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THE FENCHEL-MOREAU THEOREM FOR SET FUNCTIONS

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ABSTRACT. The Fenchel-Moreau theorem for set functions is proved, and some properties of subdifferential and conjugate functional of set functions are investigated.

1. Introduction. Let (X, Γ, μ) be a finite atomless measure space, and F a proper real-valued set function defined on Γ (i.e. $F(\Omega) > -\infty$ for all $\Omega \in \Gamma$ and $F \not\equiv \infty$). Let $\operatorname{Dom} F = \{\Omega \in \Gamma; F(\Omega) \text{ is finite}\}$. The conjugate function F^* of F is defined by

$$(1) \qquad F^*(f) = \sup_{\Omega \in \Gamma}[\langle f, \chi_\Omega \rangle - F(\Omega)], \qquad f \in L_1(X, \Gamma, \mu),$$

and the biconjugate function F^{**} of F is defined by

$$(2) \qquad F^{**}(\Omega) = \begin{cases} \sup_{f \in L_1(X,\Gamma,\mu)}[\langle f, \chi_\Omega \rangle - F^*(f)] & \text{if } \Omega \in \operatorname{Dom} F, \\ +\infty & \text{if } \Omega \notin \operatorname{Dom} F. \end{cases}$$

By the definitions of F^* and F^{**}, we easily get that

$$(3) \qquad F^{**}(\Omega) \le F(\Omega) \quad \text{for all } \Omega \in \Gamma.$$

The question arises that under what conditions, the equality in (3) holds. In [**1**] the classical Fenchel-Moreau theorem shows that a function g, defined on a topological vector space U, is convex and lower semicontinuous if and only if $g(x) = g^{**}(x)$ for all $x \in U$.

It is known that the Fenchel-Moreau theorem plays an important role in the theory of optimization; many authors investigate this theorem in more general cases, for example, one can consult Lai [**6**], Koshi and Komuro [**7**], Koshi, Lai, and Komuro [**8**], and Zowe [**12**]. All of these papers showed that the Fenchel-Moreau theorem holds for the functions defined on linear spaces. In this note, the function is considered on a σ-algebra Γ of a measure space rather than on a linear space. There is a good deal of difference between the Fenchel-Moreau theorem for the set function on a σ-algebra and for the usual function on a linear space. In this note, some properties of subdifferential and conjugate functions of set functions are also established.

2. Preliminaries. Throughout this note, we assume that (X, Γ, μ) is a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable. Under these assumptions, for any $\Omega \in \Gamma$ with $\mu(\Omega) > 0$, there exist a measurable set $\Lambda \subset \Omega$ with $\mu(\Lambda) > 0$, and

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a countable sequence $\{\Omega_n\}$ in Γ such that $\{c_n\chi_{\Omega_n}\}$ is dense in $L_1(\Omega, \Gamma, \mu)$, where $c_n \in \mathbf{R}$, and χ_{Ω_n} is the characteristic function of $\Omega_n \in \Gamma$.

DEFINITION 2.1. A set function $F\colon \Gamma \to \mathbf{R}$ is called *convex* if for any given $\lambda \in [0,1]$ and $\Omega, \Lambda \in \Gamma$, there exist sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ with $\chi_{\Omega_n} \xrightarrow{w^*} \lambda\chi_{\Omega\setminus\Lambda}$ and $\chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda\setminus\Omega}$ such that

$$\overline{\lim_{n\to\infty}} F(\Omega_n \cup \Lambda_n \cup (\Omega\cap\Lambda)) \le \lambda F(\Omega) + (1-\lambda)F(\Lambda),$$

where $\xrightarrow{w^*}$ stands for convergence in the weak*-topology. Since (X, Γ, μ) is a finite measure space, any set $\Omega \in \Gamma$ can be identified with a characteristic function χ_Ω in $L_1(\chi, \Gamma, \mu)$. Hence one can regard Γ as a subset $\chi_\Gamma = \{\chi_\Omega \colon \Omega \in \Gamma\}$ of $L_1(X, \Gamma, \mu)$. For each $f \in L_1(X, \Gamma, \mu)$, the integral $\int_\Omega f\, d\mu$ is identified with the dual pair $\langle f, \chi_\Omega\rangle$.

DEFINITION 2.2 [2]. A subfamily $\mathscr{S}$ of measurable subsets in Γ is *convex* if for any $(\Omega, \Lambda, \lambda) \in \mathscr{S} \times \mathscr{S} \times [0,1]$, associated with sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ in Γ with $\chi_{\Omega_n} \xrightarrow{w^*} \lambda\chi_{\Omega\setminus\Lambda}$ and $\chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda\setminus\Omega}$, there exist subsequences $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ and $\{\Lambda_{n_k}\}$ of $\{\Lambda_n\}$ such that $\Omega_{n_k} \cup \Lambda_{n_k} \cup (\Omega\cap\Lambda) \in \mathscr{S}$ for all k.

DEFINITION 2.3. Let $\mathscr{S} \subset \Gamma$ be a convex subfamily of measurable sets. A set function $F\colon \mathscr{S} \to \mathbf{R}$ is called *convex* on $\mathscr{S}$ if for any given $\lambda \in [0,1]$ and $\Omega, \Lambda \in \mathscr{S}$, the following inequality holds:

$$\overline{\lim_{n\to\infty}} F(\Omega_n \cup \Lambda_n \cup (\Omega\cap\Lambda)) \le \lambda F(\Omega) + (1-\lambda)F(\Lambda)$$

for any sequences $\{\Omega_n\}$, $\{\Lambda_n\}$ with $\Omega_n \cup \Lambda_n \cup (\Omega\cap\Lambda) \in \mathscr{S}$ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda\chi_{\Omega\setminus\Lambda}, \qquad \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda\setminus\Omega}.$$

We define w^*-lower (-upper) semicontinuous and w^*-continuous of a set function at a point Ω in $\operatorname{Dom} F$ as follows:

DEFINITION 2.4. Let $F\colon \Gamma \to \mathbf{R}\cup\{+\infty\}$ be a set function with $\operatorname{Dom} F = \mathscr{S} \subset \Gamma$.

(i) F is called *w^*-lower semicontinuous* at $\Omega \in \mathscr{S}$ if

$$-\infty < F(\Omega) \le \underline{\lim_{n\to\infty}} F(\Omega_n)$$

for any sequence $\Omega_n \in \mathscr{S}$ with $\chi_{\Omega_n} \xrightarrow{w^*} \chi_\Omega$.

(ii) F is called *w^*-upper semicontinuous* at $\Omega \in \mathscr{S}$ if

$$\overline{\lim_{n\to\infty}} F(\Omega_n) \le F(\Omega) < \infty$$

for any sequence $\Omega_n \in \mathscr{S}$ with $\chi_{\Omega_n} \xrightarrow{w^*} \chi_\Omega$.

(iii) F is called *w^*-continuous* at $\Omega \in \mathscr{S}$ if

$$F(\Omega) = \lim_{n\to\infty} F(\Omega_n)$$

for any sequence $\{\Omega_n\}$ in $\mathscr{S}$ with $\chi_{\Omega_n} \xrightarrow{w^*} \chi_\Omega$.

DEFINITION 2.5 [**4**]. Let $\mathscr{S}$ be a convex subfamily of subsets in Γ and $F\colon \mathscr{S} \to \mathbf{R}$ a set function; a set $[F, \mathscr{S}]$ in $\mathbf{R} \times L_1(X, \Gamma, \mu)$, defined by

$$[F,\mathscr{S}] = w^*\text{-closure}\Big\{(\gamma, \chi_\Omega) \in \mathbf{R} \times L_\infty(X,\Gamma,\mu) \,|\, \Omega \in \mathscr{S} \text{ and } \overline{\lim_{n\to\infty}} F(\Omega_n) \le \gamma \\ \text{for any sequence } \Omega_n \in \mathscr{S} \text{ with } \chi_{\Omega_n} \xrightarrow{w^*} \chi_\Omega\Big\},$$

is called the *epigraph* of F on $\mathscr{S}$.

DEFINITION 2.6 [**5**]. An element $f \in L_1(X, \Gamma, \mu)$ is called a *subgradient* of a convex set function G at $\Omega_0 \in \Gamma$ if it satisfies the inequaltiy

$$G(\Omega) \geq G(\Omega_0) + \langle f, \chi_\Omega - \chi_{\Omega_0}\rangle \quad \text{for all } \Omega \in \Gamma.$$

Note that the subgradient of a convex set function at a point Ω_0 is not unique; usually it is a set of the following form:

$$\partial G(\Omega_0) = \{f \in L_1(X, \Gamma, \mu) \colon G(\Omega) \geq G(\Omega_0) + \langle f, \chi_\Omega - \chi_{\Omega_0}\rangle \text{ for all } \Omega \in \Gamma\}.$$

The set $\partial G(\Omega_0)$ is called the *subdifferential* of G at Ω_0, and if $\partial G(\Omega_0) \neq \emptyset$, G is said to be *subdifferential* at Ω_0. The subdifferential of a conjugate functional G^* at $f_0 \in L_1(X, \Gamma, \mu)$ is defined by a subfamily of measurable subsets in Γ as follows:

$$\partial G^*(f_0) = \{\Omega \in \Gamma \colon G^*(f) \geq G^*(f_0) + \langle f - f_0, \chi_\Omega\rangle \text{ for all } f \in L_1(X, \Gamma, \mu)\}.$$

It is remarkable that ∂G^* is some different from [**5**].

3. Main results.

LEMMA 3.1 [**4**, (3)]. *If, for any given sets Ω, $\Lambda \in \Gamma$, $\lambda \in [0,1]$, and $L_\infty(X, \Gamma, \mu)$-sequences $\{\chi_{\Omega_n}\}$ and $\{\chi_{\Lambda_n}\}$,*

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Omega \setminus \Lambda}, \qquad \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda \setminus \Omega},$$

then

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_\Omega + (1-\lambda)\chi_\Lambda. \tag{4}$$

A subset $A \subset \mathbf{R} \times \chi_\Gamma$ is said to be *convex* if, for (r, χ_Ω), $(s, \chi_\Lambda) \in A$ and $\lambda \in [0,1]$, there exist sequences $V_n(\lambda) = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)$ in Γ and t_n in $\mathbf{R}$ such that (4) holds and $t_n \to \lambda r + (1-\lambda)s$.

LEMMA 3.2 [**4**, PROPOSITION 1]. *Let $\mathscr{S}$ be a convex subset of Γ and $F \colon \mathscr{S} \to \mathbf{R}$ a convex set function. Then $[F, \mathscr{S}]$ is convex.*

We modify the proof of this lemma given in [**4**] as follows:

PROOF. Let

$$A = \{(r, \chi_\Omega) | \Omega \in \mathscr{S} \text{ and } \overline{\lim} F(\Omega_n) \leq r\},$$

where $\{\Omega_n\}$ is any sequence in $\mathscr{S}$ such that $\chi_{\Omega_n} \xrightarrow{w^*} \chi_\Omega$. Then A is convex. In fact, for (r, χ_Ω), (s, χ_Λ) in A and $\lambda \in [0,1]$, since F is convex on the convex subfamily $\mathscr{S}$, there exists a sequence $V_n(\lambda) = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)$ in $\mathscr{S}$ associated with $(\Omega, \Lambda, \lambda) \in \mathscr{S} \times \mathscr{S} \times [0,1]$ such that

$$\overline{\lim_{n \to \infty}} F(V_n) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda) \leq \lambda r + (1-\lambda)s.$$

Thus there is a subsequence $\{V_{n_i}\}$ of $\{V_n\}$ such that

$$F(V_{n_i}(\lambda)) \leq \lambda r + (1-\lambda)s + 1/i \equiv t_i,$$

say. Hence for any sequence $\{U_k\}$ in $\mathscr{S}$ with $U_k \xrightarrow{w^*} V_{n_i}$, and from the w^*-upper semicontinuity of F when F is convex (see Lemma 3.3), we have $\overline{\lim}_{k \to \infty} F(U_k) \leq F(V_{n_i}) \leq t_i$. So $(t_i, V_{n_i}) \in A$ and $\chi_{V_{n_i}} \to \lambda \chi_\Omega + (1-\lambda)\chi_\Lambda$, $t_i \to \lambda r + (1-\lambda)s$ (as $i \to \infty$). This shows that A is convex. It follows that the w^*-closure convex hull of A coincides with the w^*-closure $\overline{A} = [\mathscr{S}, F]$ of A in $\mathbf{R} \times L^\infty$. Hence $[\mathscr{S}, F]$ is convex. Q.E.D.

LEMMA 3.3. *Let $\mathscr{S} \subset \Gamma$ be a convex subfamily in Γ. Then any convex set function $F: \mathscr{S} \to \mathbf{R}$ is w^*-upper semicontinuous on $\mathscr{S}$.*

PROOF. For any $\Omega \in \mathscr{S}$, let $\{\Omega_n\}$ be a sequence in $\mathscr{S}$ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \chi_\Omega = \chi_{\Omega \backslash \varnothing},$$

where $\varnothing$ is the empty set. Let each $\Lambda_n = \varnothing$. Then $\chi_{\Lambda_n} \xrightarrow{w^*} (1-1)\chi_{\varnothing \backslash \Omega}$. It follows from Lemma 3.1 that

$$\begin{aligned}\overline{\lim_{n\to\infty}} F(\Omega_n) &= \overline{\lim_{n\to\infty}} F(\Omega_n \cup \Lambda_n \cup (\Omega \cap \varnothing)) \\ &\leq F(\Omega) + (1-1)F(\varnothing) = F(\Omega).\end{aligned}$$

This shows that F is w^*-upper semicontinuous on $\mathscr{S}$. Q.E.D.

From Lemma 3.3, the following corollary is immediate.

COROLLARY 3.4. *Let $\mathscr{S} \subseteq \Gamma$ be a convex subfamily of subsets. Then any w^*-lower semicontinuous set function $F: \mathscr{S} \to \mathbf{R}$ is w^*-continuous.*

THEOREM 3.5. *Let $F: \Gamma \to \mathbf{R} \cup \{\infty\}$ be a proper convex set function on Γ, w^*-lower semicontinuous on its convex domain $\mathscr{S}$. Then $\partial F(\Omega) \neq \varnothing$ and $\operatorname{Dom} F^* \neq \varnothing$.*

PROOF. It follows from Corollary 3.4 that F is w^*-continuous on $\mathscr{S}$. Thus for any $\Lambda \in \mathscr{S}$, there is a sequence Λ_n in $\mathscr{S}$ with $\chi_{\Lambda_n} \xrightarrow{w^*} \chi_\Lambda$ such that $\overline{\lim}_{n\to\infty} F(\Lambda_n) = \lim_{n\to\infty} F(\Lambda_n) = F(\Lambda)$ and $(F(\Lambda), \chi_\Lambda) \in [F, \mathscr{S}]$ for all $\Lambda \in \mathscr{S}$. In view of Lemma 3.2 and the definition of epigraph, it has been proved that the epigraph $[F, \mathscr{S}]$ of F is convex and w^*-closed. Since for any $r < F(\Omega)$, $(r, \chi_\Omega) \notin [F, \mathscr{S}]$, and from the separation theorem, it follows that there exists a nonzero continuous linear functional $(-\alpha, f) \in \mathbf{R} \times L_1(X, \Gamma, \mu)$ such that

$$\langle f, \chi_\Lambda \rangle - \alpha C < \langle f, \chi_\Omega \rangle - \alpha r \quad \text{for all } (C, \chi_\Lambda) \in [F, \mathscr{S}]. \tag{5}$$

In particular, when $\Lambda = \Omega$ and $C = F(\Omega)$, it is deduced that $\alpha(F(\Omega) - r) > 0$. Since $r < F(\Omega)$ is arbitrary, $\alpha > 0$. Hence

$$\langle f, \chi_\Lambda \rangle - \alpha C \leq \langle f, \chi_\Omega \rangle - \alpha F(\Omega) \quad \text{for all } (C, \chi_\Lambda) \in [F, \mathscr{S}].$$

Take $C = F(\Lambda)$. When both sides of the inequality (5) are divided by $\alpha > 0$, we then obtain

$$\langle f/\alpha, \chi_\Lambda \rangle - F(\Lambda) \leq \langle f/\alpha, \chi_\Omega \rangle - F(\Omega) \quad \text{for all } \Lambda \in \mathscr{S}. \tag{6}$$

As F is a proper set function, inequality (6) holds for all $\Omega \in \Gamma$. By taking the supremum over all $\Lambda \in \Gamma$, we get

$$F^*(f/\alpha) \leq \langle f/\alpha, \chi_\Omega \rangle - F(\Omega).$$

Since (from (6))

$$F(\Lambda) \geq F(\Omega) + \langle f/\alpha, \chi_\Lambda - \chi_\Omega \rangle \quad \text{for all } \Lambda \in \Gamma,$$

it follows that $f/\alpha \in \partial F(\Omega) \neq \varnothing$ and $f/\alpha \in \operatorname{Dom} F^* \neq \varnothing$. Q.E.D.

The following theorem is the Fenchel-Moreau theorem for set functions.

THEOREM 3.6. *Let F be a proper convex w^*-lower semicontinuous set function on its convex domain $\mathscr{S}$. Then $F(\Omega) = F^{**}(\Omega)$ for all $\Omega \in \Gamma$.*

PROOF. It follows from Corollary 3.4 that F is w^*-continuous on $\mathscr{S}$. A similar argument with the proof of Theorem 3.5, we see that $(F(\Omega), \chi_\Omega) \in [F, \mathscr{S}]$ for all $\Omega \in \mathscr{S}$.

For any $r < F(\Omega)$, then the pair $(r, \chi_\Omega) \notin [F, \mathscr{S}]$. Since $\mathscr{S}$ is a convex subfamily of Γ and $F: \mathscr{S} \to \mathbf{R}$ is a convex set function, $[F, \mathscr{S}]$ is a convex w^*-closed subset of $\mathbf{R} \times L_\infty(X, \Gamma, \mu)$. Applying the separation theorem, we can find a nonzero functional $(\alpha, f) \in \mathbf{R} \times L_1(X, \Gamma, \mu)$ which strictly separates the point (r, χ_Ω) and the epigraph $[F, \mathscr{S}]$. Thus there exists $\varepsilon > 0$ such that

$$\sup_{(\lambda, \chi_\Omega) \in [F, \mathscr{S}]} \langle (\alpha, f), (\lambda, \chi_\Lambda) \rangle \leq \langle (\alpha, f), (r, \chi_\Omega) \rangle - \varepsilon.$$

It follows that

$$\langle f, \chi_\Lambda \rangle + \lambda\alpha \leq \langle f, \chi_\Omega \rangle + \alpha r - \varepsilon \tag{7}$$

for $\Lambda \in \mathscr{S}$, $\lambda \geq F(\Lambda)$. Note that $\alpha \leq 0$; for otherwise letting $\lambda \to \infty$, it would reduce to a contradiction. Actually, $\alpha < 0$. For if $\Omega \in \operatorname{Dom} F$, then letting $\Lambda = \Omega$ and $\lambda = F(\Omega)$, we reduce from (7) that $-\alpha(F(\Omega) - r) \geq \varepsilon > 0$. Since $r < F(\Omega)$, it follows that $\alpha < 0$. Next, let $\lambda = F(\Lambda)$ and then divide both sides of (7) by $-\alpha$. We obtain

$$\langle \bar{f}, \chi_\Lambda \rangle - F(\Lambda) \leq \langle \bar{f}, \chi_\Omega \rangle - r + \varepsilon/\alpha,$$

where $\bar{f} = f/-\alpha$. By taking the supremum over $\Lambda \in \Gamma$, we obtain

$$F^*(\bar{f}) \leq \langle \bar{f}, \chi_\Omega \rangle - r + \varepsilon/\alpha < \infty,$$

and so $\bar{f} \in \operatorname{Dom} F^*$. Thus

$$r < r - \varepsilon/\alpha \leq \langle \bar{f}, \chi_\Omega \rangle - F^*(\bar{f}) \leq F^{**}(\Omega).$$

This shows that for any $r < F(\Omega)$, we have

$$r < F^{**}(\Omega). \tag{8}$$

As we take $r = F(\Omega) - \delta$ for any given $\delta > 0$, we have

$$r = F(\Omega) - \delta < F(\Omega).$$

It follows from inequality (8) that

$$F(\Omega) - \delta < F^{**}(\Omega) \quad \text{or} \quad F(\Omega) < F^{**}(\Omega) + \delta.$$

Since δ is arbitrary,

$$F(\Omega) \leq F^{**}(\Omega) \quad \text{for all } \Omega \in \mathscr{S}. \tag{9}$$

Consequently, from (9) and (3), we obtain

$$F^{**}(\Omega) = F(\Omega) \quad \text{for all } \Omega \in \operatorname{Dom} F = \mathscr{S}.$$

While $\Omega \notin \operatorname{Dom} F = \mathscr{S}$, $F(\Omega) = F^{**}(\Omega) = \infty$. Hence

$$F(\Omega) = F^{**}(\Omega) \quad \text{for all } \Omega \in \Gamma. \quad \text{Q.E.D.}$$

By the same argument used in [**5**], the following lemma is immediate.

LEMMA 3.7. *Let F be a convex set function and F^* the conjugate function of F. Let $f_0 \in \operatorname{Dom} F^*$ and $\Omega_0 \in \operatorname{Dom} F$. Then*

$$f_0 \in \partial F(\Omega_0) \quad \textit{if and only if} \quad F(\Omega_0) + F^*(f_0) = \langle f_0, \chi_{\Omega_0} \rangle.$$

Using Theorem 3.6, we obtain

THEOREM 3.8. *Let F be proper convex w^*-lower semicontinuous set function on its convex domain $\mathscr{S}$. If $f_0 \in \operatorname{Dom} F^*$ and $\Omega_0 \in \mathscr{S}$, then*

$$f_0 \in \partial F(\Omega_0) \quad \textit{if and only if} \quad \Omega_0 \in \partial F^*(f_0).$$

PROOF. It follows from Lemma 3.7 and Theorem 3.6 that

$$f_0 \in \partial F(\Omega_0) \quad \text{if and only if} \quad F(\Omega_0) + F^*(f_0) = \langle f_0, \chi_{\Omega_0} \rangle,$$

that is, $F^{**}(\Omega_0) + F^*(f_0) = \langle f_0, \chi_{\Omega_0} \rangle$ and $\Omega_0 \in \partial F^*(f_0)$. Q.E.D.

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