JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 2, April 2014, Pages 339–388
S 0894-0347(2014)00782-7
Article electronically published on January 15, 2014

THE FARRELL-JONES CONJECTURE FOR COCOMPACT LATTICES IN VIRTUALLY CONNECTED LIE GROUPS

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1. Introduction

1.1. **Motivation and summary.** The algebraic K-theory and L-theory of group rings has gained a lot of attention in the last decades, in particular since they play a prominent role in the classification of manifolds. Computations are very hard and here the *Farrell-Jones Conjecture* comes into play. It identifies the algebraic K-theory and L-theory of group rings with the evaluation of an equivariant homology theory on the classifying space for the family of virtually cyclic subgroups. This is the analogue of classical results in the representation theory of finite groups such as the induction theorem of Artin or Brauer, where the value of a functor for finite groups is computed in terms of its values on a smaller family, for instance of cyclic or hyperelementary subgroups; in the Farrell-Jones setting the reduction is to virtually cyclic groups. The point is that this equivariant homology theory is much more accessible than the algebraic K- and L-groups themselves. Actually, most of all computations for infinite groups in the literature use the Farrell-Jones Conjecture and concentrate on the equivariant homology side.

The Farrell-Jones Conjecture is not only important for calculations, but also gives structural insight, since the isomorphism occurring in its formulation has also geometric interpretations. This has the consequence that the Farrell-Jones Conjecture implies a variety of other well-known conjectures such as the ones due to *Bass*, *Borel*, *Kaplansky*, *Novikov*, and *Serre* which concern character theory for infinite groups, algebraic topology, the classification of manifolds, the ring structure of group rings, and group theory. We will discuss them in more detail in Subsection 2.2.

The main result of this paper is to prove the Farrell-Jones Conjecture for a new prominent classes of groups, namely, *cocompact lattices in almost connected Lie groups*. We mention that the operator theoretic analog of the Farrell-Jones Conjecture, the Baum-Connes Conjecture, is known only for a few groups in this class. With the exception of the Novikov Conjecture, the conjectures listed above have not been known for this class so that our result presents also new contributions to them. Since we address a general version of the Farrell-Jones Conjectures, where one allows coefficients in additive categories, very powerful inheritance properties are valid which we will describe in Subsection 2.3. For instance, if this general version of the Farrell-Jones Conjecture holds for a group, it holds automatically for

Received by the editors January 3, 2011 and, in revised form, April 16, 2013.
2010 *Mathematics Subject Classification.* 18F25, 19A31, 19B28, 19G24, 22E40, 57N99.
Key words and phrases. Farrell-Jones Conjecture, K- and L-theory of group rings, cocompact lattices in virtually connected Lie groups, fundamental groups of 3-manifolds.

all subgroups, and it passes to free and direct products and to colimits of directed systems (with not necessarily injective) structure maps.

1.2. **Statement of results.** Next we give the precise formulation of our main results, more technical explanations will follow in the main body of the text.

Theorem 1.1 (Virtually poly-$\mathbb{Z}$-groups). *Let G be a virtually poly-$\mathbb{Z}$-group (see Definition* 5.1*).*

Then both the K-theoretic and the L-theoretic Farrell-Jones Conjectures with additive categories as coefficients with respect to the family $\mathcal{VC}yc$ (see Definitions 2.1 *and* 2.2*) hold for G.*

This is the main new ingredient in proving the following results.

A virtually connected Lie group is a Lie group with finitely many path components. A subgroup $G \subseteq L$ of a Lie group L is called *lattice* if G is discrete and L/G has finite volume and is called a *cocompact lattice* if G is discrete and L/G is compact.

Theorem 1.2 (Cocompact lattices in virtually connected Lie groups). *Let G be a cocompact lattice in a virtually connected Lie group.*

Then both the K-theoretic and the L-theoretic Farrell-Jones Conjectures with additive categories as coefficients with respect to the family $\mathcal{VC}yc$ (see Definitions 2.1 *and* 2.2*) hold for G.*

An argument due to Roushon [35, 36] shows that the above results imply the corresponding result for fundamental groups of 3-manifolds.

Corollary 1.3 (Fundamental groups of 3-manifolds). *Let π be the fundamental group of a* 3*-manifold (possibly non-compact, possibly non-orientable and possibly with boundary).*

Then both the K-theoretic and the L-theoretic Farrell-Jones Conjectures with additive categories as coefficients with respect to the family $\mathcal{VC}yc$ (see Definitions 2.1 *and* 2.2*) hold for π.*

We can also handle virtually weak strongly poly-surface groups (see Remark 7.2) and virtually nilpotent groups (see Remark 2.13).

Remark 1.4 (Finite wreath products). Actually, all the results above do hold for the more general version of the Farrell-Jones Conjecture, where one allows *finite wreath products*, i.e, the "with finite wreath product" version holds for a group G, if the version above holds for the wreath product $G \wr F$ for any finite group F. The "with finite wreath product" version has the extra feature that it holds for a group G if it holds for some subgroup $H \subseteq G$ of finite index.

The paper is organized as follows. We will briefly review the Farrell-Jones Conjecture and its relevance in Section 2. In this section we also collect a number of results about the Farrell-Jones Conjecture that will be used throughout this paper. In Sections 3 we treat the case of virtual finitely generated abelian groups. Here we follow an argument of Quinn [32, Sections 2 and 3.2] and extend it to our setting. The main difference is that the present proof depends on a different control theory; we use Theorem 2.16 (which is proved in [7]) instead of results from [33]. The main work of this paper is done in Section 4 where we treat special affine groups. This section builds on ideas from Farrell-Hsiang [20] and Farrell-Jones [22]. The proof of

Theorem 1.1 is given in Section 5. Theorem 1.2 is proved in Section 6 by reducing it to Theorem 1.1 and the main results from [6, 39].

Fundamental groups of 3-manifolds are discussed in 7. In Section 8 we reduce the family of virtually cyclic subgroups to a smaller family extending previous results of Quinn [32] for untwisted coefficients in rings to the more general setting of coefficients in additive categories.

2. A brief review of the Farrell-Jones Conjecture with coefficients

We briefly review the K-theoretic and L-theoretic Farrell-Jones Conjectures with additive categories as coefficients.

2.1. The formulation of the Farrell-Jones Conjecture.

Definition 2.1 (K-theoretic Farrell-Jones Conjecture with additive categories as coefficients)**.** Let G be a group and let $\mathcal{F}$ be a family of subgroups. Then G satisfies the *K-theoretic Farrell-Jones Conjecture with additive categories as coefficients with respect to* $\mathcal{F}$ if for any additive G-category $\mathcal{A}$ the assembly map

$$\mathrm{asmb}_n^{G,\mathcal{A}}\colon H_n^G\big(E_{\mathcal{F}}(G);\mathbf{K}_{\mathcal{A}}\big) \to H_n^G\big(\mathrm{pt};\mathbf{K}_{\mathcal{A}}\big) = K_n\left(\int_G \mathcal{A}\right)$$

induced by the projection $E_{\mathcal{F}}(G) \to \mathrm{pt}$ is bijective for all $n \in \mathbb{Z}$.

Definition 2.2 (L-theoretic Farrell-Jones Conjecture with additive categories as coefficients)**.** Let G be a group and let $\mathcal{F}$ be a family of subgroups. Then G satisfies the *L-theoretic Farrell-Jones Conjecture with additive categories as coefficients with respect to* $\mathcal{F}$ if for any additive G-category with involution $\mathcal{A}$ the assembly map

$$\mathrm{asmb}_n^{G,\mathcal{A}}\colon H_n^G\big(E_{\mathcal{F}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\big) \to H_n^G\big(\mathrm{pt};\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\big) = L_n^{\langle -\infty\rangle}\left(\int_G \mathcal{A}\right)$$

induced by the projection $E_{\mathcal{F}}(G) \to \mathrm{pt}$ is bijective for all $n \in \mathbb{Z}$.

Here are some explanations.

Given a group G, a *family of subgroups* $\mathcal{F}$ is a collection of subgroups of G such that $H \in \mathcal{F}, g \in G$ implies $gHg^{-1} \in \mathcal{F}$ and for any $H \in \mathcal{F}$ and any subgroup $K \subseteq H$ we have $K \in \mathcal{F}$.

For the notion of a *classifying space $E_{\mathcal{F}}(G)$ for a family $\mathcal{F}$* we refer for instance to the survey article [29].

The natural choice for $\mathcal{F}$ in the Farrell-Jones Conjecture is the family $\mathcal{VC}\mathrm{yc}$ of virtually cyclic subgroups but sometimes it is useful to consider in between other families for technical reasons.

Notation 2.3 (Abbreviation FJC)*.* In the sequel the abbreviation FJC stands for "Farrell-Jones Conjecture with additive categories as coefficients with respect to the family $\mathcal{VC}\mathrm{yc}$."

Remark 2.4 (Relevance of the additive categories as coefficients)*.* The versions of the Farrell-Jones Conjecture appearing in Definition 2.1 and Definition 2.2 are formulated and analyzed in [5], [12]. They encompass the versions for group rings RG over arbitrary rings R, where one can build in a twisting into the group ring or treat more generally crossed product rings $R * G$ and one can allow orientation homomorphisms $w\colon G \to \{\pm 1\}$ in the L-theory case. Moreover, inheritance properties are built in and one does not have to pass to fibered versions anymore as explained in Subsection 2.3.

Example 2.5 (Torsionfree G and regular R)**.** If R is regular and G is torsionfree, then the Farrell-Jones Conjecture reduces to the claim that the classical assembly maps

$$H_n(BG;\mathbf{K}_R) \to K_n(RG);$$
$$H_n\big(BG;\mathbf{L}_R^{\langle -\infty\rangle}\big) \to L_n^{\langle -\infty\rangle}(RG),$$

are bijective for $n \in \mathbb{Z}$, where BG is the classifying space of BG and $H_*\big(-;\mathbf{K}_R\big)$ and $H_*\big(-;\mathbf{L}_R^{\langle -\infty\rangle}\big)$ are generalized homology theories with $H_n\big(\mathrm{pt};\mathbf{K}_R\big) \cong K_n(R)$ and $H_n\big(\mathrm{pt};\mathbf{L}_R^{\langle -\infty\rangle}\big) \cong L_n^{\langle -\infty\rangle}(R)$ for $n \in \mathbb{Z}$.

The original source for the (fibered) Farrell-Jones Conjecture is the paper by Farrell-Jones [23, 1.6 on page 257 and 1.7 on page 262].

2.2. **Applications.** As remarked in the Introduction, the Farrell-Jones Conjecture implies a number of other conjectures. For a detailed discussion of these applications we refer to [10] and the survey article [30]. Here we summarize these applications as follows.

- *Bass Conjecture*
  One version of the Bass Conjecture predicts the possible values of the Hattori-Stallings rank of a finitely generated RG-module extending well-known results for finite groups to infinite groups. If R is a field of characteristic zero, it follows from the K-theoretic FJC.
- *Borel Conjecture*
  The Borel Conjecture says that a closed aspherical topological manifold N is topologically rigid, i.e, any homotopy equivalence $M \to N$ with a closed topological manifold as source and N as target is homotopic to a homeomorphism. The Borel Conjecture is known to be true in dimensions ≤ 3. It holds in dimensions ≥ 5 if the fundamental group satisfies the K-theoretic FJC and the L-theoretic FJC.
- *Homotopy invariance of L^2-torsion*
  There is the conjecture that for two homotopy equivalent finite connected CW-complexes whose universal coverings are det-L^2-acyclic the L^2-torsion of their universal coverings agree. This follows from the K-theoretic FJC.
- *Kaplansky Conjecture*
  The Kaplansky Conjecture predicts for an integral domain R and a torsionfree group G that 0 and 1 are the only idempotents in RG. If R is a field of characteristic zero or if R is a skew field and G is sofic, it follows from the K-theoretic FJC.
- *Moody's Induction Conjecture*
  If R is a regular ring with $\mathbb{Q} \subseteq R$, e.g., a skew field of characteristic zero, then Moody's Induction Conjecture predicts that the map

$$\operatorname{colim}_{\mathrm{Or}_{\mathcal{F}\mathrm{in}}(G)} K_0(RH) \xrightarrow{\cong} K_0(RG)$$

is bijective. Here the colimit is taken over the full subcategory of the orbit category whose objects are homogeneous spaces G/H with finite H. It follows from the K-theoretic FJC.

If F is a skew field of prime characteristic p, then Moody's Induction Conjecture predicts that the map

$$\operatorname{colim}_{\mathrm{Or}_{\mathcal{F}\mathrm{in}}(G)} K_0(FH)[1/p] \xrightarrow{\cong} K_0(RG)[1/p]$$

is bijective. This also follows from the K-theoretic FJC.

- *Poincaré duality groups*
  Let G be a finitely presented Poincaré duality group of dimension n. Then there is the conjecture that G is the fundamental group of a compact ANR-homology manifold. This follows in dimension $n \geq 6$ if the fundamental group satisfies the K-theoretic FJC and the L-theoretic FJC (see [11, Theorem 1.2]). In order to replace ANR-homology manifolds by topological manifold, one has to deal with Quinn's resolution obstruction (see [15], [31]).
- *Novikov Conjecture*
  The Novikov Conjecture predicts for a group G that the higher G-signatures are homotopy invariants and follows from the L-theoretic FJC.
- *Serre Conjecture*
  The Serre Conjecture predicts that a group G of type FP is of type FF. It follows from the K-theoretic FJC.
- *Vanishing of the reduced projective class group*
  Let G be a torsionfree group and R a regular ring. Then there is the conjecture that the change of rings map $K_0(R) \to K_0(RG)$ is bijective. In particular the reduced projective class group $\widetilde{K}_0(RG)$ vanishes if R is a principal ideal domain. This follows from the K-theoretic FJC.
- *Vanishing of the Whitehead group*
  There is the conjecture that the Whitehead group $\mathrm{Wh}(G)$ of a torsionfree group G vanishes. This follows from the K-theoretic FJC.

2.3. **Inheritance properties.** The formulation of the Farrell-Jones Conjecture with additive categories as coefficients has the advantage that the various inheritance properties which led to and are guaranteed by the so called fibered versions are automatically built in (see [5, Theorem 0.7]). This implies the following results (see [5, Corollary 0.9, Corollary 0.10 and Corollary 0.11] and [6, Lemma 2.3]).

Theorem 2.6 (Directed colimits). *Let $\{G_i \mid i \in I\}$ be a directed system (with not necessarily injective structure maps) and let G be its colimit $\operatorname{colim}_{i\in I} G_i$. Suppose that G_i satisfy the K-theoretic FJC for every $i \in I$. Then G satisfies the K-theoretic FJC.*

The same is true for the L-theoretic FJC.

Theorem 2.7 (Extensions). *Let $1 \to K \to G \xrightarrow{p} Q \to 1$ be an extension of groups. Suppose that the group Q and for any virtually cyclic subgroup $V \subseteq Q$ the group $p^{-1}(V)$ satisfy the K-theoretic FJC. Then the group G satisfies the K-theoretic FJC.*

The same is true for the L-theoretic FJC.

Theorem 2.8 (Subgroups). *If G satisfies the K-theoretic FJC, then any subgroup $H \subseteq G$ satisfies the K-theoretic FJC.*

The same is true for the L-theoretic FJC.

Theorem 2.9 (Free and direct products)**.** *If the groups G_1 and G_2 satisfy the K-theoretic FJC, then their free amalgamated product $G_1 * G_2$ and their direct product $G_1 \times G_2$ satisfy the K-theoretic FJC.*

The same is true for the L-theoretic FJC.

Theorem 2.7 and Theorem 2.8 have also been proved in [24].

Theorem 2.10 (Transitivity Principle)**.** *Let $\mathcal{F} \subseteq \mathcal{G}$ be two families of subgroups of G. Assume that for every element $H \in \mathcal{G}$ the group H satisfies the K-theoretic Farrell-Jones Conjecture with additive categories as coefficients for the family $\mathcal{F}|_H = \{K \subseteq H \mid K \in \mathcal{G}\}$.*

Then the relative assembly map

$$\operatorname{asmb}_n^{G,\mathcal{F},\mathcal{G}} \colon H_n^G\big(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}\big) \to H_n^G\big(E_{\mathcal{G}}(G); \mathbf{K}_{\mathcal{A}}\big)$$

induced by the up to G-homotopy unique G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$ is an isomorphism for any additive G-category $\mathcal{A}$ and all $n \in \mathbb{Z}$.

In particular, G satisfies the K-theoretic Farrell-Jones Conjecture with additive categories as coefficients for the family $\mathcal{G}$ if and only if G satisfies the K-theoretic Farrell-Jones Conjecture with additive categories as coefficients for the family $\mathcal{F}$

The same is true for the L-theoretic FJC.

Proof. Given an additive G-category $\mathcal{A}$ with involution, one obtains in the obvious way a homology theory over the group G in the sense of [2, Definition 1.3] using [5, Lemma 9.5]. In Bartels-Echterhoff-Lück [2, Theorem 3.3] the Transitivity Principle is formulated for homology theories over a given group G. Its proof is a slight variation of the proof for an equivariant homology theory in Bartels-Lück [3, Theorem 2.4, Lemma 2.2] and it yields the claim. ☐

Corollary 2.11. *Let $1 \to K \to G \to Q \to 1$ be an exact sequence of groups. Suppose that Q satisfies the K-theoretic FJC and that K is finite. Then G satisfies the K-theoretic FJC.*

The same is true for the L-theoretic FJC.

We mention already here the following corollary of the Transitivity Principle 2.7, Theorem 1.1 and Lemma 5.2 (v).

Corollary 2.12. *Let $1 \to K \to G \to Q \to 1$ be an exact sequence of groups. Suppose that Q satisfies the K-theoretic FJC and that K is virtually poly-$\mathbb{Z}$. Then G satisfies the K-theoretic FJC.*

The same is true for the L-theoretic FJC.

Remark 2.13 (Virtually nilpotent groups)*.* The inheritance properties allow sometimes to prove the FJC for other interesting groups. For instance, we can show that every virtually nilpotent group satisfies both the K-theoretic and the L-theoretic FJC. This follows from the argument appearing in the proof of [10, Lemma 2.13] together with Theorem 1.1, Theorem 2.6, and Theorem 2.7.

2.4. **A strategy.** In this subsection we present a general strategy to prove the FJC. It is motivated by the paper of Farrell-Hsiang [20].

We call a simplicial G-action on a simplicial X *cell preserving* if the following holds: If σ is a simplex with interior σ° and $g \in G$ satisfy $g \cdot \sigma^\circ \cap \sigma^\circ \neq \emptyset$, then we get $g \cdot x = x$ for all $x \in \sigma$. If G acts simplicially on X, then the induced simplicial G-action on the barycentric subdivision X' is always cell preserving. The condition cell preserving guarantees that X with the filtration by its skeletons coming from the simplicial structure on X is a G-CW-complex structure on X.

Recall that a finite group H is called *p-hyperelementary* for a prime p, if there is a short exact sequence

$$0 \to C \to H \to P \to 0$$

with P a p-group and C a cyclic group of order prime to p. It is called *hyperelementary* if it is *hyperelementary* for some prime p.

We recall the following definition from [7].

Definition 2.14 (Farrell-Hsiang group)**.** Let $\mathcal{F}$ be a family of subgroups of the finitely generated group G. We call G a *Farrell-Hsiang group* with respect to the family $\mathcal{F}$ if the following holds for a fixed word metric d_G:

There exists a natural number N such that for any $R > 0$, $\epsilon > 0$ there is a surjective homomorphism $\alpha_{R,\epsilon}\colon G \to F_{R,\epsilon}$ with $F_{R,\epsilon}$ a finite group such that the following condition is satisfied. For any hyperelementary subgroup H of $F_{R,\epsilon}$ we set $\overline{H} := \alpha_{R,\epsilon}^{-1}(H)$ and require that there exists a simplicial complex E_H of dimension at most N with a cell preserving simplicial $\overline{H}$-action whose stabilizers belong to $\mathcal{F}$, and an $\overline{H}$-equivariant map $f_H\colon G \to E_H$ such that $d_G(g,h) < R$ implies $d^1_{E_H}(f(g), f(h)) < \epsilon$ for all $g, h \in G$, where $d^1_{E_H}$ is the l^1-metric on E_H.

Remark 2.15. We point out that the existence of $\overline{H}$-equivariant maps $f_H\colon G \to E_H$ as in the above definition is invariant under conjugation: If K is conjugated to H in $F_{R,\varepsilon}$ then there is $\gamma \in G$ such that $K = \alpha_{R,\varepsilon}(\gamma^{-1})H\alpha_{R,\varepsilon}(\gamma)$. Set $E_K := E_H$. There is an action of $\overline{K} = \gamma^{-1}\overline{H}\gamma$ on this simplicial complex where $k \in \overline{K}$ acts as $\gamma k \gamma^{-1}$. Finally, define $f_K\colon G \to E_K$ by $f_K(g) := f_H(\gamma g)$; this map is $\overline{K}$-equivariant and has the appropriate contracting property because $g \mapsto \gamma g$ is an isometry of G.

The next result is proved in [7].

Theorem 2.16 (Farrell-Hsiang groups and the Farrell-Jones-Conjecture)**.** *Let G be a Farrell-Hsiang group with respect to the family $\mathcal{F}$ in the sense of Definition* 2.14*.*

Then G satisfies the K-theoretic and the L-theoretic Farrell-Jones Conjectures with additive categories as coefficients with respect to the family $\mathcal{F}$ (see Definition 2.1 *and Definition* 2.2*).*

3. Virtually finitely generated abelian groups

In this section we prove the K-theoretic and the L-theoretic Farrell-Jones Conjectures with additive categories as coefficients with respect to the family $\mathcal{VC}\text{yc}$ (see Definition 2.1 and Definition 2.2) for virtually finitely generated abelian groups. This will be one ingredient in proving the K-theoretic and the L-theoretic FJC for virtually poly-$\mathbb{Z}$ groups.

Theorem 3.1 (Virtually finitely generated abelian groups)**.** *Both the K-theoretic and the L-theoretic FJC hold for virtually finitely generated abelian groups.*

Remark 3.2. Since a virtually finitely generated abelian group possesses an epimorphism with a finite kernel onto a crystallographic group (see for instance [32, Lemma 4.2.1]), it suffices to prove Theorem 3.1 for crystallographic groups because of Corollary 2.11.

A crystallographic group is obviously a CAT(0)-group. Hence it satisfies the K- and L-theoretic Farrell Jones Conjectures with additive categories as coefficients with respect to $\mathcal{VC}$yc by [6, Theorem B] and [39]. Nevertheless we give a proof using different methods in this section, because this proof for virtually finitely generated abelian groups is a good model for the proof for virtually poly-$\mathbb{Z}$-groups in Sections 4 and 5.

In this section we follow Quinn's proof of the Farrell-Jones Conjecture for virtually finitely generated abelian groups and untwisted group rings RG over commutative rings [32, Theorem 1.2.2 and Corollary 1.2.3].

3.1. **Review of crystallographic groups.** In this subsection we briefly collect some basic facts about crystallographic groups.

A *crystallographic group* Δ *of rank* n is a discrete subgroup of the group of isometries of $\mathbb{R}^n$ such that the induced isometric group action $\Delta \times \mathbb{R}^n \to \mathbb{R}^n$ is proper and cocompact. The translations in Δ form a normal subgroup isomorphic to $\mathbb{Z}^n$ which is called the *translation subgroup* and will be denoted by $A = A_\Delta$. It is equal to its own centralizer. The quotient $F_\Delta := \Delta/A_\Delta$ is called the *holonomy group* and is a finite group.

A group G is called an *abstract crystallographic group of rank* n if it contains a normal subgroup A which is isomorphic to $\mathbb{Z}^n$, has finite index, and is equal to its own centralizer in G. Such a subgroup A is unique by the following argument. The centralizer in G of any subgroup B of A, which has finite index in A, is A, since any automorphism of A which induces the identity on B is itself the identity. Suppose that A' is another normal subgroup which is isomorphic to $\mathbb{Z}^n$, has finite index, and is equal to its own centralizer in G. Then $A \cap A'$ is a normal subgroup of A and of A' of finite index. Hence $A = A'$, as both A and A' coincide with the centralizer of $A \cap A'$. In particular A is a characteristic subgroup of G, i.e., any group automorphism of G sends A to A.

Every abstract crystallographic group G of rank n is a crystallographic group of rank n whose group of translations is A and vice versa (see [16, Definition 1.9 and Proposition 1.12]). The rank of a crystallographic group is equal to its virtual cohomological dimension.

Notation 3.3. Let A be an abelian group and s be an integer. We denote by sA or $s \cdot A$ the subgroup of A given the image of the map $s \cdot \mathrm{id}_A \colon A \to A$ and by A_s the quotient A/sA.

Definition 3.4 (Expansive map)**.** Let Δ be a crystallographic group and s be an integer different from zero. A group homomorphism $\phi\colon \Delta \to \Delta$ is called *s-expansive* if it fits into the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_\Delta & \xrightarrow{i} & \Delta & \xrightarrow{\mathrm{pr}} & F_\Delta & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle s\cdot\mathrm{id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & A_\Delta & \xrightarrow{i} & \Delta & \xrightarrow{\mathrm{pr}} & F_\Delta & \longrightarrow & 1 \end{array}$$

Given an abelian group A, let $A \rtimes_{-\mathrm{id}} \mathbb{Z}/2$ be the semidirect product with respect to the automorphism $-\mathrm{id}\colon A \to A$.

Lemma 3.5. *Let Δ be a crystallographic group. Let $s \neq 0$ be an integer.*

(i) *There exists an s-expansive map $\phi\colon \Delta \to \Delta$ provided that $s \equiv 1 \mod |F_\Delta|$;*

(ii) *For every s-expansive map $\phi\colon \Delta \to \Delta$ there exists $u \in \mathbb{R}^n$ such that the affine map*

$$a_{s,u}\colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto s \cdot x + u$$

is ϕ-equivariant;

(iii) *Suppose that Δ is $\mathbb{Z}^n$ or the semidirect product $\mathbb{Z}^n \rtimes_{-\mathrm{id}} \mathbb{Z}/2$. Let $\overline{H} \subseteq \Delta$ be a subgroup with $\overline{H} \cap \mathbb{Z}^n \subseteq s\mathbb{Z}^n$.*

Then there exists an s-expansive map $\phi\colon \Delta \to \Delta$ and an element $v \in \mathbb{R}$ such that $\overline{H} \subseteq \mathrm{im}(\phi)$ and the map $a\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto s \cdot x + v$ is ϕ-equivariant.

Proof. (i) Consider A_Δ as $\mathbb{Z}[F_\Delta]$-module by the conjugation action of F_Δ on A_Δ. Since A_Δ is abelian, isomorphism classes of extensions with A_Δ as a normal subgroup and F_Δ as a quotient are in one to one correspondence with elements in $H^2(F_\Delta; A_\Delta)$ (see [14, Theorem 3.12 in Chapter IV on page 93]). Let Θ be the class associated to the extension $1 \to A_\Delta \to \Delta \to F_\Delta \to 1$. Since F_Δ is finite, $H^2(F_\Delta; A_\Delta)$ is annihilated by multiplication with $|F_\Delta|$ (see [14, Corollary 10.2 in Chapter III on page 84]). Hence multiplication with s induces the identity on $H^2(F_\Delta; A_\Delta)$ because of $s \equiv 1 \mod |F_\Delta|$. Therefore $H^2(F_\Delta; s \cdot \mathrm{id}_{A_\Delta}) = s \cdot \mathrm{id}_{H^2(F_\Delta;A_\Delta)}$ sends Θ to Θ, and the claim follows.

(ii) Since F_Δ is finite, $H^1\big(F_\Delta; A_\Delta \otimes_{\mathbb{Z}} \mathbb{R}\big)$ is trivial. Now one proceeds as in the (more difficult) proof of Lemma 4.24.

(iii) In the case $\Delta = \mathbb{Z}^n$ just take $\phi = s \cdot \mathrm{id}_{\mathbb{Z}^n}$ and $v = 0$. It remains to treat the case $\Delta = \mathbb{Z}^n \rtimes_{-\mathrm{id}} \mathbb{Z}/2$.

Let t be the generator of $\mathbb{Z}/2$. We write the multiplication in $\mathbb{Z}/2$ multiplicatively and in $\mathbb{Z}^n$ additively. For an element $u \in \mathbb{Z}^n$ we define an injective group homomorphism

$$\phi_u\colon \Delta \to \Delta \tag{3.6}$$

by $\phi_u(t) = ut$ and $\phi_u(x) = s \cdot x$ for $x \in \mathbb{Z}^n$. This is well defined as the following calculation shows for $x \in \mathbb{Z}^n$:

$$\phi_u(t)^2 = utut = utut^{-1} = u + (-u) = 0;$$

and

$$\begin{aligned}\phi_u(t)\phi_u(x)\phi_u(t)^{-1} &= ut(s \cdot x)(ut)^{-1} = ut(s \cdot x)t^{-1}(-u)\\ &= u + (-s \cdot x) + (-u) = -s \cdot x = \phi_u(-x) = \phi_u(txt^{-1}).\end{aligned}$$

Obviously ϕ_u is s-expansive.

Let $\mathrm{pr}\colon \Delta \to \mathbb{Z}/2$ be the projection. If $\mathrm{pr}(\overline{H})$ is trivial, we can choose ϕ_0. Suppose that $\mathrm{pr}(\overline{H})$ is non-trivial. Then there is $u \in \mathbb{Z}^n$ with $ut \in \overline{H}$. Consider any element $x \in \overline{H} \cap \mathbb{Z}^n$. Then by assumption we can find $y \in \mathbb{Z}^n$ with $x = s \cdot y$ and hence $\phi_u(y) = x$. Consider any element of the form xt which lies in $\overline{H}$. Then $(xt) \cdot (ut) = x - u$ lies in $\overline{H} \cap \mathbb{Z}^n$ and hence in $\mathrm{im}(\phi_u)$. Since ut and $(xt) \cdot (ut)$ lie in the image of ϕ_u, the same is true for xt. We have shown $\overline{H} \subseteq \mathrm{im}(\phi_u)$.

One easily checks that the map $a\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto s \cdot x + u/2$ is ϕ_u-equivariant. This finishes the proof of Lemma 3.5. ∎

3.2. **The Farrell-Jones Conjecture for certain crystallographic groups of rank two.** We will handle the general case of a virtually finitely generated abelian group by induction over its virtual cohomological dimension. For this purpose we have to handle in Lemmas 3.8 and 3.15 two special low-dimensional cases first.

The following elementary lemma is taken from [32, Lemma 3.3.2] (see also [21, Lemma 4.3]). Denote by d^{euc} the Euclidean metric on $\mathbb{R}^n$.

Lemma 3.7. *Let p be a prime and $C \subseteq (\mathbb{Z}/p)^2$ be a non-trivial cyclic subgroup. Then there is a homomorphism*

$$r\colon \mathbb{Z}^2 \to \mathbb{Z}$$

such that the kernel of the map $(\mathbb{Z}/p)^2 \to \mathbb{Z}/p$ given by its reduction modulo p is C and the induced map

$$r_{\mathbb{R}} = r \otimes_{\mathbb{Z}} \mathbb{R}\colon \mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}$$

satisfies

$$d^{\mathrm{euc}}\big(r_{\mathbb{R}}(x_1), r_{\mathbb{R}}(x_2)\big) \le \sqrt{2p} \cdot d^{\mathrm{euc}}(x_1, x_2)$$

for all $x_1, x_2 \in \mathbb{R}^2$.

Lemma 3.8. *Both the K-theoretic and the L-theoretic FJC hold for $\mathbb{Z}^2$ and $\mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}/2$.*

Proof. Because of Theorem 2.8 applied to $\mathbb{Z}^2 \subseteq \mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}/2$ it suffices to prove the claim for $\mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}/2$. Because of Theorem 2.16 it suffices to show that $\mathbb{Z}^2 \rtimes_{\mathrm{id}} \mathbb{Z}$ is a Farrell-Hsiang group with respect to the family $\mathcal{VC}\mathrm{yc}$ in the sense of Definition 2.14.

In the sequel we abbreviate $\Delta := \mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}/2$. We have the obvious short exact sequence

$$1 \to \mathbb{Z}^2 \xrightarrow{i} \Delta \xrightarrow{\mathrm{pr}} \mathbb{Z}/2 \to 1.$$

Fix a word metric d_Δ on Δ. The map

$$\mathrm{ev}\colon \Delta \to \mathbb{R}^2$$

given by the evaluation of the obvious isometric proper cocompact Δ-action on $\mathbb{R}^2$ is by the Švarc-Milnor Lemma (see [13, Proposition 8.19 in Chapter I.8 on page 140]) a quasi-isometry if we equip $\mathbb{R}^n$ with the Euclidean metric d^{euc}. Hence we can find constants C_1 and C_2 such that for all $g_1, g_2 \in \Delta$ we have

$$d^{\mathrm{euc}}\big(\mathrm{ev}(g_1), \mathrm{ev}(g_2)\big) \quad\le\quad C_1 \cdot d_\Delta(g_1, g_2) + C_2. \tag{3.9}$$

Consider positive real numbers R and ϵ. Choose two different odd prime numbers p and q satisfying

$$\frac{8 \cdot (C_1 \cdot R + C_2)^2}{\epsilon^2} \le p, q. \tag{3.10}$$

For a natural number s define Δ_s to be $\Delta/s\mathbb{Z}^2$. We have the obvious exact sequence

$$1 \to \mathbb{Z}^2/s\mathbb{Z}^2 = \big(\mathbb{Z}/s\big)^2 \to \Delta_s \xrightarrow{\mathrm{pr}_s} \mathbb{Z}/2 \to 1.$$

The canonical projection $\alpha_{pq}\colon \Delta \to \Delta_{pq}$ will play the role of the map $\alpha_{R,\epsilon}$ appearing in Definition 2.14.

Let $H \subseteq \Delta_{pq}$ be a l-hyperelementary subgroup for some prime l. Since p and q are different, we can assume without loss of generality $l \ne p$. Then the canonical projection $\pi\colon \Delta_{pq} \to \Delta_p$ sends $H \cap \mathbb{Z}^2/pq\mathbb{Z}^2$ to a cyclic subgroup C of $\mathbb{Z}^2/p\mathbb{Z}^2$. Let $r\colon \mathbb{Z}^2 \to \mathbb{Z}$ be the homomorphism appearing in Lemma 3.7 if C is non-trivial and to be the projection on the first factor if C is trivial. Let $\overline{H}$ be the preimage of H under the projection $\alpha_{pq} \colon \Delta \to \Delta_{pq}$. In all cases we get for $x_1, x_2 \in \mathbb{R}^2$

$$d^{\mathrm{euc}}\big(r_{\mathbb{R}}(x_1), r_{\mathbb{R}}(x_2)\big) \leq \sqrt{2p} \cdot d^{\mathrm{euc}}(x_1, x_2) \tag{3.11}$$

and

$$r\big(\overline{H} \cap \mathbb{Z}^2\big) \subseteq p\mathbb{Z}. \tag{3.12}$$

The homomorphism r extends to a group homomorphism

$$\overline{r} := r \rtimes_{\mathrm{id}} \mathrm{id}_{\mathbb{Z}/2} \colon \Delta = \mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}/2 \to D_\infty = \mathbb{Z} \rtimes_{-\mathrm{id}} \mathbb{Z}/2.$$

We conclude $\overline{r}(\overline{H}) \cap \mathbb{Z} = r\big(\overline{H} \cap \mathbb{Z}^2\big) \subseteq p\mathbb{Z}$ from (3.12). Because of Lemma 3.5 (iii) we can find a p-expansive map

$$\phi \colon D_\infty \to D_\infty$$

and an affine map

$$a_{p,u} \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto p \cdot x + u$$

such that $a_{p,u}$ is ϕ-equivariant and

$$\overline{r}(\overline{H}) \subseteq \mathrm{im}(\phi). \tag{3.13}$$

Let E_H be the simplicial complex with underlying space $\mathbb{R}$ whose set of zero-simplices is $\{n/2 \mid n \in \mathbb{Z}\}$. Equip $\mathbb{R}$ with the standard D_∞-action given by translation with integers and $-\mathrm{id}_{\mathbb{R}}$. Then the D_∞-action on $E_H = \mathbb{R}$ is a cell preserving simplicial action. If d^{l^1} is the l^1-metric on E_H, we get for all y_1, y_2 in E_H

$$d^{l^1}(y_1, y_2) \leq 2 \cdot d^{\mathrm{euc}}(y_1, y_2). \tag{3.14}$$

Define a map

$$f_H \colon \Delta \xrightarrow{\mathrm{ev}} \mathbb{R}^2 \xrightarrow{r_{\mathbb{R}}} \mathbb{R} \xrightarrow{(a_{p,u})^{-1}} E = \mathbb{R}.$$

The map $\mathrm{ev} \colon \Delta \to \mathbb{R}^2$ is Δ-equivariant. The map $r_{\mathbb{R}} \colon \mathbb{R}^2 \to \mathbb{R}$ is $\overline{r} \colon \mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}/2 \to D_\infty = \mathbb{Z} \rtimes_{-\mathrm{id}} \mathbb{Z}/2$-equivariant. Because of (3.13) we can define an $\overline{H}$-action on $\mathbb{R}$ by requiring that $\overline{h} \in \overline{H}$ acts by multiplication with the element $u \in D_\infty$ which is uniquely determined by $\overline{r}(\overline{h}) = \phi(u)$. With respect to this $\overline{H}$-action and the obvious $\overline{H}$-action on Δ the map f_H is $\overline{H}$-equivariant. All isotropy groups of the $\overline{H}$-action on E are virtually cyclic. We estimate for g_1, $g_2 \in \Delta$ with $d_\Delta(g_1, g_2) \leq R$ using (3.9), (3.10), (3.11), and (3.14)

$$\begin{aligned}
d^{l^1}\big(f_H(g_1), f_H(g_2)\big) &\leq 2 \cdot d^{\mathrm{euc}}\big(f_H(g_1), f_H(g_2)\big) \\
&= 2 \cdot d^{\mathrm{euc}}\big(a_{p,u}^{-1} \circ r_{\mathbb{R}} \circ \mathrm{ev}(g_1), a_{p,u}^{-1} \circ r_{\mathbb{R}} \circ \mathrm{ev}(g_2)\big) \\
&= \frac{2}{p} \cdot d^{\mathrm{euc}}\big(r_{\mathbb{R}} \circ \mathrm{ev}(g_1), r_{\mathbb{R}} \circ \mathrm{ev}(g_2)\big) \\
&\leq \frac{2}{p} \cdot \sqrt{2p} \cdot d^{\mathrm{euc}}\big(\mathrm{ev}(g_1), \mathrm{ev}(g_2)\big) \\
&\leq \frac{2 \cdot \sqrt{2}}{\sqrt{p}} \cdot \big(C_1 \cdot d_\Delta\big(g_1, g_2\big) + C_2\big) \\
&\leq \frac{2 \cdot \sqrt{2}}{\sqrt{p}} \cdot (C_1 \cdot R + C_2) \\
&\leq \epsilon.
\end{aligned}$$

We conclude that Δ is a Farrell-Hsiang group in the sense of Definition 2.14 with respect to the family $\mathcal{VC}$yc. Hence Lemma 3.8 follows from Theorem 2.16. □

Lemma 3.15. *Let Δ be a crystallographic group of rank two which possesses a normal infinite cyclic subgroup. Then both the K-theoretic and the L-theoretic FJC hold for Δ.*

Proof. We will use induction over the order of $F = F_\Delta$. If F is trivial, then $\Delta = \mathbb{Z}^2$ and the claim follows from Lemma 3.8. The induction step for $|F| \geq 2$ is done as follows.

Because of Lemma 3.8 we can assume in the sequel that Δ is different from $\mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}/2$. Let $\mathcal{F}$ be the family of subgroups $K \subseteq \Delta$ which are virtually cyclic or satisfy $\mathrm{pr}_\Delta(K) \neq F_\Delta$ for the projection $\mathrm{pr}_\Delta \colon \Delta \to F_\Delta$. Because of the induction hypothesis, the Transitivity Principle 2.10, and Theorem 2.16 it suffices to show that Δ is a Farrell-Hsiang group with respect to the family $\mathcal{F}$ in the sense of Definition 2.14.

We have the canonical exact sequence associated to a crystallographic group

$$1 \to A = A_\Delta \xrightarrow{i} \Delta \xrightarrow{\mathrm{pr}} F = F_\Delta \to 1.$$

Next we analyze the conjugation action $\rho\colon F \to \mathrm{aut}(A)$. Since Δ is crystallographic, $\rho\colon F \to \mathrm{aut}(A)$ is injective. By assumption $A \cong \mathbb{Z}^2$ and we can find a normal infinite cyclic subgroup $C \subset \Delta$.

Next we show that A contains precisely two maximal infinite cyclic subgroups which are F-invariant.

By rationalizing we obtain a two-dimensional rational representation $A_\mathbb{Q} := A \otimes_\mathbb{Z} \mathbb{Q}$ of F. It contains a one-dimensional F-invariant $\mathbb{Q}$-subspace, namely $C_\mathbb{Q} := (C \cap A) \otimes_\mathbb{Z} \mathbb{Q}$. Hence $A_\mathbb{Q}$ is a direct summand of two one-dimensional rational representations $V_1 \oplus V_2$. For each V_i there must be a homomorphism $\sigma_i \colon F \to \{\pm 1\}$ such that $f \in F$ acts on V_i by multiplication with $\sigma_i(f)$. Hence we can find two elements x_1 and $x_2 \in A$ such that x_1 and x_2 are $\mathbb{Z}$-linearly independent and the cyclic subgroups generated by them are F-invariant. Let C_i be the unique maximal infinite cyclic subgroups of A which contains x_i. Then C_1 and C_2 are F-invariant and

$$A = C_1 \oplus C_2.$$

The F-action on C_i is given by the homomorphism $\sigma_i \colon F \to \{\pm 1\}$. Since $\rho\colon F \to \mathrm{aut}(A)$ is injective and Δ is not isomorphic to $\mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}/2$, the homomorphisms σ_1 and σ_2 from F to $\{\pm 1\}$ must be different and F is isomorphic to $\mathbb{Z}/2$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

It remains to show that any maximal infinite cyclic subgroup D which is F-invariant is equal to C_1 or C_2. Given such D, we obtain an F-invariant $\mathbb{Q}$-subspace $D_\mathbb{Q} \subseteq A_\mathbb{Q}$. Since C_1, C_2, and D are maximal infinite cyclic subgroups of A, it suffices to show $D_\mathbb{Q} = (C_i)_\mathbb{Q}$ for some $i \in \{1,2\}$. Suppose the contrary. Then for $i = 1,2$ the projection $A_\mathbb{Q} \to (C_i)_\mathbb{Q}$ induces an isomorphism $D_\mathbb{Q} \to (C_i)_\mathbb{Q}$. Hence $(C_1)_\mathbb{Q}$ and $(C_2)_\mathbb{Q}$ are isomorphic. This implies $\sigma_1 = \sigma_2$, a contradiction. Hence we have shown that A contains precisely two maximal infinite cyclic subgroups which are F-invariant.

If $C \subseteq A$ is a maximal infinite cyclic subgroup which is invariant under the F-action, then it is normal in Δ and we can consider the projection

$$\widehat{\xi_C} \colon \Delta \to \Delta/C.$$

We obtain a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{i} & \Delta & \xrightarrow{\mathrm{pr}} & F & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \xi_C} & & \Big\downarrow{\scriptstyle \widehat{\xi}_C} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & A/C & \xrightarrow{\overline{i}} & \Delta/C & \xrightarrow{\overline{\mathrm{pr}_C}} & F & \longrightarrow & 1
\end{array}$$

where the vertical maps are the obvious projections.

Since Δ/C is virtually abelian with virtual cohomological dimension one, we can find an epimorphism $\overline{\mu_C}\colon \Delta/C \to \Delta'_C$ to a crystallographic group of rank one whose kernel is finite. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A/C & \xrightarrow{\overline{i}} & \Delta/C & \xrightarrow{\overline{\mathrm{pr}_C}} & F & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mu_C} & & \Big\downarrow{\scriptstyle \widehat{\mu}_C} & & \Big\downarrow & & \\
1 & \longrightarrow & A_{\Delta'_C} & \longrightarrow & \Delta'_C & \longrightarrow & F_{\Delta'_C} & \longrightarrow & 1
\end{array}$$

The map μ_C is injective and Δ'_C is either $\mathbb{Z}$ or $D_\infty = \mathbb{Z} \rtimes_{-\mathrm{id}} \mathbb{Z}/2$. Define homomorphisms

$$\widehat{\nu}_C := \widehat{\mu}_C \circ \widehat{\xi}_C \colon \Delta \to \Delta'_C;$$
$$\nu_C := \mu_C \circ \xi_C \colon A \to A_{\Delta'_C}.$$

Consider word metrics d_Δ and $d_{\Delta'_C}$. Recall that $\widehat{\nu}_C$ is a surjective group homomorphism and the quasi-isometry type of a word metric is independent of the choice of a finite set of generators. Hence we can find constants C_1 and C_2 such that for every (of the finitely many) maximal infinite cyclic subgroups $C \subseteq A$ which are invariant under the F-action and for all $g_1, g_2 \in \Delta$ we get

$$d_{\Delta'_C}\big(\widehat{\nu}_C(g_1), \widehat{\nu}_C(g_1)\big) \le C_1 \cdot d_\Delta\big(g_1, g_2\big) + C_2. \tag{3.16}$$

Equip $\mathbb{R}$ with the standard action of Δ'_C. Let E be the simplicial complex whose underlying space is $\mathbb{R}$ and whose set of zero-simplices is $\{n/2 \mid n \in \mathbb{Z}\}$. The Δ'_C-action above is a cell preserving simplicial action on E. If d^{l^1} is the l^1-metric on E, we get for $y_1, y_2 \in \mathbb{R}$

$$d^{l^1}(y_1, y_2) \le 2 \cdot d^{\mathrm{euc}}(y_1, y_2). \tag{3.17}$$

Let the map

$$\mathrm{ev}_C \colon \Delta'_C \to \mathbb{R}$$

be given by the evaluation of the isometric proper cocompact Δ'_C-action on $\mathbb{R}$. By the Švarc-Milnor Lemma (see [13, Proposition 8.19 in Chapter I.8 on page 140]) we can find constants C_3 and C_4 such that for every (of the finitely many) maximal infinite cyclic subgroups $C \subseteq A$ which are invariant under the F-action and all $g_1, g_2 \in \Delta'_C$ we have

$$d^{\mathrm{euc}}\big(\mathrm{ev}_C(g_1), \mathrm{ev}_C(g_2)\big) \le C_3 \cdot d_{\Delta'_C}(g_1, g_2) + C_4. \tag{3.18}$$

We conclude from (3.16), (3.17), and (3.18) that we can find constants $D_1 > 0$ and $D_2 > 0$ such that for every maximal infinite cyclic subgroup $C \subseteq A$ which is invariant under the F-action and all $g_1, g_2 \in \Delta$ we have

$$d^{l^1}\big(\mathrm{ev}_C \circ \widehat{\nu}_C(g_1), \mathrm{ev}_C \circ \widehat{\nu}_C(g_1)\big) \le D_1 \cdot d_\Delta(g_1, g_2) + D_2. \tag{3.19}$$

Consider positive real numbers R and ϵ. We can choose an odd prime p satisfying

$$p \geq \frac{2 \cdot (D_1 \cdot R + D_2)}{\epsilon}. \tag{3.20}$$

Put $A_p = A/pA$ and $\Delta_p = \Delta/pA$. We obtain an exact sequence

$$1 \to A_p \to \Delta_p \xrightarrow{\mathrm{pr}_p} F \to 1.$$

The projection $\alpha_p \colon \Delta \to \Delta_p$ will play the role of the map $\alpha_{R,\epsilon}$ appearing in Definition 2.14.

Let $H \subseteq \Delta_p$ be a hyperelementary subgroup. If $\mathrm{pr}_p(H)$ is not F, then H belongs to $\mathcal{F}$ and we can take for f_H the map $\Delta \to \{\bullet\}$. Hence it remains to treat the case $\mathrm{pr}_p(H) = F$.

Next show that $A_p \cap H$ is cyclic. Choose a prime q and an exact sequence $1 \to D \to H \to P \to 1$ for a q-group P and a cyclic group of order prime to q. If p and q are different, $A_p \cap H$ embeds into D and is hence cyclic. Suppose that $p = q$. It suffices to show that $A_p \cap H$ is different from A_p, or, equivalently, $H \neq \Delta_p$. Suppose the contrary, i.e., $H = \Delta_p$. Because the order of F is 2 or 4 and p is odd, this implies that the composite $A_p \to H \to P$ is an isomorphism. Hence there is a retraction for the inclusion $A_p \to \Delta_p$. This implies that the conjugation action of F on A_p is trivial. We have already explained that there are homomorphisms $\sigma_i \colon F \to \{\pm 1\}$ such that $f \in F$ acts on C_i by multiplication with $\sigma_i(f)$ and that these two homomorphisms must be different. The induced F-action on $A_p = (C_1)_p \oplus (C_2)_p$ is analogous. Since p is odd, this leads to a contradiction. Hence $H \cap A_p$ is cyclic.

Since $\mathrm{pr}_p(H) = F$, the cyclic subgroup $H \cap A_p$ is invariant under the F-action on $A_p = (C_1)_p \oplus (C_2)_p$. Hence $A_p \cap H$ must be contained in $(C_i)_p = \alpha_p(C_i)$ for some $i \in \{1, 2\}$. We put $C = C_i$ and $\Delta' = \Delta'_{C_i}$ in the sequel.

Let $\overline{H}$ be the preimage of H under the projection $\alpha_p \colon \Delta \to \Delta_p$. Then we get for the homomorphism $\xi_C \colon A \to A/C$

$$\xi_C(\overline{H} \cap A) \subseteq p(A/C).$$

Since the map $\mu_C \colon A/C \to A_{\Delta'}$ is injective, we conclude for the homomorphism $\nu_C \colon A \to A_{\Delta'}$

$$\nu_C(\overline{H} \cap A) \cap A_{\Delta'} \subseteq pA_{\Delta'}.$$

Because p is odd and $|F|$ is 2 or 4, this implies that

$$\widehat{\nu}_C(\overline{H}) \cap A_{\Delta'} \subseteq pA_{\Delta'}.$$

Because of Lemma 3.5 (ii) and (iii) we can find a p-expansive map

$$\phi \colon \Delta' \to \Delta'$$

and an affine map

$$a_{p,u} \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto p \cdot x + u,$$

such that $a_{p,u}$ is ϕ-equivariant and

$$\nu_C(\overline{H}) \subseteq \mathrm{im}(\phi). \tag{3.21}$$

Let E_H be the simplicial complex whose underlying space is $\mathbb{R}$ and whose set of zero-simplices is $\{n/2 \mid n \in \mathbb{Z}\}$. The standard Δ'_C-action is a cell preserving simplicial action on E_H. We define the map

$$f_H \colon \Delta \xrightarrow{\widehat{\nu}_C} \Delta' \xrightarrow{\mathrm{ev}} \mathbb{R} \xrightarrow{a_{p,u}^{-1}} E_H = \mathbb{R}.$$

From (3.17), (3.19), and (3.20) and we conclude for $g_1, g_2 \in \Delta$ satisfying $d_\Delta(g_1, g_2) \leq R$

$$\begin{aligned} d^{l^1}\big(f_H(g_1), f_H(g_1)\big) &= 2 \cdot d^{\mathrm{euc}}\big(f_H(g_1), f_H(g_1)\big) \\ &= 2 \cdot d^{\mathrm{euc}}\big(a_{p,u}^{-1} \circ \mathrm{ev}_C \circ \widehat{\nu}_C(g_1), a_{p,u}^{-1} \circ \mathrm{ev}_C \circ \widehat{\nu}_C(g_1)\big) \\ &= \frac{2}{p} \cdot d^{\mathrm{euc}}\big(\mathrm{ev}_C \circ \widehat{\nu}_C(g_1), \mathrm{ev}_C \circ \widehat{\nu}_C(g_1)\big) \\ &\leq \frac{2}{p} \cdot (D_1 \cdot d_\Delta(g_1, g_2) + D_2) \\ &\leq \frac{2 \cdot (D_1 \cdot R + D_2}{p} \\ &\leq \epsilon. \end{aligned}$$

Because of (3.21) we can define an $\overline{H}$-action on E_H by requiring that $\overline{h} \in \overline{H}$ acts by the unique element $g \in \Delta'$ which is mapped under the injective homomorphism $\phi\colon \Delta' \to \Delta'$ to $\nu_C(\overline{h})$. Then the map $f_H \colon \Delta \to E$ is $\overline{H}$-equivariant and all isotropy groups of the $\overline{H}$-action on E are virtually cyclic.

We conclude that Δ is a Farrell-Hsiang group in the sense of Definition 2.14 with respect to the family $\mathcal{VC}$yc. Hence Lemma 3.15 follows from Theorem 2.16. □

Lemma 3.22. *Let $1 \to V \to G \to Q \to 1$ be an exact sequence of groups. Suppose that Q satisfies the K-theoretic FJC and that V is virtually cyclic. Then G satisfies the K-theoretic FJC*

The same is true for the for the L-theoretic FJC.

Proof. By the Transitivity Principle 2.10 it suffices to show that G satisfies the K-theoretic FJC in the special case that Q is virtually cyclic. Since this is obvious for finite V, we can assume that V is infinite. Let C' be an infinite cyclic subgroup of V. Let C be the intersection $\bigcap_{\phi \in \mathrm{aut}(V)} \phi(V)$. Since V contains only finitely many subgroups of a given index, this is a finite intersection and hence C is a characteristic subgroup of V which is infinite cyclic and has finite index. Hence C is a normal infinite cyclic subgroup of the virtually finitely generated abelian group G and the virtual cohomological dimension of G is two. There exists a homomorphism with a finite kernel onto a crystallographic group $G \to G'$ (see for instance [32, Lemma 4.2.1]). The rank of G' is two and G' contains a normal infinite cyclic subgroup. Hence G satisfies the K-theoretic FJC because of Corollary 2.11 and Lemma 3.15. □

3.3. **The Farrell-Jones Conjecture for virtually finitely generated abelian groups.** In this subsection we finish the proof of Theorem 3.1.

Proof of Theorem 3.1. We use induction over the virtual cohomological dimension n of the virtually finitely generated abelian group Δ and subinduction over the minimum of the orders of finite groups F for which there exists an exact sequence

$1 \to \mathbb{Z}^n \to \Delta \to F \to 1$. The induction beginning $n \leq 1$ is trivial since then Δ is virtually cyclic.

In the induction step we can assume that Δ is a crystallographic group of rank n because of Corollary 2.11 since a virtually finitely generated abelian group possesses an epimorphism with a finite kernel onto a crystallographic group (see for instance [32, Lemma 4.2.1]). Hence we have to prove that a crystallographic group Δ of rank $n \geq 2$ satisfies both the K- and L-theoretic FJC provided that every virtually finitely generated abelian group Δ' satisfies both the K- and L-theoretic FJC if $\operatorname{vcd}(\Delta') < n$ or if there exists an extension $1 \to \mathbb{Z}^n \to \Delta' \to F \to 1$ for a finite group F with $|F| < |F_\Delta|$.

Because of the induction hypothesis and Lemma 3.22 we can assume from now on that Δ does not contain a normal infinite cyclic subgroup C.

Let $\mathcal{F}$ be the family of subgroups of Δ which contains all subgroups $\Delta' \subseteq \Delta$ such that $\operatorname{vcd}(\Delta') < \operatorname{vcd}(\Delta)$ holds or that both $\operatorname{vcd}(\Delta') = \operatorname{vcd}(\Delta)$ and $|F_{\Delta'}| < |F_\Delta|$ hold. By the induction hypothesis, the Transitivity Principle 2.10, and Theorem 2.16 it suffices to show that Δ is a Farrell-Hsiang group in the sense of Definition 2.16 for the family $\mathcal{F}$.

Fix a word metric d_Δ on Δ. Let the map

$$\operatorname{ev}\colon \Delta \to \mathbb{R}^n$$

be given by the evaluation of the cocompact proper isometric Δ-operation on $\mathbb{R}^n$. It is by the Švarc-Milnor Lemma (see [13, Proposition 8.19 in Chapter I.8 on page 140]) a quasi-isometry if we equip $\mathbb{R}^n$ with the Euclidean metric d^{euc}. Hence we can find constants C_1 and C_2 such that for all $g_1, g_2 \in \mathbb{Z}^2$ we have

$$d^{\text{euc}}\big(\operatorname{ev}(g_1), \operatorname{ev}(g_2)\big) \leq C_1 \cdot d_\Delta(g_1, g_2) + C_2. \tag{3.23}$$

Consider real numbers $R > 0$ and $\epsilon > 0$. Since Δ acts properly, smoothly and cocompactly on $\mathbb{R}^n$, we can equip $\mathbb{R}^n$ with the structure of a simplicial complex such that the Δ-action is cell preserving and simplicial. Denote this simplicial complex by E. The induced l^1-metric d^{l^1} and the Euclidean metric d^{euc} induce the same topology since E is bounded locally finite. Hence we can find $\delta > 0$ such that

$$d^{\text{euc}}\big(y_1, y_2) \leq \delta \implies d^{l^1}\big(y_1, y_2)\big) \leq \epsilon \tag{3.24}$$

holds for all y_1, y_2.

We can write $|F_\Delta| = 2^k \cdot l$ for some odd natural number l and non-negative integer k. By Dirichlet's Theorem (see [37, Lemma 3 in III.2.2 on page 25]) there exist infinitely many primes which are congruent to -1 modulo $4l$. Hence we can choose a prime number p satisfying

$$p \geq \frac{C_1 \cdot R + C_2}{\delta} \qquad \text{and} \qquad p \equiv -1 \mod 4l.$$

Now choose r such that

$$p^r \equiv 1 \mod |F_\Delta|.$$

Since $A = A_\Delta$ is a characteristic subgroup of Δ, also $p^r \cdot A$ is characteristic and hence a normal subgroup of Δ. Define groups

$$A_{p^r} := A/(p^r \cdot A);$$
$$\Delta_{p^r} := \Delta/(p^r \cdot A).$$

Let $\mathrm{pr}\colon \Delta \to F_\Delta$ and $\mathrm{pr}_{p^r}\colon \Delta_{p^r} \to F_\Delta$ be the canonical projections. Let the epimorphism

$$\alpha_{p^r}\colon \Delta \to \Delta_{p^r}$$

be the canonical projection. It will play the role of the map $\alpha_{R,\epsilon}$ appearing in Definition 2.14.

Consider a hyperelementary subgroup $H \subseteq \Delta_{p^r}$. Let $\overline{H}$ be the preimage of H under $\alpha_{p^r}\colon \Delta \to \Delta_{p^r}$. Suppose that $\mathrm{pr}_{p^r}(H) \neq F_\Delta$. Then $\overline{H}$ is a crystallographic group with $\mathrm{vcd}(\overline{H}) = \mathrm{vcd}(\Delta)$ and $|F_{\overline{H}}| < F_\Delta$. By induction hypothesis $\overline{H}$ satisfies both the K- and L-theoretic FJC and hence belongs to $\mathcal{F}$. Therefore we can take as the desired $\overline{H}$-map in this case the projection to the one-point-space

$$f_H\colon \overline{H} \to E_H := \{\bullet\}.$$

It remains to consider the case, where $\mathrm{pr}_{p^r}(H) = F_\Delta$. We conclude from [32, Proposition 2.4.2] that $H \cap A_{p^r} = \{0\}$ since Δ contains no infinite normal cyclic subgroup and the prime number p satisfies $p \equiv 3 \mod 4$ and $p \not\equiv 1 \mod q$ for any odd prime q dividing $|F_\Delta|$ and hence l.

Since $p^r \equiv 1 \mod |F_\Delta|$ we can choose by Lemma 3.5 (i) a p^r-expansive homomorphism $\phi\colon \Delta \to \Delta$. Consider the composite $\alpha_{p^r} \circ \phi\colon \Delta \to \Delta_{p^r}$. Its restriction to $A = A_\Delta$ is trivial. Hence there is a map $\overline{\phi}\colon F_\Delta \to \Delta_{p^r}$ satisfying

$$\alpha_{p^r} \circ \phi = \overline{\phi} \circ \mathrm{pr};$$
$$\mathrm{pr}_{p^r} \circ \overline{\phi} = \mathrm{id}_{F_\Delta}.$$

Hence $\overline{\phi}$ is a splitting of the exact sequence $1 \to A_{p^r} \to \Delta_{p^r} \to F_\Delta \to 1$. The homomorphism $\mathrm{pr}_{p^r}|_H\colon H \to F_\Delta$ is an isomorphism and hence defines a second splitting. Since the order of the finite group A_{p^r} and the order of F_Δ are prime, $H^1(F_\Delta; A_{p^r})$ vanishes (see [14, Corollary 10.2 in Chapter III on page 84]). Hence the subgroups H and $\mathrm{im}(\alpha_{p^r} \circ \phi) = \mathrm{im}(\overline{\phi})$ are conjugated in Δ_{p^r} (see [14, Corollary 3.13 in Chapter IV on page 93]). Because of Remark 2.15 we can assume without loss of generality that $H = \mathrm{im}(\alpha_{p^r} \circ \phi)$. Next we show for $\overline{H} := \alpha_{p^r}^{-1}(H)$

$$\overline{H} = \mathrm{im}(\phi). \tag{3.25}$$

Because of $H = \mathrm{im}(\alpha_{p^r} \circ \phi)$ it suffices to prove $\alpha_{p^r}^{-1}\big(\mathrm{im}(\alpha_{p^r} \circ \phi)\big) \subseteq \mathrm{im}(\phi)$. Consider $g \in \alpha_{p^r}^{-1}\big(\mathrm{im}(\alpha_{p^r} \circ \phi)\big)$. Choose $g_0 \in \Delta$ with $\alpha_{p^r}(g) = \alpha_{p^r}\big(\phi(g_0)\big)$. We conclude that $\alpha_{p^r}\big(g \cdot \phi(g_0)^{-1}\big)$ is trivial. Hence we can find $a \in A$ with $\phi(a) = p^r \cdot a = g \cdot \phi(g_0)^{-1}$. This implies $g = \phi(a \cdot g_0)$. Hence (3.25) is true.

By Lemma 3.5 (ii) there exists an element $u \in \mathbb{R}$ such that the affine map

$$a_{p^r,u}\colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto p^r \cdot y + u,$$

is ϕ-linear. Consider the composite

$$f_H\colon \Delta \xrightarrow{\mathrm{ev}} \mathbb{R}^n \xrightarrow{(a_{p^r,u})^{-1}} E_H := \mathbb{R}^n.$$

We get from (3.23) for $g_1, g_2 \in \Delta$ with $d_\Delta(g_1, g_2) \leq R$

$$\begin{aligned} d^{\mathrm{euc}}\bigl(f_H(g_1), f_H(g_2)\bigr) &= d^{\mathrm{euc}}\bigl((a_{p^r,u})^{-1} \circ \mathrm{ev}(g_1), (a_{p^r,u})^{-1} \circ \mathrm{ev}(g_2)\bigr) \\ &= \frac{1}{p^r} \cdot d^{\mathrm{euc}}\bigl(\mathrm{ev}(g_1), \mathrm{ev}(g_2)\bigr) \\ &\leq \frac{1}{p^r} \cdot (C_1 \cdot d_\Delta(g_1, g_2) + C_2) \\ &\leq \frac{1}{p^r} \cdot (C_1 \cdot R + C_2)\,. \end{aligned}$$

Our choice of p guarantees

$$\frac{1}{p^r} \cdot (C_1 \cdot R + C_2) \leq \delta,$$

where δ is the number appearing in (3.24). We conclude from (3.24) for all $g_1, g_2 \in \Delta$

$$d_\Delta(g_1, g_2) \leq R \implies d^{l^1}\bigl(f_H(g_1), f_H(g_2)\bigr) \leq \epsilon.$$

Because of (3.25) we can define a cell preserving simplicial $\overline{H}$-action on the simplicial complex E_H by requiring that the action of $h \in \overline{H}$ is given by the action of $g \in \Delta$ for the element uniquely determined by $\phi(g) = h$. The isotropy groups of this H-action on E are all finite and hence belong to $\mathcal{F}$. The map $f_H \colon \Delta \to E_H$ is H-equivariant.

We conclude that Δ is a Farrell-Hsiang group in the sense of Definition 2.14 with respect to the family $\mathcal{F}$. Now Theorem 3.1 follows from Theorem 2.16. □

4. Irreducible special affine groups

In this section we prove the K-theoretic and the L-theoretic Farrell-Jones Conjectures with additive categories as coefficients with respect to $\mathcal{VC}$yc for irreducible special affine groups. This will be the key ingredient and step in proving the K-theoretic and the L-theoretic Farrell-Jones Conjectures with coefficients in an additive category with respect to $\mathcal{VC}$yc for virtually poly-$\mathbb{Z}$-groups.

The irreducible special affine groups will play in the proof for virtually poly-$\mathbb{Z}$-groups the analogous role as the crystallographic groups of rank two which contain a normal infinite cyclic subgroup played in the proof for virtually finitely generated abelian groups. The general structure of the proof for virtual poly-$\mathbb{Z}$-group is similar but technically much more sophisticated and complicated than in the case of virtually finitely generated abelian groups. It relies on the fact that we do know the claim already for virtually finitely generated abelian groups. Our proof is inspired by the one appearing in Farrell-Hsiang [20] and Farrell-Jones [22].

4.1. **Review of (irreducible) special affine groups.** In this subsection we briefly collect some basic facts about (irreducible) special affine groups. We will denote by vcd the virtual cohomological dimension of a group (see [14, Section 11 in Chapter VIII]). The following definition is equivalent to Definition 4.7 in [23].

Definition 4.1 ((Irreducible) special affine group)**.** A group Γ is called a *special affine group* of rank $(n + 1)$ if there exists a short exact sequence

$$1 \to \Delta \to \Gamma \to D \to 1$$

and an action $\rho'\colon \Gamma \times \mathbb{R}^n \to \mathbb{R}^n$ by affine motions of $\mathbb{R}^n$ satisfying:

(i) D is either the infinite cyclic group $\mathbb{Z}$ or the infinite dihedral group D_∞;
(ii) The restriction of ρ' to Δ is a cocompact isometric proper action of Δ.

We call a special affine group *irreducible* if for any epimorphism $\Gamma \to \Gamma'$ onto a virtually finitely generated abelian group Γ' we have $\operatorname{vcd}(\Gamma') \le 1$.

Notice that the group Δ appearing in Definition 4.1 is a crystallographic group of rank n. Let

$$\rho''\colon D \times \mathbb{R} \to \mathbb{R}$$

be the isometric cocompact proper standard action which is given by translations with integers and $-\operatorname{id}$. We will consider the action

$$\rho\colon \Gamma \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \tag{4.2}$$

given by the diagonal action for the Γ-action ρ' on $\mathbb{R}^n$ and the Γ-action on $\mathbb{R}$ coming from the epimorphism $\Gamma \to D$ and the D-action ρ'' on $\mathbb{R}$. This Γ-action ρ is a proper cocompact action by affine motions and is not necessarily an isometric action.

4.2. **Some homological computations.** Let $A = A_\Delta$ be the unique and hence characteristic subgroup of Δ which is abelian, normal, and equal to its own centralizer in Δ. Since it is a characteristic subgroup, it is normal in both Δ and Γ. Define

$$Q := \Gamma/A.$$

Then we obtain exact sequences

$$1 \to A \to \Gamma \xrightarrow{\mathrm{pr}} Q \to 1; \tag{4.3}$$

$$1 \to F_\Delta \to Q \xrightarrow{\pi} D \to 1, \tag{4.4}$$

where F_Δ is the finite group Δ/A. In particular Q is an infinite virtually cyclic group.

Lemma 4.5. *Let Γ be a special affine group. Consider A as $\mathbb{Z}Q$-module by the conjugation action associated to the exact sequence* (4.3). *Then*

(i) *Γ is irreducible if and only if for any subgroup $\Gamma_0 \subseteq \Gamma$ of finite index $\operatorname{rk}_{\mathbb{Z}}(H_1(\Gamma_0)) \le 1$ holds;*
(ii) *The order of $H^2(Q;A)$ is finite;*
(iii) *If Γ is irreducible, then the order of $H^1(Q;A)$ is finite.*

Proof. (i) "$\Longrightarrow$" Let $\Gamma_0 \subseteq \Gamma$ be a subgroup of finite index. We have to show $\operatorname{rk}_{\mathbb{Z}}(H_1(\Gamma_0)) \le 1$, provided that Γ is irreducible. We can find a normal subgroup $\Gamma_1 \subseteq \Gamma$ with $[\Gamma : \Gamma_1] < \infty$ and $\Gamma_1 \subseteq \Gamma_0$. Since the image of the map $H_1(\Gamma_1) \to H_1(\Gamma_0)$ induced by the inclusion has a finite cokernel, it suffices to show $\operatorname{rk}_{\mathbb{Z}}(H_1(\Gamma_1)) \le 1$. Since $[\Gamma_1, \Gamma_1]$ is a characteristic subgroup of Γ_1 and Γ_1 is a normal subgroup of Γ, the subgroup $[\Gamma_1, \Gamma_1]$ of Γ is normal in Γ. Let $f\colon \Gamma \to V := \Gamma/[\Gamma_1, \Gamma_1]$ be the projection. Its restriction to Γ_1 factorizes over the projection $f_1\colon \Gamma_1 \to H_1(\Gamma_1)$ to a homomorphism $i\colon H_1(\Gamma_1) \to V$. One easily checks that i is an injection whose image has finite index in V. Hence V is virtually finitely generated abelian. Since Γ is by assumption irreducible, the virtual cohomological dimension of V and hence of $H_1(\Gamma_1)$ is at most one. This implies $\operatorname{rk}_{\mathbb{Z}}(H_1(\Gamma_0)) \le 1$.

"$\Longleftarrow$" Consider an epimorphism $f\colon \Gamma \to V$ to a virtually finitely generated abelian group V. Put $n = \operatorname{vcd}(V)$. Choose a subgroup $V_0 \subseteq V$ with $V_0 \cong \mathbb{Z}^n$ and $[V : V_0] < \infty$. Let $\Gamma_0 \subseteq \Gamma$ be the preimage of V_0 under f and denote by $f_0\colon \Gamma_0 \to V_0$

the restriction of f to Γ_0. Then f_0 is an epimorphism and Γ_0 is a subgroup of Γ with $[\Gamma : \Gamma_0] < \infty$. The map f_0 factorizes over the projection $\Gamma_0 \to H_1(\Gamma_0)$ to an epimorphism $\overline{f_0}\colon H_1(\Gamma_0) \to V_0$. Hence $n \leq \mathrm{rk}_{\mathbb{Z}}(H_1(\Gamma_0))$. Since by assumption $\mathrm{rk}_{\mathbb{Z}}(H_1(\Gamma_0)) \leq 1$, we conclude $\mathrm{vcd}(V) \leq 1$. Hence Γ is irreducible.

(ii) Since Q is infinite and virtually cyclic, there exists an exact sequence $1 \to \mathbb{Z} \to Q \to F \to 1$ for some finite group F. Recall that the cohomology group of finite groups is finite for any coefficient module in dimensions ≥ 1 (see [14, Corollary 10.2 in Chapter III on page 84]). Obviously the cohomology of $\mathbb{Z}$ vanishes for any coefficient module in dimensions ≥ 2. Now the claim follows from the Hochschild-Serre spectral sequence (see [14, Section 6 in Chapter VII]) applied to the exact sequence above.

(iii) Because of the Hochschild-Serre spectral sequence (see [14, Section 6 in Chapter VII]) applied to the exact sequence above, it suffices to prove that $H^1(\mathbb{Z}; A)$ is finite. This is equivalent to the statement that $H_0(\mathbb{Z}; A)$ is finite since $H^1(\mathbb{Z}; A) \cong H_0(\mathbb{Z}; A)$. Let Γ_0 be the preimage of $\mathbb{Z} \subseteq Q$ under the projection $\mathrm{pr}\colon \Gamma \to Q$. It is a normal subgroup of finite index in Γ and fits into an exact sequence $1 \to A \to \Gamma_0 \to \mathbb{Z} \to 0$. From the Hochschild-Serre spectral sequence we obtain a short exact sequence

$$0 \to H_0(\mathbb{Z}; A) \to H_1(\Gamma_0) \to H_1(\mathbb{Z}) \to 0.$$

Hence it remains to show that the rank of the finitely generated abelian group $H_1(\Gamma_0)$ is at most one. This follows from assertion (i). □

4.3. **Finding the appropriate finite quotient groups.** Fix a special affine group Γ of rank $(n+1)$. For any positive integer s the subgroup $sA \subseteq A$ is characteristic and hence is normal in both A and Γ. Put

$$A_s := A/sA;$$
$$\Gamma_s := \Gamma/sA.$$

Then A_s is isomorphic to $(\mathbb{Z}/s)^n$ and we obtain an exact sequence

$$1 \to A_s \to \Gamma_s \xrightarrow{\mathrm{pr}_s} Q \to 1. \tag{4.6}$$

Let

$$p_s\colon \Gamma \to \Gamma_s$$

be the canonical projection.

Definition 4.7 (Pseudo s-expansive homomorphism)**.** Let s be an integer. We call a group homomorphism $\phi\colon \Gamma \to \Gamma$ *pseudo s-expansive* if it fits into the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & \Gamma & \xrightarrow{\mathrm{pr}} & Q & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle s\cdot\mathrm{id}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & A & \longrightarrow & \Gamma & \xrightarrow{\mathrm{pr}} & Q & \longrightarrow & 1 \end{array}$$

where both the upper and the lower horizontal exact sequence is the one of (4.3).

Recall that $|H^2(Q; A)|$ is finite by Lemma 4.5 (ii).

Lemma 4.8.

(i) *For any integer s with $s \equiv 1 \mod |H^2(Q; A)|$ there exists a pseudo s-expansive homomorphism $\phi\colon \Gamma \to \Gamma$;*

(ii) *For any integer s with $s \equiv 1 \mod |H^2(Q;A)|$ the exact sequence $1 \to A_s \to \Gamma_s \xrightarrow{\mathrm{pr}_s} Q \to 1$ of* (4.6) *splits.*

Proof. (i) Since A is abelian, isomorphism classes of extensions with A as normal subgroup and Q as quotient are in one to one correspondence with elements in $H^2(Q;A)$ (see [14, Theorem 3.12 in Chapter IV on page 93]). Let Θ be the class associated to the extension (4.3). Since by assumption $s \equiv 1 \mod |H^2(Q;A)|$, the homomorphism

$$H^2(Q; s \cdot \mathrm{id}_A) = s \cdot \mathrm{id}_{H^2(Q;A)} \colon H^2(Q;A) \to H^2(Q;A)$$

is the identity and sends Θ to Θ, and the claim follows.

(ii) Let $\phi\colon \Gamma \to \Gamma$ be a pseudo s-expansive map. The composite $p_s \circ \phi\colon \Gamma \to \Gamma_s$ sends A to the trivial group and hence factorizes through $\mathrm{pr}\colon \Gamma \to Q$ to a homomorphism $\overline{\phi}\colon Q \to \Gamma_s$ whose composite with $\mathrm{pr}_s\colon \Gamma_s \to Q$ is the identity. □

The group Q is virtually cyclic. Hence we can choose a normal infinite cyclic subgroup $C \subseteq Q$. Fix an integer s satisfying $s \equiv 1 \mod |H^2(Q;A)|$ and a positive integer r such that the order of $\mathrm{aut}(A_s) = \mathrm{GL}_n(\mathbb{Z}/s)$ divides r. Put

$$Q_r = Q/rC.$$

Let

$$\rho_s\colon Q \to \mathrm{aut}(A_s)$$

be the group homomorphism given by the conjugation action associated to the exact sequence (4.6). It factorizes through the projection $Q \to Q_r$ to a homomorphism

$$\rho_{r,s}\colon Q_r \to \mathrm{aut}(A_s).$$

By Lemma 4.8 (ii) we can choose a splitting

$$\sigma\colon Q \to \Gamma_s$$

of the projection $\mathrm{pr}_s\colon \Gamma_s \to Q$. It yields an explicit isomorphism $\Gamma_s \xrightarrow{\cong} A_s \rtimes_{\rho_s} Q$. Its composition with the group homomorphism $A_s \rtimes_{\rho_s} Q \to A_s \rtimes_{\rho_{r,s}} Q_r$, which comes from the identity on A_s and the projection $Q \to Q_r$, is denoted by

$$q_{r,s}\colon \Gamma_s \to A_s \rtimes_{\rho_{r,s}} Q_r.$$

We obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_s & \longrightarrow & \Gamma_s & \xrightarrow{\mathrm{pr}_s} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle q_{r,s}} & & \downarrow & & \\ 1 & \longrightarrow & A_s & \xrightarrow{\overline{i_s}} & A_s \rtimes_{\rho_{r,s}} Q_r & \longrightarrow & Q_r & \longrightarrow & 1 \end{array}$$

where the upper exact sequence is the one of (4.6), the lower exact sequence is the obvious one associated to a split extension, and the right vertical arrow is the canonical projection $Q \to Q_r$.

Define an epimorphism of groups by the composite

(4.9) $$\alpha_{r,s}\colon \Gamma \xrightarrow{p_s} \Gamma_s \xrightarrow{q_{r,s}} A_s \rtimes_{\rho_{r,s}} Q_r.$$

It will play the role of the map $\alpha_{R,\epsilon}$ appearing in Definition 2.14.

4.4. **Hyperelementary subgroups and index estimates.** This subsection is devoted to the proof of the following proposition. Recall that $\mathrm{pr}\colon \Gamma \to Q$ and $\pi\colon Q \to D$ are the canonical projections and that we consider A as a $\mathbb{Z}Q$-module by the conjugation action coming from the exact sequence (4.3).

Proposition 4.10. *Let Γ be an irreducible special affine group. Consider any natural number τ. Then we can find natural numbers s and r with the following properties:*

(i) $s \equiv 1 \mod |H^2(Q;A)|$;
(ii) *The order of* $\mathrm{aut}(A_s)$ *divides* r;
(iii) *For every hyperelementary subgroup $H \subseteq A_s \rtimes_{\rho_{r,s}} Q_r$ one of the following two statements is true if $\overline{H}$ is the preimage of H under the epimorphism $\alpha_{r,s}\colon \Gamma \to A_s \rtimes_{\rho_{r,s}} Q_r$:*
  (a) *The homology groups $H^1(\mathrm{pr}(\overline{H});A)$ and $H^2(\mathrm{pr}(\overline{H});A)$ are finite and there exists a natural number k satisfying*

$$\begin{aligned} &k \text{ divides } s;\\ &k \equiv 1 \mod |H^2(Q;A)|;\\ &k \equiv 1 \mod \left|H^1\big(\mathrm{pr}(\overline{H});A\big)\right|;\\ &k \equiv 1 \mod \left|H^2\big(\mathrm{pr}(\overline{H});A\big)\right|;\\ &k \geq \tau;\\ &\overline{H} \cap A \subseteq kA;\end{aligned}$$

  (b) $\big[D : \pi \circ \mathrm{pr}(\overline{H})\big] \geq \tau$.

It will provide us with the necessary index estimates when we later show that Γ is a Farrell-Hsiang group in the sense of Definition 2.14.

In order to prove Proposition 4.10 we first reduce from special affine groups of rank $(n+1)$ to the special case of semidirect products $\mathbb{Z}^n \rtimes_\phi \mathbb{Z}$. Notice that every special affine group of rank $(n+1)$ contains a subgroup of finite index which is isomorphic to $\mathbb{Z}^n \rtimes_\phi \mathbb{Z}$.

Definition 4.11. Consider $M \in \mathrm{GL}_n(\mathbb{Z})$. We will say that M is *hyper-good* if the following holds: Given natural numbers o and ν, there are natural numbers s and r satisfying

(i) $s \equiv 1 \mod o$;
(ii) The order of $\mathrm{GL}_n(\mathbb{Z}/s)$ divides r. In particular we can consider the group $(\mathbb{Z}/s)^n \rtimes_{M_s} \mathbb{Z}/r$, where M_s is the reduction of M modulo s. (We will consider $(\mathbb{Z}/s)^n$ as a subgroup of this group and denote by $\mathrm{pr}_{r,s}$ the canonical projection from this group to $\mathbb{Z}/r$.)
(iii) If H is a hyperelementary subgroup of $(\mathbb{Z}/s)^n \rtimes_{M_s} \mathbb{Z}/r$, then at least one of the following two statements is true:
  (a) There exists a natural number k satisfying

$$\begin{aligned} &k \text{ divides } s;\\ &k \equiv 1 \mod o;\\ &k \geq \nu;\\ &H \cap (\mathbb{Z}/s)^n \subseteq k(\mathbb{Z}/s)^n;\end{aligned}$$

  (b) $\big[\mathbb{Z}/r : \mathrm{pr}_{r,s}(H)\big] \geq \nu$.

In the sequel we will use the following elementary facts about indices of subgroups. Given a group G with two subgroups G_0 and G_1 of finite index, we get

$$[G_0 : (G_0 \cap G_1)] \leq [G : G_1].$$

If $f\colon G \to G'$ is an epimorphism with finite kernel K and $G_0 \subseteq G$ is a subgroup, then

$$[G' : f(G_0)] \leq [G : G_0] \leq [G' : f(G_0)] \cdot |K|.$$

Lemma 4.12. *In order to prove Proposition* 4.10 *it suffices to show that any matrix* $M \in \mathrm{GL}_n(\mathbb{Z})$ *is hyper-good.*

Proof. Recall that we have already chosen a normal infinite cyclic subgroup $C \subseteq Q$. The index $[Q : C]$ is finite. Let $\rho\colon Q \to \mathrm{aut}(A)$ be the conjugation action associated to the exact sequence $1 \to A \to \Gamma \xrightarrow{\mathrm{pr}} Q \to 1$ introduced in (4.3). Fix a generator t of C. Let

$$\eta\colon A \to A$$

be the automorphism given by $\rho(t)$. Put

$$\widehat{\Gamma} := \mathrm{pr}^{-1}(C).$$

Then $\widehat{\Gamma}$ is a normal subgroup in Γ of finite index $[\Gamma : \widehat{\Gamma}] = [Q : C]$ and fits into an exact sequence

$$1 \to A \to \widehat{\Gamma} \xrightarrow{\widehat{\mathrm{pr}}} C \to 1,$$

where $\widehat{\mathrm{pr}}$ is the restriction of pr to $\widehat{\Gamma}$.

Let τ be any natural number. Let $\Gamma' \subseteq \Gamma$ be a subgroup of finite index. The exact sequence $1 \to \Delta \to \Gamma \xrightarrow{z} D \to 1$ appearing in Definition 4.1 yields an exact sequence

$$1 \to \Gamma' \cap \Delta \to \Gamma' \to z(\Gamma') \to 1$$

for subgroups $\Gamma' \cap \Delta \subseteq \Delta$ and $z(\Gamma') \subseteq D$ of finite index. Hence Γ' is again an special affine group. We conclude from Lemma 4.5 (i) that Γ' is irreducible. The exact sequence (4.3) yields the exact sequence

$$1 \to A' := A \cap \Gamma' \to \Gamma' \to Q' := \mathrm{pr}(\Gamma') \to 1$$

for subgroups $A' \subseteq A$ and $Q' \subseteq Q$ of finite index. Since $A' = A_{\Gamma' \cap \Delta}$ this is just the version of (4.3) for Γ'. Hence $H^1(Q'; A')$ and $H^2(Q'; A')$ are finite by Lemma 4.5 (ii) and (iii). The obvious sequence of abelian groups $1 \to A' \to A \to A/A' \to 0$ is an exact sequence of $\mathbb{Z}Q'$-modules. It yields the long exact Bockstein sequence

$$\begin{aligned} \cdots \to H^1(Q'; A') \to H^1(Q'; A) \to H^1(Q'; A/A') \to H^2(Q'; A') \\ \to H^2(Q'; A) \to H^2(Q'; A/A') \to \cdots . \end{aligned}$$

Since A/A' is finite, $H^1(Q'; A/A')$ and $H^2(Q'; A/A')$ are finite. Since $H^1(Q'; A')$ and $H^2(Q'; A')$ are finite, we conclude that $H^1(Q'; A)$ and $H^2(Q'; A)$ are finite.

In particular we get for any subgroup $Q' \subseteq Q$ of finite index that $H^1(Q'; A)$ and $H^2(Q'; A)$ are finite; just apply the argument above to the special case, where Γ' is the preimage of Q' under $\mathrm{pr}\colon \Gamma \to Q$.

Let I be the set of subgroups Q' of Q of finite index such that $[D : \pi(Q')] < \tau$. We conclude for $Q' \in I$

$$[Q : Q'] \leq |F_\Delta| \cdot [D : \pi(Q')] \leq |F_\Delta| \cdot \tau$$

from the exact sequence (4.4). Since Q contains only finitely many subgroups of finite index bounded by $|F_\Delta| \cdot \tau$, the set I is finite. Apply the assumption hypergood to the matrix M describing the automorphism η after identifying $A = \mathbb{Z}^n$ for the constants

$$\nu = \tau \cdot |F_\Delta|; \tag{4.13}$$

$$o = \prod_{Q' \in I} \left(\left| H^1(Q';A) \right| \cdot \left| H^2(Q';A) \right| \right). \tag{4.14}$$

Let r, s, and k be the resulting natural numbers.

Recall that we have chosen a splitting $\sigma\colon Q \to \Gamma_s$ of the projection $\mathrm{pr}_s\colon \Gamma_s \to Q$. Let $\gamma \in \Gamma$ be any element which is mapped under $p_s\colon \Gamma \to \Gamma_s$ to $\sigma(t)$. Conjugation with γ induces on A just the automorphism $\eta\colon A \to A$ since $\mathrm{pr}\colon \Gamma \to Q$ maps γ to t. The choice of γ yields an explicit identification

$$\widehat{\Gamma} = A \rtimes_\eta C.$$

Put

$$C_r = C/rC.$$

The epimorphism $\alpha_{r,s} := q_{r,s} \circ p_s\colon \Gamma \to A_s \rtimes_{\rho_{r,s}} Q_r$ restricted to $\widehat{\Gamma}$ is the composite of the inclusion $A_s \rtimes_{\rho_{r,s}|_{C_r}} C_r \to A_s \rtimes_{\rho_{r,s}} Q_r$ with the obvious projection $\widehat{\alpha_{r,s}}\colon A \rtimes_\eta C \to A_s \rtimes_{\eta_s} C_r$, where $\eta_s\colon A_s \to A_s$ is the automorphism induced by $\eta\colon A \to A$.

Consider any hyperelementary subgroup $H \subseteq A_s \rtimes_{\rho_{r,s}} Q_r$. Put

$$\widehat{H} := H \cap (A_s \rtimes_{\eta_s} C_r).$$

This is a hyperelementary subgroup of $A_s \rtimes_{\eta_s} C_r$ and we get

$$\begin{aligned}
\alpha_{r,s}^{-1}(H) \cap \widehat{\Gamma} &= \widehat{\alpha_{r,s}}^{-1}(\widehat{H});\\
\alpha_{r,s}^{-1}(H) \cap A &= \widehat{\alpha_{r,s}}^{-1}(\widehat{H}) \cap A;\\
\mathrm{pr}\big(\alpha_{r,s}^{-1}(H)\big) \cap C &= \widehat{\mathrm{pr}}\big(\widehat{\alpha_{r,s}}^{-1}(\widehat{H})\big).
\end{aligned}$$

Since the kernel of the epimorphism $\pi\colon Q \to D$ is the finite group F_Δ, we get

$$\begin{aligned}
\big[D : \pi \circ \mathrm{pr}\big(\alpha_{r,s}^{-1}(H)\big)\big] &\geq \frac{\big[Q : \mathrm{pr}\big(\alpha_{r,s}^{-1}(H)\big)\big]}{|F_\Delta|};\\
\big[Q : \mathrm{pr}\big(\alpha_{r,s}^{-1}(H)\big)\big] &\geq \Big[C : \widehat{\mathrm{pr}}\big(\widehat{\alpha_{r,s}}^{-1}(\widehat{H})\big)\Big].
\end{aligned}$$

This implies

$$\begin{aligned}
\alpha_{r,s}^{-1}(H) \cap A \subseteq kA &\iff \widehat{\alpha_{r,s}}^{-1}(\widehat{H}) \cap A \subseteq kA;\\
\big[D : \pi \circ \mathrm{pr}\big((\alpha_{r,s}^{-1}(H)\big)\big] &\geq \frac{\Big[C : \widehat{\mathrm{pr}}\big(\widehat{\alpha_{r,s}}^{-1}(\widehat{H})\big)\Big]}{|F_\Delta|}.
\end{aligned}$$

Since the projection $C \to C_r$ maps $\widehat{\mathrm{pr}}\big(\widehat{\alpha_{r,s}}^{-1}(\widehat{H})\big)$ to $\mathrm{pr}_{r,s}(\widehat{H})$, we get

$$\begin{aligned}
\widehat{\alpha_{r,s}}^{-1}(\widehat{H}) \cap A \subseteq kA &\iff \widehat{H} \cap A_s \subseteq kA_s;\\
\Big[C : \widehat{\mathrm{pr}}\big(\widehat{\alpha_{r,s}}^{-1}(\widehat{H})\big)\Big] &\geq \Big[C_r : \mathrm{pr}_{r,s}(\widehat{H})\Big],
\end{aligned}$$

where $\mathrm{pr}_{r,s}\colon A_s \rtimes_\eta C_r \to C_r$ is the canonical projection. We conclude

$$\alpha_{r,s}^{-1}(H) \cap A \subseteq kA \Longleftrightarrow \widehat{H} \cap A_s \subseteq kA_s; \tag{4.15}$$

$$\big[D : \pi \circ \mathrm{pr}\big((\alpha_{r,s}^{-1}(H)\big)\big] \geq \frac{\Big[C_r : \mathrm{pr}_{r,s}(\widehat{H})\Big]}{|F_\Delta|}. \tag{4.16}$$

Now we can show that one of two conditions appearing in assertion (iii) of Proposition 4.10 holds with respect to the number k.

Suppose that $\Big[C_r : \mathrm{pr}_{r,s}(\widehat{H})\Big] \geq \nu$. Then $\big[D : \pi \circ \mathrm{pr}\big((\alpha_{r,s}^{-1}(H)\big)\big] \geq \tau$ by (4.16) and our choice of ν in (4.13). Hence condition (iii)b appearing in Proposition 4.10 is true. Hence it remains to show that condition (iii)a in Proposition 4.10 holds provided that $\big[D : \pi \circ \mathrm{pr}\big((\alpha_{r,s}^{-1}(H)\big)\big] < \tau$ holds. This implies $\Big[C_r : \mathrm{pr}_{r,s}(\widehat{H})\Big] < \nu$. Recall that the number k satisfies

$$\begin{aligned}&k \text{ divides } s;\\ &k \equiv 1 \mod o;\\ &k \geq \tau;\\ &\widehat{H} \cap A_s \subseteq kA_s.\end{aligned}$$

The group $\overline{H} := \alpha_{r,s}^{-1}(H)$ has the property that $\mathrm{pr}(\overline{H})$ belongs to the set I appearing in the definition of o in (4.14). Now condition (iii)a appearing in Proposition 4.10 follows from our choice of o in (4.14) and from (4.15). This finishes the proof of Proposition 4.12. ☐

Next we reduce the problem from hyperelementary groups to cyclic subgroups.

Definition 4.17. Let $M \in \mathrm{GL}_n(\mathbb{Z})$. We will say that M is *cyclic-good* if the following holds: Given positive integers o and ν, there are prime numbers p_1 and p_2 such that the following holds.

Set

$$s := p_1 p_2 \quad \text{and} \quad r := s \cdot |\,\mathrm{GL}_n(\mathbb{Z}/s)|.$$

In particular we can consider the group $(\mathbb{Z}/s)^n \rtimes_{M_s} \mathbb{Z}/r$, where M_s is the reduction of M modulo s. (We will consider $(\mathbb{Z}/s)^n$ as a subgroup of this group and denote by $\mathrm{pr}_{r,s}$ the canonical projection from this group to $\mathbb{Z}/r$.) We require that

(i)
$$\begin{aligned}&p_1 \neq p_2;\\ &p_i \equiv 1 \mod o \qquad \text{for } i = 1,2;\\ &p_i \geq \nu \quad \text{for } i = 1,2;\end{aligned}$$

(ii) If C is a cyclic subgroup of $(\mathbb{Z}/s)^n \rtimes_{M_s} \mathbb{Z}/r$, then *at least one* of the following two statements is true:
  (a) $C \cap (\mathbb{Z}/s)^n = \{0\}$;
  (b) There is $i \in \{1,2\}$ such that p_i divides both $|C|$ and $[\mathbb{Z}/r : \mathrm{pr}_{r,s}(C)]$.

Lemma 4.18. *Assume that $M \in \mathrm{GL}_n(\mathbb{Z})$ is cyclic-good. Then M is hyper-good.*

Proof. Suppose that $M \in \mathrm{GL}_n(\mathbb{Z})$ is cyclic-good. We want to show that it is hyper-good. Let $\nu > 0$ be given. Pick p_1 and p_2 and put $s = p_1p_2$ and $r = s \cdot |\,\mathrm{GL}_n(\mathbb{Z}/s)|$ as in Definition 4.17. Obviously conditions (i) and (ii) appearing in Definition 4.11 are satisfied for s and r. It remains to show that condition (ii) appearing in Definition 4.17 implies condition (iii) appearing in Definition 4.11.

Let H be a hyperelementary subgroup of $(\mathbb{Z}/s)^n \rtimes_{M_s} \mathbb{Z}/r$. There is an exact sequence $1 \to C \to H \xrightarrow{f} L \to 1$ where C is a cyclic group and L is an l-group for a prime l not dividing the order of C. It follows that $[(\mathbb{Z}/s)^n \cap H : (\mathbb{Z}/s)^n \cap C]$ and $[\mathrm{pr}_{r,s}(H) : \mathrm{pr}_{r,s}(C)]$ are both l-powers since $\big((\mathbb{Z}/s)^n \cap H\big)/\big((\mathbb{Z}/s)^n \cap C\big)$ is a subgroup of L and $\mathrm{pr}_{r,s}(H)/\mathrm{pr}_{r,s}(C)$ is a quotient of L.

Suppose that condition (ii)a appearing in Definition 4.17 is satisfied, i.e., $C \cap (\mathbb{Z}/s)^n = \{0\}$. Then $H \cap (\mathbb{Z}/s)^n$ is an l-group. If $H \cap (\mathbb{Z}/s)^n$ is trivial, condition (iii)a appearing in Definition 4.11 is obviously satisfied for $k = s$. Suppose that $H \cap (\mathbb{Z}/s)^n$ is non-trivial. Since $s = p_1 p_2$, the prime l must be p_1 or p_2. Let k be p_1 if $l = p_2$, and be p_2 if $l = p_1$. Then $H \cap (\mathbb{Z}/s)^n \subseteq k \cdot (\mathbb{Z}/s)^n$, i.e., condition (iii)a appearing in Definition 4.11 is satisfied.

Suppose that condition (ii)b appearing in Definition 4.17 is satisfied, i.e., for some $i \in \{1,2\}$ the prime p_i divides both $|C|$ and $[\mathbb{Z}/r : \mathrm{pr}_{r,s}(C)]$. We have $p \geq \nu$. Since l does not divide $|C|$, p_i must be different from l. Since $[\mathrm{pr}_{r,s}(H) : \mathrm{pr}_{r,s}(C)]$ is a power of l, the prime p_i divides $[\mathbb{Z}/r : \mathrm{pr}_{r,s}(H)]$. This implies $\nu \leq p_i \leq [\mathbb{Z}/r : \mathrm{pr}_{r,s}(H)]$. Hence condition (iii)b appearing in Definition 4.11 is satisfied. □

Finally we show that every element in $\mathrm{GL}_n(\mathbb{Z})$ is cyclic-good.

Lemma 4.19. *Let $M \in \mathrm{GL}_n(\mathbb{Z})$. Let s be any natural number. Let r be a multiple of the order of the reduction $M_s \in \mathrm{GL}_n(\mathbb{Z}/s)$ of M. Let $t \in \mathbb{Z}/r$ be the generator.*

Then for any $v \in (\mathbb{Z}/s)^n$ and $r', s', j \in \mathbb{Z}$ and j we have

$$(vt^j)^{s'r'} = t^{js'r'} \in (\mathbb{Z}/s)^n \rtimes_{M_s} \mathbb{Z}/r$$

provided that $s'v = 0 \in (\mathbb{Z}/s)^n$ and $M_s^{jr'} = \mathrm{id} \in \mathrm{GL}_n(\mathbb{Z}/s)$.

Proof. We have

$$(vt^j)^{s'r'} = \left(\sum_{i=0}^{s'r'-1} (M_s^j)^i v \right) t^{js'r'}$$

and

$$\sum_{i=0}^{s'r'-1} (M_s^j)^i v = \sum_{k=0}^{s'-1} \sum_{l=0}^{r'-1} (M_s^j)^{l+kr'} v = \sum_{k=0}^{s'-1} \sum_{l=0}^{r'-1} (M_s^j)^l v$$
$$= s' \sum_{l=0}^{r'-1} (M_s^j)^l v = \sum_{l=0}^{r'-1} (M_s^j)^l s' v = 0. \qquad \square$$

Lemma 4.20. *Let $M \in \mathrm{GL}_n(\mathbb{Z})$. Let s be any natural number. Let r' be a multiple of the order of $M_s \in \mathrm{GL}_n(\mathbb{Z}/s)$. Let $r := r's$. Let C be a cyclic subgroup of $(\mathbb{Z}/s)^n \rtimes_{M_s} \mathbb{Z}/r$ that has a non-trivial intersection with $(\mathbb{Z}/s)^n$.*

Then there is a prime power p^N $(N \geq 1)$ such that

(i) *p^N divides $r = r's$;*
(ii) *p^N does not divide $|\mathrm{pr}_{r,s}(C)|$;*
(iii) *p divides $|C \cap (\mathbb{Z}/s)^n|$.*

Proof. Let $t \in \mathbb{Z}/r$ be the generator. Let vt^j be a generator of C. Clearly $v \neq 0$ and $w := (vt^j)^{|\mathrm{pr}_{r,s}(C)|}$ is a non-trivial element of $C \cap (\mathbb{Z}/s)^n$ (otherwise $C \cap (\mathbb{Z}/s)^n$ would be trivial). Let s' be the order of $w \in (\mathbb{Z}/s)^n$. Lemma 4.19 implies that $(vt^j)^{sr'}$ is a power of t. If K is any integer with $(K, s') = 1$, then $Kw = (vt^j)^{|\mathrm{pr}_{r,s}(C)| \cdot K} \neq$

$0 \in C \cap (\mathbb{Z}/s)^n$ and hence Kw is not a power of t. Using Lemma 4.19 again, this implies that $s'r'$ does not divide $|\operatorname{pr}_{r,s}(C)| \cdot K$ for any integer K with $(K, s') = 1$. Therefore there is a prime p dividing s' and a number $N \geq 1$ such that p^N divides $s'r'$, but not $|\operatorname{pr}_{r,s}(C)|$. Clearly s' divides $|C \cap (\mathbb{Z}/s)^n|$. Thus p divides $|C \cap (\mathbb{Z}/s)^n|$. Because s' divides s, p^N divides $r = r's$. □

Lemma 4.21. *All $M \in \mathrm{GL}_n(\mathbb{Z})$ are cyclic-good.*

Proof. Let o and ν be any positive integers. By Dirichlet's Theorem (see [37, Lemma 3 in III.2.2 on page 25]) there exists infinitely many primes which are congruent 1 modulo o. Hence we can find primes p_1 and p_2 satisfying condition (i) appearing in Definition 4.17. It remains to show that condition (ii) appearing in Definition 4.17 holds.

Let C be a cyclic subgroup of $(\mathbb{Z}/s)^n \rtimes_{M_s} \mathbb{Z}/r$. We have to show condition (ii)b appearing in Definition 4.17 holds, provided that $C \cap (\mathbb{Z}/s)^n \neq 0$. We can apply Lemma 4.20 with $r' = |\mathrm{GL}_n(\mathbb{Z}/s)|$. Thus there is a prime p and a number N such that

(i) p^N divides $r = |\mathrm{GL}_n(\mathbb{Z}/s)| \cdot s$;
(ii) p^N does not divide $|\operatorname{pr}_{r,s}(C)|$;
(iii) p divides $|C \cap (\mathbb{Z}/s)^n|$.

We deduce from (i) and (ii) that p divides $[\mathbb{Z}/r : \operatorname{pr}_{r,s}(C)]$. We deduce from (iii) that p divides $|C|$ and s. Because $s = p_1 \cdot p_2$ it follows that p is either p_1 or p_2 Therefore condition (ii)b appearing in Definition 4.17 holds. □

Now Proposition 4.10 follows from Lemma 4.12, Lemma 4.18 and Lemma 4.21.

4.5. **Contracting maps for irreducible special affine groups.**

Proposition 4.22. *Let Γ be an irreducible special affine group. Fix a finite set of generators of Γ and let d_Γ be the associated word metric on Γ. Then there is a natural number N and such that for any given real numbers $R > 0$ and $\epsilon > 0$ there exists a sequence of real numbers $(\xi_m)_{m\geq 1}$ and a natural number μ such that the following is true:*

(i) *Let $\overline{H} \subseteq \Gamma$ be any subgroup of finite index such that $\left|H^1\big(\operatorname{pr}(\overline{H}); A\big)\right|$ and $\left|H^2\big(\operatorname{pr}(\overline{H}); A\big)\right|$ are finite. Suppose that there exists an integer k satisfying*

$$\begin{aligned} &k \geq \xi_{[D:\pi\circ\operatorname{pr}(\overline{H})]};\\ &k \equiv 1 \mod |H^2(Q;A)|;\\ &k \equiv 1 \mod \left|H^1\big(\operatorname{pr}(\overline{H}); A\big)\right|;\\ &k \equiv 1 \mod \left|H^2\big(\operatorname{pr}(\overline{H}); A\big)\right|;\\ &A \cap \overline{H} \subseteq kA.\end{aligned}$$

Then there is a simplicial complex E of dimension $\leq N$ with a simplicial cell preserving the $\overline{H}$-action whose isotropy groups are virtually cyclic, and an $\overline{H}$-equivariant map $f\colon \Gamma \to E$ satisfying

$$d_\Gamma\big(\gamma_1, \gamma_2\big) \leq R \implies d^{l^1}\big(f(\gamma_1), f(\gamma_2)\big) \leq \epsilon$$

for $\gamma_1, \gamma_2 \in \Gamma$;

(ii) *If $\overline{H} \subseteq \Gamma$ is any subgroup such that $[D : \pi \circ \operatorname{pr}(\overline{H})] \geq \mu$, then there exists a one-dimensional simplicial complex E with a simplicial cell preserving $\overline{H}$-action such that for every $e \in E$ the isotropy group $\overline{H}_e$ satisfies $A \subseteq \overline{H}_e$*

and $[\overline{H}_e : A] < \infty$ *and is in particular a virtually finitely generated abelian subgroup of* Γ*, and an* $\overline{H}$*-equivariant map* $f\colon \Gamma \to E$ *satisfying*

$$d_\Gamma\bigl(\gamma_1,\gamma_2\bigr) \le R \implies d^{l^1}\bigl(f(\gamma_1), f(\gamma_1)\bigr) \le \epsilon$$

for $\gamma_1, \gamma_2 \in \Gamma$.

The proposition above will provide us with the necessary contracting maps when we later show that Γ is a Farrell-Hsiang group in the sense of Definition 2.14. Its proof needs some preparation.

Lemma 4.23. *Let* k *be a natural number and* $\overline{H} \subseteq \Gamma$ *be a subgroup with* $A \cap \overline{H} \subseteq kA$*. Assume that* $k \equiv 1 \mod \bigl|H^i\bigl(\mathrm{pr}(\overline{H}); A\bigr)\bigr|$ *for* $i = 1, 2$*. Let* $\phi\colon \Gamma \to \Gamma$ *be a pseudo* k*-expansive map.*

Then $\overline{H}$ *is subconjugated to* $\mathrm{im}(\phi)$.

Proof. Recall that $p_k\colon \Gamma \to \Gamma_k := \Gamma/kA$ is the canonical projection. As explained in the proof of Lemma 4.8 (ii), the composite $p_k \circ \phi\colon \Gamma \to \Gamma_k$ factorizes through the projection $\mathrm{pr}\colon \Gamma \to Q$ to a homomorphism $\overline{\phi}\colon Q \to \Gamma_k$ whose composite with the projection $\mathrm{pr}_k\colon \Gamma_k \to Q$ is the identity. Let H' be the image of $\overline{H}$ under the projection $p_k\colon \Gamma \to \Gamma_k$. The exact sequence $1 \to A_k := A/kA \to \Gamma_k \xrightarrow{\mathrm{pr}_k} Q \to 1$ yields by restriction the exact sequence

$$1 \to A_k \to \mathrm{pr}_k^{-1}\bigl(\mathrm{pr}_k(H')\bigr) \xrightarrow{\mathrm{pr}_k'} \mathrm{pr}_k(H') \to 1.$$

The section $\overline{\phi}$ of pr_k restricts to a section $\overline{\phi}'\colon\ \mathrm{pr}_k(H') \to \mathrm{pr}_k^{-1}\bigl(\mathrm{pr}_k(H')\bigr)$ of pr_k'. The restriction of pr_k' to H' yields an isomorphism $H' \to \mathrm{pr}(H')$ since $H' \cap A_k = \{1\}$. Hence its inverse defines a second section of pr_k'. Since $\mathrm{pr}(\overline{H}) = \mathrm{pr}_k(H')$, we get by assumption for $i = 1, 2$

$$k \equiv 1 \mod H^i\bigl(\mathrm{pr}_k(H'); A).$$

Hence multiplication with k induces isomorphisms on $H^i\bigl(\mathrm{pr}_k(H'); A)$ for $i = 1, 2$. The Bockstein sequence associated to the exact sequence $0 \to A \xrightarrow{k\cdot\mathrm{id}} A \to A_k \to 0$ of $\mathbb{Z}[\mathrm{pr}(H')]$-modules implies $H^1(\mathrm{pr}_k(H'); A_k) = 0$. Hence any two sections of pr_k' are conjugated (see [14, Proposition 2.3 in Chapter IV on page 89]). This implies that H' and $\mathrm{im}(\overline{\phi}')$ are conjugated in $\mathrm{pr}_k^{-1}\bigl(\mathrm{pr}_k(H')\bigr)$. Hence H' and $\mathrm{im}(\overline{\phi}')$ are conjugated in Γ_k.

In order to show that $\overline{H}$ is subconjugated to $\mathrm{im}(\phi)$ it suffices to show that $p_k^{-1}(H')$ is subconjugated to $\mathrm{im}(\phi)$ since obviously $\overline{H} \subseteq p_k^{-1}(H')$. Choose an element $\gamma \in \Gamma$ such that $p_k(\gamma)H'p_k(\gamma)^{-1} = \mathrm{im}(\overline{\phi}')$. Since $\gamma p_k^{-1}(H')\gamma^{-1} = p_k^{-1}\bigl(p_k(\gamma)H'p_k(\gamma)^{-1}\bigr)$, we can assume without loss of generality that $H' \subseteq \mathrm{im}(\overline{\phi}')$, otherwise replace H' by $p_k(\gamma)(H')p(\gamma)^{-1}$. This implies $H' \subseteq \mathrm{im}(\overline{\phi})$. It remains to show

$$p_k^{-1}(H') \subseteq \mathrm{im}(\phi).$$

Consider $\gamma_0 \in p_k^{-1}(H')$. Because of $H' \subseteq \mathrm{im}(\overline{\phi})$ we can find $\gamma_1 \in \Gamma$ such that $p_k(\gamma_0) = p_k \circ \phi(\gamma_1)$. Hence there is $a \in kA$ with $\gamma_0 = \phi(\gamma_1)\cdot a$ since $\ker(p_k) = kA$. Since ϕ induces $k \cdot \mathrm{id}$ on A, the element a lies in the image of ϕ and hence γ_0 lies in the image of ϕ. ☐

Lemma 4.24. *Let* Γ *be an irreducible special affine group. Let* $\phi\colon \Gamma \to \Gamma$ *be a pseudo* s*-expansive group homomorphism.*

Then there exists $u \in \mathbb{R}^n$ *such that the affine diffeomorphism*

$$f\colon \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n, \quad x \mapsto s \cdot x + u$$

is ϕ*-equivariant.*

Proof. Given an element $\gamma \in \Gamma$, let $M_\gamma\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear automorphism and $v_\gamma \in \mathbb{R}^n$ uniquely determined by the property that γ is the affine map $\mathbb{R}^n \to \mathbb{R}^n$ sending x to $M_\gamma(x) + v_\gamma$. One easily checks that $M_{\gamma_1\gamma_2} = M_{\gamma_1} \circ M_{\gamma_2}$ and $v_{\gamma_1\gamma_2} = v_{\gamma_1} + M_{\gamma_1}(v_{\gamma_2})$ hold for all γ_1, γ_2 in Γ and $M_a = \mathrm{id}$ and $v_a = a$ hold for $a \in A$. Consider $\gamma \in \Gamma$. Then there exists $a \in A$ such that $a \cdot \phi(\gamma) = \gamma$ holds in Γ. This implies that for all $x \in \mathbb{R}^n$ we have

$$M_{\phi(\gamma)}(x) + a + v_{\phi(\gamma)} = M_{a\cdot\phi(\gamma)}(x) + v_{a\cdot\phi(\gamma)} = M_\gamma(x) + v_\gamma.$$

We conclude

$$M_\gamma = M_{\phi(\gamma)}.$$

Consider the function

$$d\colon \Gamma \to \mathbb{R}^n, \quad \gamma \mapsto v_{\phi(\gamma)} - s \cdot v_\gamma.$$

It factorizes through the projection $\mathrm{pr}\colon \Gamma \to Q$ to a function $\overline{d}\colon Q \to \mathbb{R}^n$, since for any $a \in A$ and $\gamma \in \Gamma$ we have

$$\begin{aligned} v_{\phi(a\cdot\gamma)} - s \cdot v_{a\cdot\gamma} &= v_{\phi(a)\cdot\phi(\gamma)} - s \cdot v_{a\cdot\gamma} = \phi(a) + v_{\phi(\gamma)} - s \cdot (a + v_\gamma) \\ &= s \cdot a + v_{\phi(\gamma)} - s \cdot (a + v_\gamma) = v_{\phi(\gamma)} - s \cdot v_\gamma. \end{aligned}$$

The conjugation action of $\gamma \in \Gamma$ on A is given by M_γ by the following calculations:

$$M_{\gamma a \gamma^{-1}} = M_\gamma \circ M_a \circ M_{\gamma^{-1}} = M_\gamma \circ \mathrm{id} \circ M_{\gamma^{-1}} = M_{\gamma\gamma^{-1}} = M_1 = \mathrm{id}$$

and

$$\begin{aligned} v_{\gamma a \gamma^{-1}} &= v_\gamma + M_\gamma(v_{a\gamma^{-1}}) = v_\gamma + M_\gamma(v_a + v_{\gamma^{-1}}) = v_\gamma + M_\gamma(v_a) + M_\gamma(v_{\gamma^{-1}}) \\ &= M_\gamma(v_a) + v_\gamma + M_\gamma(v_{\gamma^{-1}}) = M_\gamma(a) + v_{\gamma\gamma^{-1}} = M_\gamma(a). \end{aligned}$$

This action extends to an action on $\mathbb{R}^n = A \otimes \mathbb{R}$ and is used in the following calculation, that shows that $\overline{d}$ is a derivation.

$$\begin{aligned} \overline{d}\big(\mathrm{pr}(\gamma_1)\,\mathrm{pr}(\gamma_2)\big) &= d(\gamma_1\gamma_2) \\ &= v_{\phi(\gamma_1\gamma_2)} - s \cdot v_{\gamma_1\gamma_2} \\ &= v_{\phi(\gamma_1)\phi(\gamma_2)} - s \cdot v_{\gamma_1\gamma_2} \\ &= v_{\phi(\gamma_1)} + M_{\phi(\gamma_1)}\big(v_{\phi(\gamma_2)}\big) - s \cdot v_{\gamma_1} - s \cdot M_{\gamma_1}\big(v_{\gamma_2}\big) \\ &= v_{\phi(\gamma_1)} - s \cdot v_{\gamma_1} + M_{\gamma_1}\big(v_{\phi(\gamma_2)}\big) - s \cdot M_{\gamma_1}\big(v_{\gamma_2}\big) \\ &= v_{\phi(\gamma_1)} - s \cdot v_{\gamma_1} + M_{\gamma_1}\big(v_{\phi(\gamma_2)} - s \cdot v_{\gamma_2}\big) \\ &= d(\gamma_1) + \gamma_1 \cdot d(\gamma_2) \\ &= \overline{d}\big(\mathrm{pr}(\gamma_1)\big) + \mathrm{pr}(\gamma_1) \cdot \overline{d}\big(\mathrm{pr}(\gamma_2)\big). \end{aligned}$$

Since $H^1(Q;\mathbb{R}^n) = H^1\big(Q; A \otimes_{\mathbb{Z}} \mathbb{R}\big) = H^1(Q;A) \otimes_{\mathbb{Z}} \mathbb{R}$ and $H^1(Q;A)$ is finite by Lemma 4.5 (iii), we conclude $H^1(Q;\mathbb{R}^n) = 0$. The description of the cocycles as derivations and coboundaries as principal derivations (see [14, Exercise 2 in III.1 on page 60]) implies that there exists $u \in \mathbb{R}^n$ such that for all $\gamma \in \Gamma$

$$u - M_\gamma(u) = v_{\phi(\gamma)} - s \cdot v_\gamma$$

holds. Hence the affine map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ sending x to $sx+u$ is ϕ-linear by the following calculation:

$$\begin{aligned}\phi(\gamma)\cdot f(x) &= M_{\phi(\gamma)}(f(x)) + v_{\phi(\gamma)} \\ &= M_\gamma(s\cdot x+u) + v_{\phi(\gamma)} \\ &= M_\gamma(s\cdot x) + M_\gamma(u) + v_{\phi(\gamma)} \\ &= s\cdot M_\gamma(x) + s\cdot v_\gamma + u \\ &= f\big(M_\gamma(x)+v_\gamma\big) \\ &= f(\gamma\cdot x).\end{aligned}$$

□

Lemma 4.25. *Let N be a natural number and $\epsilon > 0$. Then there exists a number D_N depending only on N such that the following holds:*

Let X be a simplicial complex of dimension $\leq N$ and let X' be its barycentric subdivision. Then we get for every $x, y \in X$

$$d^{l^1}_X(x,y) \leq D_N \cdot d^{l^1}_{X'}(x,y),$$

where $d^{l^1}_X$ and $d^{l^1}_{X'}$ denote the l^1-metric on X and X'

Proof. If X is the standard $(2N+1)$-simplex Δ_{2N+1}, a direct inspection shows the existence of a number D_N such that for every $x, y \in \Delta_{2N+1}$ we have

$$d^{l^1}_{\Delta_{2N+1}}(x,y) < D_N \cdot d^{l^1}_{(\Delta_{2N+1})'}(x,y).$$

Now consider $x, y \in X$. There is a subcomplex $Y \subseteq X$ with at most $2(\dim(X)+2)$ vertices containing these four points. We can identify Y with a simplicial subcomplex of Δ_{2N+1}. Now the claim follows for the number D_N above since the l^1-metric is preserved under inclusions of simplicial subcomplexes and the barycentric subdivision is compatible with inclusions of simplicial subcomplexes. □

Since Γ acts properly and cocompactly on $\mathbb{R}^n \times \mathbb{R}$, we can choose a Γ-invariant Riemannian metric b^Γ. Let d^Γ be the associated metric on $\mathbb{R}^n \times \mathbb{R}$. Notice that d^Γ is Γ-invariant, whereas the standard Euclidean metric on $\mathbb{R}^n \times \mathbb{R}$ is not necessarily Γ-invariant. We will denote by $B^\Gamma_r(x,s)$ the closed ball of radius r around the point $(x,s) \in \mathbb{R}^n \times \mathbb{R}$ with respect to the metric d^Γ. By $B^{\mathrm{euc}}_r(x)$ we denote the closed ball of radius r around $x \in \mathbb{R}^n$ with respect to the Euclidean metric.

In the sequel we fix a word metric d_Γ on Γ. The Švarc-Milnor Lemma (see [13, Proposition 8.19 in Chapter I.8 on page 140]) implies

Lemma 4.26. *Let* $\mathrm{ev}\colon \Gamma \to \mathbb{R}^n \times \mathbb{R}$ *be the map given by evaluating the Γ-action on the origin. There exists positive real numbers C_1 and C_2 such that for $\gamma_2, \gamma_2 \in \Gamma$*

$$d^\Gamma\big(\mathrm{ev}(\gamma_1), \mathrm{ev}(\gamma_2)\big) \leq C_1 \cdot d_\Gamma\big(\gamma_1, \gamma_2\big) + C_2.$$

Lemma 4.27. *If D is $\mathbb{Z}$, denote by t a generator of $\mathbb{Z}$ and equip D with the associated word metric d_D. If D is D_∞, consider the standard presentation $\langle s, t \mid sts = t^{-1}, s^2 = 1\rangle$ and equip D with the associated word metric d_D.*

Then there exists a constant $C_3 > 0$ such that for all $\gamma_1, \gamma_2 \in \Gamma$ we get

$$d_D\big(\pi\circ\mathrm{pr}(\gamma_1), \pi\circ\mathrm{pr}(\gamma_1)\big) \leq C_3 \cdot d_\Gamma\big(\gamma_1, \gamma_2\big).$$

Proof. The word metrics for two different sets of generators are Lipschitz equivalent. Hence it suffices to prove the claim for a particular choice of finite set of generators on Γ. Fix a set of generators of Γ such that each generator is sent under the epimorphism $\pi \circ \mathrm{pr} \colon \Gamma \to D$ to the unit element in D, to t or to s. Equip Γ with the associated word metric. Then we get for $\gamma_1, \gamma_2 \in \Gamma$

$$d_D\big(\pi \circ \mathrm{pr}(\gamma_1), \pi \circ \mathrm{pr}(\gamma_2)\big) \le d_\Gamma(\gamma_1, \gamma_2). \quad \square$$

Let $\mathcal{W}$ be an open cover of $\mathbb{R}^n \times \mathbb{R}$ which is Γ-invariant, i.e., for $W \in \mathcal{W}$ and $\gamma \in \mathcal{W}$ we have $\gamma \cdot W = \{\gamma \cdot w \mid w \in W\} \in \mathcal{W}$. Recall that points in the realization of the nerve $|\mathcal{W}|$ of the open cover $\mathcal{W}$ are formal sums $z = \sum_{W \in \mathcal{W}} z_W \cdot W$, with $z_W \in [0,1]$ such that $\sum_{W \in \mathcal{W}} z_W = 1$ and the intersection of all the W with $z_W \neq 0$ is non-empty, i.e., $\{W \mid z_W \neq 0\}$ is a simplex in the nerve of $\mathcal{W}$. There is a map

$$\beta^{\mathcal{W}} \colon \mathbb{R}^n \times \mathbb{R} \to |\mathcal{W}|, \quad x \mapsto \sum_{W \in \mathcal{W}} (\beta^{\mathcal{W}})_W(x) \cdot W, \tag{4.28}$$

where

$$(\beta^{\mathcal{W}})_W(x) = \frac{a_W(x)}{\sum_{W \in \mathcal{W}} a_W(x)}$$

if we define

$$a_W(x) := d^\Gamma\big(x, (\mathbb{R}^n \times \mathbb{R}) \setminus W\big) = \inf\{d^\Gamma(x, w) \mid w \notin W\}.$$

Since $\mathcal{W}$ is Γ-invariant, the Γ-action on $\mathcal{W}$ induces a simplicial Γ-action on $|\mathcal{W}|$. Since d^Γ is Γ-invariant, the map $\beta^{\mathcal{W}}$ is Γ-equivariant. Let $d^{l^1}_{|\mathcal{W}|}$ be the l^1-metric on $|\mathcal{W}|$.

Lemma 4.29. *Consider a natural number N and a real number $\omega > 0$. Suppose that for every $(x,s) \in \mathbb{R}^n \times \mathbb{R}$ there exists $W \in \mathcal{W}$ such that $B^\Gamma_\omega(x,s)$ lies in W. Suppose that the dimension of $\mathcal{W}$ is less or equal to N.*

Then we get for $(x,s), (y,t) \in \mathbb{R}^n \times \mathbb{R}$ with $d^\Gamma\big((x,s),(y,t)\big) \le \frac{\omega}{8N}$

$$d^{l^1}_{|\mathcal{W}|}\big(\beta^{\mathcal{W}}(x,s), \beta^{\mathcal{W}}(y,t)\big) \le \frac{64 \cdot N^2}{\omega} \cdot d^\Gamma\big((x,s),(y,t)\big).$$

Proof. This follows from [9, Proposition 5.3]. $\square$

Lemma 4.30. *Consider a real number $\omega > 0$ and a compact subset $I \subseteq \mathbb{R}$. Then there are positive real numbers σ and α such that for all $x \in \mathbb{R}^n$ and $s \in I$*

$$B^\Gamma_\omega(x,s) \subseteq B^{\mathrm{euc}}_\sigma(x) \times [s - \alpha/2, s + \alpha/2].$$

Proof. Choose a compact subset $K \subseteq \mathbb{R}^n$ such that $\Delta \cdot K = \mathbb{R}^n$. Since $B^\Gamma_\omega\big(K \times I\big)$ is a compact subset of $\mathbb{R}^n \times \mathbb{R}$, we can find $\sigma_0 > 0$ and $\alpha_0 > 0$ such that $B^\Gamma_\omega(K \times I) \subseteq B^{\mathrm{euc}}_{\sigma_0}(0) \times [-\alpha_0/2, \alpha_0/2]$ holds. Choose $\sigma_1 > 0$ and $\alpha_1 > 0$ such that $K \subseteq B^{\mathrm{euc}}_{\sigma_1}(0)$ and $I \subseteq [-\alpha_1/2, \alpha_1/2]$. Put $\sigma := \sigma_0 + \sigma_1$ and $\alpha = \alpha_0 + \alpha_1$. Then we get for all $(x,s) \in K \times I$ by the triangle inequality

$$B^{\mathrm{euc}}_{\sigma_0}(0) \times [-\alpha_0/2, \alpha_0/2] \subseteq B^{\mathrm{euc}}_\sigma(x) \times [s - \alpha/2, s + \alpha/2].$$

Hence we get for all $(x,s) \in K \times I$

$$B^\Gamma_\omega(x,s) \subseteq B^{\mathrm{euc}}_\sigma(x) \times [s - \alpha/2, s + \alpha/2]\big).$$

Since $\Delta \cdot K = \mathbb{R}^n$ and Δ acts isometrically with respect to both d^{euc} and d^Γ, Lemma 4.30 follows. $\square$

Consider the following setup. Let $\mathrm{pr}\colon \Gamma \to Q$ and $\pi\colon Q \to D$ be the canonical projections appearing in (4.3) and (4.4). Choose an element $\sigma \in \Gamma$ such that the action $\rho''\colon D \times \mathbb{R} \to \mathbb{R}$ of $\pi(\sigma)$ is given by the map $\mathbb{R} \to \mathbb{R}$, $t \mapsto t+1$. Put $\Gamma_0 := \mathrm{pr}^{-1}(C)$, where $C := \langle \mathrm{pr}(\sigma)\rangle$ is the infinite cyclic subgroup of Q generated by $\mathrm{pr}(\sigma)$. We obtain an exact sequence

$$1 \to A \to \Gamma_0 \xrightarrow{\mathrm{pr}|_{\Gamma_0}} C \to 1.$$

Obviously σ lies in Γ_0 and is mapped under $\mathrm{pr}|_{\Gamma_0}\colon \Gamma_0 \to C$ to a generator of C. The subgroup Γ_0 of Γ has finite index.

Consider on $\mathbb{R}^n \times \mathbb{R}$ the flow $\Phi_\tau(x,t) = (x, t+\tau)$. Fix an integer $l \geq 1$. Let $\mathcal{L}_{\leq l}$ be the set of non-constant orbits under Φ whose Γ_0-period is bounded by l, i.e., there exists $(x,s) \in \mathbb{R}^n \times \mathbb{R}$ which lies in the orbit, an element $\gamma \in \Gamma_0$ and an element $\tau \in \mathbb{R}$ such that $\gamma \cdot (x,s) = \Phi_\tau(x,s)$ and $0 < \tau \leq l$. The Γ_0-action on $\mathbb{R}^n \times \mathbb{R}$ induces a Γ-action on $\mathcal{L}_{\leq l}$. We want to prove

Lemma 4.31. *The set $\mathcal{L}_{\leq l}/\Gamma_0$ is finite.*

Proof. For an integer $k \geq 1$ put

$$\mathcal{L}'_k := \bigl\{x \in \mathbb{R}^n \bigm| \exists\, \gamma \in \Gamma_0 \text{ with } \gamma \cdot x = x \text{ and } \mathrm{pr}(\gamma) = \mathrm{pr}(\sigma)^k\bigr\}.$$

We obtain a surjection

$$\coprod_{k=1}^{l} \mathcal{L}'_k \twoheadrightarrow \mathcal{L}_{\leq l}$$

by sending $x \in \mathcal{L}'_k$ to the orbit through $(x,0) \in \mathbb{R}^n \times \mathbb{R}$. This surjection is compatible with the obvious A-action on the source and the restriction of the Γ_0-action on the target to A. Hence it suffices to show that $\mathcal{L}'_k/A$ is finite for $k = 1,2,3\ldots$.

An element $\gamma \in \Gamma$ satisfies $\mathrm{pr}(\gamma) = \mathrm{pr}(\sigma)^k$ if and only if there exists $a \in A$ with $\gamma = a\sigma^k$. Let $\phi\colon A \to A$ be the $\mathbb{Z}$-automorphism given by conjugation with σ^k. We will consider A as a subgroup of $\mathbb{R}^n$ so that the action of a on $\mathbb{R}^n$ is given by $x \mapsto x + a$. Let $\alpha\colon \mathbb{R} \otimes_{\mathbb{Z}} A \xrightarrow{\cong} \mathbb{R}^n$ be the $\mathbb{R}$-isomorphisms which sends $\lambda \otimes a$ to $\lambda \cdot a$. Let $\phi_{\mathbb{R}}\colon \mathbb{R}^n \to \mathbb{R}^n$ be the $\mathbb{R}$-automorphism for which the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} \otimes_{\mathbb{Z}} A & \xrightarrow{\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} \phi} & \mathbb{R} \otimes_{\mathbb{Z}} A \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ \mathbb{R}^n & \xrightarrow{\phi_{\mathbb{R}}} & \mathbb{R}^n \end{array}$$

Notice that the element σ^k acts on $\mathbb{R}^n \to \mathbb{R}^n$ by some affine motion. One easily checks that there is an element $w \in \mathbb{R}^n$ such that $\sigma^k \cdot x = \phi_{\mathbb{R}}(x) + w$ holds for all $x \in \mathbb{R}^n$. We conclude for $a \in A$ and $x \in \mathbb{R}^n$

$$a\sigma^k \cdot x = x \Leftrightarrow (\mathrm{id} - \phi_{\mathbb{R}})(x) - w = a.$$

Hence we get

$$\mathcal{L}'_k = \bigl\{x \in \mathbb{R}^n \bigm| (\mathrm{id} - \phi_{\mathbb{R}})(x) - w \in A\bigr\}.$$

Let Γ_1 be the preimage of $k \cdot C$ under $\mathrm{pr}\colon \Gamma \to Q$. Since Γ_1 has finite index in Γ and Γ is special affine by assumption, Γ_1 is special affine. Since Γ is irreducible by assumption, we conclude from Lemma 4.5 (i) that Γ_1 is irreducible. Lemma 4.5 (iii) implies that $H^1(k \cdot C; A) = \mathrm{coker}\bigl((\mathrm{id}_A - \phi)\colon A \to A\bigr)$ is finite. We conclude that

$(\mathrm{id} - \phi_{\mathbb{R}})\colon \mathbb{R}^n \to \mathbb{R}^n$ is bijective. Choose $v \in \mathbb{R}^n$ with $(\mathrm{id} - \phi_{\mathbb{R}})(v) = w$. Obviously A is a subgroup of $(\mathrm{id} - \phi_{\mathbb{R}})^{-1}(A)$. We obtain a bijection of A-sets

$$\mathcal{L}'_k \xrightarrow{\cong} (\mathrm{id} - \phi_{\mathbb{R}})^{-1}(A)$$

by sending x to $x + v$. Hence it remains to show that $(\mathrm{id} - \phi_{\mathbb{R}})^{-1}(A)/A$ is finite.

Since $\operatorname{coker}\bigl(\mathrm{id}_A - \phi)\colon A \to A\bigr)$ is finite, we can find an integer $d \geq 1$ such that $d \cdot A$ lies in the image of $(\mathrm{id}_A - \phi)$. This implies that $(\mathrm{id} - \phi_{\mathbb{R}})^{-1}(d \cdot A) \subseteq A$. Hence we obtain an epimorphism

$$A/(d \cdot A) \to (\mathrm{id} - \phi_{\mathbb{R}})^{-1}(A)/A \to \quad \overline{a} \mapsto \overline{(\mathrm{id} - \phi_{\mathbb{R}})^{-1}(a)}.$$

Since A/dA is finite, $(\mathrm{id} - \phi_{\mathbb{R}})^{-1}(A)/A$ is finite. This finishes the proof of Lemma 4.31. $\square$

Now we are ready to give the proof of Proposition 4.22.

Proof of Proposition 4.22. Consider the setup introduced before Lemma 4.31. Using Lemma 4.31 one checks that the condition appearing in [8, Theorem 1.4] is satisfied for Γ_0 and the flow Φ. Hence we obtain a natural number N such that for every $\alpha > 0$ there exists a $\mathcal{VC}$yc-cover $\mathcal{U}$ of $\mathbb{R}^n \times \mathbb{R}$ with the following properties:

(i) $\dim \mathcal{U} \leq N/[\Gamma : \Gamma_0]$;

(ii) For every $x \in X$ there exists $U \in \mathcal{U}$ such that

$$\Phi_{[-\alpha,\alpha]}(x,t) := \{\Phi_\tau(x,t) \mid \tau \in [-\alpha,\alpha]\} = \{x\} \times [t-\alpha, t+\alpha] \subseteq U;$$

(iii) $\Gamma_0 \backslash \mathcal{U}$ is finite.

The number N above is the number N we are looking for in Proposition 4.22.

Let D_N, C_1, C_2, and C_3 be the constants appearing in Lemma 4.25, Lemma 4.26, and Lemma 4.27. Consider any real numbers $R > 0$ and $\epsilon > 0$. Put

$$\omega := \max\left\{\frac{64 \cdot D_N \cdot N^2 \cdot (C_1 \cdot R + C_2)}{\epsilon}, 8 \cdot N \cdot (C_1 \cdot R + C_2)\right\}. \tag{4.32}$$

Next we show that the statement (i) appearing in Proposition 4.22 is true. Fix a natural number m. Let σ_m and $\alpha_m > 0$ be the real numbers coming from Lemma 4.30 for ω defined in (4.32) and for $I = [-m,m] \subseteq \mathbb{R}$. Hence we get

$$B^{\Gamma}_{\omega}(x,s) \subseteq B^{\mathrm{euc}}_{\sigma_m}(x) \times [s - \alpha_m/2, s + \alpha_m/2] \quad \text{for } x \in \mathbb{R}^n, s \in [-m,m]. \tag{4.33}$$

For this α_m choose the $\mathcal{VC}$yc-cover $\mathcal{U}_m$ of $\mathbb{R}^n \times \mathbb{R}$ as above. Recall that a $\mathcal{VC}$yc-cover $\mathcal{U}_m$ is an open cover such that for $U \in \mathcal{U}_m$, and $\gamma \in \Gamma$ we have $\gamma U \in \mathcal{U}_m$ and $\gamma U \cap U \neq \emptyset \implies \gamma U = U$ and for every $U \in \mathcal{U}_m$ the subgroup $\Gamma_U := \{\gamma \in U \mid \gamma U = U\}$ of Γ is virtually cyclic.

Choose elements $\gamma_1, \gamma_2 \ldots, \gamma_{[\Gamma:\Gamma_0]}$ in Γ such that $\{\overline{\gamma_1}, \overline{\gamma_2} \ldots, \overline{\gamma_{\Gamma:\Gamma_0}}\} = \Gamma/\Gamma_0$. Put

$$\mathcal{V}_m := \{\gamma_i \cdot U \mid U \in \mathcal{U}_m, i = 1, 2, \ldots, [\Gamma : \Gamma_0]\}.$$

Then $\mathcal{V}_m$ is an open cover satisfying

(i) $\mathcal{V}_m$ is a Γ-invariant cover, i.e., $\gamma \in \Gamma, V \in \mathcal{V}_m \implies \gamma V \in \mathcal{V}_m$;

(ii) $\dim \mathcal{V}_m \leq N$;

(iii) For every $(x,y) \in \mathbb{R}^n \times \mathbb{R}$ there exists $V \in \mathcal{V}_m$ such that

$$\Phi_{[-\alpha_m,\alpha_m]}(x,y) := \{\Phi_\tau(x,y) \mid \tau \in [-\alpha_m,\alpha_m]\} \subseteq V;$$

(iv) $\Gamma \backslash \mathcal{V}_m$ is finite.

Next we show that we can find $\eta_m > 0$ such that for every $(x,s) \in \mathbb{R}^n \times [-m,m]$ there exists $V \in \mathcal{V}_m$ such that

$$B^{\mathrm{euc}}_{\eta_m}(x) \times [s - \alpha_m/2, s + \alpha_m/2] \subseteq V. \tag{4.34}$$

Suppose the contrary. Then we can find sequences $(x_i)_i$ in $\mathbb{R}^n$ and $(s_i)_i$ in $[-m,m]$ such that for no $i \geq 1$ there exists $V \in \mathcal{V}_m$ with the property $B^{\mathrm{euc}}_{1/i}(x_i) \times [s_i - \alpha_m/2, s_i + \alpha_m/2] \subset V$. Since the Γ-action on $\mathbb{R}^n \times \mathbb{R}$ is proper and cocompact, there is a compact subset $K \subseteq \mathbb{R}^n \times \mathbb{R}$ with $\Gamma \cdot K = \mathbb{R}^n \times \mathbb{R}$. Hence we can find a sequence $(\gamma_i)_i$ in Γ and an element (x,s) in $\mathbb{R}^n \times \mathbb{R}$ such that

$$\lim_{i\to\infty} \gamma_i \cdot (x_i, s_i) = (x,s).$$

Recall that Γ acts diagonally on $\mathbb{R}^n \times \mathbb{R}$, where the action on $\mathbb{R}$ comes from the epimorphism $\Gamma \to D$ with Δ as kernel and a proper D-action on $\mathbb{R}$. Since $[-m,m]$ is compact, the set $\{\gamma_i\Delta \mid i \geq 1\} \subseteq \Gamma/\Delta$ is finite. By passing to a subsequence, we can arrange that it consists of precisely one element; in other words, there exists an element $\gamma \in \Gamma$ and a sequence (δ_i) of elements in Δ such that $\gamma_i = \gamma \cdot \delta_i$ holds for $i \geq 1$. Hence we can assume

$$\lim_{i\to\infty} \delta_i \cdot (x_i, s_i) = (x,s),$$

otherwise replace (x,s) by $\gamma^{-1} \cdot (x,s)$.

Choose $V \in \mathcal{V}_m$ such that $\{x\} \times [s - \alpha_m, s + \alpha_m] \in V$. Since $\{x\} \times [s-\alpha_m, s+\alpha_m]$ is compact and V is open, we can find $\xi > 0$ with $B^{\mathrm{euc}}_{\xi}(x) \times [s - \alpha_m, s + \alpha_m] \subseteq V$. We can choose i such that $(\delta_i \cdot x_i, s_i) \in B^{\mathrm{euc}}_{\xi/2}(x) \times [s - \alpha_m/2, s + \alpha_m/2]$ and $1/i \leq \xi/2$. Hence $B^{\mathrm{euc}}_{1/i}(\delta_i \cdot x_i) \times [s_i - \alpha_m/2, s_i + \alpha_m/2]$ is contained in $B^{\mathrm{euc}}_{\xi}(x) \times [s - \alpha_m, s + \alpha_m]$. We conclude

$$B^{\mathrm{euc}}_{1/i}(\delta_i \cdot x_i) \times [s_i - \alpha_m/2, s_i + \alpha_m/2] \subseteq V.$$

Since Δ acts isometrically on $\mathbb{R}^n$, we obtain

$$B^{\mathrm{euc}}_{1/i}(x_i) \times [s_i - \alpha_m/2, s_i + \alpha_m/2] \subseteq \delta_i^{-1} \cdot V.$$

Since $\delta_i^{-1} V \in \mathcal{V}_m$, we get a contradiction. Hence (4.34) is true.

Now we define the desired number

$$\xi_m := \frac{\sigma_m}{\eta_m}. \tag{4.35}$$

Next consider a subgroup $\overline{H} \subseteq \Gamma$ of finite index, and a natural number k satisfying the assumptions appearing in assertion (i) of Proposition 4.22. From now on put $m = [D : \pi \circ \mathrm{pr}(\overline{H})]$. We can choose a pseudo k-expansive map

$$\phi\colon \Gamma \to \Gamma$$

by Lemma 4.8 (i). Because of Lemma 4.23 we can assume

$$\overline{H} \subseteq \mathrm{im}(\phi), \tag{4.36}$$

since the desired claim holds for $\overline{H}$ if it holds for some conjugate of $\overline{H}$, compare Remark 2.15. There exists $u \in \mathbb{R}$ such that the affine map $a\colon \mathbb{R}^n \to \mathbb{R}^n$ sending x to $k \cdot x + u$ is ϕ-equivariant (see Lemma 4.24). Since

$$a \times \mathrm{id}_{\mathbb{R}}\bigl(B^{\mathrm{euc}}_{\eta_m}(x) \times [s - \alpha_m/2, s + \alpha_m/2]\bigr) = B^{\mathrm{euc}}_{k\cdot\eta_m}(a(x)) \times [s - \alpha_m/2, s + \alpha_m/2],$$

and a is bijective, we conclude from (4.34) that for every $x \in \mathbb{R}^n$ and $s \in [-m, m]$ there exists $V \in \mathcal{V}_m$ satisfying

$$B^{\text{euc}}_{k\cdot\eta_m}(x) \times [s - \alpha_m/2, s + \alpha_m/2] \subseteq a \times \text{id}_{\mathbb{R}}(V).$$

Since $k \geq \xi_m$ implies $k \cdot \eta_m \geq \sigma_m$ by our choice (4.35) of ξ_m we conclude for every $x \in \mathbb{R}^n$ and $s \in [-m, m]$

$$B^{\text{euc}}_{\sigma_m}(x) \times [s - \alpha_m/2, s + \alpha_m/2] \subseteq a \times \text{id}_{\mathbb{R}}(V).$$

Now (4.33) implies that for every $x \in \mathbb{R}^n$ and $s \in [-m, m]$ there exists $V \in \mathcal{V}_m$ satisfying

$$B^{\Gamma}_{\omega}(x, s) \subseteq a \times \text{id}_{\mathbb{R}}(V). \tag{4.37}$$

Next consider the open covering $\mathcal{W}_m := \{a \times \text{id}(V) \mid V \in \mathcal{V}_m\}$ of $\mathbb{R}^n \times \mathbb{R}$. This is an $\text{im}(\phi)$-invariant covering, since the diffeomorphism $a \times \text{id}\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ is ϕ-equivariant and $\mathcal{V}_m$ is Γ-invariant. By (4.36) we can consider $\mathcal{W}_m$ as a $\overline{H}$-invariant open covering of $\mathbb{R}^n \times \mathbb{R}$. Since by definition $m = [D : \pi \circ \text{pr}(\overline{H})]$, we conclude that $\pi \circ \text{pr}(\overline{H})$ contains $m \cdot \mathbb{Z}$. This implies $\pi \circ \text{pr}(\overline{H}) \cdot [-m, m] = \mathbb{R}$. Since for every $\gamma \in \overline{H}$ we have $\gamma \cdot B^{\Gamma}_{\omega}(x, s) = B^{\Gamma}_{\omega}\big(\gamma \cdot (x, s)\big)$, we conclude from (4.37) that for every $(x, s) \in \mathbb{R}^n \times \mathbb{R}$ there exists $W \in \mathcal{W}$ with $B^{\Gamma}_{\omega}\big((x, s)\big) \subseteq W$. Hence Lemma 4.29 (applied in the case $\Gamma = \overline{H}$) implies that the $\overline{H}$-equivariant map

$$\beta^{\mathcal{W}}\colon \mathbb{R}^n \times \mathbb{R} \to |\mathcal{W}|$$

defined in (4.28) has the property that for $(x, s), (y, t) \in \mathbb{R}^n \times \mathbb{R}$ with $d^{\Gamma}\big((x, s), (y, t)\big) \leq \frac{\omega}{8N}$ we get

$$d^{l^1}_{|\mathcal{W}|}\big(\beta^{\mathcal{W}}(x, s), \beta^{\mathcal{W}}(y, t)\big) \leq \frac{64 \cdot N^2}{\omega} \cdot d^{\Gamma}\big((x, s), (y, t)\big). \tag{4.38}$$

Now consider the composite

$$f\colon \Gamma \xrightarrow{\text{ev}} \mathbb{R}^n \times \mathbb{R} \xrightarrow{\beta^{\mathcal{W}}} |\mathcal{W}| \xrightarrow{\text{id}} |\mathcal{W}'|.$$

The map ev is Γ-invariant and in particular $\overline{H}$-equivariant. Hence f is $\overline{H}$-equivariant. The $\overline{H}$-action on $|\mathcal{W}|$ is simplicial. Hence the $\overline{H}$-action on the barycentric subdivision $|\mathcal{W}'|$ is simplicial and cell preserving.

Next we show that the isotropy groups of the $\overline{H}$-action on the space $|\mathcal{W}| = |\mathcal{W}'|$ are all virtually cyclic. Consider $z \in |\mathcal{W}|$. Choose a simplex σ such that z lies in its interior. Let the simplex σ be given by $\{W_0, W_1, \dots W_l\}$ for pairwise distinct elements $W_i \in \mathcal{W}$. Then for every γ in the isotropy group $\overline{H}_z$ we must have

$$\gamma \cdot \{W_0, W_1, \dots W_l\} = \{W_0, W_1, \dots W_l\}.$$

Hence $\overline{H}_z$ operates on the finite set $\{W_0, W_1, \dots W_r\}$. We conclude that $\overline{H}_z$ contains a subgroup $\overline{H}'_z$ of finite index such that $\gamma \cdot W_0 = W_0$ holds for all $\gamma \in \overline{H}'_z$. By construction there is $U \in \mathcal{U}$ such that $W = f(U)$ or $W = f(\overline{s} \cdot U)$ for some fixed element $\overline{s} \in \Gamma$. Let $\Gamma''_z \subseteq \Gamma$ be the preimage of $\overline{H}'_z$ under the isomorphism $\phi\colon \Gamma \to \text{im}(\phi)$. Hence either $\gamma'' \cdot U = U$ for all $\gamma'' \in \Gamma''_z$ or $\overline{s}^{-1}\gamma''\overline{s} \cdot U = U$ for all $\gamma'' \in \Gamma''_z$. Since $\mathcal{U}$ is a $\mathcal{VC}$yc-covering, the group Γ''_z is virtually cyclic. Since it is isomorphic to a subgroup of finite index of $\overline{H}_z$, the isotropy group $\overline{H}_z$ is virtually cyclic.

Consider γ_1, γ_2 in Γ with $d_\Gamma(\gamma_1,\gamma_2) \le R$. We want to show

$$d^{l^1}_{|\mathcal{W}'|}\big(f(\gamma_1), f(\gamma_2)\big) \le \epsilon. \tag{4.39}$$

Lemma 4.26 implies

$$d^\Gamma\big(\mathrm{ev}(\gamma_1), \mathrm{ev}(\gamma_2)\big) \le C_1 \cdot d_\Gamma\big(\gamma_1,\gamma_2\big) + C_2 \le C_1 \cdot R + C_2.$$

Our choice of ω in (4.32) guarantees $C_1 \cdot R + C_2 \le \frac{\omega}{8N}$. Hence

$$d^\Gamma\big(\mathrm{ev}(\gamma_1), \mathrm{ev}(\gamma_2)\big) \le \frac{\omega}{8N}.$$

We conclude from (4.38)

$$\begin{aligned} d^{l^1}_{|\mathcal{W}|}\big(\beta^{\mathcal{W}} \circ \mathrm{ev}(\gamma_1), \beta^{\mathcal{W}} \circ \mathrm{ev}(\gamma_2)\big) &\le \frac{64 \cdot N^2}{\omega} \cdot d^\Gamma\big(\mathrm{ev}(\gamma_1), \mathrm{ev}(\gamma_2)\big) \\ &\le \frac{64 \cdot N^2}{\omega} \cdot \big(C_1 \cdot R + C_2\big). \end{aligned}$$

Lemma 4.25 implies

$$d^{l^1}_{|\mathcal{W}'|}\big(f(\gamma_1), f(\gamma_2)\big) \le \frac{64 \cdot N^2 \cdot D_N \cdot \big(C_1 \cdot R + C_2\big)}{\omega}.$$

Our choice of ω in (4.32) implies

$$\frac{64 \cdot N^2 \cdot D_N \cdot \big(C_1 \cdot R + C_2\big)}{\omega} \le \epsilon.$$

This finishes the proof of (4.39).

Since $\overline{H}$ acts simplicially on $|\mathcal{W}|$, it acts simplicially and cell preserving on $|\mathcal{W}'|$. Put $E := |\mathcal{W}'|$. We have already shown that all isotropy groups of the $\overline{H}$-action on F are virtually cyclic. The $\overline{H}$-map $f\colon \Gamma \to E$ has the desired properties because of (4.39). This finishes the proof of statement (i) appearing in Proposition 4.22.

Next we prove statement (ii) of Proposition 4.22. Choose an integer m satisfying

$$m \ge \frac{2 \cdot C_3 \cdot R}{\epsilon}. \tag{4.40}$$

We conclude from Lemma 4.27 that for all $\gamma_1, \gamma_2 \in \Gamma$

$$d_D\big(\pi \circ \mathrm{pr}(\gamma_1), \pi \circ \mathrm{pr}(\gamma_2)\big) \le C_3 \cdot d_\Gamma(\gamma_1,\gamma_2)$$

holds. Let $\mathrm{ev}\colon D \to \mathbb{R}$ be the map given by evaluation of the standard group action of D on the origin 0. One easily checks for $\delta_1, \delta_2 \in D$

$$d^{\mathrm{euc}}\big(\mathrm{ev}(\delta_1), \mathrm{ev}(\delta_2)\big) \le d_D\big(\delta_1, \delta_2\big).$$

Let the desired map f be the composite

$$f\colon \Gamma \xrightarrow{\mathrm{pr}} Q \xrightarrow{\pi} D \xrightarrow{\mathrm{ev}} \mathbb{R} \xrightarrow{\frac{1}{m}\cdot \mathrm{id}} \mathbb{R}.$$

It satisfies for all $\gamma_1, \gamma_2 \in \Gamma$

$$d_{\mathbb{R}}\big(f(\gamma_1), f(\gamma_2)\big) \le \frac{C_3}{m} \cdot d_\Gamma(\gamma_1,\gamma_2).$$

Let E be the simplicial complex whose underlying space is $\mathbb{R}$ and for which the set of zero-simplices is $\frac{1}{2} \cdot \mathbb{Z}$. Then we get for $x, y \in \mathbb{R}$

$$d^{l^1}_E(x,y) \le 2 \cdot d_{\mathbb{R}}(x,y).$$

Hence we obtain for all $\gamma_1, \gamma_2 \in \Gamma$

$$d_E^{l^1}\big(f(\gamma_1), f(\gamma_2)\big) \leq \frac{2 \cdot C_3}{m} \cdot d_\Gamma(\gamma_1, \gamma_2).$$

The choice of the integer m in (4.40) guarantees

$$\frac{2 \cdot C_3 \cdot R}{m} \leq \epsilon.$$

Hence

$$d_\Gamma(x,y) \leq R \implies d_E^{l^1}(f(x), f(y)) \leq \epsilon$$

for $\gamma_1, \gamma_2 \in \Gamma$.

The standard operation of D on $\mathbb{R}$ is simplicial and cell preserving. Consider the group homomorphism $\phi_m \colon D \to D$ which sends t to t^m and, if $D = D_\infty$, s to s, where we use the standard presentations of $\mathbb{Z}$ and D_∞. The map $m \cdot \mathrm{id} \colon \mathbb{R} \to \mathbb{R}$ is ϕ_m-equivariant if we equip source and target with the standard D-action.

Now consider any subgroup $\overline{H} \subseteq \Gamma$ with $[D : \pi \circ \mathrm{pr}(\overline{H})] \geq 2 \cdot m$. We conclude $\pi \circ \mathrm{pr}(\overline{H}) \subseteq \mathrm{im}(\phi_m)$. Since ϕ_m is injective, we can define an $\overline{H}$-action on E by defining $\overline{h} \cdot e = \delta \cdot e$ for $e \in E$ and any $\delta \in D$ for which $\phi_m(\delta) = \pi \circ \mathrm{pr}(\overline{h})$ holds. One easily checks that the map f is $\overline{H}$-equivariant. Since the isotropy groups of the standard D-action on $\mathbb{R}$ are finite and the epimorphism $\pi \colon Q \to D$ has a finite kernel and the kernel of pr is A, the isotropy group $\overline{H}_e$ of any $e \in E$ satisfies $A \subseteq \overline{H}_e$ and $[\overline{H}_e : A] < \infty$. Now define the desired natural number μ by $\mu = 2m$. This finishes the proof of Proposition 4.22. ☐

4.6. **Proof of the Farrell-Jones Conjecture for irreducible special affine groups.**

Proposition 4.41 (The Farrell-Jones Conjecture for irreducible special affine groups)**.** *Both the K-theoretic and the L-theoretic FJC hold for all irreducible special affine groups.*

Proof. Because of Theorem 2.10, Theorem 2.16, and Theorem 3.1 it suffices to show that a special affine group G is a Farrell-Hsiang group with respect to the family $\mathcal{F}$ of virtually finitely generated abelian groups in the sense of Definition 2.14.

Let N be the natural number appearing in Proposition 4.22. Consider any real numbers $R > 0$ and $\epsilon > 0$. Let μ be the natural number and $(\xi_n)_{n \leq 1}$ be the sequence appearing in Proposition 4.22. Now choose a natural number τ such that $\mu < \tau$ and $\xi_n \leq \tau$ for all $n \leq \mu$. For this choice of τ we choose r, s as appearing in Proposition 4.10. Let $\alpha_{r,s} \colon \Gamma \to A_s \rtimes_{\rho_{r,s}} Q_r$ be the epimorphism appearing in Proposition 4.10. The map $\alpha_{r,s}$ will play the role of the map $\alpha_{R,\epsilon}$ appearing in Definition 2.14.

Let H be a hyperelementary subgroup of $A_s \rtimes_{\rho_{r,s}} Q_r$. Recall that $\overline{H}$ is the preimage of H under $\alpha_{r,s}$. We have to construct the desired simplicial complex E_H and the map $f_H \colon G \to E_H$ as demanded in Definition 2.14. If $[D : \pi \circ \mathrm{pr}(\overline{H})] \geq \mu$, then we obtain the desired pair (E_H, f_H) from assertion (ii) of Proposition 4.22. Suppose that $[D : \pi \circ \mathrm{pr}(\overline{H})] \leq \mu$. Then by our choice of τ we have $\tau \geq \xi_{[D : \pi \circ \mathrm{pr}(\overline{H})]}$ and $\mu < \tau$. In particular $[D : \pi \circ \mathrm{pr}(\overline{H})] \geq \tau$ is not true. Hence by Proposition 4.10 we obtain an integer k such that the assumption appearing in assertion (i) of Proposition 4.22 is satisfied and the conclusion of assertion (i) of Proposition 4.22 gives the desired pair (E_H, f_H). Hence G is a Farrell-Hsiang group with respect to the family $\mathcal{F}$ of virtually finitely generated abelian groups. This finishes the proof of Proposition 4.41. $\square$

5. Virtually poly-$\mathbb{Z}$-groups

This section is devoted to the proof of Theorem 1.1. It will be done by induction over the virtual cohomological dimension. We will need the following ingredients.

Definition 5.1 ((Virtually) poly-$\mathbb{Z}$). We call a group G' *poly-$\mathbb{Z}$* if there exists a finite sequence

$$\{1\} = G'_0 \subseteq G'_1 \subseteq \cdots \subseteq G'_n = G'$$

of subgroups such that G'_{i-1} is normal in G'_i with infinite cyclic quotient G'_i/G'_{i-1} for $i = 1, 2, \ldots, n$.

We call a group G virtually poly-$\mathbb{Z}$ if it contains a subgroup G' of finite index such that G' is poly-$\mathbb{Z}$.

Let G be a virtually poly-$\mathbb{Z}$-group. Let $G' \subseteq G$ be any subgroup of finite index, for which there exists a finite sequence $\{1\} = G'_0 \subseteq G'_1 \subseteq \cdots \subseteq G'_n = G'$ of subgroups such that G'_{i-1} is normal in G'_i with an infinite cyclic quotient G'_i/G'_{i-1} for $i = 1, 2, \ldots, n$. We call the number $r(G) := n$ the *Hirsch rank* of G. We will see that it depends only on G but not on the particular choice of subgroup $G' \subseteq G$ and the filtration $\{1\} = G'_0 \subseteq G'_1 \subseteq \cdots \subseteq G'_n = G'$.

Lemma 5.2 (Virtual cohomological dimension of virtually poly-$\mathbb{Z}$-groups). *Let G be a virtually poly-$\mathbb{Z}$-group. Then*

- (i) $r(G) = \operatorname{vcd}(G)$*;*
- (ii) *We get $r(G) = \max\{i \mid H_i(G'; \mathbb{Z}/2) \neq 0\}$ for one (and hence all) poly-$\mathbb{Z}$ subgroup $G' \subset G$ of finite index;*
- (iii) *There exists a finite $r(G)$-dimensional model for the classifying space of proper G-actions $\underline{E}G$ and for any model $\underline{E}G$ we have $\dim(\underline{E}G) \geq r(G)$;*
- (iv) *Subgroups and a quotient groups of virtually poly-$\mathbb{Z}$-groups are again virtually poly-$\mathbb{Z}$;*
- (v) *Consider an extension of groups*

$$1 \to G_0 \to G_1 \to G_2 \to 1.$$

Suppose that two of them are virtually poly-$\mathbb{Z}$. Then all of them are virtually poly-$\mathbb{Z}$ and we get for their cohomological dimensions

$$\operatorname{vcd}(G_1) = \operatorname{vcd}(G_0) + \operatorname{vcd}(G_2).$$

Proof. Assertions (i), (ii), and (iii) are proved in [29, Example 5.2.6]. The proof of the other assertions is now obvious using induction over the Hirsch rank. $\square$

The next result is taken from [23, Lemma 4.4].

Lemma 5.3. *Let G be a virtually poly-$\mathbb{Z}$-group. Then there exists an exact sequence*

$$1 \to G_0 \to G \to \Gamma \to 1$$

satisfying

- (i) *The group G_0 is either finite or a virtually poly-$\mathbb{Z}$-group with* $\operatorname{vcd}(G_0) \leq \operatorname{vcd}(G) - 2$*;*
- (ii) Γ *is either a crystallographic or special affine group.*

Now we are ready to prove Theorem 1.1.

Proof. We use induction over the virtual cohomological dimension of the virtually poly-$\mathbb{Z}$-group G. The induction beginning $\operatorname{vcd}(G) \leq 1$ is trivial since in this case G must be virtually cyclic by Lemma 5.2. For the induction step choose an extension

$$1 \to G_0 \to G \xrightarrow{\mathrm{pr}} \Gamma \to 1$$

as appearing in Lemma 5.3. Consider any virtually cyclic subgroup $V \subseteq \Gamma$. Then we obtain an exact sequence

$$1 \to G_0 \to \mathrm{pr}^{-1}(V) \to V \to 1.$$

Since G_0 is virtually poly-$\mathbb{Z}$ with $\operatorname{vcd}(G_0) \leq \operatorname{vcd}(G) - 2$, we conclude from Lemma 5.2 that $\mathrm{pr}^{-1}(V)$ is virtually poly-$\mathbb{Z}$ with $\operatorname{vcd}\bigl(\mathrm{pr}^{-1}(V)\bigr) < \operatorname{vcd}(G)$. Hence both the K-theoretic and the L-theoretic FJC hold for $\mathrm{pr}^{-1}(V)$. Because of Theorem 2.7 it remains to prove that both the K-theoretic and the L-theoretic FJC hold for Γ. If Γ is crystallographic or an irreducible special affine group, this follows from Theorem 3.1 and Proposition 4.41. Hence it remains to prove both the K-theoretic and the L-theoretic FJC for the special affine group Γ provided that it admits an epimorphism $p\colon \Gamma \to \Gamma'$ to some virtually finitely generated abelian group Γ' with $\operatorname{vcd}(\Gamma') \geq 2$. If K is the kernel of p, we obtain the exact sequence $1 \to K \to \Gamma \xrightarrow{p} \Gamma' \to 1$. We conclude from Lemma 5.2 that K is a virtually poly-$\mathbb{Z}$-group with $\operatorname{vcd}(K) \leq \operatorname{vcd}(\Gamma) - 2 \leq \operatorname{vcd}(G) - 2$. Hence for any virtually cyclic subgroup V of Γ' the preimage $p^{-1}(V)$ is a virtually poly-$\mathbb{Z}$-group with $\operatorname{vcd}\bigl(p^{-1}(V)\bigr) < \operatorname{vcd}(G)$ by Lemma 5.2. By the induction hypothesis $p^{-1}(V)$ satisfies both the K-theoretic and the L-theoretic FJC. Since the same is true for Γ' by Theorem 3.1, we conclude from Theorem 2.7 that Γ satisfies both the K-theoretic and the L-theoretic FJC. This finishes the proof of Theorem 1.1. □

6. Cocompact lattices in virtually connected Lie groups

In this section we prove Theorem 1.2.

The main work which remains to be done is to give the proof of Proposition 6.1 below which is very similar to the one of [23, pages 264–265]. We call a Lie group *semisimple* if its Lie algebra is semisimple. A subgroup $G \subseteq L$ of a Lie group L is called a *cocompact lattice* if G is discrete and L/G compact.

Proposition 6.1. *In order to prove Theorem* 1.2 *it suffices to prove that every virtually poly-$\mathbb{Z}$-group and every group which operates cocompactly, isometrically, and properly on a complete, simply connected Riemannian manifold with non-positive sectional curvature satisfy the K- and L-theoretic FJC.*

Its proof needs some preparation.

Lemma 6.2. *Let L be a virtually connected Lie group. Let K be the maximal connected normal compact subgroup of L. Let $G \subseteq L$ be a cocompact lattice. Let $\overline{G}$ be the image of G under the projection $L \to L/K$. Then*

(i) *If L is semisimple, then L/K is semisimple;*
(ii) *Every connected normal compact subgroup of L/K is trivial;*
(iii) *$\overline{G} \subseteq L/K$ is a cocompact lattice;*
(iv) *If $\overline{G}$ satisfies the FJC, then G satisfies the FJC.*

Proof. (i) Any quotient of a semisimple Lie algebra is again semisimple.

(ii) If H is a normal compact connected subgroup of L/K, then its preimage under the projection $L \to L/K$ is a normal compact connected subgroup of L.

(iii) Since K is compact, $G \cap K$ is a finite group.

(iv) We have the exact sequence $1 \to G \cap K \to G \to \overline{G} \to 1$. Now apply Corollary 2.11. □

In the sequel we denote by L^e the component of the identity,

Lemma 6.3. *Proposition* 6.1 *is true provided that G is a cocompact lattice in a virtually connected semisimple Lie group L.*

The statement of this lemma unravels as follows. Let G be a cocompact lattice in a virtually connected semisimple Lie group L. Assume that every virtually poly-$\mathbb{Z}$-group and every group which operates cocompactly, isometrically, and properly on a complete, simply connected Riemannian manifold with non-positive sectional curvature satisfies the K- and L-theoretic FJC. Then G satisfies the K- and L-theoretic FJC.

Proof of Lemma 6.3. Because of Lemma 6.2 we can assume without loss of generality that L is a virtually connected semisimple Lie group for which every connected normal compact subgroup $K \subset L$ is trivial. Let $Z \subseteq L$ be the normal subgroup of elements in L which commute with every element in L^e. Put $\overline{L} := L/Z$. Let G_Z be the intersection $G \cap Z$ and $\overline{G}$ the image of G under the projection $\mathrm{pr}\colon L \to \overline{L}$. Then the following statements are true:

(i) G_Z is virtually finitely generated abelian;
(ii) $\overline{G}$ is a cocompact lattice of $\overline{L}$;
(iii) $\overline{L}$ is a virtually connected semisimple Lie group whose center is trivial.

For the proof of assertion (i) we can assume without loss of generality that L is connected since L is virtually connected and since a group is already finitely generated if it contains a finitely generated subgroup of finite index. Then Z is just the center of L and in particular an abelian Lie group. The intersection G_Z of G and Z is a cocompact discrete subgroup of an abelian Lie group Z and hence a finitely generated abelian group.

Assertion (ii) follows by inspecting the proof of [34, Corollary 5.17 on page 84] which applies directly to our case since all compact connected normal subgroups of L are trivial.

Next we prove assertion (iii). Obviously $\overline{L}$ is virtually connected and semisimple since the quotient of a semisimple Lie algebra is again semisimple. Let $\overline{Z} \subseteq \overline{L}$ be the center of $\overline{L}$. Let $Z' \subseteq L$ be its preimage under the projection $L \to \overline{L}$. Consider $g \in L^e$ and $g' \in Z'$. Then $g'gg'^{-1}g^{-1}$ belongs to Z. Choose a path w in L connecting 1 and g in L^e. Then $g'w(t)(g')^{-1}w(t)^{-1}$ is a path in Z connecting 1 and $g'gg'^{-1}g^{-1}$. Since L is semisimple, $Z \subseteq L$ is discrete. Hence $g'gg'^{-1}g^{-1} = 1$. This implies $g' \in Z$. Hence $Z = Z'$ and we conclude that the center of $\overline{L}$ is trivial.

Because of Corollary 2.12 it suffices to show that $\overline{G}$ satisfies the FJC.

By [1, Theorem A.5] there exists a maximal compact subgroup $K \subseteq \overline{L}$ and the space $\overline{L}/K$ is contractible. Then $K \cap \overline{L}^e$ is a maximal compact subgroup of $\overline{L}^e$ and $\overline{L}/K = \overline{L}^e/(K \cap \overline{L}^e)$. Since $\overline{L}$ is semisimple, its Lie algebra contains no compact ideal and its center is finite, the quotient

$$M := \overline{L}/K = \overline{L}^e/(K \cap \overline{L}^e)$$

equipped with a $\overline{L}$-invariant Riemannian metric is a symmetric space of a non-compact type such that $\overline{L}^e = \mathrm{Isom}(M)^e$ and $K \cap \overline{L}^e = (\mathrm{Isom}(M)^e)_x$ for $\mathrm{Isom}(M)$

the group of isometries (see [19, Section 2.2 on page 70]). Hence M has non-positive sectional curvature (see [25, Proposition 4.2 in V.4 on page 244, Theorem 3.1 in V.3 on page 241]). Obviously $\overline{G}$ acts properly cocompactly and isometrically on M. By assumption $\overline{G}$ satisfy the FJC. This finishes the proof of Lemma 6.3. □

Lemma 6.4. *Let G be a lattice in a virtually connected Lie group L. Assume that every compact connected normal subgroup of L is trivial. Let N be the nilradical in L. Then $G_N := G \cap N$ is a lattice in N.*

Proof. Let S be the semisimple part of L^e. By [38, Theorem 1.6 on page 106] it suffices to show that S has no non-trivial compact factors that act trivially on R and L. Assume that K is such a factor. Let $L^e = RS$ be the Levi decomposition of L^e. (We mention as a caveat that S is not necessarily a *closed* subgroup of L^e; nor is $R \cap S$ necessarily discrete; although $R \cap S$ is countable.) Since K is a factor of S it is a normal subgroup of S and therefore $sKs^{-1} \subseteq K$ for all $s \in S$. Because K acts trivially on R we have $rKr^{-1} \subseteq K$ for all $r \in R$. Since $L^e = RS$, we conclude that K is a normal subgroup of L^e. Consequently, K is a normal compact connected subgroup of L^e and therefore contained in the unique maximal normal compact connected subgroup K_{max} of L^e. This subgroup K_{max} is a characteristic subgroup of L^e. Thus K_{max} is in addition normal in L and therefore trivial. Hence K is trivial. □

Proof of Proposition 6.1. We proceed by induction on (the manifold) dimension of L, i.e., we assume that Proposition 6.1 is true for all virtually connected Lie groups L' where $\dim L' < \dim L$. We may assume, because of Lemma 6.2, that every compact connected normal subgroup of L is trivial. Consider the sequence of normal subgroups of L,

$$N \lhd R \lhd L^e \lhd L$$

where L^e is the connected component of L containing the identity, R is the radical of L, and N is the nilradical of L. And let

$$G_N := G \cap N.$$

By Lemma 6.4 G_N is a cocompact lattice in N. Therefore G/G_N is a cocompact lattice in L/N as well.

We now distinguish two cases. First consider the case that N is non-trivial. Then $\dim L/N < \dim L$ and G/G_N satisfies the FJC by our inductive assumption. Now consider the following exact sequence:

$$1 \to G_N \to G \to G/G_N \to 1$$

and observe that G_N is a virtually poly-$\mathbb{Z}$-group by a result of Mostow (This follows from Theorem 5.2 (iv) and [34, Proposition 3.7 on page 52].) Hence G satisfies the FJC because of Corollary 2.12

Next consider the case that N is trivial. Then $R = R/N$ is abelian. Hence $R = N = 1$. Therefore L is semisimple and G satisfies the FJC because of Lemma 6.3. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We have proved the FJC for virtually poly-$\mathbb{Z}$-groups in Theorem 4.41. Since every group G which acts cocompactly, isometrically, and properly on a complete, simply connected Riemannian manifold with non-positive sectional curvature is a cocompact CAT(0)-group, it satisfies the FJC by the main results from [6, 39]. Now apply Proposition 6.1. □

7. Fundamental groups of 3-manifolds

In this section we sketch the proof of Corollary 1.3

Remark 7.1 (Pseudo-isotopy). Let π be the fundamental group of a 3-manifold. Roushon (see [35], [36]) gives a proof of the Farrell-Jones Conjecture for pseudo-isotopy with a wreath product for the family $\mathcal{VC}$yc for π. Its proof relies on the assumption that the Farrell-Jones Conjecture for pseudo-isotopy is true for poly-$\mathbb{Z}$-groups as stated in Farrell-Jones [23]. Unfortunately that proof depends on [23, Theorem 4.8] whose proof in turn has never appeared. Hence the proof of the Farrell-Jones Conjecture for pseudo-isotopy with a wreath product for the family $\mathcal{VC}$yc for π is not complete.

Discussion of proof of Corollary 1.3. In this paper we have proved both the K-theoretic and the L-theoretic FJC for virtually poly-$\mathbb{Z}$-groups in Theorem 1.1. One can check that the rather involved argument by Roushon (see [35], [36]) for pseudo-isotopy goes through in our setting.

This check above has been carried out in detail and in a comprehensible way in the Diplom-Arbeit by Philipp Kühl [27] axiomatically. A group G satisfies *the FJC with wreath products* if for any finite group F the wreath product $G \wr F$ satisfies the FJC. Kühl proves following Roushon that the FJC with wreath products holds for the fundamental group of every 3-manifold, if the following is true:

- The FJC with wreath products holds for $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ for any automorphism $\phi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$;
- The FJC holds for fundamental groups of closed Riemannian manifolds with a non-positive sectional curvature;
- Theorem 2.6 and Theorem 2.7 are true.

Since a wreath product $G \wr F$ for a finite group F and a group which is virtually poly-$\mathbb{Z}$ is again virtually poly-$\mathbb{Z}$, the FJC with wreath product holds for all virtually poly-$\mathbb{Z}$-groups if and only if the FJC holds for all virtually poly-$\mathbb{Z}$-groups. Hence the axioms above are satisfied. □

Remark 7.2 (Virtually weak strongly poly-surface groups). Roushon defines weak strongly poly-surface groups in [36, Definition 1.2.1]. His argument in the proof of [36, Theorem 1.2.2] carries over to our setting and shows that virtually weak strongly poly-surface groups satisfy the K- and L-theoretic Farrell-Jones Conjectures with additive categories as coefficients with respect to the family $\mathcal{VC}$yc (see Definitions 2.1 and 2.2).

8. Reducing the family $\mathcal{VC}$yc

In this subsection we explain how one can reduce the family of subgroups in our setting of equivariant additive categories as coefficients.

Definition 8.1 (Hyperelementary group)**.** Let l be a prime. A (possibly infinite) group G is called *l-hyperelementary* if it can be written as an extension $1 \to C \to G \to L \to 1$ for a cyclic group C and a finite group L whose order is a power of l.

We call G *hyperelementary* if G is l-hyperelementary for some prime l.

If G is finite, this reduces to the usual definition. Notice that for a finite l-hyperelementary group L one can arrange that the order of the finite cyclic group C appearing in the extension $1 \to C \to G \to L \to 1$ is prime to l. Subgroups and quotient groups of l-hyperelementary groups are l-hyperelementary again. For a group G we denote by $\mathcal{H}$ the family of hyperelementary subgroups of G.

The following result has been proved for K-theory and untwisted coefficients by Quinn [32] and our proof is strongly motivated by his argument.

Theorem 8.2 (Hyperelementary induction). *Let G be a group and let $\mathcal{A}$ be an additive G-category (with involution). Then both relative assembly maps*

$$\operatorname{asmb}_n^{G,\mathcal{H},\mathcal{VC}yc}\colon H_n^G\big(E_{\mathcal{H}}(G);\mathbf{K}_{\mathcal{A}}\big) \to H_n^G\big(E_{\mathcal{VC}yc}(G);\mathbf{K}_{\mathcal{A}}\big)$$

and

$$\operatorname{asmb}_n^{G,\mathcal{H},\mathcal{VC}yc}\colon H_n^G\big(E_{\mathcal{H}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\big) \to H_n^G\big(E_{\mathcal{VC}yc}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\big)$$

induced by the up to G-homotopy unique G-map $E_{\mathcal{H}}(G) \to E_{\mathcal{VC}yc}(G)$ are bijective for all $n \in \mathbb{Z}$.

Proof. Because of the Transitivity Principle 2.10 we can assume without loss of generality that G is virtually cyclic. If G is finite, the claim follows from Bartels-Lück [4, Theorem 2.9 and Lemma 4.1]. There only the case of the fibered FJC for coefficients in a ring (without G-action) is treated but the proof carries directly over to the case of coefficients in an additive G-category with coefficients. Notice that action of the Swan group of a group G which is well known in the untwisted case carries directly over to the case of additive G-category with coefficients. Hence we can assume in the sequel that G is an infinite virtual cyclic group and Theorem 8.2 holds for all finite groups.

The *holonomy number* $h(G)$ of an infinite virtually cyclic group G is the minimum over all integers $n \geq 1$ such that there exists an extension $1 \to C \to G \to Q \to 1$ for an infinite cyclic group C and a finite group Q with $|Q| = n$. We will use induction over the holonomy number $h(G)$. The induction beginning $h(G) = 1$ is trivial since in this case G is infinite cyclic and both $\mathcal{H}$ and $\mathcal{VC}$yc consists of all subgroups. It remains to explain the induction step.

So fix an infinite virtually cyclic subgroup G with holonomy number $h(G) \geq 2$. We have to prove Theorem 8.2 for G under the assumption that we know Theorem 8.2 already for all finite groups and for all infinite virtually cyclic subgroups whose holonomy number is smaller than $h(G)$.

Fix an extension

$$1 \to C \xrightarrow{i} G \xrightarrow{\operatorname{pr}} Q \to 1. \tag{8.3}$$

for an infinite cyclic group C and a finite group Q with $|Q| = h(G)$. Let $\mathcal{F}$ be the family of subgroups of G which are either finite, infinite virtually cyclic groups $H \subseteq G$ with holonomy number $h(H) < h(G)$, or hyperelementary. This is indeed a family since for any infinite virtually cyclic subgroups $H \subseteq K$ we have $h(H) \leq h(K)$ and subgroups of hyperelementary groups are hyperelementary. Since the claim holds for all finite groups and all infinite virtually cyclic groups whose holonomy number is smaller than the one of G, it suffices because of the Transitivity Principle 2.10 to prove that G satisfies the Farrell-Jones Conjecture with respect to the family $\mathcal{F}$. This will be done by proving that G is a Farrell-Hsiang group with respect to the family $\mathcal{F}$ in the sense of Definition 2.14 (see Theorem 2.16).

For an integer s define $C_s := C/sC$ and $G_s := G/sC$. We obtain an induced exact sequence

$$1 \to C_s \xrightarrow{i_s} G_s \xrightarrow{\mathrm{pr}_s} Q \to 1.$$

Denote by

$$\alpha_s \colon G \to G_s$$

the projection.

In the sequel we abbreviate $\overline{H} := \alpha_s^{-1}(H)$ for a subgroup $H \subseteq G_s$.

Lemma 8.4. *In order to prove Theorem* 8.2 *it suffices to find for given real numbers $R, \epsilon > 0$ a natural number s with the following property: For every hyperelementary subgroup $H \subseteq G_s$ there exists a one-dimensional simplicial complex E_H with a cell preserving simplicial $\overline{H}$-action and an $\overline{H}$-map $f_H \colon G \to E$ such that $d_G(g_1, g_2) \leq R$ implies $d^{l^1}\big(f_H(g_1), f_H(g_1)\big) \leq \epsilon$ and all $\overline{H}$-isotropy groups of E_H belong to $\mathcal{F}$.*

Proof. This follows from Theorem 2.16. □

In the next step we reduce the claim further to a question about indices. Choose an epimorphism $\pi^G \colon G \to \Delta$ with a finite kernel onto a crystallographic group (see [32, Lemma 4.2.1]). Then Δ is either D_∞ or $\mathbb{Z}$. The subgroup A_Δ is infinite cyclic. If $\Delta = \mathbb{Z}$, then $\Delta = A_\Delta$. If $\Delta = D_\infty$, then $A_\Delta \subseteq \Delta$ has index two.

Lemma 8.5. *In order to prove Theorem* 8.2 *it suffices to find for a given natural number i a natural number s with the following property: For every hyperelementary subgroup $H \subseteq G_s$ we have $\overline{H} \in \mathcal{F}$ or $[\Delta : \pi^G(\overline{H})] \geq i$.*

Proof of Lemma 8.5. We show that the assumptions in Lemma 8.5 imply the ones appearing in Lemma 8.4. We only treat the difficult case $\Delta = D_\infty$, the case $\Delta = \mathbb{Z}$ is then obvious.

We have fixed a word metric d_G on G. Equip D_∞ with respect to the word metric with respect to the standard presentation. Since π^G is surjective, we can find constants C_1 and C_2 such that for $g_1, g_2 \in G$ we get

$$d_{D_\infty}\big(\pi^G(g_1), \pi^G(g_2)\big) \leq C_1 \cdot d_G(g_1, g_2) + C_2. \tag{8.6}$$

Fix real numbers $r, \epsilon > 0$. Put

$$i := \frac{2C_1R + 2C_2}{\epsilon}. \tag{8.7}$$

Now choose s such that we have $\overline{H} \in \mathcal{F}$ or $[\Delta : \pi^G(\overline{H}] \geq i$ for every hyperelementary subgroup $H \subseteq G_s$. If $\overline{H} \in \mathcal{F}$, we can choose f_H to be the projection $G \to \mathrm{pt}$. Hence we can assume in the sequel $[\Delta : \pi^G(\overline{H}] \geq i$.

By Lemma 3.5 (iii) we can find an i-expansive map $\phi \colon D_\infty \to D_\infty$ with $\pi^G(\overline{H}) \subseteq \mathrm{im}(\phi)$ and an element $u \in \mathbb{R}$ such that the affine map $a \colon \mathbb{R} \to \mathbb{R}$ sending x to $i \cdot x + u$ is ϕ-invariant. Let E_H be the simplicial complex whose underlying space is $\mathbb{R}$ and whose set of zero-simplices is $\{m/2 \mid m \in \mathbb{Z}\}$. The standard D_∞-action on $\mathbb{R}$ yields a cell preserving simplicial action on E_H with finite stabilizers. Define a map

$$f_H \colon G \xrightarrow{\pi^G} D_\infty \xrightarrow{\mathrm{ev}} E \xrightarrow{a^{-1}} E,$$

where ev is given by evaluating the D_∞-action on $0 \in \mathbb{R}$.

One easily checks for $d_1, d_2 \in D_\infty$

$$d^{\mathrm{euc}}\big(\mathrm{ev}(d_1), \mathrm{ev}(d_2)\big) \leq d_{D_\infty}(d_1, d_2).$$

We get for $x_1, x_2 \in E$

$$d^{l^1}(x_1,x_2) \le 2 \cdot d^{\mathrm{euc}}(x_1,x_2).$$

This implies together with (8.6) and (8.7) for $g_1, g_2 \in G$ with $d_G(g_1,g_2) \le R$

$$\begin{aligned} d^{l^1}\big(f_H(g_1), f_H(g_2)\big) &= d^{l^1}\big(a^{-1}\circ \mathrm{ev} \circ \pi^G(g_1), a^{-1}\circ \mathrm{ev}\circ\pi^G(g_2)\big) \\ &\le 2\cdot d^{\mathrm{euc}}\big(a^{-1}\circ \mathrm{ev}\circ\pi^G(g_1), a^{-1}\circ\mathrm{ev}\circ\pi^G(g_2)\big) \\ &\le \frac{2}{i}\cdot d^{\mathrm{euc}}\big(\mathrm{ev}\circ\pi^G(g_1), \mathrm{ev}\circ\pi^G(g_2)\big) \\ &\le \frac{2}{i}\cdot d_{D_\infty}\big(\pi^G(g_1),\pi^G(g_2)\big) \\ &\le \frac{2}{i}\cdot\big(C_1\cdot d_G(g_1,g_2)+C_2\big) \\ &\le \frac{2C_1R+2C_2}{i} \\ &= \epsilon. \end{aligned}$$

Since $\pi^G(\overline{H}) \subseteq \mathrm{im}(\phi)$, we can define an $\overline{H}$-action on E by requiring that $\overline{h} \in \overline{H}$ acts on E_H by the standard D_∞-action for the element $d \in D_\infty$ which is uniquely determined by $\phi(d) = \pi^G(h)$. This $\overline{H}$-action is a cell preserving simplicial action and the map f_H is $\overline{H}$-equivariant. Hence f_H has all the desired properties. This finishes the proof of Lemma 8.5. □

Now we continue with the proof of Theorem 8.2. We will show that the assumptions appearing in Lemma 8.5 are satisfied.

If the group Q appearing in (8.3) is a p-group, then G itself is hyperelementary and the claim is true because $G \in \mathcal{F}$. Hence we can assume in the sequel that we can fix two different primes p and q which divide the order of Q.

Let i be a given natural number. Let $\log_p(|Q|)$ be the integer n for which $|Q| = p^n \cdot m$ for some natural number m prime to p holds. Choose a natural number r satisfying

$$\begin{aligned} \frac{p^{r-\log_p(|Q|)}}{\big|\ker(\pi^G\colon G\to\Delta)\big|} &\ge i; \\ \frac{q^{r-\log_q(|Q|)}}{\big|\ker(\pi^G\colon G\to\Delta)\big|} &\ge i; \\ r &\ge \log_p(|Q|); \\ r &\ge \log_q(|Q|). \end{aligned}$$

Our desired number s will be

$$s = p^r q^r.$$

We have to show for any hyperelementary subgroup $H \subseteq G_s$

$$H \in \mathcal{F} \text{or} \big[A_\Delta : (\pi^G(\overline{H}) \cap A_\Delta)\big] \ge i. \tag{8.8}$$

Consider an l-hyperelementary subgroup $H \subseteq G_s$. Since p and q are different we can assume without loss of generality that $p \neq l$. Denote by $H_p \subseteq H$ the p-Sylow subgroup of H. Since H is l-hyperelementary and $l \neq p$, the subgroup H_p is normal in H and a cyclic p-group. Denote by $Q_p \subseteq Q$ the image of H_p under the projection $\mathrm{pr}_s\colon G_s \to Q$. Suppose that $\mathrm{pr}_s(H) \neq Q$. Then the holonomy number of $\overline{H}$ is smaller than the one of G and $\overline{H}$ belongs by the induction hypothesis to $\mathcal{F}$. Hence we can assume in the sequel

$$\mathrm{pr}_s(H) = Q. \tag{8.9}$$

This implies that $Q_p \subseteq Q$ is a normal cyclic p-subgroup of Q and is the p-Sylow subgroup of Q.

Denote by $\overline{Q_p}$ the preimage of Q_p under $\mathrm{pr}\colon G \to Q$. The conjugation action $\rho\colon Q \to \mathrm{aut}(C)$ of Q on C associated to the exact sequence (8.3) yields by restriction a Q_p-action.

We begin with the case, where this Q_p-action is non-trivial. Then we must have $p = 2$ and the target of the epimorphism $\pi^G\colon G \to \Delta$ is $\Delta = D_\infty = \mathbb{Z} \rtimes_{-\mathrm{id}} \mathbb{Z}/2$. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \xrightarrow{i} & G & \xrightarrow{\mathrm{pr}} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \pi^G} & & \downarrow{\scriptstyle \pi^Q} & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \rtimes_{-\mathrm{id}} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1 \end{array}$$

where j is injective and both π^G and π^Q are surjective. Let H' be the image of H under the composite $G_s = G/p^rq^rC \to G/q^rC \xrightarrow{\overline{\pi^G}} \mathbb{Z}/j(q^rC) \rtimes_{-\mathrm{id}} \mathbb{Z}/2 \to \mathbb{Z}/q^r\mathbb{Z} \rtimes_{-\mathrm{id}} \mathbb{Z}/2$, where $\overline{\pi^G}$ is induced by π^G. Then H' agrees with the image of $\overline{H}$ under the composite $G \xrightarrow{\pi^G} \mathbb{Z} \rtimes_{-\mathrm{id}} \mathbb{Z}/2 \to \mathbb{Z}/q^r \rtimes_{-\mathrm{id}} \mathbb{Z}/2$. Since H and hence H' is l-hyperelementary for $l \neq 2$ and $\mathbb{Z}/q^r\mathbb{Z} \rtimes_{-\mathrm{id}} \mathbb{Z}/2$ is 2-hyperelementary, H' is cyclic. Since $\mathrm{pr}_s(H_2) = Q_2$ and the surjectivity of $\pi^Q\colon Q \to \mathbb{Z}/2$ implies $\pi^Q(Q_2) = \mathbb{Z}/2$, the image of H' under the projection $\mathbb{Z}/q^r \rtimes_{-\mathrm{id}} \mathbb{Z}/2 \to \mathbb{Z}/2$ is $\mathbb{Z}/2$. Since H' is cyclic, q is different from $p = 2$ and hence odd, we conclude $\mathbb{Z}/q^r \cap H' = \{0\}$. Hence

$$\left[\mathbb{Z} : (\mathbb{Z} \cap \pi^G(\overline{H}))\right] \geq \left[\mathbb{Z}/q^r : (\mathbb{Z}/q^r \cap H')\right] = q^r.$$

Since by our choice of r we have $q^r \geq i$, assertion (8.8) holds.

Hence it remains to treat the case where Q_p acts trivially on C. By restriction the exact sequence (8.3) yields the exact sequence

$$1 \to C \xrightarrow{i} \overline{Q_p} \xrightarrow{\mathrm{pr}|_{\overline{Q_p}}} Q_p \to 1.$$

In the sequel we identify C with its image $i(C)$ under the injection $i\colon C \to G$. Since Q_p acts trivially on C and is a finite cyclic p-group, the group $\overline{Q}_p$ is a finitely generated abelian group of rank one. Let $T \subseteq \overline{Q}_p$ be the torsion subgroup. Fix an infinite cyclic subgroup $\mathbb{Z} \subseteq \overline{Q_p}$ such that

$$T \oplus \mathbb{Z} = \overline{Q}_p.$$

Recall that p divides the order of Q and that Q_p is a p-Sylow subgroup of Q. In particular Q_p is non-trivial. Let $n \geq 1$ be the natural number for which $|Q_p| = p^n$. Since $\mathrm{pr}|_T\colon T \to Q_p$ is injective, T is a cyclic p-group of order p^m for some natural number $m \leq n$. Since $r \geq n$ by our choice of r holds, $p^rC \subseteq \{0\} \times \mathbb{Z}$. We get

$$T \oplus \mathbb{Z}/p^rC = \overline{Q_p}/p^rC.$$

Suppose that $\overline{H} \cap T = \{0\}$. Let K be the kernel of the composite

$$\overline{H} \xrightarrow{\mathrm{pr}|_{\overline{H}}} Q \to Q/Q_p.$$

We have $K \subseteq \overline{Q_p}$. Since $\overline{H} \cap T = \{0\}$ implies $K \cap T = \{0\}$, the restriction of the canonical projection $\overline{Q_p} = T \oplus \mathbb{Z} \to \mathbb{Z}$ to K is injective and hence K is infinite cyclic. This implies $h(\overline{H}) \leq |Q/Q_p| < |Q| = h(G)$ and hence $H \in \mathcal{F}$. Therefore we can assume in the sequel

$$\overline{H} \cap T \neq \{0\}. \tag{8.10}$$

Let H' be the image of H under the projection $G/sC \to G/p^rC$. Recall that H_p is a cyclic p-group and a normal p-Sylow group of H and is mapped under the projection $G_s \to Q$ to Q_p. Let

$$H'_p \subseteq \overline{Q_p}/p^rC = T \oplus \mathbb{Z}/p^rC$$

be the image of H_p under the projection $G/sC \to G/p^rC$. Then H'_p is normal in H' and is the p-Sylow subgroup of H'. Since $C/p^rC \subseteq \overline{Q_p}/p^rC$ is a subgroup of order p^r, we conclude

$$H' \cap C/p^rC = H'_p \cap C/p^rC.$$

The intersection $\overline{H} \cap T$ is p-torsion, because T is p-torsion. Thus $\overline{H} \cap T \subseteq \overline{H_p} := \alpha_s^{-1}(H_p)$. Therefore we can conclude $H'_p \cap T \neq \{0\}$ from (8.10). Since H'_p is cyclic and $H'_p \cap T \neq \{0\}$, we must have $H'_p \cap \mathbb{Z}/p^rC = \{0\}$. Since $|T| \cdot H'_p$ is contained in $\mathbb{Z}/p^rC$, we conclude $|T| \cdot H'_p = \{0\}$ and hence the order of $|H'_p|$ divides the order of $|T|$. This implies

$$|H'_p| \leq |T| \leq p^n. \tag{8.11}$$

We conclude

$$\begin{aligned}
[G : \overline{H}] &= [G_s : H] \\
&\geq [G/p^rC : H'] \\
&\geq [C/p^rC : (C/p^rC \cap H')] \\
&= [C/p^rC : (C/p^rC \cap H'_p)] \\
&= \frac{|C/p^rC|}{|C/p^rC \cap H'_p|} \\
&\geq \frac{|C/p^rC|}{|H'_p|} \\
&\geq \frac{p^r}{p^n} \\
&= p^{r-n}.
\end{aligned}$$

This and our choice of r implies

$$\left[\Delta : \pi^G(\overline{H})\right] \geq \frac{[G : \overline{H}]}{|\ker(\pi^G)|} \geq \frac{p^{r-n}}{|\ker(\pi^G)|} \geq i.$$

Hence assertion (8.8) is true. This finishes the proof of Theorem 8.2. □

We will use the following result from [18, Corollary 1.2, Remark 1.6]. See also [17].

Theorem 8.12. *Let G be a group. Let $\mathcal{VC}yc_I$ be the family of subgroups which are either finite or admit an epimorphism onto $\mathbb{Z}$ with a finite kernel. Obviously $\mathcal{VC}yc_I \subseteq \mathcal{VC}yc$. Then for an any additive G-category $\mathcal{A}$ the relative assembly map*

$$\operatorname{asmb}_n^{G,\mathcal{VC}yc_I,\mathcal{VC}yc}\colon H_n^G\bigl(E_{\mathcal{VC}yc_I}(G);\mathbf{K}_{\mathcal{A}}\bigr) \to H_n^G\bigl(E_{\mathcal{VC}yc}(G);\mathbf{K}_{\mathcal{A}}\bigr)$$

is bijective for all $n \in \mathbb{Z}$.

The Transitivity Principle 2.10, Theorem 8.2, and Theorem 8.12 imply

Corollary 8.13. *Let G be a group. Let $\mathcal{H}_I$ be the family of subgroups which are either finite or which are hyperelementary and admit an epimorphism onto $\mathbb{Z}$ with a finite kernel. Obviously $\mathcal{H}_I \subseteq \mathcal{VC}yc$. Then for an any additive G-category $\mathcal{A}$ the relative assembly map*

$$\operatorname{asmb}_n^{G,\mathcal{H}_I,\mathcal{VC}yc}\colon H_n^G\bigl(E_{\mathcal{H}_I}(G);\mathbf{K}_{\mathcal{A}}\bigr) \to H_n^G\bigl(E_{\mathcal{VC}yc}(G);\mathbf{K}_{\mathcal{A}}\bigr)$$

is bijective for all $n \in \mathbb{Z}$.

Every infinite p-hyperelementary group for odd p admits an epimorphism to $\mathbb{Z}$ with a finite kernel. A 2-hyperelementary group G admits an epimorphism to $\mathbb{Z}$ with a finite kernel if and only if there exists a central extension $1 \to \mathbb{Z} \to G \to P \to 1$ for a finite 2-group P.

Theorem 8.14. *Let G be a group. Let $\mathcal{VC}yc_I$ be the family of subgroups which are either finite or admit an epimorphism onto $\mathbb{Z}$ with a finite kernel. Let $\mathcal{F}in$ be the family of finite groups. Obviously $\mathcal{F}in \subseteq \mathcal{VC}yc_I$. Then for an any additive G-category $\mathcal{A}$ the relative assembly map*

$$\operatorname{asmb}_n^{G,\mathcal{F}in,\mathcal{VC}yc}\colon H_n^G\bigl(E_{\mathcal{F}in}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\bigr) \to H_n^G\bigl(E_{\mathcal{VC}yc_I}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\bigr)$$

is bijective for all $n \in \mathbb{Z}$.

Sketch of proof. The argument given in [28, Lemma 4.2] goes through since it is based on the Wang sequence for a semidirect product $F \rtimes \mathbb{Z}$ which can be generalized for additive categories as coefficients. □

Acknowledgments

The work was financially supported by the Sonderforschungsbereich 878—Groups, Geometry & Actions—and the Leibniz-Preis of the last author. And the second author was partially supported by an NSF grant.

The authors thank the referee for a long list of helpful comments.

References

[1] Herbert Abels, *Parallelizability of proper actions, global K-slices and maximal compact subgroups*, Math. Ann. **212** (1974/75), 1–19. MR0375264 (51 #11460)

[2] Arthur Bartels, Siegfried Echterhoff, and Wolfgang Lück, *Inheritance of isomorphism conjectures under colimits*, K-theory and noncommutative geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 41–70, DOI 10.4171/060-1/2. MR2513332 (2012a:55005)

[3] Arthur Bartels and Wolfgang Lück, *Isomorphism conjecture for homotopy K-theory and groups acting on trees*, J. Pure Appl. Algebra **205** (2006), no. 3, 660–696, DOI 10.1016/j.jpaa.2005.07.020. MR2210223 (2007e:19005)

[4] Arthur Bartels and Wolfgang Lück, *Induction theorems and isomorphism conjectures for K- and L-theory*, Forum Math. **19** (2007), no. 3, 379–406, DOI 10.1515/FORUM.2007.016. MR2328114 (2008j:19003)

[5] Arthur Bartels and Wolfgang Lück, *On crossed product rings with twisted involutions, their module categories and L-theory*, Cohomology of groups and algebraic K-theory, Adv. Lect. Math. (ALM), vol. 12, Int. Press, Somerville, MA, 2010, pp. 1–54. MR2655174 (2011m:19003)

[6] Arthur Bartels and Wolfgang Lück, *The Borel conjecture for hyperbolic and* CAT(0)*-groups*, Ann. of Math. (2) **175** (2012), no. 2, 631–689, DOI 10.4007/annals.2012.175.2.5. MR2993750

[7] A. Bartels and W. Lück, *The Farrell-Hsiang method revisited*, Math. Ann. **354** (2012), no. 1, 209–226, DOI 10.1007/s00208-011-0727-3. MR2957625

[8] Arthur Bartels, Wolfgang Lück, and Holger Reich, *Equivariant covers for hyperbolic groups*, Geom. Topol. **12** (2008), no. 3, 1799–1882, DOI 10.2140/gt.2008.12.1799. MR2421141 (2009d:20102)

[9] Arthur Bartels, Wolfgang Lück, and Holger Reich, *The K-theoretic Farrell-Jones conjecture for hyperbolic groups*, Invent. Math. **172** (2008), no. 1, 29–70, DOI 10.1007/s00222-007-0093-7. MR2385666 (2009c:19002)

[10] Arthur Bartels, Wolfgang Lück, and Holger Reich, *On the Farrell-Jones conjecture and its applications*, J. Topol. **1** (2008), no. 1, 57–86, DOI 10.1112/jtopol/jtm008. MR2365652 (2008m:19001)

[11] A. Bartels, W. Lück, and S. Weinberger, *On hyperbolic groups with spheres as boundary. arXiv:0911.3725v1 [math.GT], to appear in the Journal of Differential Geometry* (2009).

[12] Arthur Bartels and Holger Reich, *Coefficients for the Farrell-Jones conjecture*, Adv. Math. **209** (2007), no. 1, 337–362, DOI 10.1016/j.aim.2006.05.005. MR2294225 (2008a:19002)

[13] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR1744486 (2000k:53038)

[14] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1982. MR672956 (83k:20002)

[15] J. Bryant, S. Ferry, W. Mio, and S. Weinberger, *Topology of homology manifolds*, Ann. of Math. (2) **143** (1996), no. 3, 435–467, DOI 10.2307/2118532. MR1394965 (97b:57017)

[16] Frank Connolly and Tadeusz Koźniewski, *Rigidity and crystallographic groups. I*, Invent. Math. **99** (1990), no. 1, 25–48, DOI 10.1007/BF01234410. MR1029389 (91g:57019)

[17] James F. Davis, Qayum Khan, and Andrew Ranicki, *Algebraic K-theory over the infinite dihedral group: an algebraic approach*, Algebr. Geom. Topol. **11** (2011), no. 4, 2391–2436, DOI 10.2140/agt.2011.11.2391. MR2835234 (2012h:19008)

[18] James F. Davis, Frank Quinn, and Holger Reich, *Algebraic K-theory over the infinite dihedral group: a controlled topology approach*, J. Topol. **4** (2011), no. 3, 505–528, DOI 10.1112/jtopol/jtr009. MR2832565 (2012g:19006)

[19] Patrick B. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996. MR1441541 (98h:53002)

[20] F. T. Farrell and W. C. Hsiang, *The Whitehead group of poly-(finite or cyclic) groups*, J. London Math. Soc. (2) **24** (1981), no. 2, 308–324, DOI 10.1112/jlms/s2-24.2.308. MR631942 (83b:20041)

[21] F. T. Farrell and W. C. Hsiang, *Topological characterization of flat and almost flat Riemannian manifolds M^n ($n \neq 3, 4$)*, Amer. J. Math. **105** (1983), no. 3, 641–672, DOI 10.2307/2374318. MR704219 (84k:57017)

[22] F. T. Farrell and L. E. Jones, *The surgery L-groups of poly-(finite or cyclic) groups*, Invent. Math. **91** (1988), no. 3, 559–586, DOI 10.1007/BF01388787. MR928498 (89d:57049)

[23] F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. **6** (1993), no. 2, 249–297, DOI 10.2307/2152801. MR1179537 (93h:57032)

[24] I. Hambleton, E. K. Pedersen, and D. Rosenthal, *Assembly maps for group extensions in K- and L-theory. Preprint, arXiv:math.KT/0709.0437v1, to appear in Pure and Applied Mathematics Quarterly* (2007).

[25] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978. MR514561 (80k:53081)

[26] Nigel Higson and Gennadi Kasparov, *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. **144** (2001), no. 1, 23–74, DOI 10.1007/s002220000118. MR1821144 (2002k:19005)

[27] P. Kühl, *Isomorphismusvermutungen und 3-Mannigfaltigkeiten. Preprint, arXiv:0907.0729v1 [math.KT]* (2009).

[28] Wolfgang Lück, *K- and L-theory of the semi-direct product of the discrete 3-dimensional Heisenberg group by* $\mathbb{Z}/4$, Geom. Topol. **9** (2005), 1639–1676 (electronic), DOI 10.2140/gt.2005.9.1639. MR2175154 (2006e:19007)

[29] Wolfgang Lück, *Survey on classifying spaces for families of subgroups*, Infinite groups: geometric, combinatorial and dynamical aspects, Progr. Math., vol. 248, Birkhäuser, Basel, 2005, pp. 269–322, DOI 10.1007/3-7643-7447-0_7. MR2195456 (2006m:55036)

[30] W. Lück and H. Reich, *The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory*, In Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 703–842.

[31] Frank Quinn, *An obstruction to the resolution of homology manifolds*, Michigan Math. J. **34** (1987), no. 2, 285–291, DOI 10.1307/mmj/1029003559. MR894878 (88j:57016)

[32] Frank Quinn, *Algebraic K-theory over virtually abelian groups*, J. Pure Appl. Algebra **216** (2012), no. 1, 170–183, DOI 10.1016/j.jpaa.2011.06.001. MR2826431 (2012f:19006)

[33] Frank Quinn, *Controlled K-theory I: Basic theory*, Pure Appl. Math. Q. **8** (2012), no. 2, 329–421. MR2900172

[34] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. MR0507234 (58 #22394a)

[35] S. K. Roushon, *The Farrell-Jones isomorphism conjecture for 3-manifold groups*, J. K-Theory **1** (2008), no. 1, 49–82, DOI 10.1017/is007011012jkt005. MR2424566 (2010e:57025)

[36] Sayed K. Roushon, *The isomorphism conjecture for 3-manifold groups and K-theory of virtually poly-surface groups*, J. K-Theory **1** (2008), no. 1, 83–93, DOI 10.1017/is007011012jkt006. MR2424567 (2009m:57001)

[37] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973. Translated from the French; Graduate Texts in Mathematics, No. 7. MR0344216 (49 #8956)

[38] *Lie groups and Lie algebras. II*, Encyclopaedia of Mathematical Sciences, vol. 21, Springer-Verlag, Berlin, 2000. Discrete subgroups of Lie groups and cohomologies of Lie groups and Lie algebras; A translation of *Current problems in mathematics. Fundamental directions. Vol. 21* (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow, 1988 [MR0968444 (89f:22001)]; Translated by John Danskin; Translation edited by A. L. Onishchik and E. B. Vinberg. MR1756406 (2001a:22001)

[39] Christian Wegner, *The K-theoretic Farrell-Jones conjecture for CAT(0)-groups*, Proc. Amer. Math. Soc. **140** (2012), no. 3, 779–793, DOI 10.1090/S0002-9939-2011-11150-X. MR2869063

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