THE FEKETE-SZEGÖ PROBLEM FOR A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract

Let $\mathcal{C}_1(\beta)$ be the class of normalized functions f, which are analytic in the open unit disk $\mathcal{U}$, given by the power series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

and satisfy the inequality:

$$\operatorname{Re}\left\{\frac{zf'(z)}{\phi(z)}e^{i\beta}\right\} > 0 \quad \left(z \in \mathcal{U};\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right)$$

for some normalized univalent and convex function ϕ. In this paper we solve the Fekete-Szegő problem for the family:

$$\mathcal{C}_1 := \bigcup_{\beta} \mathcal{C}_1(\beta) \left(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\right)$$

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by proving that

$$\max_{f\in\mathcal{C}_1}\left|a_3-\lambda a_2^2\right| = \begin{cases} \dfrac{5}{3}-\dfrac{9\lambda}{4} & \left(0\leq\lambda\leq\dfrac{2}{9}\right) \\[2ex] \dfrac{2}{3}+\dfrac{1}{9\lambda} & \left(\dfrac{2}{9}\leq\lambda\leq\dfrac{2}{3}\right) \\[2ex] \dfrac{5}{6} & \left(\dfrac{2}{3}\leq\lambda\leq 1\right). \end{cases}$$

1. Introduction

Let $\mathcal{S}$ be the class of (normalized) analytic and univalent functions in the open unit disk

$$\mathcal{U}=\{z : z\in\mathbb{C} \quad \text{and} \quad |z|<1\}$$

that are given by the Taylor series:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \quad (z\in\mathcal{U}). \tag{1.1}$$

Let $\mathcal{S}^*$ and $\mathcal{K}$ denote, respectively, the subsets of $\mathcal{S}$ consisting of starlike and convex functions in $\mathcal{U}$. A function f, analytic in $\mathcal{U}$ and given by the series (1.1), is said to be close-to-convex in $\mathcal{U}$ if

$$\operatorname{Re}\left\{\frac{zf'(z)}{\phi(z)}e^{i\beta}\right\}>0 \quad \left(z\in\mathcal{U};\ \phi\in\mathcal{S}^*;\ -\frac{\pi}{2}<\beta<\frac{\pi}{2}\right). \tag{1.2}$$

We denote the family of close-to-convex functions in $\mathcal{U}$ by $\mathcal{C}$. This class was introduced and studied by Kaplan [10]. The number $e^{i\beta}$ is necessary in (1.2) for the definition of close-to-convex functions.

It is well known, for $f\in\mathcal{S}$ and given by (1.1), that

$$\left|a_3-a_2^2\right|\leq 1, \tag{1.3}$$

where the equality holds true for the Koebe function:

$$k(z)=\frac{z}{(1-z)^2}=z+\sum_{n=2}^{\infty}nz^n.$$

Since $k\in\mathcal{S}^*$ ($\supset\mathcal{C}$), the estimate (1.3) is true for $\mathcal{S}^*$ and $\mathcal{C}$ without any further refinement. However, Trimble [18] has shown that, for $f\in\mathcal{K}$ and given by (1.1),

$$\left|a_3-a_2^2\right|\leq\frac{1}{3}. \tag{1.4}$$

Earlier in 1933, Fekete and Szegő [4] made use of Lőwner's parametric method in order to prove that, if $f\in\mathcal{S}$ and is given by (1.1),

$$\left|a_3-\lambda a_2^2\right|\leq 1+2e-\frac{2\lambda}{1-\lambda} \quad (0\leq\lambda\leq 1). \tag{1.5}$$

Equality in (1.5) holds true for the Koebe function $k(z)$ only for $\lambda=0$ and $\lambda=1$. The case $0<\lambda<1$ provides an example of an extremal problem over $\mathcal{S}$ in which the Keobe function $k(z)$ fails to be extremal. The determination of sharp upper bound for the nonlinear functional $\left|a_3-\lambda a_2^2\right|$

for any given family $\mathcal{F}$ of normalized analytic functions is popularly known as the Fekete-Szegő problem for $\mathcal{F}$. The result (1.5) was also proved later by Goluzin [7], Jenkins [9], Pfluger [14], Shaffer and Spencer [15], and others.

The Fekete-Szegő problem for the families $\mathcal{K}, \mathcal{S}^*$, and $\mathcal{C}$ has been completely solved in the literature. Thus we have (*cf.* [10] and [11])

$$\max_{f\in\mathcal{K}} \left|a_3 - \lambda a_2^2\right| = \max\left\{\frac{1}{3}, |\lambda - 1|\right\},$$

$$\max_{f\in\mathcal{S}^*} \left|a_3 - \lambda a_2^2\right| = \begin{cases} |3-4\lambda| & \left(\lambda \leq \frac{1}{2} \quad \text{and} \quad \lambda \leq 1\right) \\ 1 & \left(\frac{1}{2} \leq \lambda \leq 1\right), \end{cases}$$

and

$$\max_{f\in\mathcal{C}} \left|a_3 - \lambda a_2^2\right| = \begin{cases} 3-4\lambda & \left(0 \leq \lambda \leq \frac{1}{3}\right) \\ \frac{1}{3} + \frac{4}{9\lambda} & \left(\frac{1}{3} \leq \lambda \leq \frac{2}{3}\right) \\ 1 & \left(\frac{2}{3} \leq \lambda \leq 1\right). \end{cases}$$

Many other recent works on the Fekete-Szegő problem include, for example, [1], [3], [6], [13], [14], and [17].

We now introduce the class $\mathcal{C}_1(\beta)$ of (normalized) analytic functions f in $\mathcal{U}$, which are given by (1.1) and satisfy the inequality (1.2) with $\phi \in \mathcal{K}$ (instead of $\phi \in \mathcal{S}^*$), and let

$$\mathcal{C}_1 := \bigcup_\beta \mathcal{C}_1(\beta) \quad \left(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\right).$$

Since $\mathcal{K} \subset \mathcal{S}^*$, it follows that $\mathcal{C}_1 \subset \mathcal{C}$. Also, by taking $f = \phi$ in (1.2), we have $\mathcal{K} \subset \mathcal{C}_1$. Furthermore, the choice $\phi(z) = z$ in (1.2) exhibits the fact that the class of (normalized) analytic functions satisfying the inequality:

$$\operatorname{Re}\left\{e^{i\beta} f'(z)\right\} > 0 \quad \left(z \in \mathcal{U};\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right)$$

is contained in the class $\mathcal{C}_1$.

Problems involving growth and distortion inequalities, coefficient estimates, convex hull, extreme points, and so on, for the family $\mathcal{C}_1$ were investigated by Silverman and Telage [16]. In this paper, we completely solve the Fekete-Szegő problem for the family $\mathcal{C}_1$. In particular, one of our results (Theorem 4 below) gives a refinement of (1.3) for the smaller set $\mathcal{C}_1$, and it also includes some recent results of Abdel-Gawad and Thomas [2].

Theorem 1. *Let $f \in \mathcal{C}_1$ and be given by* (1.1). *Then*

$$\left|a_3 - \frac{2}{9}a_2^2\right| \leq \frac{7}{6}$$

The result is sharp.

Theorem 2. *Let $f \in \mathcal{C}_1$ and be given by* (1.1). *Then*

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{5}{3} - \frac{9\lambda}{4} \qquad \left(\lambda \leq \frac{2}{9}\right).$$

The result is sharp.

Theorem 3. *Let $f \in \mathcal{C}_1$ and be given by* (1.1). *Then*

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{2}{3} + \frac{1}{9\lambda} \quad \left(\frac{2}{9} < \lambda < \frac{2}{3}\right).$$

The result is sharp.

Theorem 4. *Let $f \in \mathcal{C}_1$ and be given by* (1.1). *Then*

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leq \frac{5}{6}.$$

The result is sharp.

Theorem 5. *Let $f \in \mathcal{C}_1$ and be given by* (1.1). *Then*

$$\left|a_3 - a_2^2\right| \leq \frac{5}{6}.$$

The result is sharp.

Theorem 6. *Let $f \in \mathcal{C}_1$ and be given by* (1.1). *Then*

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{5}{6} \quad \left(\frac{2}{3} \leq \lambda \leq 1\right).$$

The result is sharp.

It follows from the definition that, if $f \in \mathcal{C}_1$, then f' can be written as

$$f'(z) = \frac{\phi(z)}{z}h(z)e^{-i\beta} \quad \left(z \in \mathcal{U};\ \phi \in \mathcal{K};\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right), \tag{1.6}$$

where

$$h(z) = \left(\frac{1+w(z)}{1-w(z)}\right)\cos\beta + i\sin\beta$$

for some Schwarz function w analytic in $\mathcal{U}$ such that

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1\ (z \in \mathcal{U}). \tag{1.7}$$

Thus, if f is given by (1.1),

$$\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathcal{U}) \tag{1.8}$$

and

$$w(z) = \sum_{n=1}^{\infty} \alpha_n z^n \quad (z \in \mathcal{U}) \tag{1.9}$$

then

$$2a_2 = \left(2e^{-i\beta}\cos\beta\right) a_1 + b_2 \tag{1.10}$$

and

$$3a_3 = 2e^{-i\beta}\left(\alpha_2 + \alpha_1^2 + \alpha_1 b_2\right)\cos\beta + b_3. \tag{1.11}$$

Equations (1.9) and (1.10), together, yield

$$a_3 - \lambda a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}\lambda b_2^2\right) + \frac{2}{3}e^{-i\beta}\left[\alpha_2 + \left(1 - \frac{3}{2}\lambda e^{-i\beta}\cos\beta\right)\alpha_1^2\right]\cos\beta$$
$$+ \frac{2}{3}e^{-i\beta}\left(1 - \frac{3}{2}\lambda\right)\alpha_1 b_2\cos\beta. \tag{1.12}$$

The expression (1.12) shall be used throughout this paper. We shall also need the following results for the proof of Theorems 1 to 6.

Theorem A (Keogh and Merkes [11]). *Let ϕ given by* (1.8) *be a univalent convex function in $\mathcal{U}$ (that is, $\phi \in \mathcal{K}$). Then, for any complex number s,*

$$\left|b_3 - s b_2^2\right| \leq \max\left\{\frac{1}{3}, |s-1|\right\}. \tag{1.13}$$

Theorem B (Keogh and Merkes [11]). *Let the Schwarz function w be given by* (1.7) *and the power series* (1.9)*. Then, for any complex number s,*

$$\left|\alpha_2 - s\alpha_1^2\right| \leq 1 + (|s| - 1)\left|\alpha_1\right|^2. \tag{1.14}$$

Theorem C (Silverman and Telage [16]). *Let $f \in \mathcal{C}_1$ and be given by* (1.1)*. Then*

$$|a_n| \leq 2 - \frac{1}{n} \quad (n \in \mathbb{N}\setminus\{1\}). \tag{1.15}$$

Equality in (1.15) *holds true for the function*:

$$h(z,\zeta) = \overline{\zeta}\log(1-\zeta z) + \frac{2z}{1-\zeta z} \quad (|\zeta| = 1).$$

2. Proofs of Theorems 1 to 6

Proof of Theorem 1. Putting $\lambda = \frac{2}{9}$ in (1.12), we get

$$\left|a_3 - \frac{2}{9}a_2^2\right| \leq \frac{1}{3}\left|b_3 - \frac{1}{6}b_2^2\right| + \left\{\frac{2}{3}\left|\alpha_2 - \left(\frac{1}{3}e^{-i\beta}\cos\beta - 1\right)\alpha_1^2\right| + \frac{4}{9}\left|\alpha_1 b_2\right|\right\}\cos\beta$$

Using Theorem A, Theorem B, and the inequality $|b_2| \leq 1$, we get

$$\left|a_3 - \frac{2}{9}a_2^2\right| \leq \frac{5}{18} + \left\{\frac{2}{3}\left[1 + \left(\left|\frac{1}{3}(\cos\beta - i\sin\beta)\cos\beta - 1\right| - 1\right)\left|\alpha_1\right|^2\right]\right.$$
$$\left. + \frac{4}{9}\left|\alpha_1 b_2\right|\right\}\cos\beta$$
$$\leq \frac{17}{18} + \left\{\frac{2}{3}\left(\sqrt{1 - \frac{5}{9}\cos^2\beta} - 1\right)\left|\alpha_1\right|^2 + \frac{4}{9}\left|\alpha_1 b_2\right|\right\}\cos\beta. \tag{2.1}$$

Since

$$\sqrt{1-\frac{5}{9}\cos^2\beta}-1 \le \frac{2}{3}-1=-\frac{1}{3},$$

we find from (2.1) that

$$\begin{aligned}\left|a_3-\frac{2}{9}a_2^2\right| &\le \frac{17}{18}+\left\{\frac{2}{3}\left(-\frac{1}{3}\right)|\alpha_1|^2+\frac{4}{9}|\alpha_1 b_2|\right\}\cos\beta \\ &= \frac{17}{18}+\frac{2}{9}|b_2|^2\cos\beta-\left\{\frac{2}{9}|b_2|^2+\frac{2}{9}|\alpha_1|^2-\frac{4}{9}|\alpha_1 b_2|\right\}\cos\beta \\ &\le \frac{7}{6}-\frac{2}{9}\left(|b_2|-|\alpha_1|\right)^2\cos\beta \\ &\le \frac{7}{6},\end{aligned}$$

which completes the proof of Theorem 1.

Proof of Theorem 2. We begin by considering

$$\begin{aligned}\left|a_3-\lambda a_2^2\right| &= \left|a_3-\frac{2}{9}a_2^2+\frac{2}{9}a_2^2-\lambda a_2^2\right| \\ &\le \left|a_3-\frac{2}{9}a_2^2\right|+\left(\frac{2}{9}-\lambda\right)|a_2|^2,\end{aligned}$$

which, in view of Theorem 1, yields

$$\left|a_3-\frac{2}{9}a_2^2\right| \le \frac{7}{6}.$$

Thus, using Theorem C, we have

$$\begin{aligned}\left|a_3-\lambda a_2^2\right| &\le \frac{7}{6}+\left(\frac{2}{9}-\lambda\right)\left(\frac{3}{2}\right)^2 \\ &= \frac{5}{3}-\frac{9\lambda}{4},\end{aligned}$$

which completes the proof of Theorem 2.

Equality can be shown to hold true by putting $b_2=b_3=1$, $\alpha_1=1$, $\alpha_2=0$, and $\beta=0$ in (1.12).

Proof of Theorem 3. By (1.12), we have

$$\begin{aligned}\left|a_3-\lambda a_2^2\right| \le \frac{1}{3}\left|b_3-\frac{3}{4}\lambda b_2^2\right|+\frac{2}{3}\left|\alpha_2-\left(\frac{3}{2}\lambda e^{-i\beta}\cos\beta-1\right)\alpha_1^2\right|\cos\beta \\ +\frac{2}{3}\left(1-\frac{3}{2}\lambda\right)|\alpha_1|\,|b_2|\cos\beta. \qquad (2.2)\end{aligned}$$

Now, using Theorem A, Theorem B, and the fact that $|b_2| \leq 1$, we get

$$\begin{aligned}
\left|a_3 - \lambda a_2^2\right| &\leq \frac{4-3\lambda}{12} + \frac{2}{3}\left\{1 + \left(\left|\frac{3}{2}\lambda e^{-i\beta}\cos\beta - 1\right| - 1\right)|\alpha_1|^2\right\}\cos\beta + \frac{2-3\lambda}{3}|\alpha_1|\cos\beta \\
&= \frac{4-3\lambda}{12} + \left\{\frac{2}{3} + \frac{2}{3}|\alpha_1|^2\left(\sqrt{1-\left(3\lambda - \frac{9}{4}\lambda^2\right)\cos^2\beta} - 1\right)\cos\beta \right. \qquad (2.3) \\
&\qquad \left. + \frac{2-3\lambda}{3}|\alpha_1|\right\}
\end{aligned}$$

Putting $\cos\beta = y$ and $|\alpha_1| = \alpha$ in (2.3), we get

$$\begin{aligned}
\left|a_3 - \lambda a_2^2\right| &\leq \frac{4-3\lambda}{12} + y\left\{\frac{2}{3} + \frac{2}{3}\alpha^2\left(\sqrt{1-\left(3\lambda - \frac{9}{4}\lambda^2\right)y^2} - 1\right) + \frac{2-3\lambda}{3}\alpha\right\} \\
&= F_\lambda(\alpha, y) \quad \text{(say)}. \qquad (2.4)
\end{aligned}$$

In the rest of the proof, we shall show that $F_\lambda(\alpha, y)$ attains its maximum value for $(\alpha, y) \in [0,1] \times [0,1]$ at the point $\left(\frac{2-3\lambda}{6\lambda}, 1\right)$. Note that

$$F_\lambda\left(\frac{2-3\lambda}{6\lambda}, 1\right) = \frac{2}{3} + \frac{1}{9\lambda}. \qquad (2.5)$$

We shall first show that F_λ does not have a local maximum at any interior point of the open rectangle $(0,1) \times (0,1)$. For, if F_λ had a local maximum at some point $(\alpha_0, y_0) \in (0,1) \times (0,1)$, then the partial derivatives $\frac{\partial F_\lambda}{\partial \alpha}$ and $\frac{\partial F_\lambda}{\partial y}$ must vanish at (α_0, y_0). Now

$$\frac{\partial F_\lambda}{\partial \alpha} = y\left[\frac{4\alpha}{3}\left(\sqrt{1 - y^2\left(3\lambda - \frac{9\lambda^2}{4}\right)} - 1\right) + \frac{2-3\lambda}{3}\right].$$

Thus the first requirement:

$$\left.\frac{\partial F_\lambda}{\partial \alpha}\right|_{(\alpha_0, y_0)} = 0$$

implies that

$$4\alpha_0\left(\sqrt{1 - y_0^2\left(3\lambda - \frac{9\lambda^2}{4}\right)} - 1\right) = -\frac{4}{3}\gamma \quad \left(\gamma = \frac{3}{4}(2-3\lambda)\right).$$

Note that $0 < \gamma < 1$. Therefore

$$y_0^2\left(3\lambda - \frac{9\lambda^2}{4}\right) = \frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0^2}. \qquad (2.6)$$

Similarly, we have

$$\frac{\partial F_\lambda}{\partial y} = \left[\frac{2}{3} + \frac{2\alpha^2}{3}\left(\sqrt{1 - y^2\left(3\lambda - \frac{9\lambda^2}{4}\right)} - 1\right) + \left(\frac{2 - 3\lambda}{3}\right)\alpha\right] + y\left[\frac{2\alpha^2}{3}\frac{(-y)\left(3\lambda - \frac{9\lambda^2}{4}\right)}{\sqrt{1 - y^2\left(3\lambda - \frac{9\lambda^2}{4}\right)}}\right].$$

Thus the second requirement:

$$\left.\frac{\partial F_\lambda}{\partial y}\right|_{(\alpha_0, y_0)} = 0$$

implies that

$$\frac{2}{3} + \frac{2\alpha_0^2}{3}\left(\sqrt{1 - y_0^2\left(3\lambda - \frac{9\lambda^2}{4}\right)} - 1\right) + \left(\frac{2 - 3\lambda}{3}\right)\alpha_0 = \frac{2\alpha_0^2}{3}\frac{y_0^2\left(3\lambda - \frac{9\lambda^2}{4}\right)}{\sqrt{1 - y_0^2\left(3\lambda - \frac{9\lambda^2}{4}\right)}}. \tag{2.7}$$

Substituting the value of $y_0^2\left(3\lambda - \frac{9\lambda^2}{4}\right)$ from (2.6) into (2.7), we find that

$$\frac{2}{3} + \frac{2\alpha_0^2}{3}\left(-\frac{\gamma}{3\alpha_0}\right) + \frac{4\gamma\alpha_0}{9} - \frac{2}{3}\alpha_0^2\left[\frac{\frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0^2}}{1 - \left(\frac{\gamma}{3\alpha_0}\right)}\right] = 0$$

or, equivalently, that

$$\alpha_0^2\gamma - 3\alpha_0 + \gamma = 0. \tag{2.8}$$

Solving the quadratic equation (2.8) for α_0, we have

$$\alpha_0\gamma = \frac{3 - \sqrt{9 - 4\gamma^2}}{2}. \tag{2.9}$$

The value of $F_\lambda(\alpha_0, y_0)$ written in terms of γ becomes

$$F_\lambda(\alpha_0, y_0) = \frac{6 + 4\gamma}{36} + y_0\left(\frac{2}{3} + \frac{2}{9}\gamma\alpha_0\right),$$

which, upon substituting the value of $\alpha_0\gamma$ from (2.9), yields

$$\begin{aligned} F_\lambda(\alpha_0, y_0) &= \frac{6+4\gamma}{36} + y_0\left(1 - \frac{1}{3}\sqrt{1-\frac{4}{9}\gamma^2}\right) \\ &\le \frac{6+4\gamma}{36} + 1 - \frac{1}{3}\sqrt{1-\frac{4}{9}\gamma^2} \\ &= \frac{21+2\gamma-6\sqrt{1-\frac{4}{9}\gamma^2}}{18}. \end{aligned}$$

Since $y \in (0,1)$, there exists $\eta > 0$ such that

$$\gamma = \frac{3}{2}\cos\delta \quad \text{and} \quad \sqrt{1-\frac{4}{9}\gamma^2} = \sin\delta \quad \left(0 < \eta < \delta < \frac{\pi}{2}\right).$$

Moreover, the inequality:

$$1 < 2\cos\delta + \sin\delta$$

gives

$$(1-\cos\delta)(1-\sin\delta) < \frac{\cos^2\delta}{2},$$

so that

$$\begin{aligned} (1-\cos\delta)&\left(21+2\gamma-6\sqrt{1-\frac{4}{9}\gamma^2}\right) \\ &= (1-\cos\delta)(15+3\cos\delta) + 6(1-\cos\delta)(1-\sin\delta) \\ &< 15 + 3\cos\delta - 15\cos\delta - 3\cos^2\delta + 3\cos^2\delta \\ &= 15 - 12\cos\delta = 3 + 12(1-\cos\delta). \end{aligned}$$

Therefore, we have

$$\frac{21+2\gamma-6\sqrt{1-\frac{4}{9}\gamma^2}}{18} < \frac{1}{6(1-\cos\delta)} + \frac{2}{3} = \frac{2}{3} + \frac{1}{9\lambda},$$

which shows that

$$F_\lambda(\alpha_0, y_0) < \frac{2}{3} + \frac{1}{9\lambda}.$$

Hence $F_\lambda(\alpha, y)$ does not have a local maximum in $(0,1)\times(0,1)$, so that the maximum must be attained at a boundary point. Since

$$F_\lambda(\alpha, 0) = \frac{4-3\lambda}{12} < \frac{2}{3} + \frac{1}{9\lambda},$$

there is no maximum on the line $y = 0$. Similarly, we have

$$F_\lambda(0, y) = \frac{4-3\lambda}{12} + \frac{2}{3}y \le \frac{4-3\lambda}{12} + \frac{2}{3} < \frac{2}{3} + \frac{1}{9\lambda},$$

so that there is no maximum on the line $\alpha = 0$ either. On the line $y = 1$, we get

$$F_\lambda(\alpha, 1) = 1 - \frac{\lambda}{4} - \lambda\alpha^2 + \left(\frac{2}{3} - \lambda\right)\alpha = G_\lambda(\alpha) \quad \text{(say)}.$$

Putting $\alpha = 1$, we get

$$G_\lambda(1) = \frac{5}{3} - \frac{9\lambda}{4}.$$

Since $G_\lambda(1)$ is not maximal, the local maximum of $G_\lambda(\alpha)$ is attained at

$$\alpha_0 = \frac{2 - 3\lambda}{6\lambda} \quad \text{for which} \quad \left.\frac{dG_\lambda(\alpha)}{d\alpha}\right|_{\alpha=\alpha_0} = 0.$$

This leads to the maximal value given by (2.4). The proof will be complete if we show that

$$F_\lambda(1, y) \le \frac{2}{3} + \frac{1}{9\lambda} \quad (0 < y < 1). \tag{2.10}$$

Since

$$\begin{aligned} F_\lambda(1, y) &= \frac{1}{3}\left[\frac{3 + 2\gamma}{6} + 2y\left(\sqrt{1 - y^2\left(1 - \frac{4}{9}\gamma^2\right)} + \frac{2\gamma}{3}\right)\right] \\ &= \frac{1}{3}\left[\frac{3 + 2\gamma}{6} + H_\lambda(y)\right], \end{aligned}$$

where

$$H_\gamma(y) = 2y\left(\sqrt{1 - y^2\left(1 - \frac{4}{9}\gamma^2\right)} + \frac{2\gamma}{3}\right),$$

the assertion (2.10) is equivalent to

$$H_\gamma(y) \le \frac{2}{3}\frac{(3 - \gamma)^2}{(3 - 2\gamma)} \quad (0 < y < 1). \tag{2.11}$$

Therefore, it suffices to prove (2.11) at the points $y \in (0, 1)$ for which

$$\frac{dH_\gamma(y)}{dy} = 0,$$

which implies that

$$\frac{2\gamma}{3}\sqrt{1 - y^2\left(1 - \frac{4}{9}\gamma^2\right)} = 2y^2\left(1 - \frac{4}{9}\gamma^2\right) - 1. \tag{2.12}$$

Squaring both sides in (2.12), we get

$$\left\{2y^2\left(1 - \frac{4}{9}\gamma^2\right)\right\}^2 - 2\left\{2y^2\left(1 - \frac{4}{9}\gamma^2\right)\right\}\left(1 - \frac{\gamma^2}{9}\right) + \left(1 - \frac{4}{9}\gamma^2\right) = 0.$$

Thus we have

$$2y^2\left(1 - \frac{4}{9}\gamma^2\right) = \frac{9 - \gamma^2 - \gamma\sqrt{18 + \gamma^2}}{9}, \tag{2.13}$$

since $0 \le y \le 1$.

We first square both sides of the inequality (2.11), substitute the value of y^2 from (2.13), and get the equivalent inequality :

$$(3-2\gamma)\left(9-\gamma^2-\gamma\sqrt{18+\gamma^2}\right)\left(\sqrt{18+\gamma^2}+5\gamma\right)^2 \leq 8(3+2\gamma)(3-\gamma)^4.$$

For simplicity of our calculations, we put $t=\sqrt{18+\gamma^2}$; then (after a routine calculation) this becomes

$$18t\gamma(3-2\gamma)\left(2-\gamma^2\right) \leq 729-486\gamma-270\gamma^2+324\gamma^3-30\gamma^4-28\gamma^5. \tag{2.14}$$

The right-hand side of (2.14) turns out to be positive, since

$$\begin{aligned}&729-486\gamma-270\gamma^2+324\gamma^3-30\gamma^4-28\gamma^5\\&\quad=(1-\gamma)\left(729+243\gamma-27\gamma^2+297\gamma^3+267\gamma^4\right)+239\gamma^5\\&\quad>0.\end{aligned}$$

Thus, squaring (2.14) once again, we get the equivalent inequality:

$$\begin{aligned}&324\left(18+\gamma^2\right)\gamma^2\left(36-48\gamma-20\gamma^2+48\gamma^3-7\gamma^4-12\gamma^5+4\gamma^6\right)\\&\quad\leq 729^2-2\cdot729\cdot486\gamma+\left(486^2-2\cdot729\cdot270\right)\gamma^2+\left(2\cdot729\cdot342+2\cdot486\cdot270\right)\gamma^3\\&\qquad+\left(270^2-2\cdot729\cdot30-2\cdot486\cdot324\right)\gamma^4+(-2)\cdot729\cdot28\\&\qquad+2\cdot486\cdot30-2\cdot270\cdot324)\gamma^5+\left(324^2+2\cdot486\cdot28+2\cdot270.30\right)\gamma^6\\&\qquad+(2\cdot270\cdot28-2\cdot324\cdot30)\gamma^7+(900-2\cdot324\cdot28)\gamma^8+60\cdot28\gamma^9+784\gamma^{10}.\end{aligned}$$

that is,

$$\begin{aligned}&324\gamma^2\left(648-864\gamma-324\gamma^2+816\gamma^3-146\gamma^4-168\gamma^5+65\gamma^6-12\gamma^7+4\gamma^8\right)\\&\quad\leq 729^2-2\cdot729\cdot486\gamma+\left(486^2-2\cdot729\cdot270\right)\gamma^2+(2\cdot729\cdot324+2\cdot486\cdot270)\gamma^3\\&\qquad+\left(270^2-2\cdot729\cdot30-2\cdot486\cdot324\right)\gamma^4+(-2\cdot729\cdot28\\&\qquad+2\cdot486\cdot30-2.270\cdot324)\gamma^5+\left(324^2+2\cdot486\cdot28+2\cdot270\cdot30\right)\gamma^6\\&\qquad+(2\cdot270\cdot28-2\cdot324\cdot30)\gamma^7+(900-2\cdot324.28)\gamma^8+60\cdot28\gamma^9+784\gamma^{10}.\end{aligned}$$

Thus the reformulated inequality is given by

$$\begin{aligned}&729^2-2\cdot729\cdot486\gamma+\left(486^2-2\cdot729\cdot270-324\cdot648\right)\gamma^2\\&+(2\cdot729\cdot324+2\cdot486\cdot270+324\cdot864)\gamma^3+\left(270^2-2\cdot729\cdot30\right.\\&\left.-2\cdot486\cdot324+324^2\right)\gamma^4+(-2)\cdot729\cdot28+2\cdot486\cdot30-2\cdot270\cdot324\\&-324\cdot816\gamma^5+\left(324^2+2\cdot486\cdot28+2\cdot270\cdot30+324\cdot146\right)\gamma^6\\&+(2\cdot270\cdot28-2\cdot324\cdot30+324\cdot168)\gamma^7(900-2\cdot324\cdot28-324\cdot65)\gamma^8\\&+(1680+324\cdot12)\gamma^9+(784-324\cdot4)\gamma^{10}\geq 0.\end{aligned} \tag{2.15}$$

Now the left-hand side of (2.14) is

$$\begin{aligned}
&531441 - 708588\gamma - 367416\gamma^2 + 1014768\gamma^3 - 180792\gamma^4 - 451008\gamma^5 \\
&\qquad + 195696\gamma^6 + 50112\gamma^7 - 38304\gamma^8 + 5568\gamma^9 - 512\gamma^{10} \\
&\quad = 531441(1-\gamma) - 171147\gamma(1-\gamma) - 544563\gamma^2(1-\gamma) + 470205\gamma^3(1-\gamma) \\
&\qquad + 289413\gamma^4(1-\gamma) - 161595\gamma^5(1-\gamma) + 34101\gamma^6(1-\gamma) + 84213\gamma^7(1-\gamma) \\
&\qquad + 45909\gamma^8(1-\gamma) + 51477\gamma^9(1-\gamma) + 50965\gamma^{10} \\
&\quad = (1-\gamma)(531441 - 177147\gamma - 544563\gamma^2 + 470205\gamma^3 + 289413\gamma^4 - 16595\gamma^5 \\
&\qquad + 34101\gamma^6 + 84213\gamma^7 + 45909\gamma^8 + 51477\gamma^9 0 + 50965\gamma^{10} \\
&\quad = (1-\gamma)\left[(1-\gamma)(531441 + 354294\gamma - 190269\gamma^2 + 161595\gamma^4) + 279936\gamma^3\right. \\
&\qquad \left. + 127818\gamma^4 + 34101\gamma^6 + 84213\gamma^7 + 45909\gamma^8 + 51477\gamma^9\right] + 50965\gamma^{10} \\
&\quad = (1-\gamma)[(1-\gamma)\{531441 + 164025\gamma + 190260\gamma(1-\gamma) + 161595\gamma^4\} \\
&\qquad + 279936\gamma^3 + 127818\gamma^4 + 34101\gamma^6 + 84213\gamma^7 + 45909\gamma^9 + 51477\gamma^9] + 59065\gamma^{10} \\
&\quad > 0.
\end{aligned}$$

Hence (2.15) is true. This completes the proof of the main assertion of Theorem 3. The result can be shown to be sharp by setting $b_2 = b_3 = 1$, $\beta = 0$, $\alpha_1 = \dfrac{2-3\lambda}{6\lambda}$, and $\alpha_2 = 1 - \alpha_1^2$ in (1.12).

Proof of Theorem 4. Putting $\lambda = \dfrac{2}{3}$ in (1.12), we get

$$a_3 - \frac{2}{3}a_2^2 = \frac{1}{3}\left(b_3 - \frac{1}{2}b_2^2\right) + \frac{2}{3}e^{-i\beta}\left[\alpha_2 + \left(1 - e^{-i\beta}\cos\beta\right)\alpha_1^2\right]\cos\beta.$$

Hence

$$\left|a_3 - \frac{2}{3}a_2^2\right| \le \frac{1}{3}\left|b_3 - \frac{1}{2}b_2^2\right| + \frac{2}{3}\left|\alpha_2 - \left(e^{-i\beta}\cos\beta - 1\right)\alpha_1^2\right|\cos\beta.$$

Using Theorem A and Theorem B, we get

$$\begin{aligned}
\left|a_3 - \frac{2}{3}a_2^2\right| &\le \frac{1}{3}\cdot\frac{1}{2} + \frac{2}{3}\left[1 + \left(\left|e^{-i\beta}\cos\beta - 1\right| - 1\right)|\alpha_1|^2\right]\cos\beta \\
&= \frac{1}{6} + \frac{2}{3}\left[1 + \left(\sqrt{1-\cos^2\beta} - 1\right)|\alpha_1|^2\right]\cos\beta \\
&= \frac{1}{6} + \frac{2}{3}\cos\beta + \frac{2}{3}\left(|\sin\beta| - 1\right)|\alpha_1|^2\cos\beta \\
&\le \frac{5}{6} - \frac{2}{3}\left(1 - |\sin\beta|\right)|\alpha_1|^2\cos\beta \\
&\le \frac{5}{6},
\end{aligned}$$

which completes the proof of Theorem 4.

Proof of Theorem 5. Putting $\lambda = 1$ in (1.12), we get

$$a_3 - a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}b_2^2\right) + \frac{2}{3}e^{-i\beta}\left[\alpha_2 + \left(1 - \frac{3}{2}e^{-i\beta}\cos\beta\right)\alpha_1^2\right]\cos\beta$$
$$- \frac{1}{3}\alpha_1 b_2 e^{-i\beta}\cos\beta.$$

Hence

$$|a_3 - a_2^2| \le \frac{1}{3}|b_3 - b_2^2| + \frac{|b_2|^2}{12} + \frac{2}{3}\left|\alpha_2 - \left(\frac{3}{2}e^{-i\beta}\cos\beta - 1\right)\alpha_1^2\right| + \frac{|\alpha_1 b_2|}{3}\cos\beta. \tag{2.16}$$

Using Theorem B and a result of Trimble [18], we find from (2.16) that

$$\begin{aligned}|a_3 - a_2^2| &\le \frac{1}{3}\left(\frac{1-|b_2|^2}{3}\right) + \frac{|b_2|^2}{12} + \frac{2}{3} + \frac{2}{3}\left|\left(\frac{3}{2}\cos^2\beta - 1\right.\right.\\ &\quad \left.\left.- \frac{3}{2}i\sin\beta\cos\beta\right| - 1\right)|\alpha_1|^2 + \frac{|\alpha_1 b_2|}{3}\\ &= \frac{7}{9} + \frac{|b_2|^2}{18} - \frac{|b_2|^2}{12} + \frac{2}{3}\left(\sqrt{1 - \frac{3}{4}\cos^2\beta} - 1\right)|\alpha_1|^2 + \frac{|\alpha_1 b_2|}{3}\\ &= \frac{7}{9} + \frac{|b_2|^2}{18} - \frac{1}{3}\left[\frac{|b_2|^2}{4} - 2\left(\sqrt{1 - \frac{3}{4}\cos^2\beta} - 1\right)|\alpha_1|^2 - |\alpha_1 b_2|\right].\end{aligned} \tag{2.17}$$

The elementary inequality:

$$1 - \frac{3}{4}\cos^2\beta \ge \frac{1}{4}$$

immediately yields

$$\sqrt{1 - \frac{3}{4}\cos^2\beta} \ge \frac{1}{2}.$$

Thus we find from (2.17) that

$$\begin{aligned}|a_3 - a_2^2| &\le \frac{7}{9} + \frac{1}{18} - \frac{1}{3}\left(\frac{|b_2|^4}{4} + |\alpha_1|^2 - |\alpha_1 b_2|\right)\\ &= \frac{5}{6} - \frac{1}{3}\left(\frac{|b_2|}{2} - |\alpha_1|\right)^2\\ &\le \frac{5}{6}.\end{aligned}$$

This completes the proof of Theorem 5.

Proof of Theorem 6. Observe that

$$a_3 - \lambda a_2^2 = (3\lambda - 2)\left(a_3 - a_2^2\right) + 3(1-\lambda)\left(a_3 - \frac{2}{3}a_2^2\right).$$

The main assertion of Theorem 6 follows from Theorem 4 and Theorem 5.

Equality can be shown to hold true by setting $b_2 = b_3 = 1$, $\alpha_2 = 1 - \alpha_1^2$, $\beta = 0$, and

$$\alpha_1 = \frac{(2-3\lambda) \pm i\sqrt{6\lambda - 4}}{6\lambda}$$

in (1.12).

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