THE FEICHTINGER CONJECTURE

AND

REPRODUCING KERNEL HILBERT SPACES

A Dissertation

Presented to

the Faculty of the Department of Mathematics

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In Partial Fulfillment

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Doctor of Philosophy

By

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I dedicate my thesis to my lovely family.

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Abstract

In this dissertation, we study the Feichtinger Conjecture(FC), which has been shown to be equivalent to the celebrated Kadison-Singer Problem. The FC states that every norm-bounded below Bessel sequence in a Hilbert space can be partitioned into finitely many Riesz basic sequences.

This study is divided into two parts. In the first part, we explore the FC in the setting of reproducing kernel Hilbert spaces. The second part of this study introduces two new directions to explore the FC further, which are based on a factorization of positive operators in $B(\ell^2)$. The results presented in the later part have a mixed flavor in the sense that some of them point in the direction of finding a negative answer to the FC, whereas others prove the FC for some special cases.

In the first part of the thesis, we show that in order to prove the FC it is enough to prove that in every Hilbert space, contractively contained in the Hardy space H^2, each Bessel sequence of normalized kernel functions can be partitioned into finitely many Riesz basic sequences. In addition, we examine some of these spaces and show that the above holds in them.

We also look at products and tensor products of kernels, where using Schur products we obtain some interesting results. These results allows us to prove that in the Bargmann-Fock spaces on the n-dimensional complex plane and the weighted Bergman spaces on the unit ball, the Bessel sequences of normalized kernel functions split into finitely many Riesz basic sequences. We also prove that the same result holds in the $H^2_{\alpha,\beta}$ spaces as well.

Contents

Chapter 1

Introduction

The Feichtinger conjecture originated in time-frequency analysis and currently is a topic of much interest as it has been shown to be equivalent to the celebrated Kadison-Singer Problem [21], and hence to many open problems in analysis. There is a significant body of work on this conjecture [6, 7, 8, 17, 20, 21, 18, 19, 26].

In his work on time-frequency analysis, Feichtinger observed that all the Gabor frames he was working with could be decomposed into finite union of Riesz basic sequences. This observation led to the Feichtinger conjecture.

Conjecture 1.0.1. *Every norm-bounded below frame sequence in a Hilbert space can be partitioned into finitely many Riesz basic sequences.*

The Feichtinger conjecture dates back to at least 2003 and appeared in print in [18]. In [15] and [26], the Feichtinger conjecture is considered for special frames such as wavelet and Gabor frames, and frames of translates. In these papers, the authors work with a decay assumption on the off-diagonal entries of the Grammian, where this decay condition is in terms of the indices of the corresponding frame elements. In our study, we do not assume any decay of the off-diagonal entries of the Grammians. However, our results do involve some decay estimates, which most of the time is because of the use of Theorems 3.0.6 and 3.3.14. Theorem 3.0.6 proves the Feichtinger conjecture for a class of sequences in the Hardy space H^2. The key idea in this proof is to obtain a finite partition of the involved sequence such that the off-diagonal entries of the Grammian corresponding to each subsequence are very small, where the measure of being small is constant for the sequence. Theorem 3.3.14 is a result about general self-adjoint matrices in $B(\ell^2)$ with positive entries. We apply this result to a "particular" Grammian in $B(\ell^2)$ (Theorem 3.3.15). As a consequence, we get a finite partition of $\mathbb{N}$ such that the finite sum of the compressions of the Grammian, obtained corresponding to the sets in the partition, is very small.

In general it is expected that the Kadison-Singer Problem, and hence the Feichtinger conjecture will turn out to be false. Perhaps, this is why later in [20], the following "stronger" version of the Feichtinger conjecture was introduced, and shown equivalent to the Feichtinger conjecture. We work with this new version of the Feichtinger conjecture. For our purposes, we refer to this version as the Feichtinger conjecture(FC).

Conjecture 1.0.2. Feichtinger Conjecture(FC). *Every norm-bounded below Bessel sequence in a Hilbert space can be partitioned into finitely many Riesz basic sequences.*

In [8], the FC is considered for some special Bessel sequences in two families of model subspaces of H^2. The results in [8] uses a variant of Theorem 3.0.6. More recently, two new directions are employed to further explore the FC, one using the discrete Hilbert transforms [10], and the other using syndetic sets [40]. Henceforth, we say that a Bessel sequence satisfies the FC if it can be partitioned into finitely many Riesz basic sequences.

Our work in Chapters 3 is motivated by some work of Nikolski. In his lecture at the AIM workshop "The Kadison-Singer Problem" in 2006, Nikolski proved that the FC holds true for Bessel sequences consisting of normalized kernel functions in the Hardy space H^2. Later in 2009, Baranov and Dyakonov [8] proved the FC for Bessel sequences of normalized kernel functions for two families of model subspaces of H^2.

Thus, we were motivated to seek a converse. That is, to find a sufficiently large family of reproducing kernel Hilbert spaces, so that if one verified that the FC held for each Bessel sequence of normalized kernels in those spaces, then that would guarantee the full FC.

We accomplish this in Chapter 3.1. We specialize the FC to the case where the underlying Hilbert space belongs to a special family of reproducing kernel Hilbert spaces on the unit disk, namely, the Hilbert spaces that are contractively contained in the Hardy space H^2 and require, in addition, that the norm-bounded below Bessel sequence consists of the normalized kernel functions for a sequence of points in the disk. One of our results (Theorem 3.1.9) is that this special version of the FC is equivalent to the FC.

In addition, we also prove that in order to verify the FC it is enough to test a specific family of sequences in H^2, where this family is "related" to kernel functions in H^2, and thus carries some "nice" structure.

Later in Chapter 3.2, we analyze these new equivalences. In the course of this analysis, we get some interesting results; one of them (Theorem 3.2.11) has a beautiful application in Chapter 4.3. Lastly in Chapter 3.3, we discuss how various operations on kernel functions affect the equivalent variants of the FC, obtained in the previous section. In particular, we look at the products and tensor products of kernels, along with two new operations on kernels, namely, the pull-back operation and the push-out operation, which were introduced in [39]. Using Schur products, we obtain some interesting results about products of kernel functions, which has some surprising applications in the last two sections of chapter 4. In addition, one of the results (Corollary 3.3.5) obtained in Chapter 3.3 generalizes a theorem of Baranov and Dyakonov from [8].

The results presented in Chapters 3.1 and 3.2 are joint work [28] with my adviser Prof. Vern Paulsen. The rest of the work presented in this thesis is also done under the supervision of my adviser, and would not have been possible without his helpful comments and valuable suggestions.

In Chapter 4, we study the FC for some well-known spaces, namely, the Hardy space H^2 on the unit disk, the $H^2_{\alpha,\beta}$ spaces, which were introduced in [23], the weighted Bergman spaces on the unit ball, and the Bargmann-Fock spaces on the n-dimensional complex plane. In chapter 4.2, we give Nikolski's proof of the fact that Bessel sequences of normalized kernel functions in H^2 satisfy the FC (Theorem 4.2.4), which he presented at the above mentioned AIM workshop in 2006. In Chapter 4.3, we prove that the Bessel sequences of normalized kernel functions in $H^2_{\alpha,\beta}$ spaces also satisfy the FC (Theorem 4.3.1). Lastly, Chapters 4.4 and 4.5 contains the proof of the FC for Bessel sequences of normalized kernel functions in the weighted Bergman spaces on the unit ball (Theorems 4.4.6, 4.4.7), and the Bargmann-Fock spaces on the n-dimensional complex plane (Theorem 4.5.4), respectively. Though, in the several variable case, the above result holds for a restrictive class of weighted Bergman spaces. The proofs presented in this section are some beautiful applications, as mentioned above, of results (Theorem 3.3.12 and Theorem 3.3.16) about products of kernels from Chapter 3.3.

Finally, in Chapter 5, we introduce a completely new approach to the FC, which is based on a "special" factorization of positive operators in $B(\ell^2)$ [1]. This factorizations breaks a positive operator into two pieces: one of the form UU^*, where $U \in B(\ell^2)$ is upper triangular, and the other a completely non factorizable operator in $B(\ell^2)$. As a result, the study of a general Grammian reduces to the study of Grammians of these two specific forms. In Chapter 5.1, we discuss Grammians which are completely non-factorizable. This study contains some interesting observations which point in the direction of constructing a counter example to the FC.

In Chapter 5.2, we discuss Grammians of the form UU^*, where $U \in B(\ell^2)$ is upper triangular. Here, we prove the FC in some special cases. In particular, we show that if a frame for ℓ^2 has Grammian of this later form, then it must be a Riesz basis for ℓ^2. In addition, as a corollary to the results in this section, we obtain that an orthogonal projection with finite codimension is $(r, m+2)$-pavable, where $0 < r < 1$ and m is at most the dimension of the kernel of the orthogonal projection.

Chapter 2

Preliminaries

2.1 Frame Theory

Given a set $J \subseteq \mathbb{N}$, we let $\ell^2(J)$ denote the closed linear span of $\{e_i\}_{i\in J}$ in $\ell^2 = \ell^2(\mathbb{N})$, where $\{e_i\}_{i\in\mathbb{N}}$ is the canonical orthonormal basis for ℓ^2. Further, $I_{\ell^2(J)}$ will denote the identity operator on $\ell^2(J)$, and the orthogonal projection onto $\ell^2(J)$ will be denoted by P_J. For the case $J = \mathbb{N}$, we will write the identity $I_{\ell^2(\mathbb{N})}$ simply as I. We shall identify a bounded operator on ℓ^2 as a bounded operator on $\ell^2(J)$, if it maps $\ell^2(J)$ into itself and it is zero on $(\ell^2(J))^\perp$.

A sequence $\{f_i\}_{i\in\mathbb{N}}$ in a Hilbert spaces $\mathcal{H}$ is called a **frame** for $\mathcal{H}$ if there exist constants $A, B > 0$, such that

$$A\|x\|^2 \leq \sum_{i\in\mathbb{N}} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \tag{2.1}$$

for every $x \in \mathcal{H}$. In particular, a frame is called a **Parseval frame** for $\mathcal{H}$, if $A = B = 1$, in the above equation. The constants A and B in Equation (2.1) are called **lower** and **upper frame bounds**, respectively. A sequence $\{f_i\}_{i\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called a **frame sequence** if it is a frame for $\overline{\text{span}\{f_i : i \in \mathbb{N}\}}$. If only the right hand side inequality holds in Equation (2.1), then $\{f_i\}_{i\in\mathbb{N}}$ is called a **Bessel sequence**. Thus, every frame sequence is a Bessel sequence. A Bessel sequence $\{f_i\}_{i\in\mathbb{N}}$ is called **norm-bounded below** if there exists a constant $\delta > 0$, such that $\|f_i\| \geq \delta$, for every $i \in \mathbb{N}$. Note that a Bessel sequence is always bounded above.

Further, a sequence $\{f_i\}_{i\in\mathbb{N}}$ in a Hilbert spaces $\mathcal{H}$ is called a **Riesz basis** for $\mathcal{H}$, if there exists an orthonormal basis $\{u_i\}_{i\in\mathbb{N}}$ for $\mathcal{H}$ and an invertible operator $S \in B(\mathcal{H})$, such that $S(u_i) = f_i$, for every $i \in \mathbb{N}$. It is easy to verify that a sequence $\{f_i\}_{i\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$ if and only if its linear span is dense in $\mathcal{H}$ and there exist constants $A, B > 0$, such that

$$A \sum_{i\in\mathbb{N}} |\alpha_i|^2 \leq \| \sum_{i\in\mathbb{N}} \alpha_i f_i\|^2 \leq B \sum_{i\in\mathbb{N}} |\alpha_i|^2,$$

for all ℓ^2-sequences $\{\alpha_i\}_{i\in\mathbb{N}}$. A sequence $\{f_i\}_{i\in\mathbb{N}}$ is called a **Riesz basic sequence** if it is a Riesz basis for $\overline{\text{span}\{f_i : i \in \mathbb{N}\}}$. It is well-known that every Riesz basic sequence is a frame sequence, we shall include the proof of this later in this section.

Given a Bessel sequence $\{f_i\}_{i\in J}$ in a Hilbert space $\mathcal{H}$, we define the corresponding **analysis operator** $F : \mathcal{H} \to \ell^2(J)$, by $F(x) = (\langle x, f_i\rangle)_{i\in J}$. It is easy to check that $F^* : \ell^2(J) \to \mathcal{H}$ satisfies $F^*(e_i) = f_i$ for all $i \in J$, and $FF^* = (\langle f_j, f_i\rangle)$. The operators F^* and FF^* are called the **synthesis operator** and the **Grammian operator**, respectively. Often we shall refer to the Grammian operator as Grammian. Note that:

(i) $\{f_i\}_{i\in J}$ is a Bessel sequence if and only if the corresponding analysis operator F is bounded;

(ii) $\{f_i\}_{i\in J}$ is a frame sequence if and only if $F : \overline{\mathrm{span}\{f_i : i \in J\}} \to \ell^2(J)$ is bounded and bounded below;

(iii) $\{f_i\}_{i\in J}$ is a Riesz basic sequence if and only if $F : \overline{\mathrm{span}\{f_i : i \in J\}} \to \ell^2(J)$ is invertible.

Thus, if $\{f_i\}_{i\in J}$ is a frame for a Hilbert space $\mathcal{H}$ and $F : \mathcal{H} \to \ell^2(J)$ is the corresponding analysis operator, then $F^*F : \mathcal{H} \to \mathcal{H}$ is invertible. Moreover, in this case every $x \in \mathcal{H}$ can be written as $x = \sum_{i\in J} \alpha_i f_i$, where $\alpha_i = \langle x, (F^*F)^{-1} f_i\rangle$.

Henceforth, given a Bessel sequence $\{f_i\}_{i\in J}$ in a Hilbert space $\mathcal{H}$, F is reserved for the analysis operator from $\mathcal{H}$ to $\ell^2(J)$, as defined above. Also, when it is not necessary, we shall not specify the indexing set of the sequence and simply write it as $\{f_i\}$.

Lastly, a sequence in a Hilbert space $\mathcal{H}$ is called **complete** if its closed linear span is dense in $\mathcal{H}$, otherwise it is called an **incomplete** sequence.

We now begin with some preliminary results which we will be use thoughout Chapters 3, 4 and 5.

Proposition 2.1.1. *Let $\{f_i\}$ be a Riesz basis for a Hilbert space $\mathcal{H}$. Then $\{f_i\}$ is a frame for $\mathcal{H}$.*

Proof. Let $S \in B(\mathcal{H})$ be an invertible operator and let $\{u_i\}$ be an orthonormal basis for $\mathcal{H}$ so that for each i, $S(u_i) = f_i$. Then

$$\sum_i |\langle x, f_i\rangle|^2 = \sum_i |\langle x, S(u_i)\rangle|^2 = \sum_i |\langle S^*(x), u_i\rangle|^2 = \|S^*(x)\|^2.$$

As, $S \in B(\mathcal{H})$ and is invertible, therefore

$$\|S^*(x)\| \leq \|S^*\|\|x\|, \ \|S^*(x)\| \geq \frac{\|x\|}{\|S^{*-1}\|}.$$

Hence $\{f_i\}$ is a frame for $\mathcal{H}$. □

We now note the following very useful characterization of Bessel sequences due to Nina Bari [9].

Proposition 2.1.2 (Bari, [9]). *A sequence $\{f_i\}_{i\in J}$ in a Hilbert space $\mathcal{H}$ is a Bessel sequence if and only if $FF^* = (\langle f_j, f_i\rangle) \in B(\ell^2(J))$.*

Proof. A sequence $\{f_i\}_{i\in J}$ in a Hilbert space $\mathcal{H}$ is a Bessel sequence if and only if the corresponding analysis operator $F : \mathcal{H} \to \ell^2(J)$ is bounded, and hence the result follows. □

The following is a characterization of Riesz basic sequences in terms of the associated Grammian. We shall first set some notations. Given a bounded operator T from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$, we let $Ran(T)$ denote the range of T, which is a (not necessarily closed) subspace of $\mathcal{K}$, equipped with the norm of $\mathcal{K}$, and $Ker(T)$ denote the kernel of T.

Proposition 2.1.3. *Let $\{f_i\}_{i\in J} \subseteq \mathcal{H}$. Then $\{f_i\}_{i\in J}$ is a Riesz basis for $\mathcal{H}$ if and only if it is a Bessel sequence with closed linear span equal to $\mathcal{H}$, and there exists a constant $c > 0$ such that $FF^* \geq cI_{\ell^2(J)}$.*

Proof. First, we assume that $\{f_i\}_{i\in J}$ is a Riesz basis for $\mathcal{H}$. Then there exists an orthonormal basis $\{u_i\}_{i\in J}$ for $\mathcal{H}$ and an invertible operator $S \in B(\mathcal{H})$, such that $S(u_i) = f_i$, for all $i \in J$. Clearly, the closed linear span of $\{f_i\}_{i\in J}$ is $\mathcal{H}$. Further, we note that $\sum_{i\in J} |\langle x, f_i\rangle|^2 = \sum_{i\in J} |\langle S^*(x), u_i\rangle|^2 = \|S^*(x)\|^2 \leq \|S^*\|^2\|x\|^2$, and thus $\{f_i\}_{i\in J}$ is a Bessel sequence.

To verify the other conditions, we let $U : \ell^2(J) \to \mathcal{H}$ be the unitary operator defined by $U(e_i) = u_i$. Since $F^*(e_i) = f_i = S(u_i) = SU(e_i)$, we get $FF^* = U^*S^*SU$. Thus $FF^* \in B(\ell^2(J))$ is an invertible operator, and so $(FF^*)^{1/2}$ is also invertible. Therefore, there exists a constant $a > 0$ such that $\|(FF^*)^{1/2}(x)\| \geq a\|x\|$ for every $x \in \ell^2(J)$. Hence, $FF^* \geq a^2 I_{\ell^2(J)}$.

Conversely, we assume that $\{f_i\}_{i\in J}$ satisfies the three conditions. Since it is a Bessel sequence, F and F^* are both bounded. Also $FF^* \geq cI_{\ell^2(J)}$, which implies that F^* is bounded below. Thus its range is a closed subspace of $\mathcal{H}$. But each $f_i = F^*(e_i)$ is in the range of F^*, and thus $Ran(F^*) = \mathcal{H}$. Hence F^* is one-to-one and onto. Let $U = F^*(FF^*)^{-1/2}$, then $U^*U = I_{\ell^2(J)}$, so that U is an isometry, but U is invertible also, and hence, $U \in B(\ell^2(J), \mathcal{H})$ is a unitary. Let $u_i = U(e_i)$. Since U is a unitary, $\{u_i\}_{i\in J}$ is an orthonormal basis of $\mathcal{H}$. Finally note that, $S = F^*U^* \in B(\mathcal{H})$ is invertible and $S(u_i) = f_i$. This completes the proof. $\square$

Corollary 2.1.4. *A sequence $\{f_i\}_{i\in J}$ in a Hilbert space $\mathcal{H}$ is a Riesz basic sequence* *if and only if it is a Bessel sequence and there exists a constant* $c > 0$, *such that* $FF^* \geq cI_{\ell^2(J)}$

Proof. Let $\mathcal{H}_0$ be the closed linear span of $\{f_i\}_{i\in J}$. Let F_0 be the restriction of F to $\mathcal{H}_0$. Then $F_0(x) = 0$ for all $x \in H_0^\perp$, and $F_0^*(e_i) = f_i$ for all i.

Now, $\{f_i\}_{i\in J}$ is a Riesz basic sequence if and only if $\{f_i\}_{i\in J}$ is a Riesz basis for $\mathcal{H}_0$, which by Proposition 2.1.3 is equivalent to $\{f_i\}_{i\in J}$ being a Bessel sequence and $F_0F_0^* \geq cI_{\ell^2(J)}$ for some $c > 0$. Finally, note that for each $i \in J$, $FF^*(e_i) = F(f_i) = F_0(f_i) = F_0F_0^*(e_i)$. Thus $FF^* = F_0F_0^*$, and hence the result follows. □

As a direct consequence of the above corollary we get the following reformulation of the FC.

Corollary 2.1.5. *A Bessel sequence* $\{f_i\}_{i\in\mathbb{N}}$ *can be partitioned into* n *Riesz basic sequences if and only if there exist a partition* $A_1, \ldots, A_n$ *of* $\mathbb{N}$ *and constants* $c_1, \ldots, c_n > 0$, *such that* $P_{A_i}FF^*P_{A_i} \geq c_iP_{A_i}$, *for all* $1 \leq i \leq n$.

Recall that we say that a Bessel sequence in a Hilbert space satisfies the FC if can be partitioned into finitely many Riesz basic sequences.

Proposition 2.1.6. *Let* $\{f_i\}_{i\in J} \subseteq \mathcal{H}$ *and* $\{g_i\}_{i\in J} \subseteq \mathcal{K}$ *be two sequences, such that*

$$\alpha\big(\langle f_j, f_i\rangle\big) \leq D\big(\langle g_j, g_i\rangle\big)D^* \leq \beta\big(\langle f_j, f_i\rangle\big),$$

where $\alpha, \beta > 0$ *and* D *is an invertible, diagonal operator in* $B(\ell^2(J))$. *Then:*

(i) $\{f_i\}_{i\in J}$ *is a Bessel sequence if and only if* $\{g_i\}_{i\in J}$ *is a Bessel sequence,*

(ii) $\{f_i\}_{i\in J}$ *is a frame sequence if and only if* $\{g_i\}_{i\in J}$ *is a frame sequence,*

(iii) $\{f_i\}_{i\in J}$ *satisfies the FC if and only if* $\{g_i\}_{i\in J}$ *satisfies the FC.*

Proof. Note that (i) follows directly from Proposition 2.1.2, since D is invertible.

We now prove (ii). First, note that it is enough to prove only one implication, since D is invertible. Now suppose $\{f_i\}_{i\in J}$ is a frame sequence. Then $\{f_i\}_{i\in J}$ is a Bessel sequence, and the corresponding analysis operator $F : \overline{\operatorname{span}\{f_i : i \in J\}} \to \ell^2(J)$ is bounded below. This yields that $\{g_i\}_{i\in J}$ is a Bessel sequence, using (i). Further by Douglas' factorization [24], there exists a bounded invertible operator $T : \overline{\operatorname{span}\{f_i : i \in J\}} \to \overline{\operatorname{span}\{g_i : i \in J\}}$ such that $TF^* = G^*D^*$, where G is the analysis operator associated with the sequence $\{g_i\}_{i\in J}$. Then $FT^* = DG$. Thus G is bounded below on $\overline{\operatorname{span}\{g_i : i \in J\}}$. Hence $\{g_i\}_{i\in J}$ is a frame sequence.

Finally, (iii) follows from (i) and Corollary 2.1.5, using the facts that D is invertible and $P_A D = DP_A$, for every $A \subseteq J$. □

2.2 Reproducing Kernel Hilbert Spaces

2.2.1 Introduction and Notations

In this section, we recall some basic definitions and results from the theory of reproducing kernel Hilbert spaces. We will consider Hilbert spaces over the field of complex numbers, $\mathbb{C}$. Given a set X, if we equip the set of all functions from X to $\mathbb{C}$, $\mathcal{F}(X, \mathbb{C})$ with the usual operations of addition, $(f+g)(x) = f(x) + g(x)$, and scalar multiplication, $(\lambda f)(x) = \lambda(f(x))$, then $\mathcal{F}(X, \mathbb{C})$ is a vector space over $\mathbb{C}$.

Definition 2.2.1. *Given a set X, we say that $\mathcal{H}$ is a* **reproducing kernel Hilbert space(RKHS)** *on X over $\mathbb{C}$, provided that:*

(i) *$\mathcal{H}$ is a vector subspace of $\mathcal{F}(X, \mathbb{C})$,*

(ii) *$\mathcal{H}$ is endowed with an inner product, $\langle , \rangle$ making it into a Hilbert space,*

(iii) *for every $y \in X$, the* **linear evaluation functional**, *$E_y : \mathcal{H} \to \mathbb{C}$, defined by $E_y(f) = f(y)$, is bounded.*

If $\mathcal{H}$ is a RKHS on X, then since every bounded linear functional is given by the inner product with a unique vector in $\mathcal{H}$, we have that for every $y \in X$, there exists a unique vector, $k_y \in \mathcal{H}$, such that for every $f \in \mathcal{H}$, $f(y) = \langle f, k_y \rangle$.

Definition 2.2.2. *The function k_y is called the reproducing kernel for the point y. The 2-variable function defined by*

$$K(x, y) = k_y(x)$$

is called the reproducing kernel for H.

Henceforth, for simplicity, we shall refer to the reproducing kernel (for a point) as the **kernel function (for a point)**, and sometimes simply as the **kernel**. Further, if K is a kernel for a RKHS $\mathcal{H}$ on a set X, then we will occasionally say that K is a kernel on X. At this point, we introduce another notation, which we shall use throughout Chapters 3 and 4. To do so, let $\mathcal{H}$ be a RKHS on a set X. Then given a non zero kernel function for a point y, k_y, in $\mathcal{H}$, we denote the corresponing normalized function $\frac{k_y}{\|k_y\|_{\mathcal{H}}}$ by $\widetilde{k}_y$ and call it the **normalized kernel function** for the point y or simply the normalized kernel function corresponing to the kernel function k_y. Henceforth, whenever we use the term "normalized kernel function for a point", the corresponing kernel function for the point is always assumed to be non zero. Following are some standard facts about RKHS's. For proofs of the following results and more detailed discussion on RKHS's we refer the reader to [5] and [39].

Proposition 2.2.3. *Let $H_i, i = 1, 2$ be RKHS's on X with kernels, $K_i(x, y)$, respectively. If $K_1(x, y) = K_2(x, y)$ for all $x, y \in X$, then $H_1 = H_2$ and $\|f\|_1 = \|f\|_2$ for every f.*

Proposition 2.2.4. *Let $\mathcal{H}$ be a RKHS of functions on a set X with Kernel K. Then every closed subspace of $\mathcal{H}$ is also a RKHS on X. Moreover, if $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$, then the kernel for $\mathcal{H}_0$ is given by $K_0(x, y) = (Pk_y)(x)$, where $P \in B(\mathcal{H})$ is the orthogonal projection onto $\mathcal{H}_0$ and k_y is the kernel function in $\mathcal{H}$ for the point y.*

Definition 2.2.5. *Let X be a set and let $K : X \times X \to \mathbb{C}$ be a function of two variables. Then K is called a* **positive definite function** *(written : $K \geq 0$) provided that for every n and for every choice of n points, $\{x_1, \ldots, x_n\} \subseteq X$, the $n \times n$ matrix, $(K(x_i, x_j)) \geq 0$.*

Proposition 2.2.6. *Let X be a set and let $\mathcal{H}$ be a RKHS on X with kernel K. Then K is a positive definite function.*

The following converse of the above proposition lead to a characterization of kernel functions.

Theorem 2.2.7 (Moore). *Let X be a set and let $K : X \times X \to \mathbb{C}$ be a function. If K is a positive definite function, then there exists a RKHS, $\mathcal{H}$ of functions on X such that K is the kernel of $\mathcal{H}$.*

The above two results implies that a function $K : X \times X$ is a kernel function for a RKHS on X if and only if K is a positive definite function. This allows us to compare kernel functions. Given two kernel functions K_1 and K_2 on a set X, we say $K_1 \leq K_2$, if $K_2 - K_1$ is also a kernel function.

In Chapters 3 and 4, we shall be working with many RHKS's, so to avoid confusion and to keep track of which kernel function represents which RKHS, we adopt some notations here. Let $\mathcal{H}$ be a RKHS of functions on a set X. Then we denote the kernel function for $\mathcal{H}$ by $K_{\mathcal{H}}$, the kernel function for a point $z \in X$ by $k_z^{\mathcal{H}}$, and the corresponding normalized kernel function $\frac{k_z^{\mathcal{H}}}{\|k_z^{\mathcal{H}}\|_{\mathcal{H}}} = \frac{k_z^{\mathcal{H}}}{\sqrt{K_{\mathcal{H}}(z,z)}}$ by $\widetilde{k}_z^{\mathcal{H}}$. In some cases, the notation for an RKHS is not so simple. It might have too many indices, in which case, keeping the above notation for the kernel can be very inconvenient. To avoid this, we assign special notation to the kernel function. In such a scenario, we will use a different notation (based on the notation of the kernel) for the kernel function for a point and the corresponding normalized kernel function, as we will explain below. Suppose K_S is a kernel function on some set X, where S is an indexing set and can represent anything (e.g. a set, a function or a tuple). Then we will denote the corresponding kernel function for a point $w \in X$ by k_w^S, and the corresponding normalized kernel function $\frac{k_w^S}{\|k_w^S\|}$ by $\widetilde{k}_w^S$.

2.2.2 H^2 and Hilbert Spaces Contractively Contained in H^2

The Hardy space H^2 on the unit disk $\mathbb{D}$, of the complex plain $\mathbb{C}$, is a "reference point" for our study in Chapters 3 and 4. It is an RKHS, consisting of analytic functions on $\mathbb{D}$, with the Kernel function (Szegö Kernel)

$$K(z,w) = \frac{1}{1 - z\overline{w}}, \quad z,\ w \in \mathbb{D}.$$

The set $\{z^n : n \geq 0\}$ is an orthonormal basis for H^2. In the special case of H^2, we will denote the kernel function for a point $w \in \mathbb{D}$ simply by k_w, and in case $k_w \neq 0$, the corresponding normalized kernel function will be denoted by $\widetilde{k}_w(\cdot) = \frac{K(\cdot,w)}{\sqrt{K(w,w)}}$. Also, we will simply use $\|\cdot\|$ for the norm on H^2, unless it is necessary, in which case we will write it as $\|\cdot\|_2$. Apart from the reproducing kernel Hilbert space properties of H^2, its function-theoretic properties are also important for us. We will very often be using two classes of functions from H^2, which are **inner functions** and **Blaschke products**. We do not attempt to define inner functions here, as as it would require some more facts and terminology from function theory. Instead we just note that multiplication with an inner function defines an isometry on H^2.

A Blaschke product is an inner function with some extra structure. The following is the definition of a Blashke product.

Definition 2.2.8. *Let $\{z_i\}$ be a sequence in the unit disk $\mathbb{D}$, such that $\sum_i (1 - |z_i|) < \infty$. Let m be the number of z_i equal 0. Then the Blaschke product with zeroes at $\{z_i\}$ is defined by*

$$B(z) = z^m \prod_{z_i \neq 0} \frac{\overline{z_i}}{|z_i|} \frac{z_i - z}{1 - \bar{z}_i z}.$$

The condition $\sum_i (1 - |z_i|) < \infty$, on $\{z_i\} \subseteq \mathbb{D}$, is known as the **Blaschke condition**. A Blaschke product is called **infinite (finite) Blaschke product** if it has infinitely (finitely) many zeroes, that is, the sequence $\{z_i\}$ in the definition of the Blaschke product contains infinitely (finitely) many distinct $z_i^{'}s$.

At this point we would also like to set a notation, for future. Given an inner function ϕ in H^2, we denote the closed subspace $\{\phi f : f \in H^2\}$ of H^2, by ϕH^2.

Lastly, we introduce a class of Hilbert spaces, which is crucial for the new equivalences of the FC, which we will see in Chapter 3.1. A Hilbert space $\mathcal{H}$ is said to be **contractively contained** in H^2 if it is a vector subspace of H^2 and the canonical inclusion of $\mathcal{H}$ into H^2 is a contraction, that is, $\|x\|_{H^2} \leq \|x\|_{\mathcal{H}}$, for all $x \in \mathcal{H}$.

Proposition 2.2.9. *If a Hilbert space is contractively contained in H^2, then it is a RKHS.*

Proof. Let $\mathcal{H}$ be a Hilbert space, which is contractively contained in H^2, with norm $\|\cdot\|_{\mathcal{H}}$. Then by definition, $\mathcal{H}$ is a vector subspace of H^2 and $\|x\|_2 \leq \|x\|_{\mathcal{H}}$, for all $x \in \mathcal{H}$. Thus, the linear evaluation functionals on $\mathcal{H}$ are bounded, since the linear evaluation functionals on H^2 are bounded. Hence $\mathcal{H}$ is a RKHS. $\square$

For simplicity, we will call a Hilbert space a **contractive Hilbert space** if it is contractively contained in H^2. Next is a characterization of these spaces, which brings to light their connection with positive contractions on H^2.

Let $\mathcal{H}$ be a contractive Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$. Let $T : \mathcal{H} \to H^2$ be the inclusion map, then T and T^* are both contractions. Thus, $P = TT^*$ is a positive contraction in $B(H^2)$. This give rise to another Hilbert space, the **range space** $\mathcal{R}(P^{1/2})$, which one obtains by equipping the range of $P^{1/2}$ with the norm, $\|y\|_P = \|x\|_{H^2}$, where x is the unique vector in the orthogonal complement of the kernel of $P^{1/2}$, such that $y = P^{1/2}x$. We claim that $\mathcal{H} = \mathcal{R}(P^{1/2})$ as sets, and the two norms coincide. To settle the claim, first note that $Ran(P)$ is dense in $\mathcal{H}$. Further, by using the definition of $\|\cdot\|_P$, we can deduce that $\langle P^{1/2}x, P^{1/2}y\rangle_P = \langle x, y\rangle_{H^2}$, whenever $x,\ y\ (Ker(P^{1/2}))^{\perp}$. Then it follows that $Ran(P)$ is dense in $\mathcal{R}(P^{1/2})$, as well. Furthermore, note that if $x \in H^2$, then $\|Px\|_P^2 = \|P^{1/2}x\|_{H^2}^2 = \|T^*x\|_{\mathcal{H}}^2 = \|Px\|_{\mathcal{H}}^2$, since $P^{1/2}x \in Ker(P^{1/2})^{\perp}$, $P = TT^*$, and $TT^*x = T^*x$. Thus the two norms, $\|\cdot\|_P$ and $\|\cdot\|_{\mathcal{H}}$, coincide on $Ran(P)$. Also, $\mathcal{H}$ and $\mathcal{R}an(P^{1/2})$ are contained in H^2, and the norms on both these Hilbert spaces dominate the norm on H^2. Hence $\mathcal{H} = \mathcal{R}(P^{1/2})$ as sets, and the two norms coincide. Thus, for every contractive Hilbert space $\mathcal{H}$, there exists a positive contraction $P \in B(H^2)$ such that $\mathcal{H}$ is the range space, $\mathcal{R}(P^{1/2})$.

On the other hand, given a positive contraction $P \in B(H^2)$, the range space $\mathcal{R}(P^{1/2})$, as defined above, is always a contractive Hilbert space.

For future reference, we formally state the above characterization of contractive Hilbert spaces as follows.

Theorem 2.2.10. *A vector subspace $\mathcal{H}$ of H^2, equipped with another Hilbert space norm $\|\cdot\|_{\mathcal{H}}$, is a contractive Hilbert space if and only if there exists a positive contraction $P \in B(H^2)$, such that $\mathcal{H} = \mathcal{R}(P^{1/2})$, as Hilbert spaces.*

Henceforth, given a positive contraction $P \in B(H^2)$, we shall denote the contractive Hilbert space $\mathcal{R}(P^{1/2})$ by $\mathcal{H}(P)$ and the corresponding kernel function by K_P. Recall that in this case, the kernel function for a point $w \in \mathbb{D}$ is denoted by k_w^P, and when $k_w^P \neq 0$,the corresponding normalized kernel function is denoted by $\widetilde{k}_w^P$. Lastly, we note that $k_w^P = Pk_w$ for $w \in \mathbb{D}$. For more details on these spaces, we refer the reader to [1].

2.3 Schur Product

In this section, we recall the Schur product and some of its important properties. At the end of this section, we will use these properties to get a result (Theorem 2.3.2) about the Schur product of infinite matrices, which we use in Chapters 3.3 and 4.4.

Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a linear map $\phi : B(\mathcal{H}) \to B(\mathcal{K})$ is said to be **positive** if it maps positive operators in $B(\mathcal{H})$ to positive operators in $B(\mathcal{K})$. A positive map is not assumed to be bounded in the definition, but it automatically turns out to be a bounded map [38, Proposition 2.1]. Let $A = (a_{ij})$ and $B = (b_{ij})$ be infinite matrices. We define the **Schur product** of A and B, by

$$A * B = (a_{ij}b_{ij}).$$

If A and B are both bounded operators on ℓ^2, then so is $A * B$. Moreover, if A and B are both positive operators, then $A * B$ is also a positive operator. In both the above statements $\|A * B\| \leq \|A\|\|B\|$, the proof of this inequality along with the proofs of the above two statements can be found in [38]. Thus, for a fixed bounded (positive) operator $A = (a_{ij}) \in B(\ell^2)$, the Schur product give rise to a bounded (positive) linear operator $S_A : B(\ell^2) \to B(\ell^2)$, via $S_A(B) = A * B$ with $\|S_A\| \leq \|A\|$. In general, to guarantee the boundedness and positivity of the Schur product, we do not require the infinite matrix A to come from $B(\ell^2)$. The following remarkable result about when can a Schur product defines a positive map is proved in [38].

Theorem 2.3.1. *Let $A = (a_{ij})$ be an infinite matrix. Then the following are equivalent:*

(i) *$S_A : B(\ell^2) \to B(\ell^2)$ is positive,*

(ii) *there exist a Hilbert space $\mathcal{H}$ and a bounded sequence of vectors $\{x_i\}$ in $\mathcal{H}$, such that $a_{ij} = \langle x_j, x_i \rangle$, for all i, j.*

As a consequence we get the following result.

Proposition 2.3.2. *Let $T \in B(\ell^2)$ be a positive operator, such that there exist a partition $A_1, \ldots, A_N$ of $\mathbb{N}$ and constants $c_1, \ldots, c_N$ with $P_{A_i} T P_{A_i} \geq c_i P_{A_i}$, for all $1 \leq i \leq N$. If $\{x_i\}$ is a bounded sequence in a Hilbert space $\mathcal{H}$, with $\|x_i\| \geq \delta > 0$ for all i, then*

$$P_{A_m}\Big(T * (\langle x_j, x_i \rangle)\Big)P_{A_m} \geq \delta^2 c_m P_{A_m},$$

for all $1 \leq m \leq N$.

Proof. For a fixed m, $1 \leq m \leq N$, note that

$$P_{A_m}\big(T * (\langle x_j, x_i \rangle)\big)P_{A_m} = P_{A_m} T P_{A_m} * \big(\langle x_j, x_i \rangle \big).$$

Also, $P_{A_m}TP_{A_m} \geq c_m P_{A_m}$. Thus, by Theorem 2.3.1, we get

$$P_{A_m}(T * (\langle x_j, x_i \rangle)P_{A_m} \geq c_m P_{A_m} * (\langle x_j, x_i \rangle).$$

But $P_{A_m} * (\langle x_j, x_i \rangle) \geq \delta^2 P_{A_m}$, since $P_{A_m} * (\langle x_j, x_i \rangle)$ is a diagonal matrix with i^{th} diagonal entry $\langle x_i, x_i \rangle = \|x_i\|^2 \geq \delta^2$, if $i \in A_m$ and 0, otherwise. Hence,

$$P_{A_m}(T * (\langle x_j, x_i \rangle))P_{A_m} \geq \delta^2 c_m P_{A_m}.$$

$\square$

Chapter 3

The Feichtinger Conjecture and Reproducing Kernel Hilbert Spaces

In this chapter, we study the Feichtinger conjecture(FC) with a new approach, which involves RKHS's. For convenience, we recall the statement of the FC.

Conjecture 3.0.3. Feichtinger Conjecture(FC) *Every norm-bounded below Bessel sequence in a Hilbert space can be partitioned into finitely many Riesz basic sequences.*

The first section shows some new and interesting equivalences of the FC, the second section analyzes these equivalences, and the final section demonstrates how various operations on kernel functions affect the FC. We begin with a brief history and motivation behind our approach.

The study of Bessel sequences of normalized kernel functions was initiated by Shapiro and Shields in 1961 [48]. They analyzed these sequences purely in the context of interpolation problems in the corresponding RKHS. In the course of this study, they proved a beautiful result about interpolating sequences in the Hardy space H^2, which in the late 60's was reformulated by Nikolski and Pavlov [36, 35] as follows:

Theorem 3.0.4. *A sequence $\{\widetilde{k}_{z_i}\}$ of normalized kernel functions in H^2 is a Riesz basic sequence if and only if there exists a constant $\delta > 0$, such that*

$$(C) \qquad \prod_{i \neq j} \left| \frac{z_i - z_j}{1 - \bar{z}_i z_j} \right| \geq \delta, \qquad j = 1, 2, \ldots$$

In the late 70's, independent of the work of Nikolski and Pavlov, McKenna was also studying kernel functions. In [31] McKenna proved some partial converses to Shapiro and Shields results [48], and thereby brought some more insight to the area. In particular, he proved the following interesting result:

Theorem 3.0.5. *Let $\{\widetilde{k}_{z_i}\}$ be a Bessel sequence of normalized kernel functions in H^2. Then $\{z_i\}$ can be partitioned into finitely many subsequences each of which satisfies the condition (C).*

Nikolski gave a completely different proof of the above theorem which he included in [32]. The FC motivated Nikolski to combine the above two results as follows:

Theorem 3.0.6. *Every Bessel sequence $\{\widetilde{k}_{z_i}\}$ of normalized kernel functions in H^2 can be partitioned into finitely many Riesz basic sequences.*

This introduced methods from reproducing kernel Hilbert space theory to the FC.

3.1 Feichtinger Conjecture for Kernel Functions

In this section, we shall prove some new equivalences of the FC which involves kernel functions. The key idea is to associate a general norm-bounded below Bessel sequence to a sequence of kernel functions in H^2. Thus, kernel functions in H^2 will prove to be very crucial objects in our study of the FC. We start with some important properties of these kernel functions. The following is a simple, yet interesting observation about kernel functions in H^2. Recall that the kernel function for H^2 is $K(z,w) = \frac{1}{1-z\overline{w}},\ z,w \in \mathbb{D}$.

Proposition 3.1.1. *Let $\{\widetilde{k}_{z_i}\}$ be a Bessel sequence in H^2. Then the sequence $\{z_i\}$ satisfies the Blaschke condition, $\sum_i (1-|z_i|) < \infty$.*

Proof. Since $\{\widetilde{k}_{z_i}\}$ is a Bessel sequence, there exists a constant $B > 0$ such that

$$\sum_i |\langle \widetilde{k}_{z_j}, \widetilde{k}_{z_i}\rangle|^2 \leq B,$$

for all j. Note that,

$$\begin{aligned} |\langle \widetilde{k}_{z_j}, \widetilde{k}_{z_i}\rangle|^2 &= \frac{(1-|z_j|^2)(1-|z_i|^2)}{|1-z_i\overline{z_j}|^2} \\ &\geq \frac{(1-|z_j|^2)}{4}(1-|z_i|^2) \\ &\geq \frac{(1-|z_j|^2)}{4}(1-|z_i|). \end{aligned}$$

Thus, $\frac{(1-|z_j|^2)}{4}\sum_i (1-|z_i|) \leq B$, for all j. Hence, $\sum_i (1-|z_i|) < \infty$. □

The next result is due to Nikolski [33].

Proposition 3.1.2. *Let* $\{z_i\} \subseteq \mathbb{D}$ *satisfy the Blaschke condition and let* B *be the Blaschke product with zeroes at the* $z_i's$. *Then* $\overline{\text{span}\{k_{z_i}\}} = H^2 \ominus BH^2$, *and there is no kernel function contained in* $H^2 \ominus BH^2$, *other than* $\{k_{z_i}\}$.

Proof. Clearly each k_{z_i} is in $H^2 \ominus BH^2$. Also, if $f \in H^2$ is orthogonal to each k_{z_i}, then $f \in BH^2$. Hence, $\overline{\text{span}\{k_{z_i}\}} = H^2 \ominus BH^2$. Further, if there is a kernel function k_w in $H^2 \ominus BH^2$, then B must have a zero at w, and thus $w = z_i$ for some i. Hence, ${k_{z_i}}'s$ are the only kernel functions in $H^2 \ominus BH^2$. □

The above two results can be combined to deduce the following property of Bessel sequences of normalized kernel functions in H^2.

Corollary 3.1.3. *There does not exist any Bessel sequence of normalized kernel functions in* H^2 *which is complete.*

Proof. Let $\{\widetilde{k}_{z_i}\}$ be a Bessel sequence of normalized kernel functions in H^2. Then $\overline{\text{span}\{k_{z_i}\}} = H^2 \ominus BH^2$, using Proposition 3.1.1 and 3.1.2, where B is the Blaschke product with zeroes at the $z_i's$. But $H^2 \ominus BH^2 \neq H^2$, since $B \neq 0$. Hence $\{\widetilde{k}_{z_i}\}$ is not complete in H^2 □

The following result gives a glimpse of the rich structure of kernel functions in H^2. In addition, it also indicates why we have focused on the Bessel version of the FC.

Proposition 3.1.4. *Let* $\{\widetilde{k}_{z_i}\}_{i\in J}$ *be a Bessel sequence of normalized kernel functions in* H^2. *Then* $Ker(F^*) = \{0\}$, *where* F *is the analysis operator associated to the sequence* $\{\widetilde{k}_{z_i}\}_{i\in J}$.

Proof. Let $\{\lambda_i\}_{i\in J} \in Ker(F^*)$. Then $\sum_{i\in J} \lambda_i \widetilde{k}_{z_i} = 0$, which implies that

$$\langle f, \sum_{i\in J} \lambda_i \widetilde{k}_{z_i} \rangle = 0$$

for all $f \in H^2$. This further implies that $\sum_{i\in J} \overline{\lambda}_i \frac{f(z_i)}{\|k_{z_i}\|} = 0$, for all $f \in H^2$.

Now since $\{\widetilde{k}_{z_i}\}_{i\in J}$ is a Bessel sequence, therefore by Proposition 3.1.1, $\{z_i\}_{i\in J}$ satisfies the Blaschke condition. Let f_j denote the Blaschke product with zeroes at $\{z_i : i \neq j\}$. Then each f_j is in H^2, and so $\sum_{i\in J} \overline{\lambda}_i \frac{f_j(z_i)}{\|k_{z_i}\|} = 0$ for all $j \in J$. This forces $\lambda_j = 0$, for all $j \in J$. Hence $Ker(F^*) = 0$. □

As an immediate consequence we get the following interesting result about kernel functions in H^2.

Theorem 3.1.5. *A sequence $\{\widetilde{k}_{z_i}\}_{i\in J}$ of normalized kernel functions in H^2 is a frame sequence if and only if it is a Riesz basis sequence. Moreover, in this case there is no other kernel function in the closed linear span of $\{\widetilde{k}_{z_i}\}_{i\in J}$.*

Proof. Let $\mathcal{H}$ be the closed linear span of $\{\widetilde{k}_{z_i} : i \in J\}$ in H^2. If $\{\widetilde{k}_{z_i}\}_{i\in J}$ is a Riesz basis for $\mathcal{H}$, then by Proposition 2.1.1, it is a frame for $\mathcal{H}$. Conversely, suppose that $\{\widetilde{k}_{z_i}\}_{i\in J}$ is a frame for $\mathcal{H}$. Then the analysis operator $F : \mathcal{H} \to \ell^2(J)$, given by $F(x) = (\langle x, \widetilde{k}_{z_i} \rangle)$, is bounded and F^* is onto. Also by Proposition 3.1.4, $Ker(F^*) = 0$. Thus $F^* : \ell^2(J) \to \mathcal{H}$ is an invertible operator, and hence $\{\widetilde{k}_{z_i}\}_{i\in J}$ is a Riesz basis for $\mathcal{H}$. The moreover part follows from Proposition 3.1.2. □

The following theorem is a stepping-stone to our main result (Theorem 3.1.9) of this section. As a consequence of this result arises a beautiful interplay between general norm-bounded below Bessel sequences and kernel functions in H^2.

Theorem 3.1.6. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a Riesz basic sequence in a Hilbert space $\mathcal{H}$, and let $Q \in B(\ell^2)$ be a positive operator such that there exists a constant $\delta > 0$ with $\langle Qe_i, e_i\rangle \geq \delta$ for each i. Then there exists a positive operator $P \in B(\mathcal{H})$ such that*

$$Q = \big(\langle Pf_j, Pf_i\rangle_{\mathcal{H}}\big)$$

with $\|Pf_i\|^2_{\mathcal{H}} \geq \delta$, for all i.

Proof. Let $\mathcal{H}_0$ be the closed linear span of $\{f_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$. So, the analysis operator $F : \mathcal{H}_0 \to \ell^2$, given by $F(x) = (\langle x, f_i\rangle)$, is invertible with $F^*(e_i) = f_i$ for each i. Set $R = F^{-1}Q(F^{-1})^*$. Then $R : \mathcal{H}_0 \to \mathcal{H}_0$ is a bounded, positive operator. We extend R to $\mathcal{H}$ by defining it be 0 on ${\mathcal{H}_0}^\perp$. We claim that $P = R^{1/2}$ satisfies the required conditions. To prove the claim, we fix $i,\ j \in \mathbb{N}$, and consider

$$\begin{aligned}
\langle Pf_j, Pf_i\rangle_{\mathcal{H}} &= \langle R^{1/2}f_j, R^{1/2}f_i\rangle_{\mathcal{H}} \\
&= \langle Rf_j, f_i\rangle_{\mathcal{H}} \\
&= \langle Q(F^{-1})^*f_j, (F^{-1})^*f_i\rangle \\
&= \langle Qe_j, e_i\rangle.
\end{aligned}$$

Hence,

$$Q = \left(\langle Pf_j, Pf_i\rangle_{\mathcal{H}}\right).$$

Also, as obtained above $\|Pf_i\|^2_{\mathcal{H}} = \langle Pf_i, Pf_i\rangle_{\mathcal{H}} = \langle Qe_i, e_i\rangle \geq \delta$, for all i. This completes the proof. $\square$

Remark 3.1.7. *Note that in Theorem 3.1.6, we have a great deal of freedom in the choice of the frame sequence $\{f_i\}_{i\in\mathbb{N}}$ and hence on the Hilbert space $\mathcal{H}_0 = \overline{\mathrm{span}\{f_i : i \in \mathbb{N}\}}$, and also on the behavior of P on ${\mathcal{H}_0}^\perp$.*

Corollary 3.1.8. *Fix a sequence* $\{z_i\}$ *in* $\mathbb{D}$ *so that* $\{\widetilde{k}_{z_i}\}$ *is a frame sequence in* H^2. *Let* $Q \in B(\ell^2)$ *be a positive operator such that there exists a constant* $\delta > 0$ *with* $\langle Qe_i, e_i \rangle \geq \delta$ *for each* i. *Then there exists a positive operator* $P \in B(H^2)$ *such that*

$$Q = \left(\langle P\widetilde{k}_{z_j}, P\widetilde{k}_{z_i} \rangle\right)$$

with $\|P\widetilde{k}_{z_i}\|^2 \geq \delta$, *for all* i.

Proof. The result follows immediately by taking $\mathcal{H} = H^2$ in the above theorem and using Theorem 3.1.5. □

We are now ready for our main theorem. Before stating the theorem, we recall the characterization of contractive Hilbert spaces given by Theorem 2.2.10, which we will use in its proof. A contractive Hilbert space is a range space $\mathcal{R}(P^{1/2})$, for some positive contraction $P \in B(H^2)$, in which case, we denote it by $\mathcal{H}(P)$. Furthermore, in a contractive Hilbert space $\mathcal{H}(P)$, k_w^P denote the kernel function for a point $w \in \mathbb{D}$, and $\widetilde{k}_w^P$ denote the corresponding normalized kernel function. Lastly, recall that for $w \in \mathbb{D}$, $k_w^P = Pk_w$, where k_w is the kernel function in H^2 for the point w, and $\langle Px, Py \rangle_P = \langle P^{1/2}x, P^{1/2}y \rangle_2$, for every $x, y \in H^2$.

Though the above characterization of contractive Hilbert spaces is essential for the proof of the following theorem, we avoid it in the statement of the theorem, to make the statement more readable.

Theorem 3.1.9. *The following are equivalent:*

(i) *every norm-bounded below Bessel sequence in a Hilbert space can be partitioned into finitely many Riesz basic sequences (**FC**),*

(ii) *for every positive operator* $P \in B(H^2)$, *and for every sequence* $\{z_i\} \subseteq \mathbb{D}$, *if* $\{P\widetilde{k}_{z_i}\}$ *is a norm-bounded below Bessel sequence in* H^2, *then it can be partitioned into finitely many Riesz basic sequences,*

(iii) *for every contractive Hilbert space* $\mathcal{H}$, *and for every sequence* $\{z_i\} \subseteq \mathbb{D}$, *if* $\{\widetilde{k}_{z_i}^{\mathcal{H}}\}$ *is a Bessel sequence of normalized kernel functions in* $\mathcal{H}$, *then it can be partitioned into finitely many Riesz basic sequences.*

Moreover, in the last two equivalences we can assume that $\{z_i\}$ *satisfies the Blaschke condition. In fact, we can assume a much more restrictive condition on* $\{z_i\}$ *which is that* $\{\widetilde{k}_{z_i}\}$ *is a Riesz basic sequence in* H^2.

Proof. (i) implies (iii) is trivially true. We now prove (iii) implies (ii). Let $P \in B(H^2)$ be a positive operator and let $\{z_i\}$ be a sequence in $\mathbb{D}$ such that $\{P\widetilde{k}_{z_i}\}$ is a norm-bounded below Bessel sequence in H^2 with $\|P\widetilde{k}_{z_i}\|_2 \geq \delta > 0$, for all i. Then $T = P^2/\|P^2\|$ is a positive contraction in $B(H^2)$ and thus it give rise to a contractive Hilbert space $\mathcal{H}(T) = \mathcal{R}(T^{1/2})$.

Now for fixed $i,\ j$,

$$\begin{aligned}
\langle \widetilde{k}_{z_j}^T, \widetilde{k}_{z_i}^T \rangle_T &= \left\langle \frac{Tk_{z_j}}{\|Tk_{z_j}\|_T}, \frac{Tk_{z_i}}{\|Tk_{z_i}\|_T} \right\rangle_T \\
&= \left\langle \frac{T^{1/2}k_{z_j}}{\|T^{1/2}k_{z_j}\|_2}, \frac{T^{1/2}k_{z_i}}{\|T^{1/2}k_{z_i}\|_2} \right\rangle_2 \\
&= \left\langle \frac{Pk_{z_j}}{\|Pk_{z_j}\|_2}, \frac{Pk_{z_i}}{\|Pk_{z_i}\|_2} \right\rangle_2 \\
&= \frac{\|k_{z_j}\|_2}{\|Pk_{z_j}\|_2} \langle P\widetilde{k}_{z_j}, P\widetilde{k}_{z_i} \rangle_2 \frac{\|k_{z_i}\|_2}{\|Pk_{z_i}\|_2}.
\end{aligned}$$

Hence,

$$\left(\langle \widetilde{k}^T_{z_j}, \widetilde{k}^T_{z_i}\rangle_T\right) = D\left(\langle P\widetilde{k}_{z_j}, P\widetilde{k}_{z_i}\rangle_2\right) D^*,$$

where $D \in B(l^2)$ is an invertible, diagonal operator with i^{th} diagonal entry $\frac{\|k_{z_i}\|_2}{\|Pk_{z_i}\|_2}$, since $P \in B(H^2)$ and $\|Pk_{z_i}\|_2 \geq \delta\|k_{z_i}\|_2$, for all i. Then (i) of Proposition 2.1.6 implies that $\{\widetilde{k}^T_{z_i}\}$ is a Bessel sequence in $\mathcal{H}(T)$, since $\{P\widetilde{k}_{z_i}\}$ is a Bessel sequence in H^2. Thus, by assuming (iii), we have that $\{\widetilde{k}^T_{z_i}\}$ satisfies the FC. Hence we conclude that $\{P\widetilde{k}_{z_i}\}$ also satisfies the FC, by using (iii) of Proposition 2.1.6. This completes the proof of (iii) implies (ii).

For (ii) implies (i), let $\{f_i\}_{i\in\mathbb{N}}$ be a norm-bounded below Bessel sequence in a Hilbert space $\mathcal{H}$ with $\|f_i\|_{\mathcal{H}} \geq \delta > 0$ for each $i \in \mathbb{N}$. Then FF^* is a bounded, positive operator in $B(\ell^2)$, where $F : \mathcal{H} \to \ell^2$ is the analysis operator associated with $\{f_i\}_{i\in\mathbb{N}}$. Also, $\langle FF^*(e_i), e_i\rangle_{\ell^2} = \|f_i\|^2_{\mathcal{H}} \geq \delta^2$, for all $i \in \mathbb{N}$. Thus, by Corollary 3.1.8, there exists a positive operator P in $B(H^2)$ with $\|P\widetilde{k}_{z_i}\|_2 \geq \delta$ for each i, such that

$$FF^* = (\langle f_j, f_i\rangle_{\mathcal{H}}) = \left(\langle P\widetilde{k}_{z_j}, P\widetilde{k}_{z_i}\rangle_2\right),$$

where $\{\widetilde{k}_{z_i}\}_{i\in\mathbb{N}}$ is a frame sequence of normalized kernel functions in H^2. Now since $FF^* \in B(\ell^2)$, therefore $\{P\widetilde{k}_{z_i}\}_{i\in\mathbb{N}}$ is a Bessel sequence. Also, $\{P\widetilde{k}_{z_i}\}_{i\in\mathbb{N}}$ is norm-bounded below. Thus by assuming (ii), we have that it satisfies the FC. Hence, by (iii) of proposition 2.1.6, $\{f_i\}_{i\in\mathbb{N}}$ also satisfies the FC. This completes the proof of (ii) implies (i).

The moreover part follows immediately from the fact that in Corollary 3.1.8 we chose $\{z_i\}$ so that $\{\widetilde{k}_{z_i}\}$ is a frame and hence is a Riesz basic sequence in H^2. $\square$

The above theorem motivates the following conjectures.

Conjecture 3.1.10. Feichtinger Conjecture for Kernel Functions(FCKF). *Every Bessel sequence of normalized kernel functions in each RKHS satisfies the FC.*

Conjecture 3.1.11. Feichtinger Conjecture for Contractive Kernel Functions (FCCKF). *Every Bessel sequence of normalized kernel functions in each contractive Hilbert space satisfies the FC.*

Conjecture 3.1.12. Restrictive Feichtinger Conjecture for Kernel Functions (RFCKF). *For every contractive Hilbert space $\mathcal{H}(P)$, every Bessel sequence $\{\widetilde{k}^P_{z_i}\}$ of normalized kernel functions in $\mathcal{H}(P)$ satisfies the FC, where $\{\widetilde{k}_{z_i}\}$ is a Riesz basic sequence of normalized kernel functions in H^2 and $\|k^P_{z_i}\|_P \geq \delta\|k_{z_i}\|$, for all i.*

The third conjecture, that is, the RFCKF, in comparison to the other two conjectures, namely FCKF and FCCKF, demands the testing of a much more restrictive class. What is interesting is that all these three conjectures are equivalent to the FC, and hence are also equivalent to each other. We prove this in the following result.

Theorem 3.1.13. *The following are equivalent:*

(i) *the FC is true,*

(ii) *the FCKF is true,*

(iii) *the FCCKF is true,*

(iv) *the RFCKF is true.*

Proof. Clearly, (i) implies (ii), and (ii) implies (iii). Also, since FCCKF is exactly the statement (iii) of Theorem 3.1.9, therefore (i) and (iii) are equivalent. Hence, (i), (ii) and (iii) are equivalent.

To complete the result, we show that (i) and (iv) are equivalent. Clearly, (i) implies (iv). To prove (iv) implies (i), we will prove that (iv) implies the statement (ii) of Theorem 3.1.9. Let $P \in B(H^2)$ be a positive operator and let $\{P\widetilde{k}_{z_i}\}$ be a norm-bounded below Bessel sequence in H^2, where $\{\widetilde{k}_{z_i}\}$ is a Riesz basic sequence of normalized kernel functions in H^2. Let $\delta > 0$, so that $\|P\widetilde{k}_{z_i}\|_2 \geq \delta$, for all i. Then $T = P^2/\|P^2\|$ is a positive contraction in $B(H^2)$, and thus we get a contractive Hilbert space $\mathcal{H}(T)$. Now, using the same ideas as used in (iii) implies (ii) of Theorem 3.1.9, we get that $\{\widetilde{k}^T_{z_i}\}$ is a Bessel sequence in $\mathcal{H}(T)$. Also,

$$\|k^T_{z_i}\|_T = \|T^{1/2}k_{z_i}\|_2 = \frac{1}{\|P\|}\|Pk_{z_i}\|_2 \geq \frac{\delta}{\|P\|}\|k_{z_i}\|_2,$$

for all i. Thus, $\{\widetilde{k}^T_{z_i}\}$ is a Bessel sequence of normalized kernel functions in $\mathcal{H}(T)$, where $\{\widetilde{k}_{z_i}\}$ is a Riesz basic sequence of normalized kernel functions in H^2, and $\|k^T_{z_i}\|_T \geq \frac{\delta}{\|P\|}\|k_{z_i}\|_2$, for all i. Therefore, assuming the RFCKF, we get that $\{\widetilde{k}_{z_i}\}$ splits into finitely many Riesz basic sequences. From this it follows easily that $\{P\widetilde{k}_{z_i}\}$ splits into finitely many Riesz basic sequences,using the same ideas as used in (iii) implies (ii) of Theorem 3.1.9 . Hence, the FC holds true, using the moreover part and (ii) of Theorem 3.1.9. □

From this point forward, we will say that a particular RKHS $\mathcal{H}$ satisfies the FCKF if every Bessel sequence of normalized kernel functions in $\mathcal{H}$ can be partitioned into finitely many Riesz basic sequences. Further, we will say that a contractive Hilbert space $\mathcal{H}(P)$ satisfies the RFCKF if every Bessel sequence $\{\widetilde{k}^P_{z_i}\}$ of normalized kernel functions in $\mathcal{H}(P)$ satisfies the FC, where $\{\widetilde{k}_{z_i}\}$ is a Riesz basic sequence in H^2 and $\|k^P_{z_i}\|_P \geq \delta\|k_{z_i}\|$, for all i. Clearly, if a contractive Hilbert space satisfies the FCKF or FCCKF, then it also satisfies the RFCKF. But we do not know if the converse is true.

We conclude this section with the following observation.

Proposition 3.1.14. *Let $\mathcal{H}(P)$ be a contractive Hilbert space and let $\{\widetilde{k}_{z_i}\}$ be a Bessel sequence of normalized kernel functions in H^2 such that there exists a $\delta > 0$ with $\|k^P_{z_i}\|_P \geq \delta\|k_{z_i}\|$, for all i. Then $\{\widetilde{k}^P_{z_i}\}$ is a Bessel sequence in $\mathcal{H}(P)$.*

3.2 Analysis of New Equivalences of the Feichtinger Conjecture

We can easily verify that statement (ii) of Theorem 3.1.9 can be reduced to the case of positive operators which are contractions. Thus, Theorem 3.1.9 motivates the study of sequences $\{P\widetilde{k}_{z_i}\}$, $\{\widetilde{k}^P_{z_i}\}$, where $P \in B(H^2)$ is a positive contraction, and $\{\widetilde{k}_{z_i}\}$, $\{\widetilde{k}^P_{z_i}\}$ are sequences of normalized kernel functions in H^2 and $\mathcal{H}(P)$, respectively. By considering positive operators and kernel functions, we have much more structure to exploit and thereby we can expect some interesting and fruitful research in this direction. However, the presence of the FC in Theorem 3.1.9 suggests that it might not be easy to make any general statement about the whole family of these sequences. But at the same time, we have so many known positive operators floating around, and investigating the FC for the corresponding sequences might be interesting in itself. In this direction we have the following results.

The first class of operators we look at is the class of positive, invertible operators in $B(H^2)$.

Proposition 3.2.1. *Let $P \in B(H^2)$ be a positive, invertible operator, and let $\{z_i\}$ be a sequence in $\mathbb{D}$. Then:*

(i) $\|P^{-1}\|^{-1}\|k_{z_i}\| \leq \|Pk_{z_i}\| \leq \|P\|\|k_{z_i}\|$ *for all* i,

(ii) $\{P\widetilde{k}_{z_i}\}$ *is a Bessel sequence if and only if* $\{\widetilde{k}_{z_i}\}$ *is a Bessel sequence,*

(iii) $\{P\widetilde{k}_{z_i}\}$ *is a frame sequence if and only if it is a Riesz basic sequence,*

(iv) $\{P\widetilde{k}_{z_i}\}$ *is a Riesz basic sequence if and only if* $\{\widetilde{k}_{z_i}\}$ *is a Riesz basic sequence.*

Theorem 3.2.2. *Let $P \in B(H^2)$ be a positive, invertible operator. Given a sequence $\{z_i\} \subseteq \mathbb{D}$, if $\{P\widetilde{k}_{z_i}\}$ is a Bessel sequence, then it satisfies the FC.*

Proof. Suppose that $\{P\widetilde{k}_{z_i}\}$ is a Bessel sequence. Then by (iii) of Proposition 3.2.1, $\{\widetilde{k}_{z_i}\}$ is a Bessel sequence and thus satisfies the FC, by Theorem 3.0.6. Hence $\{P\widetilde{k}_{z_i}\}$ satisfies the FC, using (iv) of Proposition 3.2.1. $\square$

Remark 3.2.3. *When $P \in B(H^2)$ is a positive, invertible contraction, then $\mathcal{H}(P) = H^2$ and the two norms are equivalent. This implies that $\{P\widetilde{k}_{z_i}\}$ is a Bessel (frame or Riesz basic) sequence if and only if $\{\widetilde{k}^P_{z_i}\}$ is a Bessel (frame or Riesz basic) sequence. Thus by Theorem 3.2.2, every Bessel sequence of normalized kernel functions in $\mathcal{H}(P)$ satisfies the FC, and hence the contractive Hilbert space $\mathcal{H}(P)$ satisfies the FCKF.*

We now focus on some well-understood orthogonal projections.

Theorem 3.2.4. *Let ϕ be an inner function and let P_ϕ be the orthogonal projection onto ϕH^2. Then the contractive Hilbert space $\mathcal{H}(P_\phi)$ satisfies the FCKF.*

Proof. For $\{z_i\} \subseteq \mathbb{D}$, let $\{\widetilde{k}_{z_i}^{P_\phi}\}$ be a Bessel sequence of normalized kernel functions in $\mathcal{H}(P_\phi)$. To prove that this sequence satisfies the FC, we first note that $P_\phi = T_\phi {T_\phi}^*$, where T_ϕ is the Toeplitz operator with symbol ϕ and ${T_\phi}^* k_{z_i} = \overline{\phi(z_i)} k_{z_i}$. Also, $\mathcal{H}(P_\phi)$ coincides with the range of P_ϕ and the two norms are equal, since P_ϕ is an orthogonal projection. To simplify notation, we set $P = P_\phi$. Then,

$$\langle \widetilde{k}_{z_j}^P, \widetilde{k}_{z_i}^P \rangle_P = \left\langle \frac{Pk_{z_j}}{\|Pk_{z_j}\|}, \frac{Pk_{z_i}}{\|Pk_{z_i}\|} \right\rangle = \frac{\overline{\phi(z_j)}}{|\phi(z_j)|} \langle \widetilde{k}_{z_j}, \widetilde{k}_{z_i} \rangle \frac{\phi(z_i)}{|\phi(z_i)|}.$$

Hence,

$$\left(\langle \widetilde{k}_{z_j}^P, \widetilde{k}_{z_i}^P \rangle_P \right) = D \left(\langle \widetilde{k}_{z_j}, \widetilde{k}_{z_i} \rangle \right) D^*,$$

where $D \in B(l^2)$ is an invertible, diagonal operator with i^{th} diagonal entry $\frac{\phi(z_i)}{|\phi(z_i)|}$. Finally, using Proposition 2.1.5 and Theorem 3.0.6, we conclude that $\{\widetilde{k}_{z_i}^{P_\phi}\}$ satisfies the FC. Hence $\mathcal{H}(P_\phi)$ satisfies the FCKF. $\square$

The following example illustrate the above result for a class of finite codimensional orthogonal projections.

Example 3.2.5. *Let $P \in B(H^2)$ be an orthogonal projection such that $Ker(P) =$ span$\{k_{w_1}, \ldots, k_{w_n}\}$, for some $w_1, \ldots, w_n \in \mathbb{D}$. Then $Ran(P) = \phi H^2$ and $P = T_\phi T_\phi^*$, where ϕ is the finite Blaschke with zeroes at $w_1, \ldots, w_n$. Hence, $\mathcal{H}(P)$ satisfies the FCKF, using Theorem 3.2.4.*

By taking a closer look at the proof of (iii) implies (ii) in Theorem 3.1.9, we observe that when P is an orthogonal projection, then to show that a norm-bounded below Bessel sequence $\{P\widetilde{k}_{z_i}\}$ satisfies the FC, all we need is that the corresponding sequence $\{\widetilde{k}^P_{z_i}\}$ in $\mathcal{H}(P)$, for the same P, satisfies the FC. As an immediate consequence, we get the following result.

Theorem 3.2.6. *Let ϕ be an inner function and let P_ϕ be the orthogonal projection onto ϕH^2. If $\{z_i\}$ is a sequence in $\mathbb{D}$, such that $\{P_\phi\widetilde{k}_{z_i}\}$ is a norm-bounded below Bessel sequence in H^2, then $\{P_\phi\widetilde{k}_{z_i}\}$ satisfies the FC.*

Next, we present a different proof of Theorem 3.2.6. This detour is worth looking at, as it not only reveals some interesting properties of the sequences $\{P_\phi\widetilde{k}_{z_i}\}$, but also motivates a generalization of Theorem 3.2.6. We start with the following proposition.

Proposition 3.2.7. *Let ϕ be an inner function and let P_ϕ be the orthogonal projection onto ϕH^2. Then for every sequence $\{z_i\} \subseteq \mathbb{D}$, such that there exists a $\delta > 0$ with $|\phi(z_i)| \geq \delta$ for all i, the following hold true:*

(i) *for each i, $\delta\|k_{z_i}\| \leq \|P_\phi k_{z_i}\| \leq \|k_{z_i}\|$,*

(ii) *$\{P_\phi\widetilde{k}_{z_i}\}$is a Bessel sequence if and only if $\{\widetilde{k}_{z_i}\}$ is a Bessel sequence,*

(iii) *$\{P_\phi\widetilde{k}_{z_i}\}$ is a frame sequence if and only if it is a Riesz basic sequence,*

(iv) *$\{P_\phi\widetilde{k}_{z_i}\}$ is a Riesz basic sequence if and only if $\{\widetilde{k}_{z_i}\}$ is a Riesz basic sequence.*

Proof. Let $\{z_i\}$ be a sequence in $\mathbb{D}$ and let $\delta > 0$ be a constant such that $|\phi(z_i)| \geq \delta$, for all i.

As noted earlier, $P_\phi = T_\phi {T_\phi}^*$, where T_ϕ is the Toeplitz operator with symbol ϕ, and ${T_\phi}^* k_{z_i} = \overline{\phi(z_i)} k_{z_i}$. Thus,

$$\langle P_\phi \widetilde{k}_{z_j}, P_\phi \widetilde{k}_{z_i} \rangle = \langle {T_\phi}^* \widetilde{k}_{z_j}, {T_\phi}^* \widetilde{k}_{z_i} \rangle = \overline{\phi(z_j)} \langle \widetilde{k}_{z_j}, \widetilde{k}_{z_i} \rangle \phi(z_i). \tag{3.1}$$

Hence,

$$\left(\langle P_\phi \widetilde{k}_{z_j}, P_\phi \widetilde{k}_{z_i} \rangle \right) = D \left(\langle \widetilde{k}_{z_j}, \widetilde{k}_{z_i} \rangle \right) D^*, \tag{3.2}$$

where $D \in B(\ell^2)$ is an invertible, diagonal operator with $\phi(z_i)$ as the i^{th} diagonal entry, since $\delta \leq |\phi(z_i)| \leq 1$, for all i.

Clearly, (i) follows from Equation (2) and (ii), (iii) and (iv) follows from Equation (3), using Proposition 2.1.6 and Theorem 3.1.5. □

Alternate proof of Theorem 3.2.6. Let $\{z_i\}$ be a sequence in $\mathbb{D}$ such that $\{P_\phi \widetilde{k}_{z_i}\}$ is a norm-bounded below Bessel sequence. Then there exists a constant $\delta > 0$ such that, for each i, $\delta \|k_{z_i}\| \leq \|P_\phi k_{z_i}\| \leq \|k_{z_i}\|$. Now as obtained in Proposition 3.2.7, we get

$$\|P_\phi k_{z_i}\| = |\phi(z_i)| \|k_{z_i}\|.$$

Thus, $|\phi(z_i)| \geq \delta$, for all i. Then by (iii) of Proposition 3.2.7, the sequence $\{\widetilde{k}_{z_i}\}$ is a Bessel sequence and thus satisfies the FC, using Theorem 3.0.6. Hence by part (iv) of Proposition 3.2.7, $\{P_\phi \widetilde{k}_{z_i}\}$ satisfies the FC.

We can generalize Proposition 3.2.7 and Theorem 3.2.6 as follows.

Proposition 3.2.8. *Let $P \in B(H^2)$ be an orthogonal projection. Given a sequence $\{z_i\}$ in $\mathbb{D}$, if there exists an inner function ϕ such that $|\phi(z_i)| \geq \delta$ for all i and $\phi H^2 \subseteq Ran(P)$, then:*

(i) *for each* i, $\delta\|k_{z_i}\| \leq \|Pk_{z_i}\| \leq \|k_{z_i}\|$,

(ii) $\{P\widetilde{k}_{z_i}\}$ *is a Bessel sequence if and only if* $\{\widetilde{k}_{z_i}\}$ *is a Bessel sequence,*

(iii) $\{P\widetilde{k}_{z_i}\}$ *is a frame sequence if and only if it is a Riesz basic sequence,*

(iv) $\{P\widetilde{k}_{z_i}\}$ *is a Riesz basic sequence if and only if* $\{\widetilde{k}_{z_i}\}$ *is a Riesz basic sequence,*

Proof. Let P_ϕ denote the orthogonal projection onto ϕH^2. Then $P_\phi \leq P$ and thus, $\|P_\phi k_{z_i}\| \leq \|Pk_{z_i}\| \leq \|k_{z_i}\|$. This proves (i), since $\delta\|k_{z_i}\| \leq \|P_\phi k_{z_i}\|$.

To prove (ii), we first note that for any $x \in H^2$,

$$\langle x, P_\phi \widetilde{k}_{z_i}\rangle = \langle P_\phi x, P\widetilde{k}_{z_i}\rangle.$$

Thus, if $\{P\widetilde{k}_{z_i}\}$ is a Bessel sequence, then $\{P_\phi\widetilde{k}_{z_i}\}$ is a Bessel sequence. Hence, $\{\widetilde{k}_{z_i}\}$ is a Bessel sequence, using Proposition 3.2.7. The other implication follows trivially from the fact that $P \in B(H^2)$.

We shall now prove (iii) and (iv). Note that if $\{\widetilde{k}_{z_i}\}$ is a Bessel sequence, then

$$\left(\langle P_\phi\widetilde{k}_{z_j}, P_\phi\widetilde{k}_{z_i}\rangle\right) \leq \left(\langle P\widetilde{k}_{z_j}, P\widetilde{k}_{z_i}\rangle\right) \leq \left(\langle \widetilde{k}_{z_j}, \widetilde{k}_{z_i}\rangle\right). \tag{3.3}$$

To prove (iii), we first assume that $\{P\widetilde{k}_{z_i}\}$ is a frame sequence. Then it is a Bessel sequence, and thus $\{\widetilde{k}_{z_i}\}$ is also a Bessel sequence. So Equation (3.3) holds and we get $F_\phi F_\phi^* \leq F_P F_P^* \leq FF^*$, where F_ϕ, F_P and F are the analysis operators corresponding to the sequences $\{P_\phi\widetilde{k}_{z_i}\}$, $\{P\widetilde{k}_{z_i}\}$ and $\{\widetilde{k}_{z_i}\}$, respectively.

Since $\{P\widetilde{k}_{z_i}\}$ is a frame sequence, therefore it is enough to show that $Ker(F_P^*) = 0$. For this, let $a \in Ker(F_P^*)$. Then $F_P^*(a) = 0$, which forces $F_\phi^*(a) = 0$, using

Equation (3.3). But from Equation (3.2) we have $F_\phi F_\phi^* = DFF^*D^*$, where $D \in B(\ell^2)$ is an invertible, diagonal operator with $\phi(z_i)$ as the i^{th} diagonal entry. Hence $DFF^*D^*(a) = 0$. Now since $\{\widetilde{k}_{z_i}\}$ is a Bessel sequence, therefore by Proposition 3.1.4, F^* is one-to-one. Also $F,\ D$ and D^* are one-to-one. Hence $a = 0$. This yields that $Ker(F_P^*) = 0$, from which it follows that F_P^* is invertible. Hence $\{P\widetilde{k}_{z_i}\}$ is a Riesz basic sequence.

Finally, (iv) follows from Equation (3.3), using (iv) of Proposition 3.2.7 together with Proposition 2.1.4. □

Theorem 3.2.9. *Let $P \in B(H^2)$ be an orthogonal projection. Let $\{z_i\}$ be a sequence in $\mathbb{D}$ such that there exists an inner function ϕ with $|\phi(z_i)| \geq \delta$ for all i and $\phi H^2 \subseteq Ran(P)$. If $\{P\widetilde{k}_{z_i}\}$ is a Bessel sequence, then it satisfies the FC.*

Proof. The result follows immediately from Proposition 3.2.8, as before. □

Remark 3.2.10. *For the case, when P is an orthogonal projection, the Hilbert space $\mathcal{H}(P)$ coincides with $Ran(P)$. Further, in this case, if there exists a constant $\delta > 0$ such that $\delta\|k_{z_i}\| \leq \|Pk_{z_i}\| \leq \|k_{z_i}\|$ for all i, then $\{P\widetilde{k}_{z_i}\}$ is a Bessel (frame or Riesz basic) sequence if and only if $\{\widetilde{k}_{z_i}^P\}$ is a Bessel (frame or Riesz basic) sequence.*

Theorem 3.2.11. *Let $P \in B(H^2)$ be an orthogonal projection. If $\{\widetilde{k}_{z_i}^P\}$ is a Bessel sequence in $\mathcal{H}(P) = Ran(P)$, such that there exists an inner function ϕ with $|\phi(z_i)| \geq \delta$ for all i and $\phi H^2 \subseteq \mathcal{H}(P)$, then $\{\widetilde{k}_{z_i}^P\}$ satisfies the FC.*

Proof. The result follows immediately from Theorem 3.2.9 and Remark 3.2.10, using (i) of Proposition 3.2.8. □

The following examples illustrate some applications of Theorem 3.2.11.

Example 3.2.12. *Given an inner function* ϕ, $[\mathbb{C}+H^2_\phi]$ *denotes the closure of* $\mathbb{C}+\phi H^2$ *in* H^2. *These spaces were first introduced in [41]. Let* $\{z_i\}$ *be a sequence in* $\mathbb{D}$ *and let* ϕ *be an inner function, such that* $|\phi(z_i)| \geq \delta > 0$, *for all* i. *Then the orthogonal projection* P *onto* $[\mathbb{C}+H^2_\phi]$ *and* $\{z_i\}$ *satisfy the conditions of Theorem 3.2.11. Hence by using Theorem 3.2.11 it follows that, if the sequence* $\{\widetilde{k}^P_{z_i}\}$ *is a Bessel sequence in* $\mathcal{H}(P)$, *then it satisfies the FC.*

Example 3.2.13. *Let* $P \in B(H^2)$ *be an orthogonal projection, such that the kernel of* P *is spanned by* n *inner functions* $\phi_1, \ldots, \phi_n$. *Then* $\phi = z\phi_1 \cdots \phi_n$ *is an inner function, and* $\phi H^2 \subseteq Ran(P)$. *Now, if* $\{z_i\}$ *is a sequence in* $\mathbb{D}$ *such that there exists a constant* $\delta > 0$ *with* $|z_i| \geq \delta$, $|\phi_k(z_i)| \geq \delta$ *for all* i, k, *then* $|\phi(z_i)| \geq \delta^{n+1}$, *for all* i. *Thus,* P *and* $\{z_i\}$ *satisfy the conditions of Theorem 3.2.11. Hence by using Theorem 3.2.11 it follows that, if the sequence* $\{\widetilde{k}^P_{z_i}\}$ *is a Bessel sequence in* $\mathcal{H}(P)$, *then it satisfies the FC.*

Example 3.2.14. *Let* $P \in B(H^2)$ *be the orthogonal projection onto the closed linear span of* $\{z^j : j \neq j_1, \ldots, j_n\}$, $j_1 < \cdots < j_n$, *and let* $\{z_i\}$ *be a sequence in* $\mathbb{D}$ *such that there exists a constant* $\delta > 0$ *with* $|z_i| \geq \delta$, *for all* i. *Then* $\phi(z) = z^{j_n+1}$ *is an inner function,* $\phi H^2 \subseteq Ran(P)$ *and* $|\phi(z_i)| \geq \delta^{j_n+1}$, *for all* i. *Hence,* P *and* $\{z_i\}$ *satisfy the conditions of Theorem 3.2.11. Hence by using Theorem 3.2.11 it follows that, if the sequence* $\{\widetilde{k}^P_{z_i}\}$ *is a Bessel sequence in* $\mathcal{H}(P)$, *then it satisfies the FC.*

Remark 3.2.15. *Note that if* P *is the orthogonal projection as in Example 3.2.14 and* $\{\widetilde{k}^P_{z_i}\}_{i\in\mathbb{N}}$ *is a Bessel sequence in* $Ran(P) = \mathcal{H}(P)$, *where* $\{\widetilde{k}_{z_i}\}_{i\in\mathbb{N}}$ *is a Riesz basic* *sequence in* H^2, *then* $\{z_i\}$ *satisfies the Blaschke condition, and hence converges to 1. Thus, given a* $\delta > 0$ *there exists a* $N \in \mathbb{N}$ *such that* $|z_i| \geq \delta$, *for all* $i \geq N$. *Then, as noted in Example 3.2.14,* P *and* $\{z_i\}_{i\geq N}$ *satisfies the conditions of Theorem 3.2.11, which implies that the sequence* $\{\widetilde{k}_{z_i}^P\}_{i\geq N}$ *satisfies the FC, using Theorem 3.2.11. Hence,* $\{\widetilde{k}_{z_i}^P\}_{i\in\mathbb{N}}$ *also satisfies the FC. Thus, we conclude that* $Ran(P)$ *satisfies the RFCKF.*

Apart from ϕH^2, there is another interesting and quite well-studied closed subspace of H^2 that one associates with an inner function. This is the model space $H^2 \ominus \phi H^2$. In [8], Baranov and Dyakonov have considered the FC for $H^2 \ominus \phi H^2$, with some conditions on ϕ and proved the following two theorems.

Theorem 3.2.16 (BD, [8])**.** *Let* ϕ *be an inner function and let* $\{z_i\}$ *is a sequence in* $\mathbb{D}$ *such that* $\sup_i |\phi(z_i)| < 1$. *If the corresponding sequence of normalized kernel functions in* $H^2 \ominus \phi H^2$ *is a Bessel sequence, then it satisfies the FC.*

We generalize the above theorem in our next section. The second theorem of Baranov and Dyakonov uses one-component inner functions. An inner function ϕ is said to be an **one-component** inner function if the set $\{z : |\phi(z)| < \epsilon\}$ is connected for some, $0 < \epsilon < 1$.

Theorem 3.2.17 (BD, [8])**.** *Assume that* ϕ *is a one-component inner function. Then every Bessel sequence of normalized kernel functions in* $H^2 \ominus \phi H^2$ *satisfies the FC.*

Note that given an inner function ϕ, the model space $H^2 \ominus \phi H^2$ is the contractive Hilbert space $\mathcal{H}(P)$, where $P \in B(H^2)$ is the orthogonal projection onto $H^2 \ominus \phi H^2$.

Hence, the above theorems of Baranov and Dyakonov analyze the class of contractive Hilbert spaces $\mathcal{H}(P)$ for FCKF, where P belongs to the family of projections onto $H^2 \ominus \phi H^2$ and ϕ is an inner function with properties as stated in Theorem 3.2.16 and 3.2.17. In particular, their second theorem proves that when ϕ is a one-component inner function and P is the orthogonal projection onto $H^2 \ominus \phi H^2$, then the contractive Hilbert space $\mathcal{H}(P)$ satisfies FCKF.

Our last result in this section, focus on replacing a general contractive Hilbert space with a "particular" contractive Hilbert space, the weighted Hardy spaces on the unit disk. We shall briefly define these spaces here. For more details refer to [49].

Let $\{\beta_n\}$ be a sequence of positive numbers with $R = \liminf \beta_n > 0$. Then the set $\{\sum_n a_n z^n : \sum_n \beta_n^2 |a_n|^2 < \infty\}$ is a RKHS on the disk of radius R with norm $\|\sum_n a_n z^n\|_\beta^2 = \sum_n \beta_n^2 |a_n|^2$, and kernel $k_\beta(z,w) = \sum_n \frac{\bar{w}^n z^n}{{\beta_n}^2}$. This Hilbert space is called a weighted Hardy space, and is denoted by $H^2(\beta)$. To see how these are arising in our work, we let $P \in B(H^2)$ be a positive, diagonal contraction with n^{th} diagonal entry $p_n > 0$. Then the contractive Hilbert space $\mathcal{H}(P)$ coincides with the weighted Hardy space $H^2(\beta)$, where $\beta_n = \frac{1}{\sqrt{p_n}}$ for every n and the functions in $H^2(\beta)$ are restricted to the unit disk $\mathbb{D}$.

Theorem 3.2.18. *Let $P \in B(H^2)$ be a positive contraction and $D \in B(H^2)$ be a positive, diagonal contraction such that $\alpha D \leq P \leq \beta D$ for some $\alpha,\ \beta > 0$. Then the contractive Hilbert space $\mathcal{H}(P)$ satisfies the FCKF if and only if the contractive Hilbert space $\mathcal{H}(D)$ satisfies the FCKF. In fact, a sequence of normalized kernel functions in $\mathcal{H}(P)$ satisfies the FC if and only if the corresponding sequence of normalized kernel funcions in $\mathcal{H}(D)$ satisfies the FC.*

Proof. Using the given inequalities between P and D, we can easily deduce that

$$\alpha\|D^{1/2}x\|^2 \leq \|P^{1/2}x\|^2 \leq \beta\|D^{1/2}x\|^2, \tag{3.4}$$

for all $x \in H^2$.

Let $\{z_i\}$ be a sequence in $\mathbb{D}$. Then it follows immediately from the inequalities in (3.4) that

$$\alpha\big(\langle \widetilde{k}^D_{z_j}, \widetilde{k}^D_{z_i}\rangle_D\big)^n_{i,j=1} \leq E_n\big(\langle \widetilde{k}^D_{z_j}, \widetilde{k}^D_{z_i}\rangle_P\big)^n_{i,j=1} E^*_n \leq \beta\big(\langle \widetilde{k}^D_{z_j}, \widetilde{k}^D_{z_i}\rangle_D\big)^n_{i,j=1}, \tag{3.5}$$

where E_n is an $n \times n$ positive, diagonal matrix with i^{th} diagonal entry $\frac{\|P^{1/2}k_{z_i}\|}{\|D^{1/2}k_{z_i}\|}$. Note that the infinite diagonal matrix with i^{th} diagonal entry $\frac{\|P^{1/2}k_{z_i}\|}{\|D^{1/2}k_{z_i}\|}$ is an invertible operator in $B(\ell^2)$. Hence the result follows from Proposition 2.1.6 and inequalities given in (3.5), using the following two facts:

(i) an infinite matrix $A = (a_{ij})$ is a bounded operator on ℓ^2 if and only if $\sup_n \|A_n\| < \infty$, where $A_n = (a_{ij})^n_{i,j}$,

(ii) if $A = (a_{ij})$ and $C = (c_{ij})$ are bounded, self-adjoint operators on ℓ^2, then $A \leq C$ if and only if $A_n \leq C_n$ for all n, where $A_n = (a_{ij})^n_{i,j}$ and $C_n = (c_{ij})^n_{i,j}$.

□

3.3 Operations on Kernels and the Feichtinger Conjecture

In this section, we will discuss how various operations on kernel functions affect the FCKF and hence the FC. For this section, if K is a positive definite function on a set X, $\mathcal{H}(K)$ will denote the corresponding RKHS of functions on X. We start by investigating the following case of constructing a new kernel from a given kernel.

Let K be a kernel function on a set X, and let $f : X \to \mathbb{C}$ be a function, then it is easily checked that $K_f(z, w) = f(z)K(z, w)\overline{f(w)}$ is also a kernel function on X.

Theorem 3.3.1. *The Hilbert space $\mathcal{H}(K)$ satisfies the FCKF if and only if the Hilbert space $\mathcal{H}(K_f)$ satisfies the FCKF.*

Proof. Let $\{x_i\}$ be a sequence in X. Then by using the definition of the kernel K_f, we obtain

$$\langle \widetilde{k}^f_{x_j}, \widetilde{k}^f_{x_i}\rangle = \frac{f(x_i)}{|f(x_i)|}\langle \widetilde{k}_{x_j}, \widetilde{k}_{x_i}\rangle \frac{\overline{f(x_j)}}{|f(x_j)|}.$$

Therefore,

$$\left(\langle \widetilde{k}^f_{x_j}, \widetilde{k}^f_{x_i}\rangle\right) = D\left(\langle \widetilde{k}_{x_j}, \widetilde{k}_{x_i}\rangle\right) D^*,$$

where $D \in B(\ell^2)$ is an invertible, diagonal operator with i^{th} diagonal entry $\frac{f(z_i)}{|f(z_i)|}$, and hence the result follows using Proposition 2.1.6. □

Note that in the above theorem we do not require f to have any special properties. If f is an inner function in H^2, then the Hilbert space $\mathcal{H}(K_f) = fH^2$, which we have discussed in Theorem 3.2.7.

The next theorem addresses, partially, the situation of moving from a smaller kernel to a bigger one. For this, we need the following result due to Aronszajn [5, 39].

Theorem 3.3.2 (Aronszajn)**.** *Let X be a set and let $K_i : X \times X \to \mathbb{C}, i = 1, 2$ be kernels with corresponding reproducing kernel Hilbert spaces, $\mathcal{H}(K_i)$, and norms, $\|\cdot\|_i, i = 1, 2$. Then $\mathcal{H}(K_1) \subseteq \mathcal{H}(K_2)$ if and only if there exists a constant, $c > 0$ such that, $K_1(x, y) \leq c^2 K_2(x, y)$. In this case, $\|f\|_2 \leq c\|f\|_1$, for all $f \in \mathcal{H}(K_1)$. Moreover, if $\mathcal{H}(K_1)$ is a Hilbert space with the norm it inherits from $\mathcal{H}(K_2)$, and the two norms on $\mathcal{H}(K_1)$ coincides, then the constant c equals* 1.

Theorem 3.3.3. *Let K_1 and K_2 be two kernel functions on X, such that $K_1 \leq c^2 K_2$ for some constant $c > 0$. Let $\{z_i\}$ be a sequence in X, such that there exists a constant $d > 0$ with $K_2(z_i, z_i) \leq d^2 K_1(z_i, z_i)$, for all i.*

(i) *If $\{\widetilde{k^2_{z_i}}\}$ is a Bessel sequence, then $\{\widetilde{k^1_{z_i}}\}$ is a Bessel sequence.*

(ii) *If $\{\widetilde{k^1_{z_i}}\}$ satisfies the FC, then $\{\widetilde{k^2_{z_i}}\}$ satisfies the FC.*

Proof. Let $\|.\|_i$ denote the norm on $\mathcal{H}(K_i)$, the RKHS corresponding to the kernel $K_i,\ i = 1, 2$. Then using Theorem 3.3.2, $\mathcal{H}(K_1)$ is a vector subspace of $\mathcal{H}(K_2)$ with $\|f\|_2 \leq c\|f\|_1$ for all $f \in \mathcal{H}(K_1)$, since $K_1 \leq c^2 K_2$.

To prove (i), we first note that

$$\sum_i |\langle f, \widetilde{k^1_{z_i}}\rangle_1|^2 = \sum_i \frac{|f(z_i)|^2}{\|k^1_{z_i}\|_1^2} \leq d^2 \sum_i \frac{|f(z_i)|^2}{\|k^2_{z_i}\|_2^2} = d^2 \sum_i |\langle f, \widetilde{k^2_{z_i}}\rangle_2|^2,$$

for all $f \in \mathcal{H}(K_1)$, since $\mathcal{H}(K_1) \subseteq \mathcal{H}(K_2)$ and $\|k^2_{z_i}\|_2^2 \leq d^2\|k^1_{z_i}\|_1^2$, for all i. Now (i) follow easily from the above inequality, using $\|f\|_2 \leq c\|f\|_1$ for all $f \in \mathcal{H}(K_1)$.

In order to prove (ii), let $\{\widetilde{k^2_{z_i}}\}$ be a Bessel sequence in $\mathcal{H}(K_2)$. Then $\{\widetilde{k^1_{z_i}}\}$ is a Bessel sequence in $\mathcal{H}(K_1)$, using (i). Observe that

$$\left(\langle \widetilde{k}^1_{z_j}, \widetilde{k}^1_{z_i}\rangle_1\right) \leq D\left(\langle \widetilde{k}^2_{z_j}, \widetilde{k}^2_{z_i}\rangle_2\right) D^*,$$

where $D \in B(\ell^2)$ is an invertible, diagonal operator with i^{th} diagonal entry $\frac{\|k^2_{z_i}\|_2}{\|k^1_{z_i}\|_1}$. Hence, assuming that the sequence $\{\widetilde{k^1_{z_i}}\}$ satisfies the FC, (ii) follows, using the same ideas as used in the proof of Proposition 2.1.6. □

As an application of the above theorem, we get the following result. Recall (Proposition 2.2.4) that if $P \in B(H^2)$ is an orthogonal projection, then $Ran(P)$ is a RKHS on $\mathbb{D}$ with kernel $K_P(z, w) = Pk_w(z)$, where k_w is the kernel function in H^2 for the point w.

Corollary 3.3.4. *Let $\mathcal{L}$ be a Hilbert space and let $\mathcal{H}$ be a closed subspace of $\mathcal{L}$ with finite codimension. If $\mathcal{H}$ satisfies the FCKF, then $\mathcal{L}$ also satisfies the FCKF.*

Proof. Let $P \in B(\mathcal{L})$ be the orthogonal projection onto $\mathcal{H}$ and let $\{f_1, \ldots, f_N\}$ be an orthonormal basis for $Ker(P)$. Now to prove the result, we let $\{\widetilde{k}^{\mathcal{L}}_{z_i}\}_{i\in\mathbb{N}}$ be a Bessel sequence of normalized kernel functions in $\mathcal{L}$. Observe that,

$$1 = \|P\widetilde{k}^{\mathcal{L}}_{z_i}\|^2 + \sum_{m=1}^{N} \frac{|f_m(z_i)|^2}{\|k^{\mathcal{L}}_{z_i}\|^2}.$$

But $\{\widetilde{k}^{\mathcal{L}}_{z_i}\}_{i\in\mathbb{N}}$ is a Bessel sequence, and therefore

$$\sum_{i\in\mathbb{N}} \frac{|f_m(z_i)|^2}{\|k^{\mathcal{L}}_{z_i}\|^2} = \sum_{i\in\mathbb{N}} |\langle f_m, \widetilde{k}^{\mathcal{L}}_{z_i}\rangle|^2 < \infty,$$

for all $1 \leq m \leq N$. This implies that $\{\sum_{m=1}^{N} \frac{|f_m(z_i)|^2}{\|k^{\mathcal{L}}_{z_i}\|^2}\}_{i\in\mathbb{N}}$ converges to 0. Hence, $\{\|P\widetilde{k}^{\mathcal{L}}_{z_i}\|\}_{i\in\mathbb{N}}$ converges to 1. Therefore, given a $\delta > 0$, there exists i_0 such that

$\|P\widetilde{k}^{\mathcal{L}}_{z_i}\| \geq \delta$, all $i \geq i_0$, that is, $\|Pk^{\mathcal{L}}_{z_i}\| \geq \delta\|k^{\mathcal{L}}_{z_i}\|$, for all $i \geq i_0$. Since P is an orthogonal projection onto $\mathcal{H}$, therefore $Pk^{\mathcal{L}}_{z_i} = k^{\mathcal{H}}_{z_i}$ for all i, using Proposition 2.2.4. Thus we obtain a sequence $\{\widetilde{k}^{\mathcal{H}}_{z_i}\}$ of normalized kernel functions in $\mathcal{H}$, such that $\|k^{\mathcal{L}}_{z_i}\| \leq \frac{1}{\delta}\|k^{\mathcal{H}}_{z_i}\|$, for all $i \geq i_0$. Also by Theorem 3.3.2, $K_{\mathcal{H}} \leq K_{\mathcal{L}}$, since $\mathcal{H} = \mathcal{H}(P)$ is a closed subspace of $\mathcal{L}$. Hence by Theorem 3.3.3, $\{\widetilde{k}^{\mathcal{L}}_{z_i}\}_{i\in J}$ satisfies the FC, where $J = \{i : i \geq i_0\}$, using the hypothesis that $\mathcal{H}$ satisfies the FCKF. Further, every singleton is trivially a Riesz basic sequence, and hence the full sequence $\{\widetilde{k}^{\mathcal{L}}_{z_i}\}_{i\in\mathbb{N}}$ satisfies the FC. This completes the proof. $\square$

As another corollary to Theorem 3.3.3, we get the following generalization of Theorem 3.2.16.

Corollary 3.3.5. *Let $Q \in B(H^2)$ be an orthogonal projection. If $\{\widetilde{k}^{Q}_{z_i}\}$ is a Bessel sequence of normalized kernel functions in $Ran(Q)$, such that there exists an inner function ϕ with* $\sup_i|\phi(z_i)| < 1$ *and $H^2 \ominus \phi H^2 \subseteq Ran(Q)$, then $\{\widetilde{k}^{Q}_{z_i}\}$ satisfies the FC.*

Proof. Let $\{\widetilde{k}^{Q}_{z_i}\}$ be a Bessel sequence of normalized kernel functions in $Ran(Q)$, such that there exists an inner function ϕ with $\sup_i|\phi(z_i)| < 1$ and $H^2 \ominus \phi H^2 \subseteq Ran(Q)$. Now let $P \in B(H^2)$ be the orthogonal projection onto $H^2 \ominus \phi H^2$. Then $P = I - T_\phi T^*_\phi$, where I is the identity operator on H^2 and T_ϕ is the Toeplitz operator with symbol ϕ. Further, suppose K_P and K_Q denote the kernel functions for the Hilbert spaces $Ran(P)$ and $Ran(Q)$, respectively. Now since P is an orthogonal projection, therefore using Proposition 2.2.4 together with the fact that $T^*_\phi(k_{z_i}) = \overline{\phi(z_i)}k_{z_i}$, we get

$$\|k^{P}_{z_i}\|^2 = k^{P}_{z_i}(z_i) = Pk_{z_i}(z_i) = (1 - T_\phi T^*_\phi)k_{z_i}(z_i) = (1 - |\phi(z_i)|^2)\|k_{z_i}\|^2 \geq (1 - s^2)\|k_{z_i}\|^2,$$

for all i, where $s = \sup_i |\phi(z_i)| < 1$ and k_{z_i} is the kernel function in H^2 for the point z_i. Again, using Proposition 2.2.4 $\|k_{z_i}^Q\| = \|Qk_{z_i}\| \leq \|k_{z_i}\|$, since Q is an orthogonal projection. Thus $\|k_{z_i}^P\|^2 \geq (1 - s^2)\|k_{z_i}^Q\|^2$, for all i.

Lastly, since $Ran(P)$ is a closed subspace of $Ran(Q)$, therefore using Theorem 3.3.2, we get $K_P \leq K_Q$. Thus, all the hypotheses of Theorem 3.3.3 are satisfied, and hence the result follows from Theorem 3.3.3, using Theorem 3.2.16. $\square$

The next operation we would like to discuss is the **Pull-Backs** of RKHS's. This technique of constructing new kernel functions from existing ones is discussed by Paulsen in [39].

Let X be a set and let $K : X \times X \to \mathbb{C}$ be a kernel function on X. If S is any set and $\phi : S \to X$ is a function, then we let $K \circ \phi : S \times S \to \mathbb{C}$ denote the function given by, $K \circ \phi(s, t) = K(\phi(s), \phi(t))$. It is straight forward to check that $K \circ \phi$ is a positive definite function, since K being a kernel function is a positive definite function. Thus $K \circ \phi$ is a kernel function on the set S. The RKHS $\mathcal{H}(K \circ \phi)$ is the pull-back of the RKHS $\mathcal{H}(K)$ along the function ϕ, in a sense that is depicted by the following theorem from [39].

Theorem 3.3.6 (Paulsen, [39])**.** *Let X and S be sets, let $K : X \times X \to \mathbb{C}$ be a positive definite function and let $\phi : S \to X$ be a function. Then $\mathcal{H}(K \circ \phi) = \{f \circ \phi : f \in \mathcal{H}(K)\}$, and for $g \in \mathcal{H}(K \circ \phi)$ we have that $\|g\|_{\mathcal{H}(K\circ\phi)} = \inf\{\|f\|_{\mathcal{H}(K)} : g = f \circ \phi\}$.*

As we can see that the pull-back $\mathcal{H}(K \circ \phi)$ of $\mathcal{H}(K)$ along ϕ is completely characterized by $\mathcal{H}(K)$ and the function ϕ. Thus, it is natural to expect that the FCKF for $\mathcal{H}(K \circ \phi)$ might depend on both, the FCKF for $\mathcal{H}(K)$ and the function ϕ. The following theorem is interesting as it not only confirms this, but also shows that we do not require the function ϕ to have any special properties. For the purpose of the next theorem , we let $K_\phi = K \circ \phi$.

Theorem 3.3.7. *Let X and S be sets, let $K : X \times X \to \mathbb{C}$ be a positive definite function and let $\phi : S \to X$ be a function. If $\mathcal{H}(K)$ satisfies the FCKF, then so does $\mathcal{H}(K_\phi)$. Moreover, if ϕ is onto, then the converse is also true.*

Proof. By using the definition of the kernel function K_ϕ, we obtain

$$k_t^\phi(s) = K_\phi(s,t) = K(\phi(s), \phi(t)),$$

for all $s, t \in S$. The above equation also implies that $\|k_t^\phi\|^2_{\mathcal{H}(K_\phi)} = k_t^\phi(t) = K(\phi(t), \phi(t)) = \|k_{\phi(t)}\|^2_{\mathcal{H}}$, for all $t \in S$. Therefore

$$\langle \widetilde{k}_s^\phi, \widetilde{k}_t^\phi \rangle = \langle \widetilde{k}_{\phi(s)}, \widetilde{k}_{\phi(t)} \rangle,$$

for each pair $s, t \in S$. Now the result follows, using Proposition 2.1.6. $\square$

The Pull-backs can be a useful tool to have, as it can prove the FCKF for some really interesting RKHS's. The following is one such example.

Example 3.3.8. *Let $N \in \mathbb{N}$ and let $P \in B(H^2)$ be the orthogonal projection onto the closed linear span of $\{z^{kN} : k \geq 0\}$. Then using Proposition 2.2.4, the kernel function for $Ran(P)$ is given by $K_P(z,w) = Pk_w(z) = K(z^N, w^N)$, where K is the kernel function for H^2 and k_w is the kernel function in H^2 for the point w. Thus the contractive Hilbert space $\mathcal{H}(P) = Ran(P)$ is the pull-back of the Hardy space H^2 along the function $\phi(z) = z^N$. Hence it follows from Theorem 3.3.7 that $\mathcal{H}(P)$ satisfies the FCKF, since H^2 satisfies the FCKF.*

In [39], together with the pull-backs, the author discusses one other interesting operation associated to kernels, the **push-outs** of RKHS's. Unfortunately we do not have any concrete results in this direction. However, we wish to mention it here to point out that this discussion leads to the same old questions about transferring the FCKF from a space to its subspaces and vice-versa.

Given a RKHS $\mathcal{H}(K)$ on X and a surjective map $\psi : X \to S$, we would also like to induce a RKHS on S. To carry out this construction, we first consider the subspace, $\mathcal{H}_0 = \{f \in \mathcal{H}(K) : f(x_1) = f(x_2) \text{ whenever } \psi(x_1) = \psi(x_2)\}$. If $K_0(x,y)$ denotes the kernel for this subspace and we set, $k_y^0(x) = K_0(x,y)$, then it readily follows that, whenever $\psi(x_1) = \psi(x_2)$ and $\psi(y_1) = \psi(y_2)$, we have that, $k_{y_1}^0 = k_{y_2}^0$ and $k_w^0(x_1) = k_w^0(x_2)$, for all $w \in X$. Thus, for any such pair of points, $K_0(x_1,y_1) = K_0(x_2,y_2)$. It follows that there is a well-defined positive definite function on $K_\psi : S \times S \to \mathbb{C}$, given by $K_\psi(s,t) = K_0(\psi^{-1}(s), \psi^{-1}(t))$. We call the RKHS, $\mathcal{H}(K_\psi)$ on S, the push-out of $\mathcal{H}(K)$ along ψ. The definition of K_ψ leads to the following result.

Proposition 3.3.9. *The push-out $\mathcal{H}(K_\psi)$ satisfies the FCKF if and only if $\mathcal{H}_0$ satisfies the FCKF.*

Proof. For $s, t \in S$,

$$\langle k_t^\psi, k_s^\psi \rangle = K_\psi(s,t) = K_0(\psi^{-1}(s), \psi^{-1}(t)) = \langle k_{\psi^{-1}(t)}^0, k_{\psi^{-1}(s)}^0 \rangle.$$

The above equalities also imply that $\|k_t^\psi\|_{\mathcal{H}(K_\psi)} = \|k_{\psi^{-1}(t)}^0\|_{\mathcal{H}_0}$ for all $t \in S$. Thus given a sequence $\{s_i\}$ in S, $\left(\langle \widetilde{k}_{s_j}^\psi, \widetilde{k}_{s_i}^\psi \rangle\right) = \left(\langle \widetilde{k}_{\psi^{-1}(s_j)}^0, \widetilde{k}_{\psi^{-1}(s_i)}^0 \rangle\right)$, and hence the result follows, using Proposition 2.1.6. $\square$

Thus, asking the FCKF for the push-out $\mathcal{H}(K_\psi)$ is the same as asking it for the subspace $\mathcal{H}_0$. Hence, as far as the FCKF is concerened, the case of push-out operation "more or less" reduces to the case of closed subspaces, .

Lastly, we will discuss the **products** and the **tensor products** of kernels. We will see that these are closely related to the Schur products. We start with some basic facts and definitions, for detailed discussion on this topic, we refer the reader to [39].

If X and S are sets and $K_1 : X \times X \to \mathbb{C}$ and $K_2 : S \times S \to \mathbb{C}$ are kernel functions, then $K : (X \times S) \times (X \times S) \to \mathbb{C}$ given by $K((x, s), (y, t)) = K_1(x, y)K_2(s, t)$, is a kernel function.

Definition 3.3.10. *We call the 4-variable function $K((x, s), (y, t)) = K_1(x, y)K_2(s, t)$, the* **tensor product** *of the kernels K_1 and K_2.*

The RKHS $\mathcal{H}(K)$ is isometrically isomorphic to the Hilbert space $\mathcal{H}(K_1) \otimes \mathcal{H}(K_2)$, which justifies the name. For a proof of this statement we refer the reader to [39].

Further, if X is a set and $K_i : X \times X \to \mathbb{C}, i = 1, 2$ are kernel functions, then their product, $\pi : X \times X \to \mathbb{C}$ given by $\pi(x, y) = K_1(x, y)K_2(x, y)$, is a kernel function.

Definition 3.3.11. *We call the 2-variable kernel, $\pi(x, y) = K_1(x, y)K_2(x, y)$ the* **product** *of the kernels.*

Given $K_i : X \times X \to \mathbb{C},\ i = 1, 2$, we have two kernels and two RKHS's. The first is the tensor product $K : (X \times X) \times (X \times X) \to \mathbb{C}$, which gives a RKHS of functions on $X \times X$. The second is the product $\pi : X \times X \to \mathbb{C}$, which gives a RKHS of functions on X. The relationship between these two spaces can be seen as follows.

Let $\Delta : X \to X \times X$ denote the diagonal map, defined by $\Delta(x) = (x, x)$. Then $\pi(x, y) = K(\Delta(x), \Delta(y))$, that is, $\pi = K \circ \Delta$. Thus, $\mathcal{H}(\pi)$ is the pull-back of $\mathcal{H}(K) = \mathcal{H}(K_1) \otimes \mathcal{H}(K_2)$ along the diagonal map Δ.

Hence, with respect to the FCKF, it is enough to address the case of tensor products of kernels, since the case of products will follows immediately using Theorem 3.3.7.

For the rest of the section, we let X and S be two sets and $K_1 : X \times X \to \mathbb{C}$ and $K_2 : S \times S \to \mathbb{C}$ be two kernel functions. Let $K : (X \times S) \times (X \times S) \to \mathbb{C}$, given by $K((x, s), (y, t)) = K_1(x, y)K_2(s, t)$ be the tensor product of K_1 and K_2. Observe that

$$(\langle \widetilde{k}_{(x_j,s_j)}, \widetilde{k}_{(x_i,s_i)} \rangle) = (\langle \widetilde{k}^1_{x_j}, \widetilde{k}^1_{x_i} \rangle) * (\langle \widetilde{k}^2_{s_j}, \widetilde{k}^2_{s_i} \rangle). \tag{3.6}$$

Theorem 3.3.12. *Let $\mathcal{H}(K_1)$ (or $\mathcal{H}(K_2)$) satisfy the FCKF. If for every Bessel sequence $\{\widetilde{k}_{(x_i,s_i)}\}$ of normalized kernel functions in $\mathcal{H}(K)$, the corresponding sequence $\{\widetilde{k}^1_{x_i}\}$ (or $\{\widetilde{k}^2_{s_i}\}$) is a Bessel sequence in $\mathcal{H}(K_1)$ (or $\mathcal{H}(K_2)$), then $\mathcal{H}(K)$ satisfies the FCKF.*

Proof. Let $\{\widetilde{k}_{(x_i,s_i)}\}_{i\in\mathbb{N}}$ be a Bessel sequence of normalized kernel functions in $\mathcal{H}(K)$, then the corresponding sequence $\{\widetilde{k}^1_{x_i}\}_{i\in\mathbb{N}}$ of normalized kernels in $\mathcal{H}(K_1)$ is a Bessel sequence. Hence, there exist a partition $A_1, \dots, A_N$ of $\mathbb{N}$ and constants $c_1, \dots, c_N$ such that,

$$P_{A_m}\big(\langle \widetilde{k}^1_{x_j}, \widetilde{k}^1_{x_i} \rangle\big)P_{A_m} \geq c_m P_{A_m}, \tag{3.7}$$

for all $1 \leq m \leq N$. Also, $\|\widetilde{k}^2_{s_i}\| = 1$, for all $i \in \mathbb{N}$. Hence the result follows from Theorem 2.3.2, using Equations (3.6) and (3.7). $\square$

Next is a little surprising result about the tensor products. To prove this result, we need the following few results from the literature. The first result that we need is a beautiful theorem due to Schur [50].

Theorem 3.3.13 (Schur, [50])**.** *Let $A = (a_{ij})$ and $C = (c_{ij})$ be two bounded operators in $B(\ell^2)$. Then $\big(|a_{ij}c_{ij}|\big) \in B(\ell^2)$.*

Now we recall a definition from the literature, which we need for the following result. Given a bounded operator $T \in B(\ell^2)$, for each $m \in \mathbb{N}$,

$$\alpha_m(T) = \inf\left\{\|\sum_{i=1}^{m} P_{A_i}(T - E(T))P_{A_i}\| : \{A_i\}_{i=1}^{m} \text{ is a partition of } \mathbb{N}\right\},$$

where E is bounded operator on $B(\ell^2)$, which maps a $S \in B(\ell^2)$ to its diagonal, that is, to the the bounded, diagonal operator with i^{th} diagonal entry $\langle Se_i, e_i\rangle$.

The following is an interesting theorem due to Berman, Halpern, Kaftal, and Weiss from [12].

Theorem 3.3.14 (BHKW, [12])**.** *Let $T \in B(\ell^2)$ be a self adjoint operator with non-negative entries, then for every $m \in \mathbb{N}$, $\alpha_m(T) \leq \frac{1}{m}\|T\|$.*

Lastly, we need the Perron-Fronbenius theorem [51, Theorem 2.8], which states that if B and C are $n \times n$ matrices with non-negative entries, then $\|B\| \leq \|B + C\|$. Thus, given an operator $T \in B(\ell^2)$ with non-negative entries and a partitions

$A_1, \ldots, A_m$ of $\mathbb{N}$,

$$\|P_{A_j}(T - E(T))P_{A_j}\| \leq \|\sum_{i=1}^{m} P_{A_i}(T - E(T))P_{A_i}\|, \tag{3.8}$$

for all $1 \leq j \leq m$.

We exploit the above results as follows.

Proposition 3.3.15. *Let $T = (t_{ij}) \in B(\ell^2)$ be self-adjoint such that $(|t_{ij}|) \in B(\ell^2)$ and $t_{ii} > 0$ for all i with $\inf_i t_{ii} > 0$. Then there exist a partition $A_1, \ldots, A_N$ of $\mathbb{N}$ and constants $c_1, \ldots, c_N$, such that $P_{A_i}TP_{A_i} \geq c_i P_{A_i}$, for all $1 \leq i \leq N$.*

Proof. Set $S = (|t_{ij}|)$. Let $\delta = \inf_i t_{ii} > 0$. Choose a $N \in \mathbb{N}$ so that $\frac{1}{N}\|S\| < \frac{\delta}{2}$. Thus, $\alpha_N(S) < \frac{\delta}{2}$, by using Theorem 3.3.14. Hence, by the definition of $\alpha_N(S)$, there exists a partition $A_1, \ldots, A_N$ of $\mathbb{N}$ such that

$$\|\sum_{i=1}^{N} P_{A_i}(S - E(S))P_{A_i}\| < \frac{\delta}{2}.$$

This yields that $\|P_{A_i}(S - E(S))P_{A_i}\| < \frac{\delta}{2}$ for all $1 \leq i \leq N$, using Equation 3.8. Thus, $\|P_{A_i}(T - E(T))P_{A_i}\| < \frac{\delta}{2}$, since $S = (|t_{ij}|)$. Further, note that the operator $T - E(T)$ is self-adjoint, thus we can write:

$$\frac{-\delta}{2}P_{A_i} \leq P_{A_i}(T - E(T))P_{A_i} \leq \frac{\delta}{2}P_{A_i},$$

for all $1 \leq i \leq N$. Also, $\delta P_{A_i} \leq P_{A_i}E(T)P_{A_i} \leq \|T\|P_{A_i}$, for all $1 \leq i \leq N$. Hence, we get

$$\frac{\delta}{2}P_{A_i} \leq P_{A_i}TP_{A_i} \leq (\delta/2 + \|T\|)P_{A_i},$$

for all $1 \leq i \leq N$, which concludes the proof. $\square$

Theorem 3.3.16. *Given a Bessel sequence $\{\widetilde{k}_{(x_i,s_i)}\}$ of normalized kernel functions in $\mathcal{H}(K)$, if the corresponding sequences $\{\widetilde{k}^1_{x_i}\}$ and $\{\widetilde{k}^2_{x_i}\}$ of normalized kernel functions are Bessel sequence in $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$, respectively, then $\{\widetilde{k}_{(x_i,s_i)}\}$ satisfies the FC.*

Proof. Let $\{\widetilde{k}_{(x_i,s_i)}\}$ be a Bessel sequence in $\mathcal{H}(K)$ so that the corresponding sequences $\{\widetilde{k}^1_{x_i}\}$ and $\{\widetilde{k}^2_{x_i}\}$ of normalized kernel functions are Bessel sequence in $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$, respectively. Then $(\langle \widetilde{k}^1_{x_j}, \widetilde{k}^1_{x_i}\rangle)$ and $(\langle \widetilde{k}^2_{s_j}, \widetilde{k}^2_{s_i}\rangle)$ are bounded operators on ℓ^2. Hence, by Theorem 3.3.13

$$\Big(\big|\langle \widetilde{k}_{(x_j,s_j)}, \widetilde{k}_{(x_i,s_i)}\rangle\big|\Big)$$

is a bounded operator on ℓ^2. Also, the diagonal enteries $\langle \widetilde{k}_{(x_i,s_i)}, \widetilde{k}_{(x_i,s_i)}\rangle$ are positive and all equal to 1. Thus, by Proposition 3.3.15 we get a partition $A_1, \ldots, A_N$ of $\mathbb{N}$ and constants $c_1, \ldots, c_N$ such that,

$$P_{A_m}\big(\langle \widetilde{k}_{(x_j,s_j)}, \widetilde{k}_{(x_i,s_i)}\rangle\big)P_{A_m} \geq c_m P_{A_m},$$

for all $1 \leq m \leq N$. Hence by Proposition 2.1.3, each $\{\widetilde{k}_{(x_i,s_i)}\}_{i\in A_m}$ is a Riesz basic sequence. This completes the proof. $\square$

We end this section with the note that we will see some interesting applications of Theorem 3.3.12 and Theorem 3.3.16 in the last section of Chapter 4.

3.4 Summary

We conclude this chapter with a list of contractive Hilbert spaces, which satisfies the FCKF or RFCKF, completely or partially.

(1) The Hardy space H^2 satisfies the FCKF (Theorem 3.0.6). Note that $H^2 = \mathcal{H}(I)$, where I is the identity operator on ℓ^2.

(2) If $P \in B(H^2)$ is a positive, invertible contraction, then $\mathcal{H}(P)$ satisfies the FCKF.

(3) Given a one-component inner function ϕ, the contractive Hilbert space $\mathcal{H}(P)$ satisfies the FCKF, where P is the orthogonal projection onto $H^2 \ominus \phi H^2$ (Baranov and Dykanov, [8]).

(4) If ϕ is an inner function and P is the orthogonal projection onto $H^2 \ominus \phi H^2$, then every Bessel sequence $\{\widetilde{k}^P_{z_i}\}$ of normalized kernel functions in $\mathcal{H}(P)$, such that $\sup_i |\phi(z_i)| < 1$, satisfies the FC (Baranov and Dykanov, [8]).

(5) Theorem 3.3.3 generalizes the second result of Baranov and Dyakonov. This generalization states that if $Q \in B(H^2)$ is an orthogonal projection and $\{\widetilde{k}^Q_{z_i}\}$ is a Bessel of normalized kernel functions in $\mathcal{H}(Q)$, such that there exists an inner function ϕ, with $H^2 \ominus \phi H^2 \subseteq Ran(Q)$, $\sup_i |\phi(z_i)| < 1$, then $\{\widetilde{k}^Q_{z_i}\}$ satisfies the FC (Corollary 3.3.5).

(6) If ϕ is an inner function and $P \in B(H^2)$ is the orthogonal projection onto ϕH^2, then $\mathcal{H}(P)$ satisfies the FCKF (Theorem 3.2.4). As an application, we get that when P is an orthogonal projection such that $Ker(P)$ is spanned by finitely

many kernel functions in H^2, then the contractive Hilbert space $\mathcal{H}(P)$ satisfies the FCKF (Example 3.2.5).

(7) If $P \in B(H^2)$ is an orthogonal projection and $\{\widetilde{k}_{z_i}^P\}$ is a Bessel sequence of normalized kernel functions in $\mathcal{H}(P)$ such that there exists an inner function ϕ with $\phi H^2 \subseteq Ran(P),\ |\phi(z_i)| \geq \delta > 0$, for all i, then $\widetilde{k}_{z_i}^P$ satisfies the FC (Remark 3.2.10). Following are some examples of such projections.

   (a) Let $P \in B(H^2)$ be the orthogonal projection onto $[\mathbb{C} + \phi H^2]$, where ϕ is an inner function (Example 3.2.14).

   (b) Let $P \in B(H^2)$ be the orthogonal projection such that $Ker(P)$ is spanned by finitely many inner functions (Example (3.2.12)).

(8) If $P \in B(H^2)$ is an orthogonal projection such that $Ker(P) = \text{span}\{z^{i_1}, \ldots, z^{i_n}\}$, for some $i_1, \ldots, i_n$, then $\mathcal{H}(P)$ satisfies the RFCKF (Remark 3.2.15).

(9) If $P \in B(H^2)$ is an orthogonal projection onto $\text{span}\{z^{Nk} : k \geq 0\}$, for some non-negative integer N. Then $\mathcal{H}(P)$ satisfies the FCKF (Example 3.3.8).

Chapter 4

The Feichtinger Conjecture for Kernel Functions for Some Well-known Spaces

In this chapter, we prove the Feichtinger conjecture for kernel functions(FCKF) for some well-known spaces. The list includes:

(i) Hardy space H^2 on the unit disk.

(ii) $H^2_{\alpha,\beta}$ spaces on the unit disk.

(iii) Weighted Bergman Spaces on the unit ball (with a mild restriction in the several variable case).

(iv) Bargmann-Fock spaces on the n-dimensional complex plane.

Our proof of the FCKF for the above listed spaces, except for the Hardy space, are new and are based on the results presented in the last two sections of Chapter

3. In the first section, we present Nikolski's proof of the fact that H^2 satisfies the FCKF. We include the proof here as this fact is indispensable for our proofs in the next two sections, and besides, this particular proof is not known to many people. In the next section we prove the FCKF for the $H^2_{\alpha,\beta}$ spaces on the unit disk. Last two sections are devoted to the weighted Bergman spaces on the unit ball and the Bargmann-Fock spaces on the n-dimensional complex plane. Both of these classes carry a very rich theory, and one can combine results therein to prove the FCKF for them. For several variable case, these results also imply the FCKF only for a subclass of weighted Bergman spaces, which coincides with the class we obtain with our approach.

4.1 Notations and Terminology

Let $\mathbb{C}^n$ denote the n dimensional complex plane. For $z = (z_1, \ldots, z_n),\ w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w\rangle = \sum_{i=1}^n z_i\overline{w_i}$ and $|z|^2 = \sum_{i=1}^n |z_i|^2$. Let ν be the Lebesgue measure on $\mathbb{C}^n$, normalized so that $\nu(\mathbb{B}_n) = 1$, where $\mathbb{B}_n$ is the open unit ball of $\mathbb{C}^n$, and let σ be the surface measure on $\mathbb{S}_n = \{z \in \mathbb{C}^n : |z| = 1\}$.

Recall that to every β in the open unit disk of $\mathbb{C}$ corresponds an automorphism ϕ_β of the open unit disk that interchanges β and 0, namely $\phi_\beta(z) = \frac{\beta - z}{1 - z\overline{\beta}}$ (an **automorphism** of an open set $\Omega \subseteq \mathbb{C}^n$ is a holomorphic map from Ω onto itself with holomorohic inverse). The same can be done for the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. To see this, let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the one-dimensional subspace $[a]$ generated by $a \in \mathbb{B}_n,\ a \neq 0$, and let $Q_a = I - P_a$ be the orthogonal projection from

$\mathbb{C}^n$ onto the orthogonal complement of $[a]$. To be more explicit, $P_0 = 0$ and

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2}\, a \quad \text{if } a \neq 0.$$

Put $s_a = \sqrt{1-|a|^2}$ and define

$$\phi_a^n(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathbb{B}_n.$$

Note that for $n = 1$, $\phi_a^n(z) = \frac{z-a}{1-z\overline{a}} = \phi_a(z)$.

The map ϕ_a^n is an automorphism of $\mathbb{B}_n$ and it also interchanges a and 0. We will record this, along some other important properties of these maps, formally in Section 4.4. From this point forward, we shall write $z \cdot \overline{w}$ to denote $\langle z, w\rangle$, $z, w \in \mathbb{C}^n$.

These automorphisms of $\mathbb{B}_n$ lead to a metric on $\mathbb{B}_n$, as follows. For each $z, w \in \mathbb{B}_n$, let

$$\rho_n(z, w) = |\phi_w^n(z)|.$$

Then ρ_n defines a metric on $\mathbb{B}_n$, known as the **pseudo-hyperbolic metric** on $\mathbb{B}_n$.

Lastly, given a holomorphic mapping $F(z) = (f_1(z), \ldots, f_n(z))$ from $\mathbb{C}^n$ into $\mathbb{C}^n$, we shall write $F'(z)$ to denote the $n\times n$ complex matrix $\left(\frac{\partial f_i}{\partial z_j}(z)\right)$. By writing each f_m as $f_m = u_m + iv_m$, where u_m and v_m are real and imaginary parts of f_m, respectively, and writing the coordinates z_m of each $z = (z_1, \ldots, z_n)$, as $z_m = x_m + iy_m$, where x_m and y_m are real and imaginary parts of z_m, respectively, the mapping F, can also be regarded as a map that takes the real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ to real-valued functions $u_1, \ldots, u_n, v_1, \ldots, v_n$. In this case, F has a real Jacobian at z, which we shall denote by $J_{\mathbb{R}}F(z)$. Note that $J_{\mathbb{R}}F(z)$ is a $2n \times 2n$ matrix of reals. It is well-known that $\det(J_{\mathbb{R}}F(z)) = \left|\det(\mathrm{F}'(\mathrm{z}))\right|^2$, where $\det(A)$ denotes the determinant of a matrix A.

Definition 4.1.1. *A sequence* $\{z_i\}$ *in* $\mathbb{C}^n$ *is said to be* **weakly separated** *if there exists a* $\delta > 0$*, such that* $\rho_n(z_i, z_j) \geq \delta$*, for all* $i \neq j$*.*

The following enumeration technique is used by McKenna [31] in his characterization of Bessel sequences of normalized kernel functions in H^2.

Lemma 4.1.2 (McKenna, [31])**.** *Let* $\{x_i\}_{i\in\mathbb{N}}$ *be a sequence in a Hilbert spaces* $\mathcal{H}$*, such that there exists a constant* $B > 0$ *with*

$$\sum_{i\in\mathbb{N}} |\langle x_i, x_j\rangle| \leq B, \tag{4.1}$$

for all $j \in \mathbb{N}$*. Then given a constant* $\alpha,\ 0 < \alpha < 1$*, there exists a partition* $A_1, \ldots, A_N$ *of* $\{x_i\}$ *such that*

$$|\langle x_i, x_j\rangle| < \alpha, \tag{4.2}$$

for all $x_i,\ x_j \in A_m,\ x_i \neq x_j,\ 1 \leq m \leq N$*.*

Proof. Fix $\alpha,\ 0 < \alpha < 1$, and choose $N \in \mathbb{N}$ so that $N > B/\alpha$. We will partition $\{x_i\}$ into N subsequences, where each subsequence satisfies the required condition.

Let $x_{n,1} = x_n$, for $1 \leq n \leq N$. We take $x_{n,i}$ as the first term of the i^{th} subsequences. Suppose we have already partitioned $\{x_i : i \leq m\}$ into N subsequences, so that within each subsequence we have

$$|\langle x_i, x_j\rangle| < \alpha,$$

whenever $x_i \neq x_j$.

We now have to allocate x_{m+1} to one of the N subsequences. By the virtue of Inequality (4.1) and by the choice of N, we know that Inequality (4.2) can be violated for at most $N-1$ values of i, if $j=m$. Thus z_{m+1} can be made the next term of at least one of the subsequences in such a way that each subsequence still satisfies the Inequality (4.2). Now the proof follows by induction. $\square$

4.2 Hardy Space H^2

In this section, we present Nikolski's proof of the fact that H^2 satisfies the FCKF, which he gave in an AIM workshop in 2006. By collecting some results from [32], one can obtain another proof of the same result. The outline of the two proofs are the same, but perhaps, the construction of the partition is simpler in the proof presented at the workshop.

Recall that H^2 is a RKHS with kernel function $K(z,w) = \frac{1}{1-z\overline{w}},\ z,\ w \in \mathbb{D}$. There is a crucial relationship between the normalized kernel functions in H^2 and pseudo-hyperbolic metric ρ_1, namely

$$|\langle \widetilde{k}_w, \widetilde{k}_z \rangle|^2 = 1 - (\rho_1(z,w))^2 \tag{4.3}$$

A characterization of Riesz basic sequences due to Bari is an essential ingredient of the proof. To state this result, we need the following definition.

Definition 4.2.1. *A sequence $\{x_i\}$ in a Hilbert spaces is said to be* **biorthogonal** *to another sequence $\{y_i\}$ in $\mathcal{H}$ if $\langle x_i, y_j \rangle = \delta_{i,j}$ for all i,j, where $\delta_{i,j}$ is the Kronecker delta.*

Theorem 4.2.2 (Bari)**.** *Let $\{x_i\}_{i\in J}$ be a Bessel sequence in a Hilbert space $\mathcal{H}$, with $\overline{\text{span}\{x_i : i \in J\}} = \mathcal{H}$. Then $\{x_i\}_{i\in J}$ is a Riesz basis for $\mathcal{H}$ if and only if there exists a Bessel sequence $\{y_i\}_{i\in J}$ in $\mathcal{H}$, such that $\{y_i\}_{i\in J}$ is biorthogonal to $\{x_i\}_{i\in J}$.*

Proof. Let $\{x_i\}_{i\in J}$ be a Riesz basis of $\mathcal{H}$. Then there exists an orthonormal basis $\{u_i\}_{i\in J}$ of $\mathcal{H}$ and an invertible operator $S \in B(\mathcal{H})$, such that $S(u_i) = x_i$, for all i. Now for each i, set $y_i = (S^*)^{-1}(u_i)$. Then $\langle y_i, x_j\rangle = \langle u_i, S^{-1}x_j\rangle = \langle u_i, u_j\rangle = \delta_{ij}$, for all i, j. Hence it follows that the sequence $\{y_i\}_{i\in J}$ is biorthogonal to the sequence $\{x_i\}_{i\in J}$.

Conversely, let $\{y_i\}_{i\in J}$ be a Bessel sequence in $\mathcal{H}$, such that $\{y\}_{i\in J}$ is biorthogonal to $\{x_i\}_{i\in J}$. Thus for each i, j, we have $\langle y_j, x_i\rangle = \delta_{i,j}$, which implies that $\langle G^*e_j, F^*e_i\rangle = \delta_{i,j}$, where $G : \mathcal{H} \to \ell^2(J)$ and $F : \mathcal{H} \to \ell^2(J)$ are the analysis operators corresponding to the Bessel sequences $\{y_i\}_{i\in J}$ and $\{x_i\}_{i\in J}$, respectively. Therefore, $FG^* = I_{\ell^2(J)}$, and thus it follows that F is onto. Also, F is one-to-one, by definition. Hence, F is invertible, from which it follows that the Bessel sequence $\{x_i\}_{i\in J}$ is a Riesz basis for $\overline{\text{span}\{x_i : i \in J\}} = \mathcal{H}$. □

The next result is an important part of the proof, we are presenting it separately for reader's convenience.

Lemma 4.2.3. *Let $\{\widetilde{k}_{z_i}\}_{i\in J}$ be a Bessel sequence of normalized kernel functions in H^2. If there exists a $\delta > 0$, such that $\prod_{j\neq i}\left|\frac{z_i - z_j}{1-\bar{z}_i z_j}\right| \geq \delta$ for all $i \in J$, then $\{\widetilde{k}_{z_i}\}_{i\in J}$ is a Riesz basic sequence.*

Proof. Let B be the Blaschke product with zeroes at $\{z_i\}_{i\in J}$. Then $\overline{\text{span}\{\widetilde{k}_{z_i} : i \in J\}} =$

$H^2 \ominus BH^2$, using Proposition 3.1.2. We show that $\{\widetilde{k}_{z_i}\}_{i\in J}$ is a Riesz basis for $H^2 \ominus BH^2$. To accomplish this, set

$$f_i = \frac{B_i k_{z_i}}{B_i(z_i)\|k_{z_i}\|},$$

for all $i \in J$, where B_i is the Blaschke product with zeroes at $\{z_j : j \neq i\}$. Then $\{f_i\}_{i\in J} \subseteq H^2 \ominus BH^2$, and it is biorthogonal to $\{\widetilde{k}_{z_i}\}_{i\in J}$. Now to complete the proof, all we need to show is that $\{f_i\}_{i\in J}$ is a Bessel sequence. For this, let $f \in H^2 \ominus BH^2$ and consider

$$\sum_{i\in J} |\langle f, f_i\rangle|^2 = \sum_{i\in J} \left|\left\langle f, \frac{B_i k_{z_i}}{B_i(z_i)\|k_{z_i}\|}\right\rangle\right|^2 = \sum_{i\in J} \left|\left\langle \phi_i f, \frac{\phi_i B_i k_{z_i}}{B_i(z_i)\|k_{z_i}\|}\right\rangle\right|^2, \tag{4.4}$$

where $\phi_i(z) = \frac{z_i - z}{1-\overline{z_i}z}$, which is an inner function in H^2. Now note that

$$B = \frac{\overline{z_i}}{|z_i|}\frac{z_i - z}{1-\overline{z_i}z}B_i = \frac{\overline{z_i}}{|z_i|}\phi_i B_i,$$

therefore Equation (4.4) can be written as

$$\begin{aligned}
\sum_{i\in J} |\langle f, f_i\rangle|^2 &= \sum_{i\in J} \left|\left\langle \phi_i f, \frac{B k_{z_i}}{B_i(z_i)\|k_{z_i}\|}\right\rangle\right|^2 \\
&= \sum_{i\in J} \frac{|\langle (z_i - z) f k_{z_i}, B k_{z_i}\rangle|^2}{|B_i(z_i)|^2\|k_{z_i}\|^2} \\
&= \sum_{i\in J} \frac{|\langle \widetilde{k}_{z_i}, \overline{zf}B\rangle|^2}{|B_i(z_i)|^2},
\end{aligned}$$

where the last equality on the right hand side follows from the fact that $\overline{f}B$, $\overline{zf}B$, $\overline{f}Bk_{z_i}$, and $\overline{zf}Bk_{z_i}$ are all functions in H^2, and the observation that $\overline{z_i}(\overline{f}B)(z_i) = |z_i|^2(\overline{zf}B)(z_i)$. Finally, the last term on the right hand side of the above set of equations is less than or equal to a constant (independent of i) times the norm of f, since $\{\widetilde{k}_{z_i}\}$ is a Bessel sequence and $|B_i(z_i)| = \prod_{j\neq i}\left|\frac{z_j - z_i}{1-\overline{z_j}z_i}\right| \geq \delta$, for all i. Thus $\{f_i\}_{i\in J}$ is a Bessel sequence. Hence the result follows, using Theorem 4.2.2. $\square$

Now we are ready to present the Nikolski's proof of the following result.

Theorem 4.2.4. *Every Bessel sequence of normalized kernel functions in H^2 splits into finitely many Riesz basic sequences.*

Proof. Let $\{\widetilde{k}_{z_i}\}_{i\in\mathbb{N}}$ be a Bessel sequence of normalized kernel functions in H^2. Then there exists a constant $B > 0$, such that

$$\sum_{j\in\mathbb{N}} |\langle \widetilde{k}_{z_j}, \widetilde{k}_{z_i}\rangle|^2 \leq B, \tag{4.5}$$

for all $i \in \mathbb{N}$. Then using Equation (4.3), we get

$$\sum_{j\in\mathbb{N}} \left(1 - (\rho_1(z_i, z_j))^2\right) \leq B, \tag{4.6}$$

for all $i \in \mathbb{N}$. We claim that for each fixed $r > 0$ there exists $N \in \mathbb{N}$, such that each disk $D_{\rho_1}(z_i, r) = \{z : \rho_1(z, z_i) < r\}$ can have at most N $z_j's$ (including z_i). To settle the claim, we fix a z_m and a $r > 0$. Now from Equation (4.6), we can deduce that

$$\sum_{j, z_j \in D_{\rho_1}(z_m, r)} \left(1 - (\rho_1(z_m, z_j))^2\right) \leq B$$

Note that $z_j \in D_{\rho_1}(z_m, r)$ implies that $\rho_1(z_j, z_m) < r$, which further implies that $1 - (\rho_1(z_m, z_j))^2 > 1 - r^2$. Thus choosing N to be the lagest positive integer less than or equal to $\frac{B}{1-r^2}$, we get that $D_\rho(z_m, r)$ can have at most N $z_j's$. Hence each disk $D_{\rho_1}(z_i, r)$ can have at most N $z_j's$(including z_i).

Finally, we shall construct a partition $A_1, \ldots, A_t$ of $\mathbb{N}$, with $t \leq N$, so that each subsequence $\{\widetilde{k}_{z_i}\}_{i\in A_j}$ is a Riesz basic sequence. We will accomplish this in two steps. First we will obtain a partition $A_1, \ldots, A_t$ of $\mathbb{N}$, so that in each of the corresponding subsequence ρ_1- distance between any z_i, z_j is at least r, whenever $i \neq j$. Then in the second step we will show that each of the corresponding subsequence $\{\widetilde{k}_{z_i}\}_{i\in A_j}$ is a Riesz basic sequence.

Step 1. To construct the first set A_1, we start by taking 1 in A_1. Then we will put 2 in A_1 if $\rho_1(z_1, z_2) \geq r$, otherwise will move to the positive integer 3. Continuing like this, suppose we have we decided till m and out of these positive integers we have $1, i_1, \ldots, i_q$ in A_1, then we will take $m+1$ in A_1 if $\rho_1(z_{m+1}, z_j) \geq r$ for all $j = 1, i_1, \ldots, i_q$, otherwise we will move on to the next positive integer, which will be $m+2$. Proceeding like this we get our first subsequence A_1. Now to construct the second subsequence A_2, we will start by taking l in A_2, where l is the least positive integer so that $l \notin A_1$. Then we continue to add $j's$ in A_2 from the set $\{i \in \mathbb{N} : i \notin A_1\}$, as we did in A_1. This way we get A_2. We will keep constructing $A_i's$ until all the $z_j's$ get exhausted.

Lastly, we claim that by the above construction we can get at most N $A_j^{'}s$. On the contrary, suppose we get sets $A_1, \ldots, A_{N+1}$ using the above construction. Now choose a positive integer $s \in A_{N+1}$. Then $s \notin A_j,\ 1 \leq j \leq N$, and thus for each $j,\ 1 \leq j \leq N$, there exists $s_j \in A_j$ such that $\rho_1(z_s, z_{s_j}) < r$. This implies that $z_{s_j} \in D_{\rho_1}(z_s, r)$. Also $z_s \neq z_{s_j}$ for all $1 \leq j \leq N$, and therefore the disk $D_{\rho_1}(z_s, r)$ contains at least $N+1$ $z_l's$. This is a contradiction, and hence we can get at most N $A_j^{'}s$ in the above construction, and let these be $A_1, \ldots, A_t$. Thus, $A_1, \ldots, A_t$ is a partition of $\mathbb{N}$, so that $\rho_1(z_i, z_j) \geq r$, for all $i, j \in A_m, i \neq j,\ 1 \leq m \leq t$.

Step 2. We now show that each subsequence $\{\widetilde{k}_{z_i}\}_{i\in A_m}$ is a Riesz basic sequence.

Fix m and consider $\{\widetilde{k}_{z_i}\}_{i\in A_m}$. Then

$$\rho_1(z_i, z_j) \geq r, \tag{4.7}$$

for $i, j \in A_m,\ i \neq j$. Also using Equation (4.6), we get

$$\sum_{\substack{j\in A_m\\ j\neq i}} \left(1 - (\rho_1(z_i, z_j))^2\right) \leq B, \tag{4.8}$$

for all $i \in A_m$. Using the simple fact from calculus $-ln(x) \leq \frac{1-x}{x}$ for all $0 < x < 1$, and the estimates from Equations (4.7) and (4.8) we deduce that

$$\prod_{\substack{j\in A_m\\ j\neq i}} \left|\frac{z_i - z_j}{1 - \overline{z_i} z_j}\right|^2 \geq e^{-B/r^2}, \tag{4.9}$$

for all $i \in A_m$. Hence $\{\widetilde{k}_{z_i}\}_{i\in A_m}$ is a Riesz basic sequence, using Lemma 4.2.3. $\square$

4.3 $H^2_{\alpha,\beta}$ Spaces

These spaces were introduced in [23], where the authors studied some constrained Nevanlinna-Pick interpolation problems. We begin by formally defining these RKHS's.

Given complex numbers α and β with $|\alpha|^2 + |\beta|^2 = 1$, we let $H^2_{\alpha,\beta}$ denote the codimension one subspace of H^2,

$$H^2_{\alpha,\beta} = \text{span}\{\alpha + \beta z,\ z^2 H^2\}.$$

It is easily checked that $H^2_{\alpha,\beta}$ is a RKHS with kernel

$$K_{\alpha,\beta}(z, w) = (\alpha + \beta z)\overline{(\alpha + \beta w)} + \frac{z^2\overline{w}^2}{1 - z\overline{w}}.$$

Recall that for the above notation of the kernel, the kernel function for a point $w \in \mathbb{D}$ is denoted by $k_w^{\alpha,\beta}$, and in case $k_w^{\alpha,\beta} \neq 0$, the corresponding normalized kernel function $\frac{k_w^{\alpha,\beta}}{\|k_w^{\alpha,\beta}\|} = \frac{k_w^{\alpha,\beta}}{\sqrt{K_{\alpha,\beta}(w,w)}}$ is denoted by $\widetilde{k}_w^{\alpha,\beta}$.

Theorem 4.3.1. *$H^2_{\alpha,\beta}$ satisfies the FCKF.*

Proof. If $\alpha = 0$, then $|\beta| = 1$, and hence $K_{0,\beta}(z,w) = zK(z,w)\overline{w}$, where K is the kernel function for H^2. Thus in this case, the result follows from Theorem 3.3.1, since H^2 satisfies the FCKF. We now assume $\alpha \neq 0$. Let $\phi(z) = z^2$. Then $\phi \in H^2$ is an inner function, and $\phi H^2 = z^2 H^2 \subseteq H^2_{\alpha,\beta}$. We show that if $\{\widetilde{k}_{z_i}^{\alpha,\beta}\}$ is a Bessel sequence, then $\{|\phi(z_i)|\}$ is bounded away from zero, for all but finitely many $z_i's$. Once this is done, the proof follows from Theorem 3.2.11.

Let $\{\widetilde{k}_{z_i}^{\alpha,\beta}\}_{i\in\mathbb{N}}$ be a Bessel sequence in $H^2_{\alpha,\beta}$. Then there exists a constant $B > 0$, such that

$$\begin{aligned}
B &\geq \sum_{j\in\mathbb{N}} |\langle \widetilde{k}_{z_j}^{\alpha,\beta}, \widetilde{k}_{z_i}^{\alpha,\beta}\rangle|^2 \\
&= \sum_{j\in\mathbb{N}} \frac{|K_{\alpha,\beta}(z_i,z_j)|^2}{\|k_{z_i}^{\alpha,\beta}\|^2 \|k_{z_j}^{\alpha,\beta}\|^2} \\
&= \sum_{j\in\mathbb{N}} \frac{\left|(\alpha+\beta z_i)\overline{(\alpha+\beta z_j)}(1-z_i\overline{z_j}) + z_i^2\overline{z_j}^2\right|^2 (1-|z_i|^2)(1-|z_j|^2)}{|1-z_i\overline{z_j}|^2(|\alpha+\beta z_i|^2(1-|z_i|^2)+|z_i|^4)(|\alpha+\beta z_j|^2(1-|z_j|^2)+|z_j|^4)},
\end{aligned}$$

for all $i \in \mathbb{N}$. Note that $|\alpha+\beta z_m|^2(1-|z_m|^2)+|z_m|^4 \leq 1-|z_m|^4 \leq 1$, for each $m \in \mathbb{N}$, since $|\alpha|^2+|\beta|^2 = 1$. Thus the above inequality implies that

$$\sum_{j\in\mathbb{N}} \frac{\left|(\alpha+\beta z_i)\overline{(\alpha+\beta z_j)}(1-z_i\overline{z_j}) + z_i^2\overline{z_j}^2\right|^2 (1-|z_i|^2)(1-|z_j|^2)}{|1-z_i\overline{z_j}|^2} \leq B, \tag{4.10}$$

for all $i \in \mathbb{N}$. Further, note that the function $f(x) = (|\alpha| - |\beta|x)\sqrt{1-x^2} - x^2$ is a continuous function on $[0,1]$ and $f(0) = |\alpha| > 0$. Thus there exists a $0 < \delta < 1$, such that $f(x) > 0$, for all $x \in [0,\delta]$. Choose $0 < \delta_0 \leq \delta$, so that $|\alpha| - \delta_0|\beta| > 0$. Recall that we want to show that $\{z_i\}_{i\in\mathbb{N}}$ is eventually bounded away from zero. On the contrary, suppose $\{z_i\}_{i\in\mathbb{N}}$ is not eventually bounded away from zero, then there exists a subsequence $\{z_{n_m}\}_{m\in\mathbb{N}}$, such that $|z_{n_m}| \leq \delta_0$, for all $m \in \mathbb{N}$. This implies that

$$|(\alpha+\beta z_{n_i})\overline{(\alpha+\beta z_{n_j})}(1 - z_{n_i}\overline{z_{n_j}}) + z_{n_i}^2\overline{z_{n_j}}^2| \geq (|\alpha| - |\beta|\delta_0)^2(1-\delta_0^2) - \delta_0^4 > 0, \quad (4.11)$$

for all $i, j \in \mathbb{N}$, where the right-hand side term is positive, since $0 < \delta_0 \leq \delta$. Using this estimate and Equation (4.10) we obtain,

$$\begin{aligned} \frac{B}{K^2} &\geq \sum_{j\in\mathbb{N}} \frac{(1-|z_{n_i}|^2)(1-|z_{n_j}|^2)}{|1 - z_{n_i}\overline{z_{n_j}}|^2} \\ &\geq \frac{(1-|z_{n_i}|^2)}{4} \sum_{j\in\mathbb{N}} (1-|z_{n_j}|^2), \end{aligned}$$

for all $i \in \mathbb{N}$, where $K = (|\alpha| - |\beta|\delta_0)^2(1-\delta_0^2) - \delta_0^4$. This implies that the sequence $z_{n_j} \to 1$, as $j \to \infty$, which is a contradiction to the fact that $|z_{n_j}| \leq \delta_0 < 1$ for all $j \in \mathbb{N}$. Hence the sequence $\{z_i\}_{\in\mathbb{N}}$ is bounded away from zero, except may be for finitely many $z_i's$.

Let $\gamma > 0$ such that $|z_i| \geq \gamma$ for all $i \geq i_0$. Then $|\phi(z_i)| = |z_i|^2 \geq \gamma^2$. Hence by Theorem 3.2.11, the sequence $\{\widetilde{k}_{z_i}^{\alpha,\beta}\}_{i\geq i_0}$ satisfies the FC. Lastly, since every singleton is, trivially, a Riesz basic sequence, therefore the full sequence $\{\widetilde{k}_{z_i}^{\alpha,\beta}\}_{i\in\mathbb{N}}$ satisfies the FC. Hence, $H^2_{\alpha,\beta}$ satisfies the FCKF. $\square$

4.4 Weighted Bergman Spaces

In this section, we focus on weighted Bergman spaces [25, 52]. There is a lot of research on these spaces in relation to interpolation and sampling problems. For details refer to [27, 45, 43, 48] and the references therein. One interesting thing is that the notion of interpolation and sampling are just reformulation of some concepts in frame theory. More precisely, a sequence $\{z_i\} \subseteq \mathbb{C}^n$ is called a **sampling sequence** for a weighted Bergman space, if the corresponding sequence of normalized kernel functions is a frame for the weighted Bergman space. Furthermore, a sequence $\{z_i\} \subseteq \mathbb{C}^n$ is called an **interpolating sequence** for a weighted Bergman space, if the synthesis operator (adjoint of the analysis operator) for the corresponding sequence of normalized kernel functions is onto, here the sequence of normalized kernel functions is not assumed to be a Bessel sequence, that is, the synthesis operator is not assumed to be bounded. Consequently, results in [45, 46, 48] can be reformulated to conclude that weighted Bergman spaces for one variable satisfies the FCKF. For the same reason, a result from [27] can be used to conclude that a "large" class of weighted Bergman spaces for several variable also satisfies the FCKF. We specify this class later in the section.

An essential ingredient for our proof of these results is a characterization of Bessel sequences of normalized kernel functions in these spaces (Theorem 4.4.4). As we shall see, in Lemma 4.4.3, that the analytic nature of functions in these spaces is crucial to obtain this characterization. We begin by formally defining these spaces.

For $\alpha > -1$, the weighted Bergman space $A^2_{\alpha,n}$ is the RKHS consisting of functions

f analytic in the unit ball $\mathbb{B}_n$, such that

$$\|f\|_\alpha^2 = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)} \int_{\mathbb{B}_n} |f(z)|^2(1-|z|^2)^\alpha d\nu(z) < \infty,$$

where Γ denote the Gamma function. The kernel function for $A^2_{\alpha,n}$ is

$$K_{\alpha,n}(z,w) = \frac{1}{(1-z\cdot\overline{w})^{\alpha+n+1}}, \quad z,w \in \mathbb{B}_n.$$

When $n=1$ and $\alpha=0$, the RKHS $A^2_{\alpha,n}$ is better known as the Bergman space.

For the rest of this section we fix $K_{\alpha,n}$ to denote the kernel function for the weighted Bergman space $A^2_{\alpha,n}$. Recall that given a kernel function $K_{\alpha,n}$, we denote the kernel function for a point $w \in \mathbb{B}_n$ by $k_w^{\alpha,n}$, and in case $k_w^{\alpha,n} \neq 0$, the corresponding normalized kernel $\frac{k_w^{\alpha,n}}{\sqrt{K_{\alpha,n}(w,w)}} = (1-|w|^2)^{\frac{1+\alpha+n}{2}} k_w^{\alpha,n}$ is denoted by $\widetilde{k}_w^{\alpha,n}$.

From [31] and [48], we have the following characterization of Bessel sequences in the Bergman space $A^2_{0,1}$.

Theorem 4.4.1. *A sequence $\{\widetilde{k}_{z_i}^{0,1}\}$ of normalized kernel functions in the Bergman space $A^2_{0,1}$ is a Bessel sequence if and only if the sequence $\{z_i\}$ is a finite union of weakly separated sequences.*

We shall see that the same characterization holds in the weighted Bergman spaces $A^2_{\alpha,n}$, for a general n. To prove this result we need the following two lemmas. The first lemma is proved in [42] and [52], which lists some basic properties of the mappings ϕ_w^n (defined in Section 4.1).

Lemma 4.4.2. *For each $w \in \mathbb{B}_n$ the mapping ϕ_w^n satisfies the following properties:*

(i) $\phi_w^n(w) = 0$ *and* $\phi_w^n(0) = w$,

(ii) $1 - |\phi_w^n(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-z\cdot\overline{w}|^2}, \quad z \in \mathbb{B}_n,$

(iii) $\phi_w^n \circ \phi_w^n(z) = z, \quad z \in \mathbb{B}_n,$

(iv) ϕ_w^n *is an automorphism of* $\mathbb{B}_n$,

(v) $J_{\mathbb{R}}\phi_w^n(z) = |(\phi_w^n)'(z)|^2 = \left(\frac{1-|w|^2}{|1-z\cdot\overline{w}|^2}\right)^{n+1}, \quad z \in \mathbb{B}_n.$

The following lemma for the case $n = 1$ is proved in [25]. Using the same techniques as used in [25], we extend the result to higher dimensions.

Lemma 4.4.3. *Let* $\{z_i\}$ *be a weakly separated sequence in* $\mathbb{B}_n$ *with constant* $\delta > 0$. *Then for any real number* t *there exists a constant* $C > 0$ *(depending on* t *and* δ*), such that*

$$\sum_i (1-|z_i|^2)^{t+n+1}|f(z_i)|^2 \leq C\int_T (1-|z|^2)^t|f(z)|^2 d\nu(z),$$

for every function f *analytic in* $\mathbb{B}_n$, *where* $T = \{z \in \mathbb{B}_n : \inf_i \rho_n(z, z_i) < \delta\}$.

Proof. Since the sequence $\{z_i\}$ is weakly separated, any two pseudo-hyperbolic disks $D_{\rho_n}(z_i, \delta/2)$ and $D_{\rho_n}(z_j, \delta/2)$ are disjoint, whenever $i \neq j$. Also for each i, the disk $D_{\rho_n}(z_i, \delta/2) \subseteq T$. Thus,

$$\int_T (1-|z|^2)^t|f(z)|^2 d\nu(z) \geq \sum_i \int_{D_{\rho_n}(z_i,\delta/2)} (1-|z|^2)^t|f(z)|^2 d\nu(z) \tag{4.12}$$

To simplify notation, for the rest of the proof we will use $D(w)$ to denote the pseudo-hyperbolic disk $D_{\rho_n}(w, \delta/2)$. Now implementing the change of variable $z = \phi_{z_i}^n(a),\ a \in D(0) = D_{\rho_n}(0, \delta/2)$ in the i^{th} integral on the right-hand side of the above inequality, we obtain

$$\begin{aligned}
\int_{D(z_i)} (1-|z|^2)^t |f(z)|^2 d\nu(z) &= \int_{D(0)} \left(1-|\phi^n_{z_i}(a)|^2\right)^t |f(\phi^n_{z_i}(a))|^2 J_{\mathbb{R}}\phi^n_{z_i}(y) d\nu(a), \\
&= \int_{D(0)} \frac{(1-|a|^2)^t(1-|z_i|^2)^{t+n+1}}{|1-a\cdot\overline{z_i}|^{2(t+n+1)}} |f(\phi^n_{z_i}(a))|^2 d\nu(a) \\
&= (1-|z_i|^2)^{t+n+1} \int_{D(0)} \frac{(1-|a|^2)^t}{|1-a\cdot\overline{z_i}|^{2(t+n+1)}} |f(\phi^n_{z_i}(a))|^2 d\nu(a) \\
&\geq M(1-|z_i|^2)^{t+n+1} \int_{D(0)} \frac{|f(\phi^n_{z_i}(a))|^2}{|1-a\cdot\overline{z_i}|^{2(t+n+1)}} d\nu(a),
\end{aligned}$$

using (ii) and (v) of Lemma 4.4.2, where the constant M equals $(1-(\delta/2)^2)^t$, when $t>0$, and equals 1, when $t \leq 0$. Further,

$$\int_{D(0)} \frac{|f(\phi^n_{z_i}(a))|^2}{|1-a\cdot\overline{z_i}|^{2(t+n+1)}} d\nu(a) \geq |f(\phi^n_{z_i}(0))|^2 = |f(z_i)|^2,$$

using the fact that $\phi^n_{z_i}(0) = z_i$, since the integrand is a subharmonic function and $D(0) = D_{\rho_n}(0,\delta/2) = \{w \in \mathbb{C}^n : |\phi^n_w(0)| < \delta/2\} = (\delta/2)\mathbb{B}_n$. Therefore,

$$\int_{D(z_i)} (1-|z|^2)^t |f(z)|^2 d\nu(z) \geq M(1-|z_i|^2)^{t+n+1}|f(z_i)|^2 \tag{4.13}$$

Lastly, using Inequality (4.13) in Equation (4.12) we get

$$\sum_i (1-|z_i|^2)^{t+n+1}|f(z_i)|^2 \leq \frac{1}{M}\int_T (1-|z|^2)^t |f(z)|^2 d\nu(z),$$

where $M = \begin{cases} (1-(\delta/2)^2)^t & \text{if} \quad t>0 \\ 1 & \text{if} \quad t \leq 0 \end{cases}$. □

The following is an extension of Theorem 4.4.1 to all weighted Bergman spaces.

Theorem 4.4.4. *A sequence $\{\widetilde{k}^{\alpha,n}_{z_i}\}_{i\in\mathbb{N}}$ of normalized kernel functions is a Bessel sequence in $A^2_{\alpha,n}$ if and only if the sequence $\{z_i\}$ can be partitioned into finitely many weakly separated sequences.*

Proof. Let $\{\widetilde{k}_{z_i}^{\alpha,n}\}$ be a Bessel sequence in $A^2_{\alpha,n}$. Then there exists a constant $B > 0$, such that

$$\sum_{j\in\mathbb{N}} |\langle \widetilde{k}_{z_j}^{\alpha,n}, \widetilde{k}_{z_i}^{\alpha,n}\rangle|^2 \leq B,$$

for all $i \in \mathbb{N}$. Thus by Lemma 4.1.2, given a constant $0 < \beta < 1$ there exists a partition $A_1, \ldots, A_N$ of the $\mathbb{N}$, such that

$$|\langle \widetilde{k}_{z_j}^{\alpha,n}, \widetilde{k}_{z_i}^{\alpha,n}\rangle|^2 < \beta,$$

for all $i, j \in A_m,\ i \neq j,\ 1 \leq m \leq N$. Lastly, note that

$$\begin{aligned} |\langle \widetilde{k}_{z_j}^{\alpha,n}, \widetilde{k}_{z_i}^{\alpha,n}\rangle|^2 &= \left(\frac{(1-|z_i|^2)(1-|z_j|^2)}{|1-z_i\cdot\overline{z_j}|^2}\right)^{n+\alpha+1} \\ &= (1-|\phi_{z_j}^n(z_i)|^2)^{n+\alpha+1} \\ &= (1-\rho_n(z_i,z_j)^2)^{n+\alpha+1}. \end{aligned}$$

Thus, $(1-\rho_n(z_i,z_j)^2)^{n+\alpha+1} < \beta$, for all $i, j \in A_m,\ i \neq j,\ 1 \leq m \leq N$. Hence $\rho_n(z_i, z_j) > \sqrt{1-\beta^{2/(n+\alpha+1)}}$, whenever $i, j \in A_m,\ i \neq j,\ 1 \leq m \leq N$, from which it follows that each subsequence $\{\widetilde{k}_{z_i}^{\alpha,n}\}_{i\in A_m}$ is weakly separated.

Conversely, suppose $\{z_i\}$ splits into finitely many weakly separated subsequences. Then taking $t = \alpha$ in Lemma 4.4.3, it follows that each of the corresponding subsequence of normalized kernel functions is a Bessel sequence. Hence $\{\widetilde{k}_{z_i}^{\alpha,n}\}$ being a finite union of Bessel sequences is a Bessel sequence. □

Remark 4.4.5. *By Theorem 4.4.4, a sequence of normalized kernel functions in a weighted Bergman space $A^2_{\alpha,n}$ is a Bessel sequence if and only if the corresponding sequence of normalized kernel functions in every other weighted Bergman space $A^2_{\beta,n}$ (for the same n) is a Bessel sequence. In the one variable case ($n = 1$), we can* *add more to this statement. In [31], McKenna showed that a sequence $\{\widetilde{k}_{z_i}\}$ of normalized kernel functions in H^2 is a Bessel sequence if and only if $\{z_i\} \subseteq \mathbb{D}$ splits into weakly separated subsequences. Thus it follows that a sequence of normalized kernel functions in a weighted Bergman space $A^2_{\alpha,1}$ is a Bessel sequence if and only if the corresponding sequence of normalized kernel functions in H^2 is a Bessel sequence.*

Following are the main results of this section. Our first result proves the FCKF for all weighted Bergman spaces in the one variable situation. In the second result we work with the several variable case, where we prove that the weighted Bergman space $A^2_{\alpha,n}$ satisfies the FCKF, whenever $\alpha > n-1$. The main ingredient in the proof of the one variable case is that the Hardy space H^2 on the unit disk satisfies the FCKF. Since it is not known that the Hardy space on the n-dimensional unit ball $\mathbb{B}_n$ satisfies the FCKF, we could not extend the same proof for the several variable situation. To prove the result in the several variable situation we uses a slightly different idea, which unfortunately works only for the above mentioned class of weighted Bergman spaces. Both the proofs uses results about the products of kernel functions from Section 3.3 and they both rely on the fact that H^2 and the weighted Bergman spaces $A^2_{\alpha,n}$ have the same characterization of Bessel sequences of normalized kernel functions.

Note that for each fixed $\alpha > -1$ and $w \in \mathbb{D}$, the function $\frac{1}{(1-z\cdot\overline{w})^{\alpha+1}}$ is a holomorphic function on $\mathbb{D}$. We can also easily verify that the j^{th} coefficient, with $j \geq 1$, in the power series of this function is $\frac{(\alpha+1)\ldots(\alpha+j)\overline{w}^j}{j!}$, and the constant term of the power series is 1. Hence for each $\alpha > -1$, we obtain a function $\Phi_\alpha : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ given by

$$\Phi_\alpha(z,w) = \frac{1}{(1 - z \cdot \overline{w})^{\alpha+1}} = 1 + \sum_{j=1}^{\infty} \frac{(\alpha+1)\dots(\alpha+j)}{j!} z^j \overline{w}^j, \quad z, w \in \mathbb{D}$$

Now since $\alpha > -1$, all the coefficients $\frac{(\alpha+1)\dots(\alpha+j)}{j!}$ are positive. Therefore it can be checked that Φ_α is a positive definite function on $\mathbb{D} \times \mathbb{D}$, and hence it is a kernel function on $\mathbb{D}$.

Further, recall that $K(z,w) = \frac{1}{1-z\cdot\overline{w}}$ is the kernel for H^2. Thus for any fixed $\alpha > -1$, we can write the kernel function $K_{\alpha,1}$ as a product of two kernel function, namely

$$K_{\alpha,1}(z,w) = K(z,w)\Phi_\alpha(z,w), \tag{4.14}$$

Theorem 4.4.6. *The weighted Bergman space $A^2_{\alpha,1}$ satisfies the FCKF, for every $\alpha > -1$.*

Proof. For every Bessel sequence $\{\widetilde{k}^{\alpha,1}_{z_i}\}$ of normalized kernel functions in $A^2_{\alpha,1}$, the corresponding sequence $\{\widetilde{k}_{z_i}\}$ of normalized kernel functions in H^2 is also a Bessel sequence, using Remark 4.4.5. Hence the result follows from Theorem 3.3.12, using Equation (4.14) and the fact that H^2 satisfies the FCKF. □

Theorem 4.4.7. *If $\alpha > n-1$, then the weighted Bergman spaces $A^2_{\alpha,n}$ satisfies the FCKF.*

Proof. Let $\alpha > n-1$. Then $\beta = \frac{\alpha-n-1}{2} > -1$, and thus we get the weighted Bergman space $A^2_{\beta,n}$. Moreover, we can write

$$K_{\alpha,n}(z,w) = K_{\beta,n}(z,w)K_{\beta,n}(z,w).$$

Also by Remark 4.4.5, if a sequence of normalized kernel function in $A^2_{\alpha,n}$ is a Bessel sequence, then the corresponding sequence of normalized kernel functions in $A^2_{\beta,n}$ is also a Bessel sequence. Hence the result follows from Theorem 3.3.16. $\square$

Note that the proof of Theorem 4.4.7 for $n = 1$ gives an alternate proof of the fact that the weighted Bergman spaces $A^2_{\alpha,1},\ \alpha > 0$, satisfy the FCKF.

By using the reformulation of interpolating sequences in our setting, which we mentioned in the beginning of the section, we see that Corollary 4.6(d) in [27] also proves the FCKF for the weighted Bergman spaces $A^2_{\alpha,n}$, with $\alpha > n-1$. Though the proof in [27] uses completely different techniques to get this result. As far as we know, the FCKF is still open for the weighted Bergman space $A^2_{\alpha,n}$ for $n > 1,\ \alpha \leq n-1$.

4.5 Bargmann-Fock Spaces

In this section, we show that the Bargmann-Fock spaces satisfy the FCKF. There is an extensive literature on these spaces. For general information on theses spaces we refer the reader to [16, 13, 30, 37, 44, 43, 47] and the references therein. Like weighted Bergman spaces, there is a lot of research about sampling and interpolating sequences in these spaces. Again, like weighted Bergman spaces in these spaces as well, a sequence $\{z_i\} \subseteq \mathbb{C}^n$ is a **sampling sequence** for a Bargmann-Fock space if the corresponding sequence of normalized kernel functions is a frame for the Bargman-Fock space. Also, a sequence $\{z_i\} \subseteq \mathbb{C}^n$ is an **interpolating sequence** for a Bargmann-Fock space if the synthesis operator (adjoint of the analysis operator) for the corresponding sequence of normalized kernel functions is onto, here the sequence of normalized kernel functions is not assumed to be a Bessel sequence, that is the synthesis operator is not assumed to be bounded. Consequently, results from this area can easily be transferred to our setting. Interpolating sequences for Bargmann-Fock spaces for one variable have been completely characterized in [44], [47]. Later, a sufficient condition for interpolating sequences for Bargmann-Fock space in several variable is given in [30]. One can use these results to conclude that Bargmann-Fock spaces satisfies the FCKF.

Our proof of the FCKF for Bargman-Fock spaces, like in the case of the weighted Bergman spaces, is completely based on a characterization of Bessel sequences of normalized kernel functions in these spaces (Theorem 4.5.2). As we shall see that the fact that the functions in these spaces are analytic is crucial to obtain this characterization. We begin by formally defining these spaces.

For $\alpha > 0$, the Bargmann-Fock space $F^2_{\alpha,n}$ is the RKHS consisting of entire functions f in $\mathbb{C}^n$, for which

$$\|f\|^2_{\alpha,n} = \int_{\mathbb{C}^n} |f(z)|^2 e^{-\alpha|z|^2} d\nu(z) < \infty,$$

with the kernel function

$$K_{\alpha,n}(z,w) = e^{\alpha z\cdot\overline{w}}, \quad z, w \in \mathbb{C}^n.$$

For the rest of the section, we fix $K_{\alpha,n}$ to denote the kernel for the Bargmann-Fock space $F^2_{\alpha,n}$. Recall that with this notation for the kernel, the kernel function for a point $w \in \mathbb{C}^n$ is denoted by $k^{\alpha,n}_w$, and in case $k^{\alpha,n}_w \neq 0$, the corresponding normalized kernel function $\frac{k^{\alpha,n}_w}{\sqrt{K_{\alpha,n}(w,w)}} = e^{\frac{-\alpha|w|^2}{2}} k^{\alpha,n}_w$ for the point w is denoted by $\widetilde{k}^{\alpha,n}_w$.

Our proof of FCKF for the Bargmann-Fock spaces is based on a characterization of Bessel sequences in theses spaces. To explain this characterization the following definition is essential.

Definition 4.5.1. *A sequence $\{z_i\}$ in $\mathbb{C}^n$ is said be* **uniformly discrete** *if there exists a constant $\delta > 0$, such that $|z_i - z_j| \geq \delta$, whenever $i \neq j$.*

Theorem 4.5.2. *A sequence $\{\widetilde{k}_{z_i}^{\alpha,n}\}$ of normalized kernel functions in $F^2_{\alpha,n}$ is a Bessel sequence if and only if it can partitioned into finitely many uniformly discrete subsequences.*

Proof. Let $\{\widetilde{k}_{z_i}^{\alpha,n}\}_{i\in\mathbb{N}}$ be a Bessel sequence of normalized kernel functions in $F^2_{\alpha,n}$. Then there exists a constant $B > 0$, such that

$$\sum_{i\in\mathbb{N}} |\langle \widetilde{k}_{z_j}^{\alpha,n}, \widetilde{k}_{z_i}^{\alpha,n}\rangle|^2 \leq B,$$

for all $j \in \mathbb{N}$. Thus by using Lemma 4.1.2, given a $0 < \beta < 1$ there exist a $N \in \mathbb{N}$ and a partition $A_1, \ldots, A_N$ of $\mathbb{N}$, such that

$$\left|\langle \widetilde{k}_{z_j}^{\alpha,n}, \widetilde{k}_{z_i}^{\alpha,n}\rangle\right|^2 < \beta,$$

for all $i, j \in A_m,\ i \neq j,\ 1 \leq m \leq N$. Now note that $\left|\langle \widetilde{k}_{z_j}^{\alpha,n}, \widetilde{k}_{z_i}^{\alpha,n}\rangle\right|^2 = e^{-\alpha|z_i - z_j|^2}$. Therefore $|z_i - z_j|^2 > \frac{1}{\alpha} ln(\frac{1}{\beta}) > 0$, for all $i, j \in A_m,\ i \neq j,\ 1 \leq m \leq N$. Hence, each sequence $\{z_i\}_{i\in A_k}$ is uniformly discrete.

Conversely, let $\{z_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbb{C}^n$ and $A_1, \ldots, A_N$ be a partition of $\mathbb{N}$, such that each subsequence $\{z_i\}_{i\in A_m}$ is uniformly discrete. Then there exists a $\delta > 0$, such that

$$|z_i - z_j| \geq \delta,$$

for all $i, j \in A_m,\ i \neq j,\ 1 \leq m \leq N$. We show that $\{\widetilde{k}_{z_i}^{\alpha,n}\}_{i\in A_m}$ is a Bessel sequence, for every $1 \leq m \leq N$. First note that, for each $a \in \mathbb{C}^n$, the translation operator

$$(T_a f)(z) = e^{\alpha \overline{a}\cdot z - \alpha \frac{|a|^2}{2}} f(z-a)$$

acts isometrically on $F^2_{\alpha,n}$. Therefore for a fixed $z_i,\ T_{-z_i} f$ is an entire function for every $f \in F^2_{\alpha,n}$, and hence $|T_{-z_i} f|^2$ is a subharmonic function in $\mathbb{C}^n$ for every $f \in F^2_{\alpha,n}$. Thus for $f \in F^2_{\alpha,n}$ and $0 < r \leq \delta/2$ we get

$$\begin{aligned} |(T_{-z_i} f)(0)|^2 &\leq \int_{\mathbb{S}_n} |T_{-z_i} f(r\zeta)|^2 d\sigma(\zeta), \\ \text{that is, } |e^{-\alpha \frac{|z_i|^2}{2}} f(z_i)|^2 &\leq \int_{\mathbb{S}_n} |e^{-\alpha r\zeta\cdot\bar{z}_i - \alpha\frac{|z_i|^2}{2}} f(r\zeta + z_i)|^2 d\sigma(\zeta), \\ \text{that is, } e^{-\alpha|z_i|^2}|f(z_i)|^2 &\leq \int_{\mathbb{S}_n} |f(r\zeta+z_i)|^2 e^{-\alpha[2Re(r\zeta\cdot\bar{z}_i)+|z_i|^2]} d\sigma(\zeta). \end{aligned}$$

Thus,

$$\int_0^{\delta/2} r^{2n-1} e^{-\alpha r^2} e^{-\alpha|z_i|^2} |f(z_i)|^2 dr = \int_0^{\delta/2} \int_{\mathbb{S}_n} |f(r\zeta+z_i)|^2 e^{-\alpha|r\zeta+z_i|^2} r^{2n-1} d\sigma(\zeta) dr$$

Hence,

$$e^{-\alpha|z_i|^2}|f(z_i)|^2 \leq C \int_{B_n(z_i,\delta/2)} |f(z)|^2 e^{-\alpha|z|^2} d\nu(z), \tag{4.15}$$

where $C^{-1} = \int_0^{\delta/2} r^{2n-1} e^{-\alpha r^2} dr$ and $\mathbb{B}_n(z_i, \delta/2) = \{z \in \mathbb{C}^n : |z - z_i| < \delta/2\}$.

Lastly, since for any fixed $1 \leq m \leq N$, if $j, l \in A_m$ and $j \neq l$, then $|z_j - z_l| \geq \delta$, and hence $B_n(z_j, \delta/2)$ and $B_n(z_l, \delta/2)$ are disjoint sets. Thus, summing Equation (4.15) over $i \in A_m$ on both sides we get

$$\begin{aligned} \sum_{i\in A_m} e^{-\alpha|z_i|^2}|f(z_i)|^2 &\leq C \sum_{i\in A_m} \int_{B_n(z_i,\delta/2)} |f(z)|^2 e^{-\alpha|z|^2} d\nu(z) \\ &\leq C \int_{\mathbb{C}^n} |f(z)|^2 e^{-\alpha|z|^2} d\nu(z) \\ &= C\|f\|^2_{\alpha,n}\ , \end{aligned}$$

But $\sum_{i\in A_m} e^{-\alpha|z_i|^2}|f(z_i)|^2 = \sum_{i\in A_m} |\langle f, \widetilde{k}^{\alpha,n}_{z_i}\rangle|^2$, and hence it follows that $\{\widetilde{k}^{\alpha,n}_{z_i}\}_{i\in A_m}$ is a Bessel sequence. Finally, $\{\widetilde{k}^{\alpha,n}_{z_i}\}_{i\in\mathbb{N}}$ being a finite union of Bessel sequences is a Bessel sequence. $\square$

Remark 4.5.3. *By Theorem 4.5.2, a sequence $\{\widetilde{k}^{\alpha,n}_{z_i}\}_{i\in\mathbb{N}}$ of normalized kernel function in $F^2_{\alpha,n}$ is a Bessel sequence if and only if the corresponding sequence $\{\widetilde{k}^{\beta,n}_{z_i}\}_{i\in\mathbb{N}}$ of normalized kernel functions in $F^2_{\beta,n}$ (for the same n) is a Bessel sequence.*

Theorem 4.5.4. *The Bargmann-Fock spaces $F^2_{\alpha,n}$ satisfy the FCKF, for all $\alpha > 0$.*

Proof. For a fixed $\alpha > 0$, we can write

$$K_\alpha(z,w) = K_\beta(z,w)K_\gamma(z,w),$$

where $\beta, \gamma > 0$ such that $\beta + \gamma = \alpha$. Hence, $K_\alpha(z,w) = e^{\alpha z\cdot\overline{w}}$ is the product of two kernel functions, namely $K_\beta(z,w) = e^{\beta z\cdot\overline{w}}$ and $K_\gamma(z,w) = e^{\gamma z\cdot\overline{w}}$. Also using Remark 4.5.3, if $\{\widetilde{k}^{\alpha,n}_{z_i}\}_{i\in\mathbb{N}}$ is a Bessel sequence in $F^2_{\alpha,n}$, then $\{\widetilde{k}^{\beta,n}_{z_i}\}_{i\in\mathbb{N}}$ and $\{\widetilde{k}^{\gamma,n}_{z_i}\}_{i\in\mathbb{N}}$ are also Bessel sequences in $F^2_{\beta,n}$ and $F^2_{\gamma,n}$, respectively. Hence the result follows, using Theorem 3.3.16. $\square$

Chapter 5

The Feichtinger Conjecture and a Factorization of Positive Operators in $B(\ell^2)$

In this chapter, we introduce two new directions to investigate the FC. This study is motivated by a factorization of positive operators in $B(\ell^2)$, discussed in [1]. To explain these new directions, the above mentioned factorization, and how this factorization leads to these new directions, we first need to recall the following definition from [1].

Definition 5.0.5. *A positive operator R in $B(\ell^2)$ is said to be completely non-factorizable if $UU^* \leq R$ implies $U = 0$, for every upper triangular U in $B(\ell^2)$.*

In [1], it is proved that every positive operator P in $B(\ell^2)$ can be written uniquely as $P = UU^* + R$, where $U \in B(\ell^2)$ is upper triangular and $R \in B(\ell^2)$ is completely non-factorizable. We will use this factorization in our setting as follows. Given a Bessel sequence, the associated Grammian FF^* is a positive operator in $B(\ell^2)$. Then by invoking the above factorization, there exist an upper triangular $U \in B(\ell^2)$ and a completely non-factorizable $R \in B(\ell^2)$, such that $FF^* = UU^* + R$. Thus, this factorization divides an arbitrary Grammian operator into two positive operators, one completely non-factorizable and the other of the form UU^*, where $U \in B(\ell^2)$ is upper traingular. In this chapter, we focus on Grammian operators which contains only one of the two pieces. One obvious reason to study the FC for norm-bounded below Bessel sequences for which the Grammian operator has one of the above two specific forms is that in such a case we get more structure to exploit, which might help in better understanding of the FC. There is one another reason behind this approach, which we explain below.

Suppose, we could prove that a norm-bounded below Bessel sequence satisfies the FC if its Grammian operator has one of these special forms, that is, either it is UU^* for some upper triangular $U \in B(\ell^2)$ or it is a completely non-factorizable operator. Then given an arbitrary norm-bounded below Bessel sequence $\{f_i\}_{i\in\mathbb{N}}$, we write the corresponding Grammian operator FF^* as $FF^* = UU^* + R$, for some upper triangular $U \in B(\ell^2)$ and a completely non-factorizable $R \in B(\ell^2)$. Now since both the factors are positive, we get $FF^* \geq UU^*$ and $FF^* \geq R$. Further, since $UU^* \in B(\ell^2)$, $R \in B(\ell^2)$ and $UU^* = (\langle U^*e_j, U^*e_i\rangle)$, $R = (\langle R^{1/2}e_j, R^{1/2}e_i\rangle)$, therefore the sequences $\{U^*e_i\}_{i\in\mathbb{N}}$ and $\{R^{1/2}e_i\}_{i\in\mathbb{N}}$ are Bessel sequences with the Grammian operators UU^* and R, respectively. Now, if any one of these two Bessel sequences is norm-bounded below, then that sequence satisfies the FC, using the hypothesis. For instance, suppose $\{U^*e_i\}$ is norm-bounded below, then it satisfes the FC. Therefore there exists a partition $A_1, \ldots, A_N$ of $\mathbb{N}$ and positive constants $c_1, \ldots, c_N$, such that $P_{A_m}UU^*P_{A_m} \geq c_m P_{A_m}$, for all $1 \leq m \leq N$, using Corollary 2.1.5. Thus $P_{A_m}FF^*P_{A_m} \geq c_m P_{A_m}$, for all $1 \leq m \leq N$, and from this it follows that $\{f_i\}$ satisfies the FC, using Corollary 2.1.5. Hence, there is a possibility that once we prove the FC for norm-bounded below Bessel sequences for which the Grammian operator is either of the above two forms, then we can prove it for general norm-bounded below Bessel sequences, as well. However, in this process, we assumed that either $\{U^*e_i\}$ is norm-bounded below or $\{R^{1/2}e_i\}$ is norm-bounded below. But, we do not know if one can do that always or not. Because, $\|f_i\|^2 = \|U^*e_i\|^2 + \|R^{1/2}e_i\|^2$, and in general there is no reason why the existence of a lower bound for the sequence $\{\|f_i\|\}$ would guarantee a lower bounded for any of the two sequences $\{\|U^*e_i\|\}$ and $\{\|R^{1/2}e_i\|\}$.

This chapter is an attempt to initiate the investigation along these two directions. The results presented here are mainly about frame sequences, for no specific reason except that they have more structure for our use. We hope to gather more information about the FC using these directions in future. We divide the chapter into two sections. The first section deals with the case of completely non-factorizable Grammian operators, and the second section examines the situation when a Grammian operator is of the form UU^* for an upper triangular operator $U \in B(\ell^2)$.

5.1 Completely Non-factorizable Grammian

Our main result (Theorem 5.1.9) of this section brings out an interesting fact about frames with completely non-factorizable Grammian. To realize this property we state some more characterizations of Riesz basic sequences. First we recall a few definitions.

Definition 5.1.1. *A sequence $\{f_i\}$ in a Hilbert space is said be* ω**-independent** *if whenever the series $\sum_i \alpha_i f_i = 0$ for some scalars α_i, then $\alpha_i = 0$, for all i.*

Notice that a Bessel sequence is w-independent if F^* has zero kernel. Further, note that an w-independent sequence is linearly independent. Though, the converse is not true [22]. By linear independence of an infinite set, we mean every finite subset is linearly independent.

Definition 5.1.2. *A sequence $\{f_i\}$ in a Hilbert space is said be* **minimal** *if $f_j \notin \overline{\text{span}\{f_i : i \neq j\}}$, for all j.*

Definition 5.1.3. *A frame $\{f_i\}$ is said to be* **exact** *if it ceases to be a frame when an arbitrary element is removed from it.*

The following result discloses the connection of all of the above concepts with Riesz basic sequences. For a proof of this result, we refer the reader to [17, 22].

Theorem 5.1.4. *Let $\{f_i\}$ be a frame sequence for a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) *$\{f_i\}$ is a Riesz basis for $\mathcal{H}$.*

(ii) $\{f_i\}$ *is exact.*

(iii) $\{f_i\}$ *is minimal.*

(iv) $\{f_i\}$ *is ω-independent.*

Thus, in a Riesz basic sequences there is lot of "independency" among the sequence elements.

To get to the main result of this section, we first prove some preliminary results. The following is an interesting observation about 2×2 positive matrices of operators. First recall that a net $\{T_\lambda\}_{\lambda \in \Lambda}$ of bounded operators on a Hilbert space $\mathcal{H}$ converges to a bounded operator $T \in B(\mathcal{H})$ in the weak operator topology if and only if $\langle T_\lambda h, k \rangle \to \langle Th, k \rangle$, for all $h, k \in \mathcal{H}$.

Proposition 5.1.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $T \in B(\mathcal{K}, \mathcal{H})$, $V \in B(\mathcal{K})$ with V positive. Then:*

(i) $T(V + \epsilon I)^{-1} T^* \leq A$ *for all $\epsilon > 0$, and for every $A \in B(\mathcal{H})$ such* $\begin{pmatrix} A & T \\ T^* & V \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{K})$ *is a positive operator,*

(ii) *there exists a positive $X \in B(\mathcal{H})$, such that $\langle Xh, h \rangle = \lim_{\epsilon \to 0} \langle T(V + \epsilon I)^{-1} T^* h, h \rangle$, for every $h \in \mathcal{H}$,*

(iii) $X \leq A$, *for every positive $A \in B(\mathcal{H})$ such* $\begin{pmatrix} A & T \\ T^* & V \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{K})$ *is a positive operator,*

(iv) $\begin{pmatrix} X & T \\ T^* & V \end{pmatrix}$ *is the smallest positive operator in* $B(\mathcal{H}\oplus\mathcal{K})$ *of the form* $\begin{pmatrix} * & T \\ T^* & V \end{pmatrix}$.

Proof. To prove (i), let $\epsilon > 0$ and $A \in B(\mathcal{H})$ such that $\begin{pmatrix} A & T \\ T^* & V \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{K})$ is a positive operator. Note that

$$\begin{aligned}
\begin{pmatrix} A & T \\ T^* & V \end{pmatrix} \geq 0 \;\Leftrightarrow\; & \begin{pmatrix} A+\delta I & T \\ T^* & V+\epsilon I \end{pmatrix} \geq 0 \quad \forall \quad \epsilon,\ \delta > 0 \\
\Leftrightarrow\; & \begin{pmatrix} I & W \\ W^* & I \end{pmatrix} \geq 0 \quad \forall \quad \epsilon,\ \delta > 0,
\end{aligned}$$

where $W = (A+\delta I)^{-1/2}T(V+\epsilon I)^{-1/2}$. But the last inequality on the right hand side holds if and only if $\|W\| \leq 1$ [38, Lemma 3.2]. Further note that

$$\begin{aligned}
\|W\| \leq 1 \;&\Leftrightarrow\; (A+\delta I)^{-1/2}T(V+\epsilon I)^{-1}T^*(A+\delta I)^{-1/2} \leq I \quad \forall\ \epsilon,\ \delta > 0 \\
&\Leftrightarrow\; T(V+\epsilon I)^{-1}T^* \leq A + \delta I \quad \forall\ \epsilon,\ \delta > 0 \\
&\Leftrightarrow\; T(V+\epsilon I)^{-1}T^* \leq A \quad \forall\ \epsilon > 0.
\end{aligned}$$

We now prove (ii). Note that $\{T(V+\epsilon I)^{-1}T^*\}_\epsilon$ is a bounded net in $B(\mathcal{H})$ and the closed balls in $B(\mathcal{H})$ are compact in the weak operator topology. Thus there exists a $X \in B(\mathcal{H})$, such that a subnet $\{T(V+\epsilon I)^{-1}T^*\}_{\epsilon_\lambda}$ of $\{T(V+\epsilon I)^{-1}T^*\}_\epsilon$ converges to X in the weak operator topology. Hence $\langle Xh, h\rangle = \lim_{\epsilon_\lambda \to 0}\langle T(V+\epsilon I)^{-1}T^*h, h\rangle$, for every $h \in \mathcal{H}$. Now since $\{T(V+\epsilon I)^{-1}T^*\}_{\epsilon>0}$ is an increasing net, therefore it follows that $\langle Xh, h\rangle = \lim_{\epsilon \to 0}\langle T(V+\epsilon I)^{-1}T^*h, h\rangle$, for every $h \in \mathcal{H}$. In addition, we get that X is positive, since each $T(V+\epsilon I)^{-1}T^*$ is positive.

Clearly, (iii) follows from (i) and (ii). Lastly, we prove (iv). From (iii) it readily follows that the self-adjoint operator $\begin{pmatrix} X & T \\ T^* & V \end{pmatrix}$ is smaller than every positive operator of the form $\begin{pmatrix} * & T \\ T^* & V \end{pmatrix}$ in $B(\mathcal{H} \oplus \mathcal{K})$. Thus to complete the proof of (iv) all it remains is to show that $\begin{pmatrix} X & T \\ T^* & V \end{pmatrix} \geq 0$. Note that for this, it is enough to prove that $\begin{pmatrix} X & T \\ T^* & V + \delta I \end{pmatrix} \geq 0$, for all $\delta > 0$, which can be easily deduced from some simple calculations, using the fact that $\langle Xh, h \rangle = \lim_{\substack{\epsilon \to 0 \\ \epsilon < \delta}} \langle T(V + \epsilon I)^{-1} T^* h, h \rangle$, for every $h \in \mathcal{H}$ and $\delta > 0$. $\square$

For notational convenience, we shall denote the operator X, obtained in Theorem 5.1.5, by $TV^{-1}T^*$. Next is an interesting observation about Grammians.

Theorem 5.1.6. *Let $\{f_i\}_{i \in \mathbb{N}}$ be a Bessel sequence in a Hilbert space $\mathcal{H}$ and let F be the associated analysis operator. If we write,*

$$FF^* = (\langle f_j, f_i \rangle) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$$

then

$$R_{12} R_{22}^{-1} R_{21} = \big(\langle P_n f_j, P_n f_i \rangle \big)_{i,j=1}^{n},$$

where $R_{11} = (\langle f_j, f_i \rangle)_{i,j=1}^{n}$, $n \in \mathbb{N}$, and P_n is the orthogonal projection onto $\mathcal{H}_n = \overline{\operatorname{span}\{f_i : i \geq n+1\}}$.

Proof. Let $Q \in B(\ell^2)$ be the smallest positive operator of the form $\begin{pmatrix} * & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$. Then using (iv) of Theorem 5.1.5, we get

$$Q = \begin{pmatrix} R_{12}R_{22}^{-1}R_{21} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}.$$

Now since $Q \geq 0$, we can write $Q = (\langle h_j, h_i\rangle)$, for a sequence $\{h_i\}$ in ℓ^2. Let $\mathcal{K} = \overline{\operatorname{span}\{h_i : i \in J\}}$, where $J = \{i \in \mathbb{N} : i \geq n+1\}$. Then

$$\big(\langle P_{\mathcal{K}} h_j, P_{\mathcal{K}} h_i\rangle\big) \leq \big(\langle h_j, h_i\rangle\big) = Q,$$

where $P_{\mathcal{K}}$ is the orthogonal projection onto $\mathcal{K}$. But

$$\Big(\langle P_{\mathcal{K}} h_j, P_{\mathcal{K}} h_i\rangle\Big) = \begin{pmatrix} * & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$$

and Q is the smallest positive operator in $B(\ell^2)$ of this form; therefore it follows that $Q = (\langle P_{\mathcal{K}} h_j, P_{\mathcal{K}} h_i\rangle)$. This implies that $h_j \in \mathcal{K}$, for all $1 \leq j \leq n$, and hence $\overline{\operatorname{span}\{h_i : i \in \mathbb{N}\}} = \mathcal{K}$. Also, $Q \leq R$. Thus the map, $C : \overline{\operatorname{span}\{f_i : i \in \mathbb{N}\}} \to \mathcal{K}$, given by $C(\sum_{i\in\mathbb{N}} \lambda_i f_i) = \sum_{i\in\mathbb{N}} \lambda_i h_i$, is a well-defined contraction. Further, note that $C|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{K}$ is an onto isometry, since $(\langle f_j, f_i\rangle)_{i,j\in J} = (\langle h_j, h_i\rangle)_{i,j\in J}$. Therefore, $C|_{\mathcal{H}_n{}^\perp} = 0$.

Lastly, note that $(C|_{\mathcal{H}_n})^* C|_{\mathcal{H}_n}$ is the orthogonal projection onto $\mathcal{H}_n$, and $Q = (\langle h_j, h_i\rangle) = (\langle Cf_j, Cf_i\rangle)$. Hence,

$$R_{12}R_{22}^{-1}R_{21} = (\langle h_j, h_i\rangle)_{i,j}^n = (\langle P_n f_j, P_n f_i\rangle)_{i,j}^n.$$

$\square$

Next is a standard fact about frames, for proof we refer the reader to [17, 22].

Proposition 5.1.7. *Let $\{f_i\}$ be a frame for a Hilbert space $\mathcal{H}$. Then the removal of a vector from $\{f_i\}$ leaves either a frame for $\mathcal{H}$ or an incomplete set.*

The following is a useful observation about completely non-factorizable operators.

Lemma 5.1.8. *Let $R \in B(\ell^2)$ be a completely non-factorizable operator. If we decompose R as*

$$R = \begin{pmatrix} A & T \\ T^* & V \end{pmatrix},$$

then $A = TV^{-1}T^*$, where $A = (\langle Re_j, e_i \rangle)_{i,j=1}^n$.

Proof. We factor R as

$$\begin{aligned} R &= \begin{pmatrix} TV^{-1}T^* & T \\ T^* & V \end{pmatrix} + \begin{pmatrix} A - TV^{-1}T^* & 0 \\ 0 & 0 \end{pmatrix} \\ &\geq \begin{pmatrix} A - TV^{-1}T^* & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

since $\begin{pmatrix} TV^{-1}T^* & T \\ T^* & V \end{pmatrix}$ is positive, using (iv) of Theorem 5.1.5. Further, from (iii) of Theorem 5.1.5, $A - TV^{-1}T^* \geq 0$. Also it a finite matrix, therefore using the Cholesky factorization there exists an upper triangular matrix U, such that $A - TV^{-1}T^* = UU^*$. Thus

$$R \geq \begin{pmatrix} UU^* & 0 \\ 0 & 0 \end{pmatrix} = WW^*,$$

where $W = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}$ is an upper triangular operator in $B(\ell^2)$. But R is completely non-factrizable, therefore $W = 0$. Thus $U = 0$, which implies that $A - TV^{-1}T^* = UU^* = 0$. Hence $A = TV^{-1}T^*$. $\square$

We are now ready to prove the main result of the section.

Theorem 5.1.9. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a frame for a Hilbert space $\mathcal{H}$ with frame bounds A, B, such that $FF^* = (\langle f_j, f_i\rangle)$ is completely non-factorizable. Then for each $n \in \mathbb{N}$, $\{f_i\}_{i\in J_n}$ is again a frame for $\mathcal{H}$, with frame bounds $\frac{A}{B^2 n\|(F_n^*F_n)^{-1}\|^2+1}$, B, where $J_n = \{i \in \mathbb{N} : i \geq n+1\}$ and $F_n : \mathcal{H} \to \ell^2(J_n)$ is the corresponding analysis operator.*

Proof. For a fixed $n \in \mathbb{N}$, we write

$$FF^* = \begin{pmatrix} A_n & B_n \\ B_n^* & C_n \end{pmatrix},$$

where $A_n = (\langle f_j, f_i\rangle)_{i,j=1}^n$. Then by Theorem 5.1.6,

$$B_n C_n^{-1} B_n^* = \Big(\langle P_n f_j, P_n f_i\rangle\Big)_{i,j}^n,$$

where P_n is the orthogonal projection onto $\mathcal{K} = \overline{\operatorname{span}\{f_i : i \in J_n\}}$. Now since FF^* is completely non-factorizable, therfore

$$A_n = B_n C_n^{-1} B_n^*,$$

using Lemma 5.1.8. Hence, $A_n = (\langle P_n f_j, P_n f_i\rangle)_{i,j}^n$, which implies that $\|f_i\|^2 = \langle f_i, f_i\rangle = \langle P_n f_i, P_n f_i\rangle = \|P_n f_i\|^2$, for all $1 \leq i \leq n$. Thus $f_i \in \mathcal{K}$, for all $1 \leq i \leq n$.

So, $\mathcal{H} = \overline{\text{span}\{f_i : i \in J_n\}} = \mathcal{K}$, and from this it follows that $\{f_i\}_{i\in J_n}$ is a frame for $\mathcal{H}$, using Proposition 5.1.7.

In order to complete the proof, we now verify that this frame has the desired frame bounds. Recall, $F_n(x) = (\langle x, f_i\rangle)_{i\in J_n},\ x \in \mathcal{H}$. Since $\{f_i\}_{i\in J_n}$ is a frame for $\mathcal{H}$, we can write $f_i = \sum_{j\in J_n} \alpha^i_j f_j$, for all $1 \le i \le n$, where $\alpha^i_j = \langle f_i, (F_n^*F_n)^{-1} f_j\rangle$. Then for $x \in \mathcal{H}$ and $1 \le i \le n$

$$\begin{aligned}
|\langle x, f_i\rangle|^2 &= \Big|\langle x, \sum_{j\in J_n} \alpha^i_j f_j\rangle\Big|^2 \\
&= \Big|\sum_{j\in J_n} \overline{\alpha^i_j}\langle x, f_j\rangle\Big|^2 \\
&\le \Big(\sum_{j\in J_n} |\alpha^i_j|^2\Big)\Big(\sum_{j\in J_n} |\langle x, f_j\rangle|^2\Big) \\
&\le B\|(F_n^*F_n)^{-1} f_i\|^2 \sum_{j\in J_n} |\langle x, f_j\rangle|^2 \\
&\le B^2\|(F_n^*F_n)^{-1}\|^2 \sum_{j\in J_n} |\langle x, f_j\rangle|^2.
\end{aligned}$$

Thus,

$$\sum_{i=1}^{n} |\langle x, f_i\rangle|^2 \le B^2 n\|(F_n^*F_n)^{-1}\|^2 \sum_{j\in J_n} |\langle x, f_j\rangle|^2.$$

So,

$$\begin{aligned}
A\|x\|^2 &\le \sum_{i=1}^{n} |\langle x, f_i\rangle|^2 + \sum_{i\in J_n} |\langle x, f_i\rangle|^2 \\
&\le \big(B^2 n\|(F_n^*F_n)^{-1}\|^2 + 1\big) \sum_{i\in J_n} |\langle x, f_i\rangle|^2.
\end{aligned}$$

Therefore,

$$\frac{A\|x\|^2}{B^2 n\|(F_n^*F_n)^{-1}\|^2 + 1} \le \sum_{i\in J_n} |\langle x, f_i\rangle|^2.$$

Also,

$$\sum_{i\in J_n} |\langle x, f_i\rangle|^2 \le \sum_{i\in\mathbb{N}} |\langle x, f_i\rangle|^2 \le B\|x\|^2.$$

Hence $\{f_i\}_{i\in J_n}$ is a frame with frame bounds

$$\frac{A}{B^2 n\|(F_n^* F_n)^{-1}\|^2 + 1}, B.$$

$\square$

The above theorem points out that in a frame with completely non-factorizable Grammian, vectors are “dependent” on one another in such a way that no matter which finite set you remove, you end up with a frame whose span is still dense in the whole space. Thus, keeping in mind Theorem 5.1.4, Theorem 5.1.9 suggests that a frame with completely non-factorizable Grammian might be a good candidate for constructing a counter example to the FC.

Next is an application of the above theorem. For this we need the following result from [14], which points out a complementarity behavior between spanning and linear independence.

Proposition 5.1.10 (BCPS, [14])**.** *Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{f_i\}_{i\in S}$, let $P \in B(\mathcal{H})$ be the orthogonal projection onto a closed subspace of $\mathcal{H}$, and let $B \subseteq S$. Then the linear span of $\{Pf_i\}_{i\in B}$ is dense in Ran(P) if and only if the operator $\big(\langle (I-P)f_j, (I-P)f_i\rangle\big)_{i,j\in B^c}$ on $\ell^2(B^c)$ is one-to-one.*

Theorem 5.1.11. *Let $P \in B(\ell^2)$ be an orthogonal projection. If P is completely non-factorizable, then the Parseval frame $\{(I-P)e_i\}$ is linearly independent.*

Proof. Since P is an orthogonal projection, $\{Pe_i\}$ is a Parseval frame for $Ran(P)$ and P is the Grammian for $\{Pe_i\}$. Also, P is completely non-factorizable. Thus using Theorem 5.1.9, $\overline{\text{span}\{Pe_i : i \in F^c\}} = Ran(P)$, for every finite set $F \subseteq \mathbb{N}$. Then by Proposition 5.1.10, we get that the matrix $(\langle (I-P)e_j, (I-P)e_i \rangle)_{i,j\in F}$ is one-to-one, for every finite subset F of $\mathbb{N}$. This further implies that $\{(I-P)e_i\}_{i,j\in F}$ is a linearly independent set, for every finite subset F of $\mathbb{N}$. Hence $\{(I-P)e_i\}$ is a linearly independent set. $\square$

We end this section with a concrete example of a frame with completely non-factorizable Grammian. This completely non-factorizable operator is discussed in [1]. First we recall a result from [1], which characterizes completely non-factorizable operators.

In [1], it is shown that given a positive operator $A = (a_{ij}) \in B(\ell^2)$, the function $K_A(z,w) = \sum_{i,j=0}^{\infty} a_{ij} z^i \overline{w}^j$ defines a positive definite function on $\mathbb{D} \times \mathbb{D}$. Hence K_A is a kernel function on $\mathbb{D}$. We denote the RKHS corresponding to the kernel function K_A by $\mathcal{H}(A)$. We can verify that when $P \in B(H^2)$ is a positive contraction and $A = (\langle Pz^j, z^i \rangle)$, then the RKHS $\mathcal{H}(A)$ coincides with the contractive Hilbert space $\mathcal{H}(P)$. In particular, when $P \in B(H^2)$ is an orthogonal projection and $A = (\langle Pz^j, z^i \rangle)$, then $\mathcal{H}(A)$ coincides with $Ran(P)$, since in this case $\mathcal{H}(P)$ and $Ran(P)$ are the same Hilbert space.

Theorem 5.1.12 (AFMP, [1])**.** *A positive operator $A \in B(\ell^2)$ is completely non-factorizable if and only if the RKHS $\mathcal{H}(A)$ contains no polynomials.*

Example 5.1.13. *Let b be an infinite Blaschke product in H^2. Let P be the orthogonal projection onto bH^2. Set $f_i = Pz^i$, for all $i \geq 0$. Then $\{f_i\}$ is a Parseval frame for bH^2 and $FF^* = (\langle Pz^j, z^i\rangle)$. Since b has infinitely many zeroes, therefore bH^2 contains no polynomial. But, $bH^2 = Ran(P) = \mathcal{H}(FF^*)$. Hence by Theorem 5.1.12, FF^* is completely non-factorizable..*

A closer look reveals more interesting and a bit surprising facts about this example. Let $b = \sum_{i=0}^{\infty} \alpha_i z^i$. Then, $P = T_b T_b^*$, and so $f_i = bg_i$, where $g_i = \sum_{j=0}^{i} \overline{\alpha_{i-j}} z^j$ is a polynomial of degree at most i. Further, since b is an inner function, therefore $\langle f_j, f_i\rangle = \langle g_j, g_i\rangle$. Hence, we can replace our original Parseval frame $\{f_i\}$ with the Parseval frame $\{g_i\}$, which consists of polynomials. In addition, $g_i = S^*(g_{i+1})$ for all i, where $S^* \in B(H^2)$ is the backward shift operator. Moreover, if we assume that $\alpha_0 \neq 0$, then $\{g_i\}$ is a linearly independent set, and the degree of the i^{th} polynomial g_i is exactly i.

Furthermore, the linear independence of $\{g_i\}_{i\geq 0}$ forces the matrix $(\langle g_j, g_i\rangle)_{i,j\in F}$ to be one-to-one, for every finite set $F \subseteq \mathbb{N} \cup \{0\}$. This implies that the matrix $(\langle f_j, f_i\rangle)_{i,j\in F}$ is one-to-one, since $\langle f_j, f_i\rangle = \langle g_j, g_i\rangle$. Hence $\overline{\text{span}\{(I-P)z^i : i \in F^c\}} = Ran(I-P)$, for every finite subset F of $\mathbb{N}\cup\{0\}$, using Proposition 5.1.10. Also, since $FF^* = (\langle Pz^j, Pz^i\rangle)$ is completely non-factorizable, therefore using Theorem 5.1.9, $\overline{\text{span}\{Pz^i : i \in F^c\}} = Ran(P)$, for every finite set $F \subseteq \mathbb{N}\cup\{0\}$. Hence by Proposition 5.1.10, it follows that $\{(I-P)z^i\}$ is a linearly independent set.

To summarize, we have got a linearly independent (assuming $\alpha_0 \neq 0$) Parseval frame $\{g_i\}$, for $Ran(P)$, where each g_i is a polynomial of degree i. Though the frame vectors are independent, there is still a lot of "dependency" among them in the sense that $g_i = S^*(g_{i+1})$, for all i and $\overline{\text{span}\{g_i : i \in F^c\}} = Ran(P)$, for every finite subset F of $\mathbb{N} \cup \{0\}$. In addition, we have got that the Parseval frame $\{(I-P)z^i\}$, for $Ran(I-P)$, is also linearly independent, and satisfies that $\overline{\{(I-P)z^i : i \in F^c\}} = Ran(I-P)$, for every finite set $F \subseteq \mathbb{N} \cup \{0\}$.

5.2 Grammian and the Upper Triangular Operators

In this section, we will discuss some results concerning frames with Grammian of the form UU^*, where $U \in B(\ell^2)$ is upper triangular. We begin with some preliminary results, which might be interesting in their own right.

Theorem 5.2.1. *Let $\{f_i\}$ be a frame for a Hilbert space. If $dim(Ker(F^*)) = 1$, then there exists i_0 such that $\{f_i\}_{i \neq i_0}$ is a Riesz basic sequences, where F is the corresponding analysis operator. In fact, $\{f_i\}_{i \neq i_0}$ is a Riesz basis for $\mathcal{H}$.*

Proof. Let $x = \{x_i\} \in Ker(F^*),\ x \neq 0$. Let i_0 be the first index such that $x_i \neq 0$. Then, $0 = x_{i_0} f_{i_0} + \sum_{i > i_0} x_i f_i$, which implies that

$$f_{i_0} = \sum_{i > i_0} y_i f_i,$$

where $y_i = -x_i / x_{i_0}$.

Thus $\mathcal{H} = \overline{\text{span}\{f_i : i \in J\}}$, that is, $\{f_i\}_{i \in J}$ is a complete set, where $J = \{i : i \neq i_0\}$. Hence, it is a frame for $\mathcal{H}$, using Proposition 5.1.7. Let $G : \mathcal{H} \to \ell^2(J)$ be the analysis operator for the frame $\{f_i\}_{i\in J}$. We claim that $\{f_i\}_{i\in J}$ is a Riesz basis for $\mathcal{H}$. To accomplish this we will show that $Ker(G^*) = 0$.

Let $\{\alpha_i\} \in Ker(G^*)$. Then $\sum_{i\in J} \alpha_i f_i = 0$. This gives an element $\{\beta_i\}$ in $Ker(F^*)$, where $\beta_i = \alpha_i$ if $i \in J$ and $\beta_{i_0} = 0$. Then there exists a scalar c, such that $\beta_i = cx_i$ for all i, since $Ker(F^*)$ is spanned by x. In particular, $0 = \beta_{i_0} = cx_{i_0}$, which implies that $c = 0$, since $x_{i_0} \neq 0$. Thus $\beta_i = 0$, for all i. This makes $\alpha_i = 0$, for all i. Thus $Ker(G^*) = 0$, and hence $\{f_i\}_{i\in J}$ is a Riesz basis for $\mathcal{H}$. □

Theorem 5.2.2. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a frame for a Hilbert space $\mathcal{H}$. If $dim(Ker(F^*))$ is finite, then the frame splits into finitely many Riesz basic sequences. In fact, if $dim(Ker(F^*)) = m$ then there exist $i_1, i_2, \ldots, i_m \in \mathbb{N}$ such that $\{f_i\}_{i\in L}$ is a Riesz basis for $\mathcal{H}$, where $L = \{i \in \mathbb{N} : i \neq i_1, \ldots, i_m\}$.*

Proof. We prove the result by induction on $dim(Ker(F^*))$. If $dim(Ker(F^*)) = 0$, then $Ker(F^*) = \{0\}$ and so $\{f_i\}_{i\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$. By Theorem 5.2.1 the result is also true, when $dim(K(F^*)) = 1$. We assume the result is true for every frame for which $dim(Ker(F^*)) \leq m$, and we will prove the result for the case when $dim(Ker(F^*)) = m + 1$.

Let $x = \{x_i\}_{i\in\mathbb{N}} \in Ker(F^*),\ x \neq 0$. Let i_0 be the first index such that $x_i \neq 0$. Then, $0 = x_{i_0} f_{i_0} + \sum_{i>i_0} x_i f_i$ which implies that

$$f_{i_0} = \sum_{i>i_0} y_i f_i, \tag{5.1}$$

where $y_i = -x_i/x_{i_0}$.

Thus $\{f_i\}_{i\in J}$ is a complete set in $\mathcal{H}$, where $J = \{i \in \mathbb{N} : i \neq i_0\}$. Therefore,

$\{f_i\}_{i\in J}$ is a again a frame for $\mathcal{H}$, using Proposition 5.1.7. Let $G : \mathcal{H} \to \ell^2(J)$ be the corresponding analysis operator. We claim that the dimension of $Ker(G^*)$ is m.

Note that if $\{\alpha_i\}_{i\in\mathbb{N}} \in Ker(F^*)$, then the sequence $\{\alpha_i + \alpha_{i_0} y_i\}_{i\in J}$ is in $Ker(G^*)$, using equation (5.1), where $y_i = 0$, for $i < i_0$. Thus we obtain a well-defined linear map $T : Ker(F^*) \to Ker(G^*)$, given by $T(\{\alpha_i\}_{i\in\mathbb{N}}) = \{\alpha_i + \alpha_{i_0} y_i\}_{i\in J}$. Clearly, T is onto. Further, note that $Ker(T)$ is a one-dimensional subspace of $Ker(F^*)$, spanned by the vector $\{z_i\}_{i\in\mathbb{N}}$, where $z_i = -y_i$, if $i \in J$ and equals 1, if $i = i_0$. Therefore the dimension for $Ker(T)^\perp$ is m. Hence, $dim(Ker(G^*)) = m$, since $T : Ker(T)^\perp \to Ker(G^*)$ is an isomorphism. Thus by induction hypothesis there exist $i_1, \ldots, i_m \in J$, such that $\{f_i\}_{i\in J_1}$ is Riesz basis for $\mathcal{H}$, where $J_1 = \{i \in J : i \neq i_1, \ldots, i_m\}$. Clearly, a singleton is a Riesz basis for its span. Hence the frame $\{f_i\}_{i\in\mathbb{N}}$ is divided into $m+1$ Riesz basic sequences, namely $\{f_{i_0}\}, \{f_1\}, \ldots, \{f_m\}$ and $\{f_i\}_{i\in L}$, where $L = \{i \in \mathbb{N} : i \neq i_0, i_1 \ldots, i_m\}$ and $\{f_i\}_{i\in L}$ is a Riesz basis for $\mathcal{H}$. $\square$

Recall that a frame is a Riesz basis if $Ker(F^*) = 0$. Hence the above theorem is interesting as it justifies the intuition that a frame for which $Ker(F^*)$ is finite dimensional should not be too far from being a Riesz basis. Moreover, it also gives a count on the number of vectors that must be removed to make such a frame a Riesz basis. In addition, it also tells us that such frames satisfy the FC.

As we mentioned earlier, there are many equivalent conjectures to the Kadison-Singer Problem and hence to the FC. One such equivalent conjecture, perhaps one of the most famous ones, is the paving conjecture [2, 3, 4]. Our next result is related to this conjecture and it comes up as an application to our previous theorem. To state this result, we recall the following definition.

Definition 5.2.3. *A bounded operator $T \in B(\ell^2)$ is said to be (δ, r)-pavable if there exists a partition $A_1, \ldots, A_r$ of $\mathbb{N}$ such that $\|P_{A_i}(T - E(T))P_{A_i}\| \leq \delta$, for all $1 \leq i \leq r$.*

Recall that $E : B(\ell^2) \to B(\ell^2)$ is the bounded operator which maps an operator T to its diagonal, that is, to the diagonal operator with i^{th} diagonal entry $\langle Te_i, e_i \rangle$.

Theorem 5.2.4. *Let $P \in B(\ell^2)$ be an orthogonal projection with finite codimension. Then P is $(r, m+2)$-pavable for some $0 < r < 1$, where $m \leq dim(Ker(P))$.*

Proof. Let $f_i = P(e_i)$, then $\{f_i\}_{i \in \mathbb{N}}$ is a frame for $Ran(P)$. Since kernel of P is finite dimensional, therefore at most finitely many $f_i^{'}s$ can be zero. Finite codimension of P also implies that $\|f_i\| = \|P(e_i)\| \to 1$. Thus there exist a $0 < \delta < 1$ and $i_1, \ldots, i_t \in \mathbb{N}$, such that $\|Pe_i\| \geq \delta$ for all $i \neq i_1, \ldots i_t$ and $P(e_i) = 0$, for $i = i_1, \ldots, i_t$.

Let $J = \{i \in \mathbb{N} : i \neq i_1, \ldots, i_t\}$ and let $F : Ran(P) \to \ell^2(J)$ be the analysis operator for the frame $\{f_i\}_{i \in J}$. Then we can easily verify that $dim(Ker(F^*)) \leq dim(Ker(P))$. Thus Kernel of F^* is finite dimensional, let $dim(Ker(F^*)) = m$. Therefore by Theorem 5.2.2, there exist a partition $A_1, \ldots A_{m+1}$ of J and constants $0 < c_1, \ldots, c_{m+1} < 1$, such that

$$P_{A_l} F F^* P_{A_l} \geq c_l P_{A_l},$$

for all $1 \leq l \leq m+1$. Thus, $P_{A_l} P P_{A_l} \geq c_l P_{A_l}$ for all $1 \leq l \leq m+1$. Also, for each $1 \leq l \leq m+1$, $\delta^2 P_{A_l} \leq P_{A_l} E(P) P_{A_l} \leq P_{A_l}$, since $\|f_i\| \geq \delta$, for all $i \in J$. Hence,

$$-(1 - c_l) P_{A_l} \leq P_{A_l}(P - E(P)) P_{A_l} \leq (1 - \delta^2) P_{A_l},$$

for all $1 \leq l \leq m+1$. Lastly, let $r = max_l\{1 - c_l, 1 - \delta^2\}$. Then

$$-rP_{A_l} \leq P_{A_l}(P - E(P))P_{A_l} \leq rP_{A_l},$$

for all $1 \leq l \leq m+1$. Further, let $A_{m+2} = \{i_1, \dots, i_t\}$. Then $P_{A_{m+2}}(P - E(P))P_{A_{m+2}} = 0$, and therefore $-rP_{A_{m+2}} \leq P_{A_{m+2}}(P - E(P))P_{A_{m+2}} \leq rP_{A_{m+2}}$. Thus we obtain a partition $A_1, \dots, A_{m+2}$ of $\mathbb{N}$, such that

$$-rP_{A_l} \leq P_{A_l}(P - E(P))P_{A_l} \leq rP_{A_l},$$

for all $1 \leq l \leq m+2$. This implies that $\|P_{A_l}(P - E(P))P_{A_l}\| \leq r$, for all $1 \leq l \leq m+2$, and hence P is $(r, m+2)$-pavable. $\square$

Corollary 5.2.5. *Let b be a finite Blaschke product with m factors (repeated according to the multiplicity). Then the positive operator $T_bT_b^*$ is $(r, t+2)$- pavable, where $0 < r < 1$ and $0 \leq t \leq m$.*

Proof. The result follows directly from Theorem 5.2.4, using the fact that $T_bT_b^*$ is an orthogonal projection with codimension m. $\square$

In the above corollary, we are considering $T_bT_b^*$ as an operator on ℓ^2, using the canonical identification between H^2 and ℓ^2. This identification is given by the isometric isomorphism $T : \ell^2 \to H^2$, defined by $T(e_i) = z^{i-1},\ i \in \mathbb{N}$.

We are now ready to present the main result (Theorem 5.2.7) of the section, in which we prove the FC for a class of frames for which the Grammian is of the form UU^*, for some upper triangular $U \in B(\ell^2)$. Note that a Grammian of the form UU^*, where $U \in B(\ell^2)$ is upper triangular, can also be viewed as a Grammian of the form L^*L, where $L \in B(\ell^2)$ is lower triangular. We work with the later form. The following result is an essential part of the proof our main result, we are presenting it separately for reader's convenience.

Theorem 5.2.6. *Let $R \in B(\ell^2)$ be a lower triangular operator with only finitely many diagonal entries zero. Then $dim(Ker(R))$ is finite.*

Proof. Let $R = (r_{ij})$ and let $r_{nn} \neq 0$ for all $n > N$. Write

$$R = \begin{pmatrix} A & 0 \\ B & L \end{pmatrix},$$

where $A = (r_{ij})_{i,j=1}^N$. Then L is a lower triangular matrix with all diagonal entries non-zero, and thus it is one-to-one. In order to prove $dim(Ker(R))$ is finite, we will show that that $dim(Ker(R)) \leq 2dim(Ker(A))$.

To settle this, we let $\{x_i\}_{i=1}^m$ be a basis for $Ker(A)$. Further, suppose $z = \begin{bmatrix} x \\ y \end{bmatrix} \in Ker(R)$. Then $\begin{bmatrix} A(x) \\ B(x) + L(y) \end{bmatrix} = 0$, and thus $A(x) = 0$ and $L(y) = -B(x)$. Now expressing $x = \sum_{i=1}^m \alpha_i x_i$ for some scalars $\alpha_i' s$, we get $L(y) = -\sum_{i=1}^m \alpha_i B x_i$, which implies that $Ly \in \text{span}\{Bx_1, \ldots, Bx_m\}$. This defines a linear map $T : Ker(R) \to \mathcal{F}$, given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = Ly,$$

where $\mathcal{F} = \text{span}\{B(x_1), \ldots, B(x_m)\}$. Note that $\begin{bmatrix} x \\ y \end{bmatrix} \in Ker(T)$ if and only if $y = 0$, since L is one-to-one. Therefore,

$$\begin{aligned} Ker(T) &= \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : \begin{bmatrix} x \\ 0 \end{bmatrix} \in Ker(R) \right\} \\ &= \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : \begin{bmatrix} Ax \\ Bx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}. \end{aligned}$$

Thus, $dim(Ker(T)) \leq dim(Ker(A)) = m$. Also, $dim(Ran(T)) \leq dim(\mathcal{F}) \leq m$. Hence, $dim(Ker(R)) \leq 2m < \infty$. □

The following is the main result of the section.

Theorem 5.2.7. *Let $\{f_i\}$ be a frame for a Hilbert space $\mathcal{H}$ and let F be the corresponding analysis operator. If F^* is a lower triangular matrix with only finitely many diagonal entries zero, then $\{f_i\}$ satisfies the FC.*

Proof. By Theorem 5.2.6, $dim(Ker(F^*))$ is finite, Since F^* is lower triangular and has only finitely many diagonal entries zero. Thus by theorem 5.2.2, the frame $\{f_i\}$ splits into finitely many Riesz basic sequences, and hence satisfies the FC. □

Remark 5.2.8. *Notice that frames, where F^* is lower triangular with only finitely many diagonal entries zero, carry more information than it is needed in Theorem 5.2.7. Since for such frames $dim(Ker(F^*))$ is finite, therefore these can be made into Riesz basis just by removing finitely many vectors from them. Moreover, we also have an upper bound on the number of vectors to be removed in terms of the number of zeroes on the diagonal of F^*.*

The following is a direct proof of Theorem 5.2.7, when $\mathcal{H} = \ell^2$.

Theorem 5.2.9. *Let $\{f_i\}$ be a frame for ℓ^2, such that F^* is lower triangular. Then $\{f_i\}$ is a Riesz basis for ℓ^2.*

Proof. To prove $\{f_i\}$ is a Riesz basis for l^2, we will show that F^* has no kernel. For this it is enough to prove that no diagonal entry of F^* is zero, since it is lower triangular. Suppose there exists a n, such that the n^{th} diagonal entry of F^* is zero, i.e., $\langle F^*(e_n), e_n\rangle = \langle f_n, e_n\rangle = 0$.

Further, any $x \in \text{span}\{e_1, \ldots, e_n\}$ can be written as $x = \sum_i a_i f_i$, since $\{f_i\}$ is a frame for ℓ^2. Let P_n be the orthogonal projection onto $\text{span}\{e_1, \ldots, e_n\}$. Then

$$x = P_n(x) = \sum_i a_i P_n(f_i).$$

Since F^* is lower triangular, therefore $\langle f_n, e_i\rangle = 0$, for all $1 \leq i \leq n-1$, and $\langle f_j, e_i\rangle = 0$, for all $1 \leq i \leq n,\ j > n$. Also, $\langle f_n, e_n\rangle = 0$. Hence, $\langle f_j, e_i\rangle = 0$ for all $1 \leq i \leq n,\ j \geq n$, which implies that $P(f_j) = 0$ whenever $j \geq n$. Thus, $x = \sum_{i=1}^{n-1} a_i P_n(f_i)$. But then, $\text{span}\{e_1, \ldots, e_n\} \subseteq \text{span}\{P_n(f_1), \ldots, P_n(f_{n-1})\}$. , which is not possible. Hence, we conclude that each diagonal entry of F^* is non-zero. Now it follows easily that F^* has no kernel, since F^* is lower triangular. Also F^* is onto, since $\{f_i\}$ is a frame. Hence $\{f_i\}$ is a Riesz basis for ℓ^2. □

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