this identity we obtain

$$T - S^{n+1}TS^{*n+1} = \sum_0^n S^j J^* J S^{*j}$$

for all $n = 0, 1, 2, \cdots$. Therefore, if we regard J as an operator from $\mathfrak{IC}$ to $\mathfrak{IC}$, we have

$$\langle Tf, g \rangle - \langle S^{n+1}TS^{*n+1}f, g \rangle = \sum_0^n \langle JS^{*j}f, JS^{*j}g \rangle$$

$$= \left\langle \sum_0^n S^j J S^{*j}f, \sum_0^n S^k J S^{*k}g \right\rangle$$

for all $f, g \in \mathfrak{IC}$, $n = 0, 1, 2, \cdots$. Define $A \in \mathfrak{B}(\mathfrak{IC})$ by $A^* = \sum_0^\infty S^j J S^{*j}$. It is easy to see that the series converges in the strong operator topology, A is S-analytic, and by the last identity, $\langle Tf, g \rangle = \langle AA^*f, g \rangle$ for all $f, g \in \mathfrak{IC}$. Therefore the condition in the theorem is both necessary and sufficient.

Now let A and C be two S-analytic operators such that $AA^* = CC^*$. If T is the common value of AA^* and CC^* then

$$AP_0 A^* = T - STS^* = CP_0 C^*$$

as above. In particular there is a unique partial isometry $V_0 \in \mathfrak{B}(\mathfrak{C})$ with initial space $\mathfrak{C}_A = \mathrm{CL}(P_0 A^* \mathfrak{IC})$ and final space $\mathfrak{C}_C = \mathrm{CL}(P_0 C^* \mathfrak{IC})$ such that

$$V_0: P_0 A^*f \to P_0 C^*f, \quad f \in \mathfrak{IC}.$$

The operator V_0 has a unique extension to an S-analytic operator V on $\mathfrak{IC}$, determined by

$$VS^j c = S^j V_0 c, \quad c \in \mathfrak{C}, \ j = 0, 1, 2, \cdots.$$

Clearly V and V^* are S-inner with initial spaces $\sum_0^\infty \oplus S^j \mathcal{C}_A$ and $\sum_0^\infty \oplus S^j \mathcal{C}_C$ respectively. By construction $Ac = CVc$ whenever $c = P_0 A^* f$ for some $f \in \mathcal{K}$. We find at once that $Ag = CVg$ for every $g \in \sum_0^\infty \oplus S^j \mathcal{C}_A$. If $g \perp \sum_0^\infty \oplus S^j \mathcal{C}_A$ and $g = \sum_0^\infty S^j b_j$, $b_0, b_1, b_2, \cdots \in \mathcal{C}$, then

$$0 = \langle g, S^j P_0 A^* f \rangle = \langle A b_j, f \rangle$$

for all $f \in \mathcal{K}$. So $Ag = \sum_0^\infty S^j A b_j = 0 = CVg$. Hence $A = CV$. Applying a similar argument with the roles of A and C interchanged, we get also $C = A V^*$. This completes the proof.

As an application we derive the standard characterization of the invariant subspaces of S in terms of S-inner operators.

1.3. THEOREM. *A subspace $\mathfrak{M}$ of $\mathcal{K}$ is invariant under S if and only if $\mathfrak{M} = B \mathcal{K}$ for some S-inner operator B. In this case B is determined by $\mathfrak{M}$ up to a trivial right factor as in Theorem 1.2.*

1.4. LEMMA. *If Q is a projection operator acting on a separable Hilbert space $\mathcal{K}$, then*

$$\dim Q\mathcal{K} = \sum_{j \in J} \|Q\phi_j\|^2$$

for any orthonormal basis $\{\phi_j\}_{j \in J}$ for $\mathcal{K}$.

PROOF OF 1.4. Let $\{\psi_k\}_{k \in K}$ be any orthonormal basis *for* $Q\mathcal{K}$. Then

$$\sum_{j \in J} \|Q\phi_j\|^2 = \sum_{j \in J} \sum_{k \in K} |\langle Q\phi_j, \psi_k \rangle|^2$$

$$= \sum_{k \in K} \sum_{j \in J} |\langle \phi_j, \psi_k \rangle|^2$$

$$= \sum_{k \in K} \|\psi_k\|^2 = \dim Q\mathcal{K}.$$

PROOF OF 1.3. The range of an S-inner operator is clearly a closed invariant subspace for S.

Suppose $\mathfrak{M}$ is a subspace of $\mathcal{K}$ which is invariant under S, and let P be the projection mapping $\mathcal{K}$ onto $\mathfrak{M}$. We show that P satisfies the condition in Theorem 1.2. Since P is selfadjoint, idempotent, and $PSP = SP$, the operator $Q = P - SPS^*$ is selfadjoint and idempotent. Hence Q is also a projection operator. We claim that $\dim Q\mathcal{K} \leq \dim \mathcal{C}$. If $\mathcal{C}$ is infinite dimensional, then $\mathcal{K}$ has the same dimension as $\mathcal{C}$ and the inequality is trivial. If $\mathcal{C}$ is finite dimensional, then $\mathcal{K}$ is separable. Let $\{e_k\}_{k \in K}$ be any orthonormal basis for $\mathcal{C}$. Then the vectors $S^j e_k$, $k \in K$, $j = 0, 1, 2, \cdots$, form an orthonormal basis for $\mathcal{K}$. By Lemma 1.4,

$$\dim Q\mathcal{K} = \sum_{k\in K} \sum_{j=0}^{\infty} \langle QS^j e_k, S^j e_k \rangle$$

$$= \sum_{k\in K} \sum_{j=0}^{\infty} \langle (P - SPS^*)S^j e_k, S^j e_k \rangle$$

$$= \lim_{n\to\infty} \sum_{k\in K} \langle PS^n e_k, S^n e_k \rangle$$

$$\leq \sum_{k\in K} \|e_k\|^2 = \dim \mathcal{C}.$$

Hence $\dim Q\mathcal{K} \leq \dim \mathcal{C}$ in any case.

It follows that there exists an isometry V mapping $Q\mathcal{K}$ into $\mathcal{C}$. But then $P - SPS^* = Q = J^*J$ where $J = VQ \in \mathcal{B}(\mathcal{K}, \mathcal{C})$. By Theorem 1.2, $P = BB^*$ for some S-analytic operator B. Since P is a projection operator, B is S-inner. By construction $\mathfrak{M} = P\mathcal{K} = B\mathcal{K}$.

The uniqueness assertion follows from the corresponding part of Theorem 1.2.

We proceed to a characterization of products A^*A where A is an S-analytic operator on $\mathcal{K}$.

1.5. DEFINITION. A nonnegative S-Toeplitz operator T on $\mathcal{K}$ is called *factorable* if $T = A^*A$ for some S-analytic operator A on $\mathcal{K}$.

1.6. DEFINITION. Let T be any nonnegative S-Toeplitz operator on $\mathcal{K}$. We write $\mathcal{K}_T = \mathrm{CL}(T^{1/2}\mathcal{K})$ for the closure of the range of $T^{1/2}$, considered as a Hilbert space in the metric of $\mathcal{K}$. We write S_T for the unique isometry on $\mathcal{K}_T$ such that $S_T(T^{1/2}f) = T^{1/2}Sf$ for each $f \in \mathcal{K}$. We put $\mathcal{C}_T = \ker S_T^*$.

The definition of S_T is meaningful because for each f in $\mathcal{K}$, $\|T^{1/2}Sf\|^2 = \langle S^*TSf, f \rangle = \langle Tf, f \rangle = \|T^{1/2}f\|^2$.

1.7. THEOREM. *Let T be a nonnegative S-Toeplitz operator on $\mathcal{K}$. The following are equivalent*:

(i) *T is factorable,*

(ii) *S_T is a shift operator on $\mathcal{K}_T$,*

(iii) *there is a dense subspace $\mathcal{C}_0$ of $\mathcal{C}$ such that*

$$\lim_{n\to\infty} \{\sup[|\langle Tc, S^n f \rangle| : f \in \mathcal{K}, \langle Tf, f \rangle = 1]\} = 0$$

for each $c \in \mathcal{C}_0$. In this case we can write $T = A^*A$ where A is S-outer and $A_0 \geq 0$.

The theorem is a version of Lowdenslager's theorem [17] as presented by Rosenblum [25]. Technically it is different because $\mathcal{K}_T$ is now defined without a change from the norm of $\mathcal{K}$.

PROOF. (i)$\Rightarrow$(iii). Let $T = A^*A$ where A is S-analytic. If $c \in \mathcal{C}$, $f \in \mathcal{3C}$, and $\langle Tf, f \rangle = 1$, then

$$| \langle Tc, S^n f \rangle |^2 = | \langle A^*Ac, S^n f \rangle |^2 = | \langle S^{*n}Ac, Af \rangle |^2$$
$$\leq \| S^{*n}Ac \|^2 \| Af \|^2 = \| S^{*n}Ac \|^2 \langle Tf, f \rangle = \| S^{*n}Ac \|^2.$$

It follows that (iii) holds with $\mathcal{C}_0 = \mathcal{C}$.

(iii)$\Rightarrow$(ii). Suppose (iii) holds. For each $c \in \mathcal{C}_0$ and $n = 0, 1, 2, \cdots$,

$$\| S_T^{*n} T^{1/2} c \| = \sup [| \langle Tc, S^n f \rangle | : f \in \mathcal{3C}, \langle Tf, f \rangle = 1].$$

This follows from the fact that $\langle Tc, S^n f \rangle = \langle T^{1/2}c, T^{1/2}S^n f \rangle = \langle T^{1/2}c, S_T^n T^{1/2} f \rangle = \langle S_T^{*n} T^{1/2}c, T^{1/2}f \rangle$, and, the set of vectors $T^{1/2}f$, $f \in \mathcal{3C}$, $\langle Tf, f \rangle = 1$, is dense in the unit sphere in $\mathcal{3C}_T$. Therefore if (iii) holds, then

$$(2) \qquad\qquad \lim_{n \to \infty} \| S_T^{*n} g \| = 0$$

for any g in $\mathcal{3C}_T$ of the form $g = T^{1/2}c$, $c \in \mathcal{C}_0$. If $g = T^{1/2}S^k c$, for some $c \in \mathcal{C}_0$ and $k = 0, 1, 2, \cdots$, then $S_T^{*n} g = S_T^{*n} T^{1/2}S^k c = S_T^{*n} S_T^k T^{1/2}c = S_T^{*n-k} T^{1/2}c$ if only $n \geq k$; hence (2) holds for such g. By linearity and approximation, (2) holds for all g in $\mathcal{3C}_T$. Hence S_T is a shift operator and (ii) holds.

(ii)$\Rightarrow$(i). Assume that S_T is a shift operator. Define $J \in \mathcal{B}(\mathcal{C}_T, \mathcal{C})$ by $Jk = T^{1/2}k$, $k \in \mathcal{C}_T$. To see that $T^{1/2}\mathcal{C}_T \subseteq \mathcal{C}$, note that by the definition of S_T, $T^{1/2}S_T^* g = S^* T^{1/2}g$, $g \in \mathcal{3C}_T$; hence if $k \in \mathcal{C}_T = \ker S_T^*$ then $T^{1/2}k \in \mathcal{C} = \ker S^*$. Thus J is well defined. Clearly J is one-to-one. Therefore in the polar decomposition $J = RV$, $R = (JJ^*)^{1/2}$, V is an isometry mapping $\mathcal{C}_T$ into $\mathcal{C}$. Since we assume that S_T is a shift operator, each $g \in \mathcal{3C}_T$ has a unique representation

$$g = \sum_0^\infty S_T^j k_j$$

where $k_j \in \mathcal{C}_T, j = 0, 1, 2, \cdots$. Therefore we can define an isometry K mapping $\mathcal{3C}_T$ into $\mathcal{3C}$ by setting

$$Kg = \sum_0^\infty S^j(Vk_j).$$

We have then $KS_T = SK$. Now define $A \in \mathcal{B}(\mathcal{3C})$ by $Af = KT^{1/2}f$, $f \in \mathcal{3C}$. For any $f \in \mathcal{3C}$, $ASf = KT^{1/2}Sf = KS_T T^{1/2}f = SKT^{1/2}f = SAf$, and $\langle Tf, f \rangle = \langle T^{1/2}f, T^{1/2}f \rangle = \langle KT^{1/2}f, KT^{1/2}f \rangle = \langle Af, Af \rangle$. Thus A is S-analytic and $T = A^*A$, so (i) holds.

It is easily checked that for the operator A just constructed

$CL(A\mathcal{K}) = CL(KT^{1/2}\mathcal{K}) = K\mathcal{K}_T$ reduces S so A is S-outer. Also $A_0 = P_0 A P_0| \mathcal{C} = R \geq 0$. This completes the proof.

The condition $A_0 \geq 0$ was used by Masani [18] to discuss uniqueness of factors.

1.8. THEOREM. *Let T be a nonnegative S-Toeplitz operator. If T is factorable, then there is a unique S-outer operator A such that $T = A^*A$ and $A_0 \geq 0$.*

We make use of a second uniqueness result.

1.9. THEOREM. *Let T be a nonnegative S-Toeplitz operator. Suppose T is factorable and let $T = A^*A$ and $T = C^*C$ be two representations where A and C are S-outer operators. Then A and C are equivalent in the sense that $A = VC$, $C = V^*A$ where V is an operator such that both V and V^* are S-inner with initial spaces $CL(C\mathcal{K})$ and $CL(A\mathcal{K})$ respectively. Conversely if A and C are equivalent S-outer operators then $A^*A = C^*C$.*

1.10. LEMMA. *If A is an S-outer operator, then $CL(A\mathcal{K}) = \sum_0^\infty \oplus S^j CL(A_0 \mathcal{C})$.*

PROOF OF 1.10. If $f = Ag \in A\mathcal{K}$, we can write $f = Ac + ASS^*g$ for some $c \in \mathcal{C}$. Then $P_0 f = P_0 Ac \in A_0 \mathcal{C}$ because $A_0 = P_0 A P_0| \mathcal{C}$ by (1). Therefore $P_0 A\mathcal{K} \subseteq A_0 \mathcal{C}$ and $P_0\, CL(A\mathcal{K}) \subseteq CL(A_0\mathcal{C}) \subseteq P_0\, CL(A\mathcal{K})$, hence $P_0\, CL(A\mathcal{K}) = CL(A_0\mathcal{C})$. Since A is S-outer, $CL(A\mathcal{K})$ reduces S so

$$CL(A\mathcal{K}) = \sum_0^\infty \oplus S^j \mathcal{C}_0$$

where $\mathcal{C}_0 = \ker (S^*| CL(A\mathcal{K})) = P_0\, CL(A\mathcal{K}) = CL(A_0\mathcal{C})$.

PROOFS OF 1.8 AND 1.9. Let A and C be S-outer operators such that $A^*A = C^*C$. Define V on $C\mathcal{K}$ to $\mathcal{K}$ by $VCf = Af$, $f \in \mathcal{K}$. Since $A^*A = C^*C$, V is well defined and has a unique extension to a partial isometry on $\mathcal{K}$, also denoted V, with initial space $CL(C\mathcal{K})$ and final space $CL(A\mathcal{K})$. We find at once that both V and V^* commute with S, so both are S-inner. The first assertion in Theorem 1.9 follows; the second is verified by direct calculation.

Suppose now that $A_0 \geq 0$, $C_0 \geq 0$. Since $A = VC$, $C = V^*A$, we have $A_0 = V_0 C_0$, $C_0 = V_0^* A_0$. Hence $A_0^2 = C_0 V_0^* V_0 C_0 \leq C_0^2$ and $C_0^2 = A_0 V_0 V_0^* A_0 \leq A_0^2$, so $A_0^2 = C_0^2$. Since $A_0 \geq 0$, $C_0 \geq 0$, we have $A_0 = C_0$. Since $A_0 = V_0 C_0$, V_0 is the identity on $CL(C_0\mathcal{C})$. Hence V is the identity on $CL(C_0\mathcal{C})$. Since V commutes with S and $CL(C\mathcal{K}) = \sum_0^\infty \oplus S^j\, CL(C_0\mathcal{C})$ by Lemma 1.10, it follows that V is the identity on $CL(C\mathcal{K})$. But $A = VC$, so we get $A = C$. This proves the unique-

ness part of Theorem 1.8. The existence assertion follows from Theorem 1.7.

1.11. THEOREM. *Let A be an S-analytic operator. Then A has a unique factorization $A = BC$ where C is S-outer, $C_0 \geq 0$, and B is S-inner with initial space $CL(C\mathcal{K})$. For this choice of C, $C^*C = A^*A$. On the other hand, if C is any S-outer operator such that $C^*C = A^*A$, then $A = BC$ for some S-inner operator B with initial space $CL(C\mathcal{K})$.*

PROOF. Consider any S-outer operator C such that $C^*C = A^*A$. By Theorem 1.8 there is a unique such C with $C_0 \geq 0$. But for any such C there is a unique partial isometry B on $\mathcal{K}$ with initial space $CL(C\mathcal{K})$ such that $B(Cf) = Af$, $f \in \mathcal{K}$, i.e. $A = BC$. If $g = Cf \in C\mathcal{K}$ then $BSg = BCSf = ASf = SAf = SBg$. Since C is S-outer, $CL(C\mathcal{K})$ reduces S. Therefore if $g \perp C\mathcal{K}$ then $Sg \perp C\mathcal{K}$ and $BSg = 0 = SBg$. It follows that B is S-inner. The assertions of the theorem follow quickly from these facts.

The S-outer operators are distinguished from more general S-analytic operators by their extremal properties. Our first result in this direction is motivated by work of Lax [15] and Masani [18].

1.12. THEOREM. *Each of the following conditions is necessary and sufficient for an S-analytic operator C to be S-outer:*
(i) $CL(C\mathcal{K}) = \sum_0^\infty \oplus S^j CL(C_0\mathcal{C})$,
(ii) *for each $c \in \mathcal{C}$, $\langle C_0^*C_0c, c \rangle = \inf_{f \in \mathcal{K}} \langle C^*C(c - Sf), c - Sf \rangle$,*
(iii) $C_0^*C_0 \geq A_0^*A_0$ *for every S-analytic operator A such that $C^*C = A^*A$.*

PROOF. Let (o) be the condition that C is S-outer.
(o)$\Leftrightarrow$(i). Sufficiency is obvious; necessity follows from Lemma 1.10.
(o)$\Rightarrow$(ii). If C is S-outer and $c \in \mathcal{C}$, then $S^*Cc \in CL(C\mathcal{K})$ and the vector $g = S^*Cc$ minimizes $\|Cc - Sg\|$. Hence

$$\inf_{f \in \mathcal{K}} \langle C^*C(c - Sf), c - Sf \rangle = \inf_{f \in \mathcal{K}} \|Cc - SCf\|^2$$
$$= \inf_{g \in CL(C\mathcal{K})} \|Cc - Sg\|^2 = \|P_0Cc\|^2$$
$$= \langle C_0^*C_0c, c \rangle.$$

(ii)$\Rightarrow$(iii). If (ii) holds and A is S-analytic with $C^*C = A^*A$, then, for any $c \in \mathcal{C}$,

$$\langle C_0^*C_0c, c \rangle = \inf_{f \in \mathcal{K}} \langle A^*A(c - Sf), c - Sf \rangle = \inf_{f \in \mathcal{K}} \|Ac - SAf\|^2$$
$$\geq \inf_{g \in \mathcal{K}} \|Ac - Sg\|^2 = \|P_0Ac\|^2 = \langle A_0^*A_0c, c \rangle.$$

(iii)$\Rightarrow$(i). Assume (iii) holds. Factor $C = BA$ where A is S-outer and B is S-inner with initial space $\mathrm{CL}(A\mathfrak{K})$. This is possible by Theorem 1.11. Then $C^*C = A^*A$, so by (iii), $C_0^*C_0 \geq A_0^*A_0$. But $C_0 = B_0A_0$ where $\|B_0\| \leq 1$. Hence $C_0^*C_0 = A_0^*A_0$. It follows that B_0 is isometric on $A_0\mathfrak{C}$. Since $B_0 = P_0BP_0|\mathfrak{C}$, B agrees with B_0 on $A_0\mathfrak{C}$, and therefore BS^j agrees with S^jB_0 on $\mathrm{CL}(A_0\mathfrak{C})$, $j = 0, 1, 2, \cdots$. Hence

$$BS^j\,\mathrm{CL}(A_0\mathfrak{C}) = S^jB_0\,\mathrm{CL}(A_0\mathfrak{C}) = S^j\,\mathrm{CL}(C_0\mathfrak{C}),$$

$$\mathrm{CL}(C\mathfrak{K}) = B\,\mathrm{CL}(A\mathfrak{K}) = B\left(\sum_0^\infty \oplus\, S^j\,\mathrm{CL}(A_0\mathfrak{C})\right) = \sum_0^\infty \oplus\, S^j\,\mathrm{CL}(C_0\mathfrak{C}).$$

Therefore (iii) implies (i), and each of the conditions (i)–(iii) is equivalent to C being S-outer.

Some interesting variants of condition (iii) in Theorem 1.12 are suggested by a result of Robinson [23] (see Theorem 2.6). We now extend Robinson's ideas.

1.13. THEOREM. *Let C be an S-analytic operator. Let $\mathcal{W}$ be any subset of the double commutant of S such that the weakly closed linear span of $\mathcal{W}$ and the identity operator contains S. Let $\mathfrak{F}$ be any subset of $\mathfrak{K}$ such that $P_0\mathfrak{F}$ spans a dense subspace of $\mathfrak{C}$. Then C is S-outer if and only if*

$$(3) \qquad\qquad \|W^*Cf\| \leq \|W^*Af\|$$

for all $W \in \mathcal{W}$, $f \in \mathfrak{F}$, and every S-analytic operator A such that $A^*A = C^*C$.

In particular if C is S-outer, then (3) holds for every operator W in the double commutant of S, all $f \in \mathfrak{K}$, and every S-analytic operator A such that $A^*A = C^*C$. The proof shows more, and for reference purposes we isolate some additional facts in

1.14. LEMMA. *Let B be an S-inner operator with initial space $\mathfrak{M}$.*

(i) *If $g \in \mathfrak{M}$ and W is any S-analytic operator which commutes with B, then*

$$(4) \qquad\qquad \|W^*g\| \leq \|W^*Bg\|$$

with equality if and only if $BW^*g = W^*Bg$.

(ii) *Let $\mathcal{W}$ be any set of S-analytic operators which commute with B such that the weakly closed linear span of $\mathcal{W}$ and the identity operator contains S. Let $\mathcal{G}$ be any subset of $\mathfrak{M}$ such that $P_0\mathcal{G}$ spans a dense subspace of $P_0\mathfrak{M}$. If equality holds in (4) for all $W \in \mathcal{W}$ and all $g \in \mathcal{G}$, then B^* is also S-inner.*

In Theorem 1.13 or Lemma 1.14 we could choose $\mathcal{W} = \{V^t\}_{t>0}$, where $V^t = \exp[-t(I+S)(I-S)^{-1}]$ for each $t > 0$. The exponential is

defined to be the strong limit of $\exp[-t(I+rS)(I-rS)^{-1}]$ as $r \nearrow 1$. See Sz.-Nagy and Foiaş [21]. A short calculation gives

$$rS = I - 2 \int_0^\infty e^{-t} \exp[-t(I + rS)(I - rS)^{-1}] \, dt$$

for $0 < r < 1$, where the integral is taken in the weak sense. It follows that

$$S = I - 2 \int_0^\infty e^{-t} V^t \, dt.$$

The last integral can also be taken as an improper Riemann integral with convergence in the uniform operator topology. In particular, the weakly closed linear span of $\mathcal{W} = \{V^t\}_{t>0}$ and I contains S. See Theorem 2.6.

A different choice of $\mathcal{W}$ leads to

1.15. Theorem. *Let C be an S-analytic operator. Then C is S-outer if and only if*

$$(5) \qquad \sum_0^n C_j^* C_j \geqq \sum_0^n A_j^* A_j, \qquad n = 0, 1, 2, \cdots,$$

for every S-analytic operator A such that $A^*A = C^*C$. More generally, (5) holds if A and C are any S-analytic operators which are related by $A = BC$ for some S-inner operator B whose initial space contains $CL(C\mathcal{K})$.

Proof of 1.14. Let $g \in \mathfrak{M}$ and let W be any S-analytic operator which commutes with B. Then $B^*Bg = g$ and so

$$\|W^*g\| = \|W^*B^*Bg\| = \|B^*W^*Bg\| \leq \|W^*Bg\|.$$

Since B^* is a partial isometry, equality holds only if $W^*Bg = Bh$ for some $h \in \mathfrak{M}$. Then $B^*Bh = h$, and since W commutes with B, we get $W^*g = h$. Hence $W^*Bg = BW^*g$. Conversely if $W^*Bg = BW^*g$ then $\|W^*Bg\| = \|BW^*g\| \leq \|W^*g\| \leq \|W^*Bg\|$, and equality holds throughout. This proves (i).

To prove (ii) note that, by (i), $BW^*g = W^*Bg$ for all $W \in \mathcal{W}$, $g \in \mathcal{G}$. By our assumptions on $\mathcal{W}$ it follows that $BS^*g = S^*Bg$, $g \in \mathcal{G}$. Multiply this by S and use the fact that $P_0 = I - SS^*$ to get $BP_0 g = P_0 Bg$, $g \in \mathcal{G}$. If we set $\mathcal{C}_0 = P_0 \mathfrak{M}$, so that $\mathfrak{M} = \sum_0^\infty \oplus S^j \mathcal{C}_0$, then our assumptions concerning $\mathcal{G}$ imply that $Bc = P_0 Bc$ for all $c \in \mathcal{C}_0$ and hence for all $c \in \mathcal{C}$. But then B agrees with $B_0 = P_0 BP_0 | \mathcal{C}$ on $\mathcal{C}$ and so BS^j with $S^j B_0$ on $S^j \mathcal{C}$, $j = 0, 1, 2, \cdots$. It follows that B_0 is a par-

tially isometric operator on $\mathbb{C}$, and $B(\sum_0^\infty S^j c_j) = \sum_0^\infty S^j(B_0 c_j)$ for any sequence $(c_j)_0^\infty$ in $\mathbb{C}$ such that $\sum_0^\infty |c_j|^2 < \infty$. We quickly deduce (ii).

PROOF OF 1.13. Suppose that C is S-outer, and let A be any S-analytic operator such that $A^*A = C^*C$. By Theorem 1.11, $A = BC$ for some S-inner operator B with initial space $CL(C\mathcal{K})$. If W is any operator in the double commutant of S and f is any vector in $\mathcal{K}$, we get (3) from (4) on setting $g = Cf$.

Conversely suppose that (3) holds for all $W \in \mathcal{W}$, $f \in \mathcal{F}$. Factor $C = BA$ where A is S-outer and B is S-inner with initial space $\mathfrak{M} = CL(A\mathcal{K})$. By Theorem 1.12 (i), $P_0\mathfrak{M} = CL(A_0\mathbb{C})$. Set $\mathcal{G} = P_0\mathcal{F}$. Since $P_0 Af = A_0 P_0 f$, $f \in \mathcal{K}$, our assumptions on $\mathcal{F}$ imply that $P_0\mathcal{G}$ spans a dense subspace of $P_0\mathfrak{M}$. By (3), $\|W^* Bg\| \leq \|W^* g\|$ for all $W \in \mathcal{W}$, $g \in \mathcal{G}$. Since the reverse inequality holds automatically by Lemma 1.14 (i), by Lemma 1.14 (ii), B^* is also S-inner. It follows that $CL(C\mathcal{K}) = B\, CL(A\mathcal{K})$ is the initial space of an S-inner operator, hence $CL(C\mathcal{K})$ reduces S and C is S-outer.

PROOF OF 1.15. In Theorem 1.13 choose $\mathcal{W} = \{S^{n+1}\}_0^\infty$ and $\mathcal{F} = \mathbb{C}$. Thus C is S-outer if and only if

$$\|S^{*n+1}Cc\| \leq \|S^{*n+1}Ac\|$$

for each $n = 0, 1, 2, \cdots$, $c \in \mathbb{C}$, and every S-analytic operator A such that $A^*A = C^*C$. Thus C is S-outer if and only if, for each such A and $n = 0, 1, 2, \cdots$,

$$P_0 C^* S^{n+1} S^{*n+1} C P_0 \leq P_0 A^* S^{n+1} S^{*n+1} A P_0$$

or

$$P_0 C^* \left(I - \sum_0^n S^j P_0 S^{*j} \right) C P_0 \leq P_0 A^* \left(I - \sum_0^n S^j P_0 S^{*j} \right) A P_0.$$

This last inequality is equivalent to (5). It is clear from the proof that we can verify (5) for any S-analytic operators A and C such that $A = BC$ for some S-inner operator B whose initial space contains $CL(C\mathcal{K})$.

We turn again to the problem of giving conditions on a nonnegative S-Toeplitz operator T that T be factorable. Theorem 1.7 gives conditions which are both necessary and sufficient. It follows, for example, that if T is S-Toeplitz and $T \geq \delta I$ for some $\delta > 0$, then T is factorable. For then $\mathcal{K}_T = \mathcal{K}$ and $S_T = T^{1/2} S T^{-1/2}$; hence $S_T^{*n} \to 0$ strongly and S_T is a shift operator. Thus T is factorable. However, the conditions in Theorem 1.7 are not always so easy to apply. It is therefore of interest to have sufficient conditions which are useful for applications.

The next result is central in our work.

1.16. THEOREM. *Let T be a nonnegative S-Toeplitz operator. Suppose that there exists an operator $V \neq 0$ in the double commutant of S such that TV is S-analytic. Then T is factorable. Moreover if $T = A^*A$ where A is S-outer, then A^*V is also S-analytic.*

1.17. LEMMA. *Let $V \neq 0$ be an operator in the double commutant of S. Then V has a factorization $V = UW$ where U is an isometry in the double commutant of S and W is an S-outer operator in the double commutant of S.*

PROOF OF 1.17. Let $V \sim [V_{j-k}]$ where $V_j = 0$ for all $j = -1, -2, \cdots$. Since V is in the double commutant of S, there exist scalars v_j such that $V_j = v_j I$, $j = 0, \pm 1, \pm 2, \cdots$. Let $S^{(1)}$ be a shift operator on a Hilbert space $\mathcal{3C}^{(1)}$ such that $\mathcal{C}^{(1)} = \ker S^{(1)*}$ has dimension 1. Construct an $S^{(1)}$-analytic operator $V^{(1)}$ such that $V^{(1)} \sim [v_{j-k}]$. Factor $V^{(1)} = U^{(1)}W^{(1)}$ where $U^{(1)}$ is $S^{(1)}$-inner and $W^{(1)}$ is $S^{(1)}$-outer. Let $U^{(1)} \sim [u_{j-k}]$, $W^{(1)} \sim [w_{j-k}]$ where $u_j = w_j = 0$ for $j = -1, -2, \cdots$. Then there exist operators U, W on $\mathcal{3C}$ such that $U \sim [U_{j-k}]$, $W \sim [W_{j-k}]$ where $U_j = u_j I$, $W_j = w_j I$ for all $j = 0, \pm 1, \pm 2, \cdots$, and these operators have the required properties.

PROOF OF 1.16. Using Lemma 1.17 we can reduce the theorem to the case where V is an isometry.

Suppose then that $V^*V = I$. We show that $V^*TV = T$. Since $T + I \geq I$, by a previous remark we can write $T + I = A^*A$ where A is S-analytic. Since $V^*V = I$ and $AV = VA$ by hypothesis, $V^*TV = V^*A^*AV - V^*V = A^*A - I = T$, as claimed.

Since TV is S-analytic, $(TV)S^n = S^n(TV)$ and $S^{*n}V^*T = V^*TS^{*n}$ for each $n = 0, 1, 2, \cdots$. Hence if $c \in \mathcal{C}$ and $f \in \mathcal{3C}$, $\langle Tf, f \rangle = 1$, then

$$\langle Tc, S^n f \rangle = \langle S^{*n}Tc, f \rangle = \langle S^{*n}V^*TVc, f \rangle$$
$$= \langle V^*TS^{*n}Vc, f \rangle = \langle T^{1/2}S^{*n}Vc, T^{1/2}Vf \rangle,$$

so

$$|\langle Tc, S^n f \rangle| \leq \|T^{1/2}\| \, \|S^{*n}Vc\| \, \|T^{1/2}Vf\|$$
$$= \|T^{1/2}\| \, \|S^{*n}Vc\| \langle Tf, f \rangle$$
$$= \|T^{1/2}\| \, \|S^{*n}Vc\|.$$

Therefore T satisfies condition (iii) of Theorem 1.7, so T is factorable.

Let $T = A^*A$ where A is S-outer. Since $[(A^*V)S - S(A^*V)]A = A^*AVS - SA^*AV = (TV)S - S(TV) = 0$, we have $(A^*V)Sf = S(A^*V)f$ for all $f \in A\mathcal{3C}$. But if $f \perp A\mathcal{3C}$ then $Sf \perp A\mathcal{3C}$ and $VSf \perp A\mathcal{3C}$

because $\mathrm{CL}(A\mathfrak{X})$ reduces S and hence also V. Therefore $(A^*V)Sf=0$ $=S(A^*V)f$ for $f\perp A\mathfrak{X}$. Hence A^*V is S-analytic.

We prove next a comparison theorem which is adapted from B. Moore [19]. Similar results were obtained by Lowdenslager [17] and Douglas [7].

1.18. THEOREM. *Let T_1, T_2 be nonnegative S-Toeplitz operators such that $T_1\leqq T_2$. Assume that*

$$\lim_{n\to\infty}\langle T_2f_n, f_n\rangle = 0$$

for every sequence (f_n) in $\mathfrak{X}$ such that $\lim_{n,k\to\infty}\langle T_2(f_n-f_k), f_n-f_k\rangle=0$ and $\lim_{n\to\infty}\langle T_1f_n, f_n\rangle=0$. Then if T_1 is factorable so is T_2.

PROOF. Since $\|T_1^{1/2}f\|^2=\langle T_1f, f\rangle\leqq\langle T_2f, f\rangle=\|T_2^{1/2}f\|^2$, $f\in\mathfrak{X}$, there is a unique operator C mapping $\mathfrak{X}_{T_2}=\mathrm{CL}(T_2^{1/2}\mathfrak{X})$ into $\mathfrak{X}_{T_1}=\mathrm{CL}(T_1^{1/2}\mathfrak{X})$ such that

$$CT_2^{1/2}f = T_1^{1/2}f, \qquad f\in\mathfrak{X}.$$

Our assumptions imply that C has zero kernel and hence C^* has range dense in $\mathfrak{X}_{T_2}$. For any $f\in\mathfrak{X}$ and $n=0, 1, 2, \cdots$,

$$CS_{T_2}^n T_2^{1/2}f = CT_2^{1/2}S^n f = T_1^{1/2}S^n f = S_{T_1}^n T_1^{1/2}f = S_{T_1}^n CT_2^{1/2}f.$$

Hence $CS_{T_2}^n=S_{T_1}^n C$ and $S_{T_1}^{*n}C^*=C^*S_{T_2}^{*n}$ for each $n=0, 1, 2, \cdots$. If T_1 is factorable then S_{T_1} is a shift operator on $\mathfrak{X}_{T_1}$ by Theorem 1.7. But then $\|S_{T_2}^{*n}g\|\to0$ for each g in the range of C^*. Since the range of C^* is dense in $\mathfrak{X}_{T_2}$, $S_{T_2}^{*n}\to0$ strongly, so S_{T_2} is a shift operator and T_2 is factorable by Theorem 1.7.

2. **Operator valued functions on a circle or line.** In the rest of the paper, $\mathfrak{C}$ will denote a fixed separable Hilbert space. We write $|\cdot|$ for the norm in both $\mathfrak{C}$ and $\mathfrak{B}(\mathfrak{C})$. By a vector or operator valued function we mean a function with values in $\mathfrak{C}$ or $\mathfrak{B}(\mathfrak{C})$ respectively. We are mainly concerned with functions defined on the unit circle $\Gamma=\{e^{i\theta}:0\leqq\theta<2\pi\}$ in the complex plane or the real line R. Unless otherwise stated, we use the weak definitions of analyticity and measurability for vector and operator valued functions and integrals of such functions. Measure theoretic notions always refer to the Borel subsets of Γ and R, normalized Lebesgue measure on Γ, and Lebesgue measure on R. In general we do not distinguish functions which differ only on a null set.

We take for granted (1) basic complex function theory, as presented in Hoffman [12] or Privalov [22], (2) elementary properties of Fou-

rier series and integrals for vector valued functions, and (3) the definitions and basic properties of the Lebesgue and Hardy classes $L^2_\mathbb{C}$, $H^2_\mathbb{C}$ and $L^\infty_{\mathbb{B}(\mathbb{C})}$, $H^\infty_{\mathbb{B}(\mathbb{C})}$ on both Γ and R. We recall, at least, the definition of $H^\infty_{\mathbb{B}(\mathbb{C})}$: this is the class of all $F \in L^\infty_{\mathbb{B}(\mathbb{C})}$ such that $FH^2_\mathbb{C} \subseteq H^2_\mathbb{C}$. We have then $F \in H^\infty_{\mathbb{B}(\mathbb{C})}(\Gamma)$ (respectively $F \in H^\infty_{\mathbb{B}(\mathbb{C})}(R)$) if and only if F is the a.e. strong boundary value function of a bounded operator valued analytic function defined in the disk $|z| < 1$ (respectively the half-plane $y > 0$). There is a similar characterization of $H^2_\mathbb{C}$ on Γ or R. *Thus, although the elements of $H^2_\mathbb{C}$ and $H^\infty_{\mathbb{B}(\mathbb{C})}$ are functions defined on Γ or R, we can and frequently will refer to values of such functions in the disk $|z| < 1$ or half-plane $y > 0$.*

Henceforth S will denote the standard shift operator "multiplication by $e^{i\theta}$" in $H^2_\mathbb{C}(\Gamma)$ or "multiplication by $(x-i)/(x+i)$" in $H^2_\mathbb{C}(R)$. In $H^2_\mathbb{C}(\Gamma)$, ker S^* is the set of constant functions, and in $H^2_\mathbb{C}(R)$, ker S^* is the set of functions of the form $k(x) = c/(x+i)$, $c \in \mathbb{C}$. In either case S has multiplicity equal to the dimension of $\mathbb{C}$. In fact there is a natural way to identify ker S^* with $\mathbb{C}$ in either case.

An operator $T \in \mathbb{B}(H^2_\mathbb{C})$ is S-Toeplitz if and only if $Tf = PFf$, $f \in H^2_\mathbb{C}$, for some $F \in L^\infty_{\mathbb{B}(\mathbb{C})}$, where P is the projection mapping $L^2_\mathbb{C}$ onto $H^2_\mathbb{C}$. In this case F is essentially determined by T and we write $T = T_F$. We have $T^*_F = T_{F^*}$, and $T_F \geq 0$ if and only if $F \geq 0$ a.e. An operator $A \in \mathbb{B}(H^2_\mathbb{C})$ is S-analytic if and only if $A = T_G$ for some $G \in H^\infty_{\mathbb{B}(\mathbb{C})}$, in which case

$$Af = Gf, \qquad f \in H^2_\mathbb{C}.$$

If $F \in L^\infty_{\mathbb{B}(\mathbb{C})}$ and $G \in H^\infty_{\mathbb{B}(\mathbb{C})}$, the relation $F = G^*G$ a.e. is equivalent to $T_F = T^*_G T_G$. These facts are more or less well known. See, for example, Brown and Halmos [5].

A function $G \in H^\infty_{\mathbb{B}(\mathbb{C})}$ is called *inner* (resp. *outer*) if $A = T_G$ is S-inner (resp. S-outer). In connection with this it should be noted that a subspace $\mathfrak{M}$ of $H^2_\mathbb{C}$ reduces S if and only if $\mathfrak{M} = H^2_\mathbb{B}$ for some subspace $\mathbb{B}$ of $\mathbb{C}$.

For any $G \in H^\infty_{\mathbb{B}(\mathbb{C})}$, the values associated with G in the disk $|z| < 1$ or half-plane $y > 0$ have a norm-convergent representation

$$(6) \qquad G(z) = \sum_0^\infty A_j z^j, \qquad |z| < 1,$$

or

$$(7) \qquad G(z) = \sum_0^\infty A_j \left(\frac{z - i}{z + i} \right)^j, \qquad y > 0,$$

for some sequence $(A_j)_0^\infty$ in $\mathfrak{B}(\mathfrak{C})$. If we make the natural identification between $\mathfrak{C}$ and $\ker S^*$, the sequence $(A_j)_0^\infty$ also appears in the matrix representation of T_G, namely $T_G \sim [A_{j-k}]$ where $A_{-j}=0$, $j=1, 2, 3, \cdots$. In particular $A_0 = G(0)$ or $A_0 = G(i)$ depending on the case.

It is already clear how to translate many of the results in §1 into corresponding theorems for bounded operator valued functions on the circle or line. However, bounded functions do not offer the natural generality for such results. Henceforth, the functions which we treat are possibly unbounded.

2.1. DEFINITION. (i) An operator valued function G on Γ or R is said to be of *class* $\mathfrak{N}^+$ if there exists a bounded scalar valued outer function ϕ ($\not\equiv 0$) such that $\phi G \in H^\infty_{\mathfrak{B}(\mathfrak{C})}$.

(ii) Operator valued inner functions are defined as before: G is *inner* if $G \in H^\infty_{\mathfrak{B}(\mathfrak{C})}$ and multiplication by G in $H^2_\mathfrak{C}$ is a partial isometry. In this case, the initial space of multiplication by G in $H^2_\mathfrak{C}$ has the form $H^2_\mathfrak{B}$ where $\mathfrak{B}$ is a subspace of $\mathfrak{C}$, and we write $\mathfrak{B} = \mathfrak{B}_{in}(G)$.

(iii) An operator valued function G on Γ or R is called *outer* if G is of class $\mathfrak{N}^+$, and for some (and hence any) ϕ chosen for G as in (i), ϕG is outer in the sense previously defined for functions in $H^\infty_{\mathfrak{B}(\mathfrak{C})}$, i.e. $CL((\phi G)H^2_\mathfrak{C})$ reduces the standard shift operator S. In this case $CL((\phi G)H^2_\mathfrak{C}) = H^2_\mathfrak{B}$ for a subspace $\mathfrak{B}$ of $\mathfrak{C}$ which is independent of the choice of ϕ, and we write $\mathfrak{B} = \mathfrak{B}_{out}(G)$.

Definitions and theorems concerning operator valued functions apply in particular to scalar valued functions (specialize to the case dim $\mathfrak{C} = 1$ and interpret operators on $\mathfrak{C}$ as scalars). In this case our definitions of inner and outer functions are consistent with the classical definitions. In the scalar case we are permitted to think of inner and outer functions as defined on the circle Γ or line R, or on the disk $|z| < 1$ or half-plane $y > 0$. We show that the same is true for operator valued functions of class $\mathfrak{N}^+$.

An operator valued analytic function $G(z)$ defined in the disk $|z| < 1$ or half-plane $y > 0$ is said to have a *scalar outer majorant* if there exists a scalar valued outer function $\psi(z)$ such that $|G(z)| \leq |\psi(z)|$ in the disk or half-plane.

2.2. LEMMA. (i) *Let $G(z)$ be an operator valued analytic function which has a scalar outer majorant in the disk $|z| < 1$ or half-plane $y > 0$. Then $G(z)$ has an a.e. strong boundary value function*

$$(8) \qquad G(e^{i\theta}) = \lim_{r \nearrow 1} G(re^{i\theta}) \quad or \quad G(x) = \lim_{y \searrow 0} G(x + iy).$$

If $G(z) \not\equiv 0$ then $\log|G(e^{i\theta})| \in L^1(0, 2\pi)$ or $(1+x^2)^{-1} \log|G(x)|$

$\in L^1(-\infty, \infty)$, and $|G(z)| \leqq |\psi(z)|$ where

$$(9) \qquad \psi(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|G(e^{i\theta})|\,d\theta\right\}$$

for $|z| < 1$, or

$$(10) \qquad \psi(z) = \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\frac{\log|G(t)|}{1+t^2}\,dt\right\}$$

for $y > 0$ respectively.

(ii) *An operator valued function $G(e^{i\theta})$ or $G(x)$ on Γ or R is of class $\mathfrak{N}^+$ if and only if it is a.e. equal to the boundary value function of an operator valued analytic function $G(z)$ which has a scalar outer majorant in the disk $|z| < 1$ or half-plane $y > 0$ respectively.*

PROOF. We give the proof for the circle case only. If $G(z)$ is an operator valued analytic function with a scalar outer majorant in the disk $|z| < 1$, then

$$(11) \quad \log|G(re^{i\theta})| \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\log k(e^{it})\,dt$$

for $0 \leqq r < 1$, $0 \leqq \theta < 2\pi$, where $k(e^{it}) \geqq 0$ a.e. and $\log k(e^{it}) \in L^1(0, 2\pi)$. Construct a scalar valued outer function $\phi(z)$ such that

$$-\log|\phi(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\log^+ k(e^{it})\,dt$$

for $0 \leqq r < 1$, $0 \leqq \theta < 2\pi$. Then $\phi(z)$ and $\phi(z)G(z)$ are bounded functions in the disk $|z| < 1$. Since every bounded operator valued analytic function in the disk has a strong boundary value function, $G(e^{i\theta})$ exists a.e. Clearly $G(e^{i\theta})$ is of class $\mathfrak{N}^+$ on Γ. We have proved the existence of (8) in (i) and the sufficiency part of (ii).

Suppose $G(e^{i\theta})$ is any operator valued function of class $\mathfrak{N}^+$ on Γ, and set $F_\phi(e^{i\theta}) = \phi(e^{i\theta})G(e^{i\theta})$ where $\phi(e^{i\theta})$ is chosen for $G(e^{i\theta})$ as in Definition 2.1. Then $F_\phi(e^{i\theta}) \in H^\infty_{\mathfrak{G}(\mathfrak{C})}(\Gamma)$, and if $a, b \in \mathfrak{C}$, $|a| \leqq 1$, $|b| \leqq 1$,

$$\log|\langle F_\phi(re^{i\theta})a, b\rangle_{\mathfrak{C}}| \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\log|\langle F_\phi(e^{it})a, b\rangle_{\mathfrak{C}}|\,dt$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\log|F_\phi(e^{it})|\,dt$$

for $0 \leqq r < 1$, $0 \leqq \theta < 2\pi$. So if $G(e^{i\theta})$ does not vanish a.e., then

$\log|G(e^{i\theta})| = \log|F_\phi(e^{i\theta})| - \log|\phi(e^{i\theta})| \in L^1(0, 2\pi)$, and $G(e^{i\theta})$ is the boundary value function of $G(z) = \phi(z)^{-1}F_\phi(z)$, $|z| < 1$, which has the scalar outer majorant (9). The remaining parts of the lemma follow.

Let G be an operator valued function of class $\mathfrak{N}^+$ on Γ or R. The values associated with G in the disk $|z| < 1$ or half-plane $y > 0$ have norm convergent representations (6) or (7). It is not hard to see that if G is an outer function, then $\mathfrak{G}_{out}(G) = \mathrm{CL}(G(0)\,\mathfrak{C})$ or $\mathrm{CL}(G(i)\,\mathfrak{C})$ depending on whether G is defined on Γ or R. Similarly, it can be shown that if G is an inner function, then $\mathfrak{G}_{in}(G) = \mathsf{V}_{|w|<1}\, G(w)^*\mathfrak{C}$ or $\mathsf{V}_{\mathrm{Im}\,w>0}\,G(w)^*\mathfrak{C}$ depending on the case.

A consequence of Lemma 2.2 is that if $G(z)$ is an operator valued analytic function defined for $|z| < 1$ such that

$$\sup_{0\le r<1} \int_0^{2\pi} |G(re^{i\theta})|^p\, d\theta < \infty$$

for some p, $0 < p < \infty$, then $G(z)$ has a strong boundary value function which is of class $\mathfrak{N}^+$. To see this note that $u(re^{i\theta}) = \log|G(re^{i\theta})|$ is subharmonic in the unit disk, and by assumption

$$\sup_{0\le r<1} \int_0^{2\pi} f(u(re^{i\theta}))\, d\theta < \infty$$

where $f(x) = e^{px}$ for all real x. By a theorem of Gårding and Hörmander [8, p. 95], $G(z)$ satisfies an inequality of the form (11), which is equivalent to the existence of a scalar outer majorant. The analogous half-plane result uses the condition

$$\sup_{v>0} \int_{-\infty}^{\infty} |G(x + iy)|^p\, dx < \infty$$

where $0 < p < \infty$.

2.3. THEOREM. *Let G be an operator valued function of class $\mathfrak{N}^+$ on Γ or R. Then G has a unique factorization $G = G_i G_o$ where G_o is an outer function such that $G_o(0) \ge 0$ or $G_o(i) \ge 0$ in the circle and line cases respectively, and G_i is an inner function such that $\mathfrak{G}_{in}(G_i) = \mathfrak{G}_{out}(G_o)$. In this case $G_o^*G_o = G^*G$ a.e. Conversely, if G_o is any operator valued outer function and G is any operator valued function of class $\mathfrak{N}^+$ on Γ or R such that $G^*G = G_o^*G_o$ a.e., then $G = G_i G_o$ for some inner function G_i such that $\mathfrak{G}_{in}(G_i) = \mathfrak{G}_{out}(G_o)$.*

PROOF. We easily reduce the theorem to the case where $G \in H^\infty_{\mathfrak{G}(\mathfrak{C})}$. Then the result follows from Theorem 1.11 and the discussion at the beginning of this section.

2.4. Theorem. *Let G_1 and G_2 be operator valued outer functions on Γ or R.*

(i) *We have $G_1^*G_1 = G_2^*G_2$ a.e. if and only if $G_2 = UG_1$, $G_1 = U^*G_2$ a.e. where U is a (constant) partially isometric operator on $\mathbb{C}$ with initial space $\mathfrak{B}_{out}(G_1)$ and final space $\mathfrak{B}_{out}(G_2)$.*

(ii) *If $G_1^*G_1 = G_2^*G_2$ a.e. and $G_j(0) \geqq 0$ or $G_j(i) \geqq 0$ in the circle and line cases respectively, $j = 1, 2$, then $G_1 = G_2$ a.e.*

Proof. Suppose $G_1^*G_1 = G_2^*G_2$ a.e. Choose a bounded scalar valued outer function $\phi \not\equiv 0$ such that, for each $j = 1, 2$, $F_j = \phi G_j$ is an outer function in $H_{\mathfrak{B}(\mathbb{C})}^\infty$. Our assumptions imply that $T_{F_2}^*T_{F_1} = T_{F_1}^*T_{F_2}$, so by Theorem 1.9, $T_{F_2} = T_U T_{F_1}$, $T_{F_1} = T_U^* T_{F_2}$ where U is an operator valued function such that both U and U^* are inner with initial spaces $\mathfrak{B}_{out}(F_1) = \mathfrak{B}_{out}(G_1)$ and $\mathfrak{B}_{out}(F_2) = \mathfrak{B}_{out}(G_2)$ respectively. It follows that U can be taken to be a constant partially isometric operator on $\mathbb{C}$ with initial space $\mathfrak{B}_{out}(G_1)$ and final space $\mathfrak{B}_{out}(G_2)$. Moreover $F_2 = UF_1$, $F_1 = U^*F_2$ a.e. and so $G_2 = UG_1$, $G_1 = U^*G_2$ a.e. This proves the necessity of the condition in (i). The sufficiency is proved by reversing the steps in this argument. The proof of (ii) is similar, except that for (ii) we use Theorem 1.8.

If $A, B, \ldots$ are operator valued functions of class $\mathfrak{N}^+$ on Γ or R, we write $(A_j)_0^\infty$, $(B_j)_0^\infty, \ldots$ for the coefficients in the expansions (6) and (7) for the corresponding analytic functions in the disk $|z| < 1$ or half-plane $y > 0$.

2.5. Theorem. *Let C be an operator valued function of class $\mathfrak{N}^+$ on Γ. Then C is outer if*

$$(12) \qquad\qquad C_0^*C_0 \geqq A_0^*A_0$$

for every operator valued function A of class $\mathfrak{N}^+$ on Γ such that $A^*A = C^*C$ a.e. Conversely, if C is outer, then

$$(13) \qquad \sum_0^n C_j^*C_j \geqq \sum_0^n A_j^*A_j, \qquad n = 0, 1, 2, \cdots,$$

for every operator valued function A of class $\mathfrak{N}^+$ on Γ such that $A^*A = C^*C$.

Proof. Suppose (12) holds for all A as in the theorem. Choose ϕ for C as in Definition 2.1 such that $C' = \phi C \in H_{\mathfrak{B}(\mathbb{C})}^\infty(\Gamma)$. We show that the S-analytic operator $T_{C'}$ on $H_{\mathbb{C}}^2(\Gamma)$ is S-outer. Let $T_{A'}$, $A' \in H_{\mathfrak{B}(\mathbb{C})}^\infty(\Gamma)$, be any S-analytic operator such that $T_{A'}^*T_{A'} = T_{C'}^*T_{C'}$. If we define A by $A' = \phi A$, then A is of class $\mathfrak{N}^+$ and $A^*A = C^*C$ a.e. By (12) then $C_0^*C_0 \geqq A_0^*A_0$, and hence

$$C_0'^* C_0' = |\phi_0|^2 C_0^* C_0 \geqq |\phi_0|^2 A_0^* A_0 = A_0'^* A_0'.$$

It follows from Theorem 1.12 that $T_{C'}$ is S-outer (see the discussion following (6) and (7)). Hence C' and C'/ϕ are outer functions.

Suppose that C is an outer function. Choose a sequence $(\phi^{(k)})_1^\infty$ of bounded scalar valued outer functions such that $C^{(k)} = \phi^{(k)} C \in H^\infty_{\mathcal{B}(\mathfrak{C})}(\Gamma)$ for each $k = 1, 2, 3, \cdots$ and $\lim \phi^{(k)}(z) = 1$ uniformly on any disk $|z| \leqq r$, $r < 1$. For example, if ϕ is a single bounded scalar valued outer function such that $\phi C \in H^\infty_{\mathcal{B}(\mathfrak{C})}(\Gamma)$, we could choose

$$\phi^{(k)}(z) = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log[\min(k \,|\, \phi(e^{i\theta}) \,|\,, 1)] \, d\theta \right\},$$

$|z| < 1$, $k = 1, 2, 3, \cdots$. If $\gamma_0^{(k)}$, $\gamma_1^{(k)}$, $\gamma_2^{(k)}$, $\cdots$ are the Taylor coefficients of $1/\phi^{(k)}$, then

$$\lim_{k \to \infty} \gamma_j^{(k)} = 1, \quad j = 0,$$

$$= 0, \quad j > 0.$$

But for any $n = 0, 1, 2, \cdots$, $k = 1, 2, 3, \cdots$,

$$\sum_{j=0}^n C_j^{(k)*} C_j^{(k)} \geqq \sum_{j=0}^n A_j^{(k)*} A_j^{(k)}$$

by Theorem 1.15 (again, see the discussion following (6) and (7)). We now deduce (13) by letting $k \to \infty$.

The continuous analogue of Theorem 2.5 is due to Robinson [23] in the case where dim $\mathfrak{C} = 1$.

2.6. THEOREM. *Let C be an operator valued function of class $\mathfrak{N}^+$ on R such that $C(x)c \in H^2_{\mathfrak{C}}(R)$ for each $c \in \mathfrak{C}$. Then C is outer if and only if*

$$(14) \qquad \int_0^t |v(s)|^2 \, ds \geqq \int_0^t |u(s)|^2 \, ds$$

for all $t > 0$, whenever u, v are vector valued functions such that

$$(15) \qquad \begin{Bmatrix} C(x)c \\ A(x)c \end{Bmatrix} = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{ixs} \begin{Bmatrix} v(s) \\ u(s) \end{Bmatrix} \, ds$$

for some operator valued function A of class $\mathfrak{N}^+$ on R such that $A^*A = C^*C$ a.e. and some $c \in \mathfrak{C}$.

PROOF. Let $\mathcal{W} = \{V^t\}_{t>0}$, where for each $t > 0$, V^t is the operator "multiplication by e^{itx}" in $H^2_{\mathfrak{C}}(R)$. Every $f \in H^2_{\mathfrak{C}}(R)$ has a representation

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{ixs} g(s)\, ds$$

where $g \in L^2_{\mathfrak{C}}(0, \infty)$, and conversely each such g determines an $f \in H^2_{\mathfrak{C}}(R)$ by this formula. If f and g are so related, then

(16)
$$\left\| V^{t*}f \right\|^2 = \int_t^\infty |g(s)|^2\, ds$$

for all $t > 0$. By the discussion following Lemma 1.14, the weakly closed linear span of $\mathcal{W}$ and the identity operator contains the standard shift operator S.

Suppose now that C is an outer function. Let A be any operator valued function of class $\mathfrak{N}^+$ on R such that $A^*A = C^*C$ a.e. By Theorem 2.3, $A = BC$ where B is an inner function such that $\mathfrak{B}_{\mathrm{in}}(B) = \mathfrak{B}_{\mathrm{out}}(C)$. By assumption $C(x)c \in H^2_{\mathfrak{C}}(R)$, and hence $A(x)c \in H^2_{\mathfrak{C}}(R)$ for each $c \in \mathfrak{C}$. It is easy to see that $C(x)c$ is in the initial space of multiplication by B in $H^2_{\mathfrak{C}}(R)$ for any $c \in \mathfrak{C}$. So, by Lemma 1.14 (i),

$$\left\| V^{t*}Cc \right\| \leq \left\| V^{t*}Ac \right\|$$

for all $t > 0$, $c \in \mathfrak{C}$. We get (14) by combining this with (16).

Conversely suppose that (14) holds whenever u, v have the form (15). We use the inequality only for a special choice of A. Namely, choose A outer such that $C = BA$ where B is an inner function such that $\mathfrak{B}_{\mathrm{in}}(B) = \mathfrak{B}_{\mathrm{out}}(A)$. Such an A exists by Theorem 2.3. Since A is outer, we can apply what we just proved with the roles of A and C interchanged, and this shows that equality holds in (14), or equivalently

$$\left\| V^{t*}Ac \right\| = \left\| V^{t*}BAc \right\|$$

for all $t > 0$ and $c \in \mathfrak{C}$. Let $\mathcal{G}$ be the set of elements of $H^2_{\mathfrak{C}}(R)$ of the form Ac, $c \in \mathfrak{C}$. With this choice of $\mathcal{G}$ and $\mathcal{W}$ as above we apply Lemma 1.14 (ii) to the S-inner operator "multiplication by B" in $H^2_{\mathfrak{C}}(R)$. It follows from the lemma that B^* is also an inner function. Hence B can be taken to be a constant partially isometric operator on $\mathfrak{C}$. Finally, since A is an outer function, so is C.

Singularities of operator valued analytic functions are classified in exactly the same way as for scalar valued analytic functions. An operator valued function $F(z)$ is said to be *meromorphic* in a region Ω if $F(z)$ is defined and analytic except at certain isolated points which are poles. An operator valued function $F(z)$ which is meromorphic in the extended complex plane is called a *rational function*. Every opera-

tor valued rational function has the form $P(z)/q(z)$ where $P(z)$
$=A_0+A_1z+ \cdots +A_mz^m$ is a polynomial with operator coefficients
and $q(z)=b_0+b_1z+ \cdots +b_nz^n$ is a polynomial with scalar coeffi-
cients, $q(z)\not\equiv 0$, and conversely.

   2.7. DEFINITION. Let u, v be nonzero scalar valued functions of
class $\mathfrak{N}^+$ on Γ or R. An operator valued function F on Γ or R is said
to be of *class* $\mathfrak{M}(u, v)$ if uF and vF^* are of class $\mathfrak{N}^+$.

   If $u=u_iu_o$, $v=v_iv_o$ are inner-outer factorizations of u, v, then a
function F is of class $\mathfrak{M}(u, v)$ if and only if F is of class $\mathfrak{M}(u_i, v_i)$.
In other words the class $\mathfrak{M}(u, v)$ is independent of the outer factors
in u, v.

   We show that functions of class $\mathfrak{M}(u, v)$ can be characterized as
two-sided boundary value functions of certain kinds of meromorphic
functions defined off Γ or R. In particular, we can again speak of
function values off Γ or R.

   If $F(z)$ is an operator valued function defined in a region Ω of the
complex plane, we write $\tilde{F}(z)$ for one of the reflected functions
$\tilde{F}(z)=F(\bar{z})^*$ in $\tilde{\Omega}=\{z:\bar{z}\in\Omega\}$ or $\tilde{F}(z)=F(1/\bar{z})^*$ in $\tilde{\Omega}=\{z:1/\bar{z}\in\Omega\}$,
depending on context.

   2.8. THEOREM. *Let u, v be nonzero scalar valued functions of class $\mathfrak{N}^+$*
on Γ or R.

   (i) *Let $F(z)$ be an operator valued function which is defined and mero-*
morphic separately for $|z|<1$, $|z|>1$ or for $y>0$, $y<0$. Suppose that
$u(z)F(z)$, $v(z)\tilde{F}(z)$ are analytic and have scalar outer majorants in the
disk $|z|<1$ or half-plane $y>0$. Suppose that $F(re^{i\theta})$ has the same strong
limit $F(e^{i\theta})$ a.e. as $r\nearrow 1$ and $r\searrow 1$, respectively $F(x+iy)$ has the same
strong limit $F(x)$ a.e. as $y\searrow 0$ and $y\nearrow 0$. Then the boundary value func-
tion is of class $\mathfrak{M}(u, v)$ on Γ or R, and conversely every operator valued
function of class $\mathfrak{M}(u, v)$ on Γ or R arises in this way.

   (ii) *In the situation of (i), if the boundary value function is essentially*
bounded on an open arc on Γ or interval on R, and if u and v are analytic
across the arc or interval, then the meromorphic functions are analytic
continuations of each other across the arc or interval.

   In [14] Kriete proves that a square-integrable scalar valued func-
tion of class $\mathfrak{M}(u, v)$ has an integral representation analogous to the
Paley-Wiener representation theorem for entire functions of expo-
nential type which are square integrable on the real axis.[1]

   PROOF. We give this proof for the line case only.

   (i) Suppose $F(z)$ is meromorphic for $y>0$, $y<0$, and the assump-

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   [1] Kriete discovered this result in the academic year 1966-67 and circulated it
privately, before the authors began work on the present manuscript.

tions in (i) hold. By Lemma 2.2, $u(x)F(x)$ and $v(x)F(x)^*$ are functions of class $\mathfrak{N}^+$ on R, so we immediately deduce that $F(x)$ is of class $\mathfrak{M}(u, v)$ on R. Conversely, if $F(x)$ is a given function of class $\mathfrak{M}(u, v)$ on R, then by Lemma 2.2, $u(x)F(x)$ and $v(x)F(x)^*$ are a.e. strong boundary value functions of operator valued analytic functions $F_+(z)$ and $F_-(z)$ which are defined and have scalar outer majorants in the half-plane $y > 0$. The function

$$F(z) = F_+(z)/u(z), \qquad y > 0,$$
$$= \tilde{F}_-(z)/\tilde{v}(z), \qquad y < 0,$$

is meromorphic separately for $y > 0$, $y < 0$, it satisfies the conditions in (i), and it has $F(x)$ as an a.e. strong two-sided boundary value function. In fact, we get at once that $F(x) = \lim F(x+iy)$ a.e. as $y \searrow 0$ in the strong operator topology, and as $y \nearrow 0$ in the weak operator topology. But weak convergence can be improved to strong convergence here by the analogue of Lemma 2.2 for the lower half-plane.

(ii) Using notation as above, we assume now that $u(z)$ and $v(z)$ have analytic continuations across an interval (a, b) and that $F(x)$ is essentially bounded on (a, b). We may assume without loss of generality that the continuations of $u(z)$ and $v(z)$ have no zeros on (a, b). For we can contract the interval (a, b) slightly so that only a finite number of zeros are present, and these can be removed by dividing out a suitable outer factor (see the remark following Definition 2.7). Alternately, it can be shown that the inner parts of u and v are analytic and nonzero in a neighborhood of (a, b). Then, using scalar majorants for $F_+(z)$ and $F_-(z)$ of the form (10) and the fact that $F(x)$ is essentially bounded on (a, b), we can reduce to the case where $F(z)$ is bounded on and inside some rectangle $\rho: x = a$, $x = b$, $y = -\delta$, $y = \delta$ ($\delta > 0$), ignoring points on the real axis. At worst we will have to contract the interval (a, b) slightly once more. But now straightforward arguments will show that

$$G(z) = \frac{1}{2\pi i} \oint_\rho \frac{F(t)}{t - z}\, dt$$

is analytic in the interior of ρ and is the required continuation of $F(z)$ across (a, b).

An operator valued function F on Γ or R is in $H^\infty_{\mathfrak{G}(\mathfrak{C})}$ if and only if F is essentially bounded and for each $c \in \mathfrak{C}$ the scalar valued function $F_c(\cdot) = \langle F(\cdot)c, c \rangle_{\mathfrak{C}}$ is in H^∞ on Γ or R. This follows from elementary properties of Fourier series. It is convenient to have an analogous characterization of the classes $\mathfrak{N}^+$ and $\mathfrak{M}(u, v)$.

2.9. THEOREM. *Let F be an operator valued function on* Γ *or* R. *Then F is of class* $\mathfrak{N}^+$ *if and only if for each* $c \in \mathcal{C}$ *the scalar valued function* F_c *is of class* $\mathfrak{N}^+$ *and*

$$(17) \quad \int_0^{2\pi} \log^+ |F(e^{i\theta})| \, d\theta < \infty \quad or \quad \int_{-\infty}^{\infty} (1+x^2)^{-1} \log^+ |F(x)| \, dx < \infty.$$

If u, v *are nonzero scalar valued functions of class* $\mathfrak{N}^+$ *then F is of class* $\mathfrak{M}(u, v)$ *if and only if* F_c *is of class* $\mathfrak{M}(u, v)$ *for each* $c \in \mathcal{C}$ *and* (17) *holds*.

PROOF. Suppose that F_c is of class $\mathfrak{N}^+$ for all $c \in \mathcal{C}$ and (17) holds. By (17) there is a scalar valued outer function ϕ such that $|\phi|^{-1}$ $= \max(|F|, 1)$. Then ϕ and $G = \phi F$ are bounded functions, and G_c $= \phi F_c$ is of class $\mathfrak{N}^+$ for all $c \in \mathcal{C}$. Hence G_c is in H^∞ for all $c \in \mathcal{C}$, so $G \in H^\infty_{\mathfrak{B}(\mathcal{C})}$. Therefore F is of class $\mathfrak{N}^+$. We omit the easy proof of the converse statement. The second assertion follows directly from the first.

3. The factorization problem for nonnegative operator valued functions.

Let F be a nonnegative operator valued function defined on the the circle Γ or the line R. We call *F factorable* if $F = G^*G$ a.e. for some operator valued function G of class $\mathfrak{N}^+$ on Γ or R. If F is factorable, then, by Theorems 2.3 and 2.4, $F = G^*G$ a.e. for an essentially unique outer function G, or a unique outer function G such that $G(0) \geq 0$ or $G(i) \geq 0$ in the circle and line cases respectively. Here, of course, we treat functions which are equal a.e. as being identical, and "essential uniqueness" refers to constant partially isometric left factors as in Theorem 2.4.

In the special case where $F \in L^\infty_{\mathfrak{B}(\mathcal{C})}$, F is factorable if and only if the induced S-Toeplitz operator T_F on $H^2_{\mathcal{C}}$ is factorable in the sense of Definition 1.5. Therefore the results in §1 can be made to yield sufficient conditions for F to be factorable.

Suppose, for example, that F is a bounded nonnegative operator valued function on Γ or R which is of class $\mathfrak{M}(v, v)$ for some nonzero scalar valued function v of class $\mathfrak{N}^+$. By Theorem 1.16, F is factorable, and moreover if G is an outer function such that $F = G^*G$ a.e. then G is of class $\mathfrak{M}(1, v)$. This result is already sufficient for a number of interesting applications. But in fact the assumption that F is bounded is redundant.

3.1. THEOREM. *Let F be a nonnegative operator valued function defined on* Γ *or* R *which is of class* $\mathfrak{M}(v, v)$ *for some nonzero scalar valued function v of class* $\mathfrak{N}^+$. *Then F is factorable, and moreover if G is any outer function such that* $F = G^*G$ *a.e., then G is of class* $\mathfrak{M}(1, v)$.

PROOF. We give the proof in the circle case. We can assume without loss of generality that v is an inner function. By assumption vF is of class $\mathfrak{N}^+$, so there exists a bounded scalar valued outer function ϕ ($\not\equiv 0$) such that $\phi vF \in H^\infty_{\mathfrak{B}(\mathfrak{C})}(\Gamma)$. The function $K = \bar{\phi} F \phi$ is bounded and it induces an S-Toeplitz operator T_K on $H^2_\mathfrak{C}(\Gamma)$. We show that T_K is factorable as an S-Toeplitz operator.

By Theorem 1.7 it is enough to show that $\lim I_n(c) = 0$ as $n \to \infty$ for each $c \in \mathfrak{C}$, where

$$I_n(c) = \sup\{\,|\langle T_K c, S^n f\rangle| : f \in H^2_\mathfrak{C}(\Gamma), \langle T_K f, f\rangle = 1\}$$

for all $c \in \mathfrak{C}$ and $n = 0, 1, 2, \cdots$. Here $\mathfrak{C}$ is regarded as a subspace of $H^2_\mathfrak{C}(\Gamma)$ in the natural way. Let σ be normalized Lebesgue measure on Γ; let $\chi(e^{i\theta}) = e^{i\theta}$, $e^{i\theta} \in \Gamma$; and let P be the projection mapping $L^2_\mathfrak{C}(\Gamma)$ onto $H^2_\mathfrak{C}(\Gamma)$. Recall that $\phi vF \in H^\infty_{\mathfrak{B}(\mathfrak{C})}(\Gamma)$ and $|v| = 1$ a.e. Consider $c \in \mathfrak{C}$, $f \in H^2_\mathfrak{C}(\Gamma)$, and estimate as follows:

$$\left| \langle T_K c, S^n f\rangle \right| = \left| \int_\Gamma \langle Kc, \chi^n f\rangle_\mathfrak{C}\, d\sigma \right|$$

$$= \left| \int_\Gamma \left\langle \frac{\phi}{\phi^*}\, vc, \chi^n \phi^2 vFf \right\rangle_\mathfrak{C}\, d\sigma \right|$$

$$= \left| \int_\Gamma \left\langle P\chi^{-n} P \frac{\phi}{\phi^*}\, vc, v(F^{1/2}\phi)^2 f \right\rangle_\mathfrak{C}\, d\sigma \right|$$

$$= \left| \int_\Gamma \left\langle F^{1/2}\phi^* P\chi^{-n} P \frac{\phi}{\phi^*}\, vc, vF^{1/2}\phi f \right\rangle_\mathfrak{C}\, d\sigma \right|$$

$$\leqq \left(\int_\Gamma \left| F^{1/2}\phi^* P\chi^{-n} P \frac{\phi}{\phi^*}\, vc \right|^2 d\sigma \right)^{1/2} \langle T_K f, f\rangle$$

for any $n = 0, 1, 2, \cdots$. The inner products and norms appearing inside an integral apply to function values and, of course, are taken pointwise. Since $F^{1/2}\phi^*$ is a bounded function by the choice of ϕ, we get

$$I_n(c) \leqq \text{const.} \left(\int_\Gamma \left| P\chi^{-n} P \frac{\phi}{\phi^*}\, vc \right|^2 d\sigma \right)^{1/2} = \text{const.} \|S^{*n} g c\|$$

for all $c \in \mathfrak{C}$ and $n = 0, 1, 2, \cdots$, where $g_c = P(\phi/\phi^*)vc$. Since S is a shift operator, $\lim I_n(c) = 0$ as $n \to \infty$ for all $c \in \mathfrak{C}$, so T_K is factorable by Theorem 1.7.

It follows that K is factorable, and hence so is F. In fact, $K = M^*M$ a.e. where $M \in H^\infty_{\mathfrak{B}(\mathfrak{C})}(\Gamma)$ and so $F = G^*G$ a.e. where $G = \phi^{-1}M$ is of class $\mathfrak{N}^+$.

If G is an outer function such that $F = G^*G$ a.e. then $M = \phi G$ is an outer function in $H^\infty_{\mathfrak{B}(\mathfrak{C})}(\Gamma)$ such that $M^*M = K$ a.e. We show that the function $W = \phi v G^*$ is in $H^\infty_{\mathfrak{B}(\mathfrak{C})}(\Gamma)$, thus proving that vG^* is of class $\mathfrak{N}^+$ and hence G is of class $\mathfrak{M}(1, v)$. Clearly $W = vM^*\phi/\phi^*$ is bounded, and, by the choice of ϕ,

$$WM = \phi^2 vF \in H^\infty_{\mathfrak{B}(\mathfrak{C})}(\Gamma).$$

Therefore $W\,\mathrm{CL}(MH^2_{\mathfrak{C}}(\Gamma)) \subseteq \mathrm{CL}(H^2_{\mathfrak{C}}(\Gamma))$. Since M is a bounded outer function, if $g \in H^2_{\mathfrak{C}}(\Gamma) \ominus \mathrm{CL}(MH^2_{\mathfrak{C}}(\Gamma))$ then $S^j g \in H^2_{\mathfrak{C}}(\Gamma) \ominus \mathrm{CL}(MH^2_{\mathfrak{C}}(\Gamma))$ for all $j = 0, 1, 2, \cdots$.
Hence

$$\int_\Gamma \langle M^*g, \chi^{-j}u \rangle \, d\sigma = \int_\Gamma \langle \chi^j g, Mu \rangle \, d\sigma = 0$$

for all $u \in H^2_{\mathfrak{C}}(\Gamma)$, $j = 0, 1, 2, \cdots$. It follows that $M^*g = 0$ a.e. so $Wg = 0$ a.e. But then $WH^2_{\mathfrak{C}}(\Gamma) \subseteq H^2_{\mathfrak{C}}(\Gamma)$ so $W \in H^\infty_{\mathfrak{B}(\mathfrak{C})}(\Gamma)$ and the proof is complete.

Special choices of v in Theorem 3.1 lead to factorization theorems for polynomials, rational functions, and entire functions of exponential type. However, for polynomials and rational functions (bounded or unbounded on Γ or R) we do not need the full strength of Theorem 3.1. In these cases it would be possible to argue directly from Theorem 1.16.

The operator generalization of the Fejér-Riesz theorem is due to Rosenblum [25], Rosenblatt [24], and Gohberg [9]. See also Helson [10].

3.2. THEOREM. *Let* $F(e^{i\theta}) = \sum_{-n}^{n} A_j e^{ij\theta}$ *be a trigonometric polynomial whose coefficients are operators on* $\mathfrak{C}$ *and which is nonnegative on* Γ. *Then* $F = G^*G$ *where* G *is an outer function on* Γ *of the form* $G(e^{i\theta}) = \sum_{0}^{n} C_j e^{ij\theta}$ *for some operators* $C_0, \cdots, C_n$ *on* $\mathfrak{C}$.

PROOF. Let $u(e^{i\theta}) = e^{im\theta}$, $v(e^{i\theta}) = e^{in\theta}$ on Γ where m and n are nonnegative integers. A bounded weakly measurable operator valued function K on Γ is of class $\mathfrak{M}(u, v)$ if and only if $uK, vK^* \in H^\infty_{\mathfrak{B}(\mathfrak{C})}(\Gamma)$, or

$$\int_0^{2\pi} \langle K(e^{i\theta})a, b \rangle e^{ij\theta} \, d\theta = 0$$

whenever $a, b \in \mathfrak{C}$ and either $j > m$ or $j < -n$, or $K(e^{i\theta}) = \sum_{-m}^{n} B_j e^{ij\theta}$ for some operators $B_{-m}, \cdots, B_n$ on $\mathfrak{C}$. Thus the theorem follows from either Theorem 3.1 or Theorem 1.16.

3.3. THEOREM. *Let* $P(x) = \sum_{0}^{2n} P_j x^j$ *be a polynomial whose coeffi-*

cients are operators on $\mathfrak{C}$ and which is nonnegative on R. Then $P = Q^*Q$ where Q is an outer function on R of the form $Q(x) = \sum_0^n Q_j x^j$ for some operators $Q_0, \cdots, Q_n$ on $\mathfrak{C}$.

PROOF. The function $F(z) = (z^2+1)^{-n} P(z)$ is meromorphic in the complex plane. Setting $v(z) = [(z-i)/(z+i)]^n$ we see that $v(z)F(z)$ and $v(z)\tilde{F}(z)$ are analytic and bounded in the half-plane $y > 0$. By Theorem 2.8, $F(x)$ is of class $\mathfrak{M}(v, v)$ on R. By Theorem 3.1 or Theorem 1.16, $F(x) = G(x)^*G(x)$ on R where $G(x)$ is an outer function on R of class $\mathfrak{M}(1, v)$. By Theorem 2.8, $G(x)$ is the restriction to the real axis of a meromorphic function $G(z)$, defined in the complex plane and analytic at each real point, such that $G(z)$ and $v(z)\tilde{G}(z)$ are analytic and have scalar outer majorants for $y > 0$. Since $G(x)$ and $v(x)\tilde{G}(x)$ are bounded on R, actually $G(z)$ and $v(z)\tilde{G}(z)$ are bounded for $y > 0$ by Lemma 2.2. Thus, considering the upper and lower half-planes separately, we see that

$$| (z + i)^n G(z) | \leq M(| z | + 1)^n$$

for all complex z and some $M > 0$. By Cauchy's estimates, $Q(z) = (z+i)^n G(z)$ is a polynomial of degree at most n. Since $G(x)$ is an outer function on R, so is $Q(x)$. By construction $P(x) = Q(x)^*Q(x)$ on R and the theorem follows.

In §2 and the present section we have the standing assumption that $\mathfrak{C}$ is a separable Hilbert space. Actually Theorems 3.2 and 3.3 are valid for nonseparable spaces $\mathfrak{C}$ as well. Basically all that is needed to extend the proofs is to replace $H_{\mathfrak{C}}^2(\Gamma)$ by a more general Hilbert space of square summable sequences or formal power series. If we then argue directly from Theorem 1.16, we obtain a proof of Theorem 3.2 without the separability assumption. But then Theorem 3.3 can be deduced from Theorem 3.2 by a transformation of the independent variables. We chose a direct proof of Theorem 3.3 above to convey some feeling for the classes $\mathfrak{M}(u, v)$.

3.4. THEOREM. *Let $F(z)$ be an operator valued rational function which is either nonnegative at each point $e^{i\theta} \in \Gamma$ which is not a pole, or nonegative at each point $x \in R$ which is not a pole. Then $F(z) = \tilde{G}(z)G(z)$ where $G(z)$ is an operator valued rational function such that the restriction of $G(z)$ to the circle Γ or line R is an outer function. In particular, this $G(z)$ is analytic in the disk $|z| < 1$ or half-plane $y > 0$ respectively.*

PROOF. We may choose a nonzero scalar polynomial $q(z)$ such that the restriction of $q(z)$ to Γ or R is an outer function and $P(z) = \tilde{q}(z)q(z)F(z)$ is a polynomial in $1/z$ and z, respectively a polynomial in z, with nonnegative values on Γ or R. Factor $P(e^{i\theta}) = Q(e^{i\theta})^*Q(e^{i\theta})$ or $P(x) = Q(x)^*Q(x)$ on Γ or R as in Theorem 3.2 or Theorem 3.3.

The required factorization is then obtained with $G(z) = Q(z)/q(z)$.

Let f be a scalar valued entire function. We say that f is of exponential type if there is a real number τ such that $|f(z)| \leq Ke^{\tau|z|}$ for all z in the complex plane and some $K > 0$. In this case the *exact type* of f is defined to be the infimum τ_f of all such numbers τ. By Liouville's theorem $\tau_f \geq 0$ except when $f \equiv 0$, in which case $\tau_f = -\infty$. We say that f is of *exponential type* τ, where τ is a real number if f is of exponential type and $\tau_f \leq \tau$. For any entire function f, $\bar{f}$ is the entire function such that $\bar{f}(z) = \overline{f(\bar{z})}$. By a theorem of M. G. Kreĭn [13], if f is an entire function the following are equivalent:

(i) the restrictions of f and $\bar{f}$ to the upper half-plane $y > 0$ are of bounded type in the half-plane.

(ii) f is of exponential type and

$$\int_{-\infty}^{\infty} (1 + x^2)^{-1} \log^+ |f(x)| \, dx < \infty.$$

See also de Branges [3]. If f satisfies (i) and (ii) and is not identically zero, Nevanlinna's factorization gives

$$f(z) = C_+ B_+(z) e^{-i\tau + z} \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \frac{\log |f(t)|}{1 + t^2} \, dt \right\},$$

$$f(z) = C_- B_-(z) e^{-i\tau - z} \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \frac{\log |f(t)|}{1 + t^2} \, dt \right\}$$

for $y > 0$, where C_+, C_- are constants of modulus 1, τ_+, τ_- are real numbers, $B_+(z)$, $B_-(z)$ are Blaschke products, and the integrals converge absolutely. The numbers τ_+, τ_- are called the mean types of f and $\bar{f}$ respectively (with respect to the upper half-plane). One of them may be negative, but $\tau_+ + \tau_- \geq 0$ and $\tau_f = \max(\tau_+, \tau_-)$, $|\tau_\pm| \leq \tau_f$, where τ_f is the exact type of f. If $f \equiv 0$ we set $\tau_+ = \tau_- = -\infty$. In general

$$(18) \qquad \tau_\pm = \limsup_{y \to \infty} y^{-1} \log |f(\pm iy)|.$$

See Boas [2], de Branges [3]. An entire function f is said to be of Kreĭn class, more precisely of class $\mathcal{K}(\tau_1, \tau_2)$ where $\tau_1, \tau_2 \geq 0$, if f satisfies the equivalent conditions (i) and (ii), and $\tau_+ \leq \tau_1$, $\tau_- \leq \tau_2$ where τ_+, τ_- are defined for f as above.

We extend these concepts to operator valued functions. Recall that for any operator valued function F and any $c \in \mathfrak{C}$, F_c is the scalar valued function defined by $F_c(\cdot) = \langle F(\cdot)c, c \rangle_\mathfrak{C}$. An operator valued entire function F is said to be of exponential type τ if for each $c \in \mathfrak{C}$ the scalar valued entire function F_c is of exponential type τ. An opera-

tor valued entire function F is said to be of class $\mathcal{K}(\tau_1, \tau_2)$, $\tau_1 \geqq 0$, $\tau_2 \geqq 0$, if for each $c \in \mathfrak{C}$ the scalar valued entire function F_c is of class $\mathcal{K}(\tau_1, \tau_2)$, and moreover

$$\int_{-\infty}^{\infty} (1 + x^2)^{-1} \log^+ | F(x) | \, dx < \infty.$$

3.5. LEMMA. *Let $F(x)$ be a locally bounded operator valued function defined on R. Let $u(x) = e^{i\tau_1 x}$, $v(x) = e^{i\tau_2 x}$ on R where $\tau_1 \geqq 0$, $\tau_2 \geqq 0$. Then $F(x)$ is of class $\mathfrak{M}(u, v)$ if and only if $F(x)$ is the a.e. restriction to the real axis of an operator valued entire function of class $\mathcal{K}(\tau_1, \tau_2)$.*

PROOF. Suppose that $F(x) = G(x)$ a.e. on R where $G(z)$ is an operator valued entire function of class $\mathcal{K}(\tau_1, \tau_2)$. Using the Nevanlinna factorization we see at once that for each $c \in \mathfrak{C}$, F_c is of class $\mathfrak{M}(u, v)$. Also, by the definition of $\mathcal{K}(\tau_1, \tau_2)$, $(1+x^2)^{-1} \log^+ | F(x) | \in L^1(-\infty, \infty)$. Therefore by Lemma 2.9, F is of class $\mathfrak{M}(u, v)$.

Conversely assume that F is of class $\mathfrak{M}(u, v)$. Since F is locally bounded and $u(z) = e^{i\tau_1 z}$, $v(z) = e^{i\tau_2 z}$ are entire functions, by Lemma 2.8, $F(x) = G(x)$ a.e. for some operator valued entire function $G(z)$ such that $e^{i\tau_1 z} G(z)$ and $e^{i\tau_2 z} \tilde{G}(z)$ have scalar outer majorants in the half-plane $y > 0$. It follows from Kreĭn's theorem that, for each $c \in \mathfrak{C}$, the scalar valued entire function G_c is of exponential type. By (18) the mean types of G_c and $(G_c)^{\sim} = (\tilde{G})_c$ do not exceed τ_1 and τ_2 respectively. In other words G_c is of class $\mathcal{K}(\tau_1, \tau_2)$ for each $c \in \mathfrak{C}$. By Lemma 2.2, $(1+x^2)^{-1} \log^+ | G(x) | \in L^1(-\infty, \infty)$, and hence $G(z)$ is of class $\mathcal{K}(\tau_1, \tau_2)$. The result follows.

We can now state an operator generalization of a classical factorization theorem for entire functions of exponential type which are nonnegative on the real axis (Boas [2, p. 125], de Branges [3, p. 34], Levin [16, p. 437]).

3.6. THEOREM. *Let $F(z)$ be an operator valued entire function of exponential type τ, $\tau \geqq 0$, such that $F(x) \geqq 0$ for all real x, and*

$$\int_{-\infty}^{\infty} (1 + x^2)^{-1} \log^+ | F(x) | \, dx < \infty.$$

Then $F(z) = \tilde{G}(z) G(z)$ for some operator valued entire function $G(z)$ such that $e^{-(1/2)i\tau z} G(z)$ is of exponential type $\frac{1}{2}\tau$ and the restriction of $G(z)$ to the real axis is an outer function.

A similar result is proved by the authors in [26]. The theorem stated here is a stronger result, and the proof which we give is considerably simpler than our previous argument.

PROOF. The function $F(z)$ is of class $\mathcal{K}(\tau, \tau)$, so by Lemma 3.5 its restriction $F(x)$ to R is of class $\mathfrak{M}(v, v)$ where $v(x) = e^{i\tau x}$. Factor $F(x) = G(x)^*G(x)$ a.e. as in Theorem 3.1 where $G(x)$ is outer and of class $\mathfrak{M}(1, v)$. By Lemma 3.5 we can alter $G(x)$ on a set of measure zero so that it becomes the restriction to R of an operator valued entire function $G(z)$ of class $\mathcal{K}(0, \tau)$. By analyticity $F(z) = \tilde{G}(z)G(z)$ for all complex z, and $e^{-(1/2)i\tau z}G(z)$ is of class $\mathcal{K}(\frac{1}{2}\tau, \frac{1}{2}\tau)$ and hence of exponential type $\frac{1}{2}\tau$.

Theorem 3.6 has a curious application to scalar valued entire functions.

3.7. THEOREM. *Let f be a scalar valued entire function of exponential type τ, $\tau \geq 0$, such that $|f(x)| \leq 1$ for all real x. Then $f = ab + cd$ where a, b, c, d are entire functions of exponential type $\frac{1}{2}\tau$ such that $\bar{a}a + \bar{c}c = \bar{b}b + \bar{d}d = 1$ identically.*

PROOF. Apply Theorem 3.6 to the matrix valued entire function $F = \begin{bmatrix} 1 & f \\ \bar{f} & 1 \end{bmatrix}$. Write $e^{-(1/2)i\tau z} G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and compare matrix entries in the relation $F = \tilde{G}G$.

We return again to the general factorization problem for nonnegative operator valued functions on the circle or line. For functions which have invertible values a.e. there is a simple sufficient condition for factorability (Lowdenslager [17], Devinatz [6], Douglas [7]).

3.8. THEOREM. *Let F be a weakly measurable nonnegative operator valued function which has invertible values a.e. on Γ or R. Suppose that either*

$$\log^+ \left| F(e^{i\theta}) \right| \quad \text{and} \quad \log^+ \left| F(e^{i\theta})^{-1} \right| \in L^1(0, 2\pi)$$

or

$$(1 + x^2)^{-1} \log^+ \left| F(x) \right| \quad \text{and} \quad (1 + x^2)^{-1} \log^+ \left| F(x)^{-1} \right| \in L^1(-\infty, \infty).$$

Then F is factorable, i.e. $F = G^*G$ a.e. for some operator valued function G of class $\mathfrak{N}^+$ on Γ or R.

PROOF. We reduce to the case where F is bounded. In fact, consider $F_1 = F/f$ where $f(\cdot) = \max(1, |F(\cdot)|)$. Our assumptions imply that f is a factorable scalar valued function, so F and F_1 are simultaneously factorable or not. Since $0 \leq F_1 \leq I$ on Γ or R, we may assume that $0 \leq F \leq I$ on Γ or R. But now $F \geq mI$ on Γ or R where $m(\cdot) = |F(\cdot)^{-1}|^{-1}$ is a factorable scalar valued function, say $m = |\psi|^2$ where ψ is a bounded scalar valued outer function on Γ or R. The proof can now be completed by showing that the hypotheses of Theorem 1.18 are satisfied with $\mathcal{K} = H^2_{\mathfrak{E}}$, $T_1 = T_{mI}$, and $T_2 = T_F$. If $(f_n)_1^\infty$ is a sequence in

$H^2_{\mathfrak{C}}$ such that

$$\lim_{n,k\to\infty} \langle T_2(f_n - f_k), f_n - f_k \rangle = 0 \quad \text{and} \quad \lim_{n\to\infty} \langle T_1 f_n, f_n \rangle = 0,$$

then $(F^{1/2}f_n)^\infty_1$ is a Cauchy sequence in $L^2_{\mathfrak{C}}$ and $(\psi f_n)^\infty_1$ converges to 0 in the metric of $L^2_{\mathfrak{C}}$. But then $\psi f_{n_j}\to 0$ a.e. for a suitable subsequence, and hence $f_{n_j}\to 0$ a.e. and $F^{1/2}f_{n_j}\to 0$ a.e. (in the metric of $\mathfrak{C}$). It follows that $F^{1/2}f_n\to 0$ in the metric of $L^2_{\mathfrak{C}}$ and $\lim_{n\to\infty}\langle T_2 f_n, f_n \rangle = 0$. By Theorem 1.18, $T_2 = T_F$ is factorable as an S-Toeplitz operator and therefore F is factorable.

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