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The Fibonacci Numbers -- Exposed More Discretely

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In the previous article, Kalman and Mena [**5**] propose that Fibonacci and Lucas sequences, despite the mathematical favoritism shown them for their abundant patterns, are nothing more than ordinary members of a class of super sequences. Their arguments are beautiful and convinced us to present the same material from a more *discrete* perspective. Indeed, we will present a simple combinatorial context encompassing nearly all of the properties discussed in [**5**].

As in the Kalman-Mena article, we generalize Fibonacci and Lucas numbers: Given nonnegative integers a and b, the *generalized Fibonacci sequence* is

$$F_0 = 0, \quad F_1 = 1, \quad \text{and for } n \geq 2, \quad F_n = aF_{n-1} + bF_{n-2}. \tag{1}$$

The *generalized Lucas sequence* is

$$L_0 = 2, \quad L_1 = a, \quad \text{and for } n \geq 2, \quad L_n = aL_{n-1} + bL_{n-2}.$$

When $a = b = 1$, these are the celebrity Fibonacci and Lucas sequences. For now, we will assume that a and b are nonnegative integers. But at the end of the article, we will see how our methods can be extended to noninteger values of a and b.

Kalman and Mena prove the following generalized Fibonacci identities

$$F_n = F_m F_{n-m+1} + bF_{m-1}F_{n-m} \tag{2}$$

$$(a + b - 1)\sum_{i=1}^{n} F_i = F_{n+1} + bF_n - 1 \tag{3}$$

$$a\left(b^n F_0^2 + b^{n-1}F_1^2 + \cdots + bF_{n-1}^2 + F_n^2\right) = F_n F_{n+1} \tag{4}$$

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n b^{n-1} \tag{5}$$

$$\gcd(F_n, F_m) = F_{\gcd(n,m)} \tag{6}$$

$$L_n = aF_n + 2bF_{n-1} \tag{7}$$

$$L_n = F_{n+1} + bF_{n-1} \tag{8}$$

using the tool of difference operators acting on the real vector space of real sequences. In this paper, we offer a purely combinatorial approach to achieve the same results. We hope that examining these identities from different perspectives, the reader can more fully appreciate the unity of mathematics.

*Editor's Note: Readers interested in clever counting arguments will enjoy reading the authors' upcoming book, *Proofs That Really Count: The Art of Combinatorial Proof*, published by the MAA.

Fibonacci numbers—The combinatorial way

There are many combinatorial interpretations for Fibonacci and Lucas numbers [3]. We choose to generalize the "square and domino tiling" interpretation here. We show that the classic Fibonacci and Lucas identities naturally generalize to the (a, b) recurrences simply by adding a splash of color.

For nonnegative integers a, b, and n, let f_n count the number of ways to tile a $1 \times n$ board with 1×1 colored squares and 1×2 colored dominoes, where there are a color choices for squares and b color choices for dominoes. We call these objects *colored n-tilings*. For example, $f_1 = a$ since a length 1 board must be covered by a colored square; $f_2 = a^2 + b$ since a board of length 2 can be covered with two colored squares or one colored domino. Similarly, $f_3 = a^3 + 2ab$ since a board of length 3 can be covered by 3 colored squares or a colored square and a colored domino in one of 2 orders. We let $f_0 = 1$ count the empty board. Then for $n \geq 2$, f_n satisfies the generalized Fibonacci recurrence

$$f_n = af_{n-1} + bf_{n-2},$$

since a board of length n either ends in a colored square preceded by a colored $(n-1)$-tiling (tiled in af_{n-1} ways) or a colored domino preceded by a colored $(n-2)$-tiling (tiled in bf_{n-2} ways.) Since $f_0 = 1 = F_1$ and $f_1 = a = F_2$, we see that for all $n \geq 0$, $f_n = F_{n+1}$. After defining $f_{-1} = 0$, we now have a combinatorial definition for the generalized Fibonacci numbers.

THEOREM 1. *For $n \geq 0$, $F_n = f_{n-1}$ counts the number of colored $(n-1)$-tilings (of a $1 \times (n-1)$ board) with squares and dominoes where there are a colors for squares and b colors for dominoes.*

Using Theorem 1, equations (2) through (6) can be derived and appreciated combinatorially. In most of these, our combinatorial proof will simply ask a question and answer it two different ways.

For instance, if we apply Theorem 1 to equation (2) and reindex by replacing n by $n+1$ and m by $m+1$, we obtain

IDENTITY 1. *For $0 \leq m \leq n$,*

$$f_n = f_m f_{n-m} + bf_{m-1} f_{n-m-1}.$$

Question: How many ways can a board of length n be tiled with colored squares and dominoes?

Answer 1: By Theorem 1, there are f_n colored n-tilings.

Answer 2: Here we count how many colored n-tilings are *breakable* at the m-th cell and how many are not. To be breakable, our tiling consists of a colored m-tiling followed by a colored $(n-m)$-tiling, and there are $f_m f_{n-m}$ such tilings. To be *unbreakable* at the m-th cell, our tiling consists of a colored $(m-1)$-tiling followed by a colored domino on cells m and $m+1$, followed by a colored $(n-m-1)$-tiling; there are $bf_{m-1} f_{n-m-1}$ such tilings. Altogether, there are $f_m f_{n-m} + bf_{m-1} f_{n-m-1}$ colored n-tilings.

Since our logic was impeccable for both answers, they must be the same. The advantage of this proof is that it makes the identity *memorable* and visualizable. See FIGURE 1 for an illustration of the last proof.

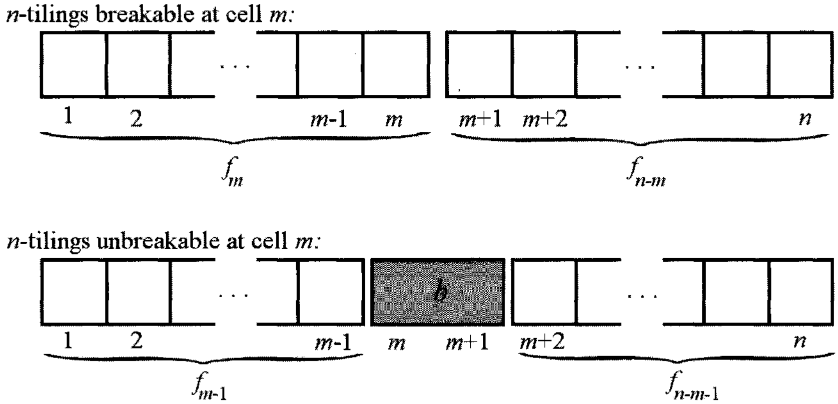

Figure 1 A colored n-tiling is either breakable or unbreakable at cell m

Equation (3) can be rewritten as the following identity.

IDENTITY 2. *For* $n \geq 0$,

$$f_n - 1 = (a-1)f_{n-1} + (a+b-1)[f_0 + f_1 + \cdots + f_{n-2}].$$

Question: How many colored n-tilings exist, excluding the tiling consisting of all white squares?

Answer 1: By definition, $f_n - 1$. (Notice how our question and answer become shorter with experience!)

Answer 2: Here we partition our tilings according to the last tile that is not a white square. Suppose the last tile that is not a white square begins on cell k. If $k = n$, that tile is a square and there are $a - 1$ choices for its color. There are f_{n-1} colored tilings that can precede it for a total of $(a-1)f_{n-1}$ tilings ending in a nonwhite square. If $1 \leq k \leq n-1$, the tile covering cell k can be a nonwhite square or a domino covering cells k and $k+1$. There are $a+b-1$ ways to pick this tile and the previous cells can be tiled f_{k-1} ways. Altogether, there are $(a-1)f_{n-1} + \sum_{k=1}^{n-1}(a+b-1)f_{k-1}$ colored n-tilings, as desired.

Notice how easily the argument generalizes if we partition according to the last tile that is not a square of color 1 or 2 or... or c. Then the same reasoning gives us for any $1 \leq c \leq a$,

$$f_n - c^n = (a-c)f_{n-1} + \big((a-c)c + b\big)[f_0c^{n-2} + f_1c^{n-3} + \cdots + f_{n-2}]. \tag{9}$$

Likewise, by partitioning according to the last tile that is not a black domino, we get a slightly different identity, depending on whether the length of the tiling is odd or even:

$$f_{2n+1} = a(f_0 + f_2 + \cdots + f_{2n}) + (b-1)(f_1 + f_3 + \cdots + f_{2n-1}),$$
$$f_{2n} - 1 = a(f_1 + f_3 + \cdots + f_{2n-1}) + (b-1)(f_0 + f_2 + \cdots + f_{2n-2}).$$

After applying Theorem 1 to equation (4) and reindexing ($n \to n+1$) we have

IDENTITY 3. *For* $n \geq 0$,

$$a\sum_{k=0}^{n} f_k^2 b^{n-k} = f_n f_{n+1}.$$

Question: In how many ways can we create a colored n-tiling and a colored $(n+1)$-tiling?

Answer 1: $f_n f_{n+1}$.

Answer 2: For this answer, we ask for $0 \le k \le n$, how many colored tiling pairs exist where cell k is the last cell for which both tilings are breakable? (Equivalently, this counts the tiling pairs where the last square occurs on cell $k+1$ in exactly one tiling.) We claim this can be done $af_k^2b^{n-k}$ ways, since to construct such a tiling pair, cells 1 through k of the tiling pair can be tiled f_k^2 ways, the colored square on cell $k+1$ can be chosen a ways (it is in the n-tiling if and only if $n-k$ is odd), and the remaining $2n-2k$ cells are covered with $n-k$ colored dominoes in b^{n-k} ways. See FIGURE 2. Altogether, there are $a\sum_{k=0}^{n} f_k^2b^{n-k}$ tilings, as desired.

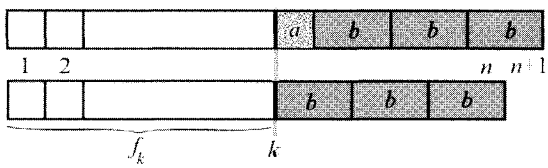

Figure 2 A tiling pair where the last mutually breakable cell occurs at cell k

The next identity uses a slightly different strategy. We hope that the reader does not *find fault* with our argument.

Consider the two colored 10-tilings offset as in FIGURE 3. The first one tiles cells 1 through 10; the second one tiles cells 2 through 11. We say that there is a *fault* at cell i, $2 \le i \le 10$, if both tilings are breakable at cell i. We say there is a fault at cell 1 if the first tiling is breakable at cell 1. Put another way, the pair of tilings has a fault at cell i for $1 \le i \le 10$ if neither tiling has a domino covering cells i and $i+1$. The pair of tilings in FIGURE 3 has faults at cells 1, 2, 5, and 7. We define the *tail* of a tiling to be the tiles that occur after the last fault. Observe that if we swap the tails of FIGURE 3 we obtain the 11-tiling and the 9-tiling in FIGURE 4 and it has the same faults.

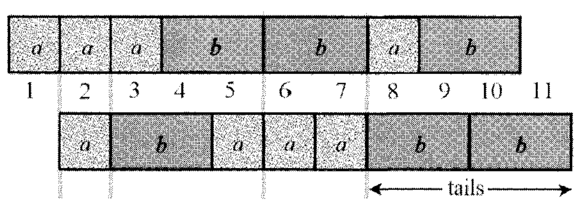

Figure 3 Two 10-tilings with their faults (indicated with gray lines) and tails

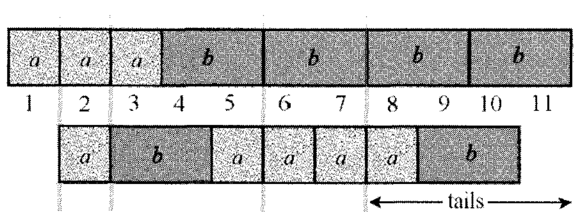

Figure 4 After tail-swapping, we have an 11-tiling and a 9-tiling with exactly the same faults

Tail swapping is the basis for the identity below, based on (5). At first glance, it may appear unsuitable for combinatorial proof due to the presence of the $(-1)^n$ term. Nonetheless, we will see that this term is merely the error term of an *almost* one-to-one correspondence between two sets whose sizes are easily counted. We use a different format for this combinatorial proof.

IDENTITY 4. $f_n^2 = f_{n+1}f_{n-1} + (-1)^n b^n$

Set 1: Tilings of two colored n-boards (a *top* board and a *bottom* board). By definition, this set has size f_n^2.

Set 2: Tilings of a colored $(n+1)$-board and a colored $(n-1)$-board. This set has size $f_{n+1}f_{n-1}$.

Correspondence: First, suppose n is odd. Then the top and bottom board must each have at least one square. Notice that a square in cell i ensures that a fault must occur at cell i or cell $i-1$. Swapping the tails of the two n-tilings produces an $(n+1)$-tiling and an $(n-1)$-tiling with the same tails. This produces a 1-to-1 correspondence between all pairs of n-tilings and all tiling pairs of sizes $n+1$ and $n-1$ that have faults. Is it possible for a tiling pair with sizes $n+1$ and $n-1$ to be fault free? Yes, with all colored dominoes in *staggered formation* as in FIGURE 5, which can occur b^n ways. Thus, when n is odd, $f_n^2 = f_{n+1}f_{n-1} - b^n$.

Similarly, when n is even, tail swapping creates a 1-to-1 correspondence between faulty tiling pairs. The only fault-free tiling pair is the all domino tiling of FIGURE 6. Hence, $f_n^2 = f_{n+1}f_{n-1} + b^n$. Considering the odd and even case together produces our identity.

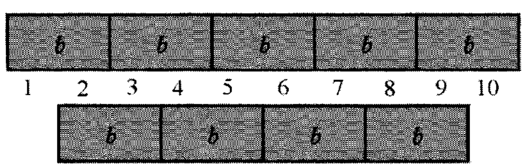

Figure 5 When n is odd, the only fault-free tiling pairs consist of all dominoes

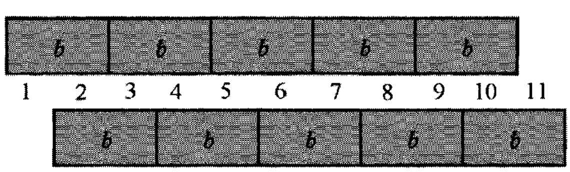

Figure 6 When n is even, the only fault-free tiling pairs consist of all dominoes

We conclude this section with a combinatorial proof of what we believe to be the most beautiful Fibonacci fact of all.

THEOREM 2. *For generalized Fibonacci numbers defined by (1) with relatively prime integers a and b,*

$$\gcd(F_n, F_m) = F_{\gcd(n,m)}. \tag{10}$$

We will need to work a little harder to prove this theorem combinatorially, but it can be done. Fortuitously, we have already combinatorially derived the identities needed to prove the following lemma.

LEMMA 1. *For generalized Fibonacci numbers defined by (1) with relatively prime integers a and b and for all $m \geq 1$, F_m and bF_{m-1} are relatively prime.*

Proof. First we claim that F_m is relatively prime to b. By conditioning on the location of the last colored domino (if any exist), equation (9) says (after letting $c = a$ and reindexing),

$$F_m = a^{m-1} + b\sum_{j=1}^{m-2} a^{j-1} F_{m-1-j}.$$

Consequently, if $d > 1$ is a divisor of F_m and b, then d must also divide a^{m-1}, which is impossible since a and b are relatively prime.

Next we claim that F_m and F_{m-1} are relatively prime. This follows from equation (5) since if $d > 1$ divides F_m and F_{m-1}, then d must divide b^{m-1}. But this is impossible since F_m and b are relatively prime.

Thus since $\gcd(F_m, b) = 1$ and $\gcd(F_m, F_{m-1}) = 1$, then $\gcd(F_m, bF_{m-1}) = 1$, as desired. ∎

To prove Theorem 2, we exploit Euclid's algorithm for computing greatest common divisors: If $n = qm + r$ where $0 \leq r < m$, then

$$\gcd(n, m) = \gcd(m, r).$$

Since the second component gets smaller at each iteration, the algorithm eventually reaches $\gcd(g, 0) = g$, where g is the greatest common divisor of n and m. The identity below shows one way that F_n can be expressed in terms of F_m and F_r. It may look formidable at first but makes perfect sense when viewed combinatorially.

IDENTITY 5. *If $n = qm + r$, where $0 \leq r < m$, then*

$$F_n = (bF_{m-1})^q F_r + F_m \sum_{j=1}^{q} (bF_{m-1})^{j-1} F_{(q-j)m+r+1}.$$

Question: How many colored $(qm + r - 1)$-tilings exist?

Answer 1: $f_{qm+r-1} = F_{qm+r} = F_n$.

Answer 2: First we count all such colored tilings that are unbreakable at every cell of the form $jm - 1$, where $1 \leq j \leq q$. Such a tiling must have a colored domino starting on cell $m - 1, 2m - 1, \ldots, qm - 1$, which can be chosen b^q ways. Before each of these dominoes is an arbitrary $(m - 2)$-tiling that can each be chosen f_{m-2} ways. Finally, cells $qm + 1, qm + 2, \ldots, qm + r - 1$ can be tiled f_{r-1} ways. See FIGURE 7. Consequently, the number of colored

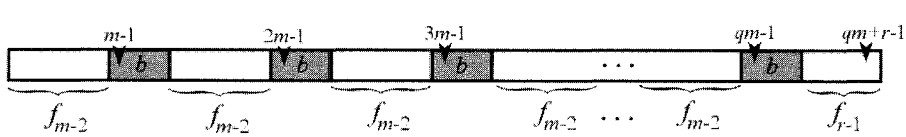

Figure 7 There are $(bF_{m-1})^q F_r$ colored $(qm + r - 1)$-tilings with no breaks at any cells of the form $jm - 1$ where $1 \leq j \leq q$

tilings with no $jm-1$ breaks is $b^q(f_{m-2})^q f_{r-1} = (bF_{m-1})^q F_r$. Next, we partition the remaining colored tilings according to the first breakable cell of the form $jm-1$, $1 \le j \le q$. By similar reasoning as before, this can be done $(bF_{m-1})^{j-1}F_m F_{(q-j)m+r+1}$ ways. (See FIGURE 8.) Altogether, the number of colored tilings is $(bF_{m-1})^q F_r + F_m \sum_{j=1}^{q}(bF_{m-1})^{j-1}F_{(q-j)m+r+1}$.

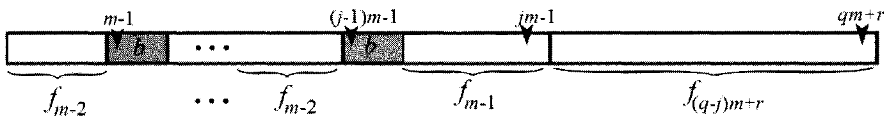

Figure 8 There are $(bF_{m-1})^{j-1}F_mF_{(q-j)m+r+1}$ colored $(qm+r-1)$-tilings that are breakable at cell $jm-1$, but not at cells of the form $im-1$ where $1 \le i < j$

The previous identity explicitly shows that F_n is an integer combination of F_m and F_r. Consequently, d is a common divisor of F_n and F_m if and only if d divides F_m and $(bF_{m-1})^q F_r$. But by Lemma 1, since F_m is relatively prime to bF_{m-1}, d must be a common divisor of F_m and F_r. Thus F_n and F_m have the same common divisors (and hence the same gcd) as F_m and F_r. In other words,

COROLLARY 1. *If $n = qm + r$, where $0 \le r < m$, then*

$$\gcd(F_n, F_m) = \gcd(F_m, F_r).$$

But wait!! This corollary is the same as Euclid's algorithm, but with F's inserted everywhere. This proves Theorem 2 by following the same steps as Euclid's algorithm. The $\gcd(F_n, F_m)$ will eventually reduce to $\gcd(F_g, F_0) = (F_g, 0) = F_g$, where g is the greatest common divisor of m and n.

Lucas numbers—the combinatorial way

Generalized Lucas numbers are nothing more than generalized Fibonacci numbers running in circles. Specifically, for nonnegative integers a, b, and n, let ℓ_n count the number of ways to tile a circular $1 \times n$ board with slightly curved colored squares and dominoes, where there are a colors for squares and b colors for dominoes. Circular tilings of length n will be called *n-bracelets*. For example, when $a = b = 1$, $\ell_4 = 7$, as illustrated in FIGURE 9. In general, $\ell_4 = a^4 + 4a^2b + 2b^2$.

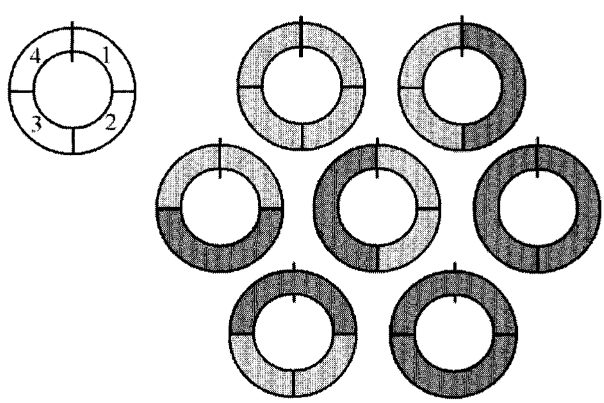

Figure 9 A circular board of length 4 and its seven 4-bracelets

From the definition of ℓ_n it follows that $\ell_n \ge f_n$ since an n-bracelet can have a domino covering cells n and 1; such a bracelet is called *out-of-phase*. Otherwise, there is a break between cells n and 1, and the bracelet is called *in-phase*. The first 5 bracelets in FIGURE 9 are in-phase and the last 2 are out-of-phase. Notice $\ell_1 = a$ and $\ell_2 = a^2 + 2b$ since a circular board of length 2 can be covered with two squares, an in-phase domino, or an out-of-phase domino. We define $\ell_0 = 2$ to allow 2 empty bracelets, one in-phase and one out-of-phase. In general for $n \geq 2$, we have

$$\ell_n = a\ell_{n-1} + b\ell_{n-2}$$

because an n-bracelet can be created from an $(n-1)$-bracelet by inserting a square to the left of the first tile or from an $(n-2)$-bracelet by inserting a domino to the left of the first tile. The *first tile* is the one covering cell 1 and it determines the phase of the bracelet; it may be a square, a domino covering cells 1 and 2, or a domino covering cells n and 1.

Since $\ell_0 = 2 = L_0$ and $\ell_1 = a = L_1$, we see that for all $n \geq 0$, $\ell_n = L_n$. This becomes our combinatorial definition for the generalized Lucas numbers.

THEOREM 3. *For all $n \geq 0$, $L_n = \ell_n$ counts the number of n-bracelets created with colored squares and dominoes where there are a colors for squares and b colors for dominoes.*

Now that we know how to combinatorially think of Lucas numbers, generalized identities are a piece of cake. Equation (7), which we rewrite as

$$L_n = af_{n-1} + 2bf_{n-2},$$

reflects the fact that an n-bracelet can begin with a square (af_{n-1} ways), an in-phase domino (bf_{n-2} ways), or an out-of-phase domino (bf_{n-2} ways). Likewise, equation (8), rewritten as

$$L_n = f_n + bf_{n-2},$$

conditions on whether or not an n-bracelet is in-phase (f_n ways) or out-of-phase (bf_{n-2} ways.)

You might even think these identities are too easy, so we include a couple more generalized Lucas identities for you to ponder along with visual hints. For more details see [**4**].

$$f_{n-1}L_n = f_{2n-1} \qquad \text{See FIGURE 10.}$$

$$L_n^2 = L_{2n} + 2\cdot(-b)^n \qquad \text{See FIGURE 11.}$$

Further generalizations and applications

Up until now, all of our proofs have depended on the fact that the recurrence coefficients a and b were nonnegative integers, even though most generalized Fibonacci identities remain true when a and b are negative or irrational or even complex numbers. Additionally, our sequences have had very specific initial conditions ($F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = a$), yet many identities can be extended to handle arbitrary ones. This section illustrates how combinatorial arguments can still be used to overcome these apparent obstacles.

Arbitrary initial conditions Let a, b, A_0, and A_1 be nonnegative integers and consider the sequence A_n defined by the recurrence, for $n \geq 2$, $A_n = aA_{n-1} + bA_{n-2}$. As

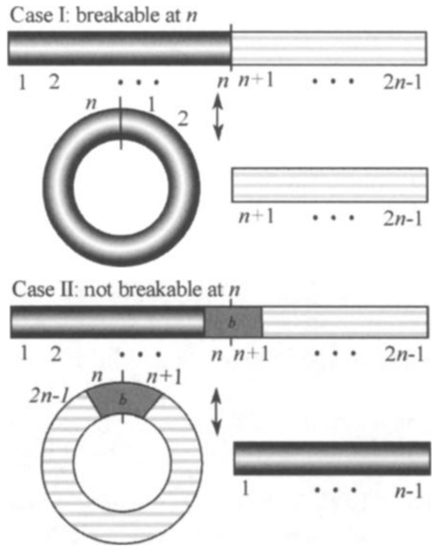

Figure 10 Picture for $f_{n-1}L_n = f_{2n-1}$

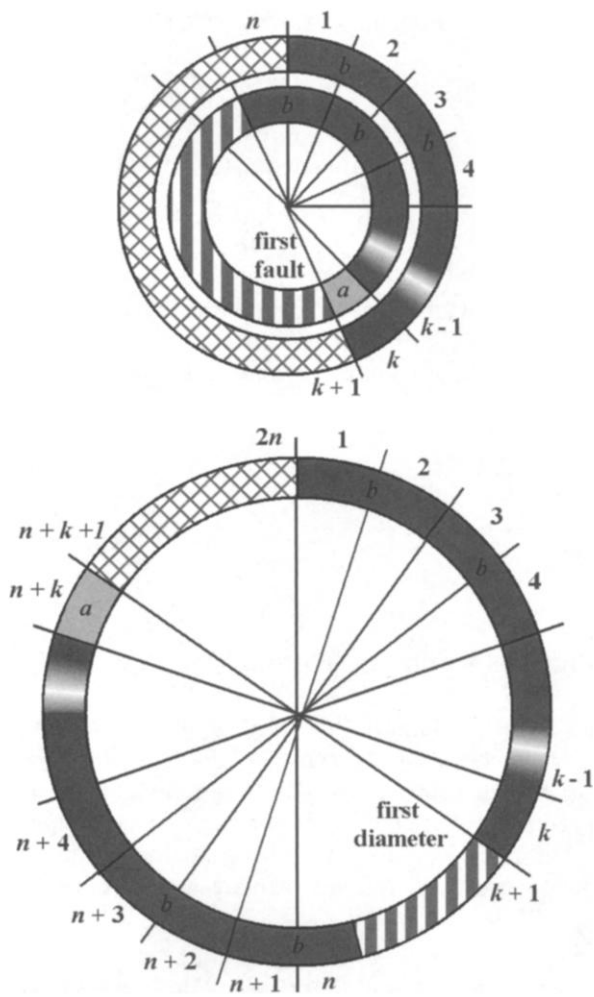

Figure 11 Picture for $L_n^2 = L_{2n} + 2 \cdot (-b)^n$ when n is even

described in [1] and Chapter 3 of [4], the *initial conditions* A_0 and A_1 determine the number of choices for the *initial tile*. Just like F_n, A_n counts the number of colored n-tilings where *except for the first tile* there are a colors for squares and b colors for dominoes. For the first tile, we allow A_1 choices for a square and bA_0 choices for a domino. So as not to be confused with the situation using ideal initial conditions, we assign the first tile a *phase* instead of a color.

For example, when $A_0 = 1$ and $A_1 = a$, the ideal initial conditions, we have a choices for the phase of an initial square and b choices for the phase of an initial domino. Since *all* squares have a choices and *all* dominoes have b choices, it follows that $A_n = f_n$. When $A_0 = 0$ and $A_1 = 1$, A_n counts the number of colored n-tilings that begin with an "uncolored" square; hence $A_n = f_{n-1} = F_n$. When $A_0 = 2$ and $A_1 = a$, A_n counts the number of colored n-tilings that begin with a square in one of a phases or a domino in one of $2b$ phases. This is equivalent to a colored n-bracelet since there are an equal number of square phases as colors and twice as many domino phases as colors (representing whether the initial domino is in-phase or out-of-phase.) Thus when $A_0 = 2$ and $A_1 = a$, we have $A_n = L_n$.

In general, there are $A_1 f_{n-1}$ colored tilings that begin with a phased square and $bA_0 f_{n-2}$ colored tilings that being with a phased domino. Hence we obtain the following identity from Kalman and Mena [5]:

$$A_n = bA_0 F_{n-1} + A_1 F_n. \tag{11}$$

Arbitrary recurrence coefficients Rather than assigning a discrete number of colors for each tile, we can assign weights. Squares have weight a and dominoes have weight b except for the initial tile, which has weight A_1 as a square and weight bA_0 as a domino. Here a, b, A_0, and A_1 do not have to be nonnegative integers, but can be chosen from the set of complex numbers (or from any commutative ring). We define the *weight of an n-tiling* to be the product of the weights of its individual tiles. For example, the 7-tiling "square-domino-domino-square-square" has weight a^3b^2 with ideal initial conditions and has weight $A_1a^2b^2$ with arbitrary initial conditions. Inductively one can prove that for $n \geq 1$, A_n is the sum of the weights of all weighted n-tilings, which we call the *total weight* of an n-board.

If X is an m-tiling of weight w_X and Y is an n-tiling of weight w_Y, then X and Y can be glued together to create an $(m+n)$-tiling of weight $w_X w_Y$. If an m-board can be tiled s different ways and has total weight $A_m = w_1 + w_2 + \cdots + w_s$ and an n-board can be tiled t ways with total weight $A_n = x_1 + x_2 + \cdots + x_t$, then the sum of the weights of all weighted $(m+n)$-tilings breakable at cell m is

$$\sum_{i=1}^{s}\sum_{j=1}^{t} w_i x_j = (w_1 + w_2 + \cdots + w_s)(x_1 + x_2 + \cdots + x_t) = A_m A_n.$$

Now we are prepared to revisit some of our previous identities using the weighted approach. For Identity 1, we find the total weights of an n-board in two different ways. On the one hand, since the initial conditions are ideal, the total weight is $A_n = f_n$. On the other hand, the total weight is comprised of the total weight of those tilings that are breakable at cell m ($f_m f_{n-m}$) plus the total weight of those tilings that are unbreakable at cell m ($f_{m-1} b f_{n-m-1}$). Identities 2, 3, and 5 can be argued in similar fashion.

For Identity 4, we define the weight of a tiling pair to be the product of the weights of all the tiles, and define the total weight as before. Next we observe that tail swapping preserves the weight of the tiling pair since no tiles are created or destroyed in the process. Consequently, the total weight of the set of *faulty* tiling pairs (X, Y) where

X and Y are n-tilings equals the total weight of the faulty tiling pairs (X', Y'), where X' is an $(n+1)$-tiling and Y' is an $(n-1)$-tiling. The fault-free tiling pair, for the even and odd case, will consist of n dominoes and therefore have weight b^n. Hence identity 4 remains true even when a and b are complex numbers.

Kalman and Mena [**5**] prove Binet's formulas for Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \tag{12}$$

and for more general sequences.

These can also be proved combinatorially [**2**]. Binet's formula follows from considering a random tiling of an infinitely long strip with cells $1, 2, 3, \ldots,$ where squares and dominoes are randomly and independently inserted from left to right. The probability of inserting a square is $1/\phi$ and the probability of inserting a domino is $1/\phi^2$, where $\phi = (1+\sqrt{5})/2$. (Conveniently, $1/\phi + 1/\phi^2 = 1$.) By computing the probability of being breakable at cell $n-1$ in two different ways, Binet's formula instantly appears. This approach can be extended to generalized Fibonacci numbers and beyond, as described in [**1**].

Finally, we observe that the Pythagorean Identity presented in [**5**] for traditional Fibonacci numbers, which can be written as

$$(f_{n-1}f_{n+2})^2 + (2f_nf_{n+1})^2 = f_{2n+2}^2$$

can also be proved combinatorially. For details, see [**4**].

We hope that this paper illustrates that Fibonacci and Lucas sequences are members of a very special class of sequences satisfying beautiful properties, namely sequences defined by second order recurrence relations. But why stop there? Combinatorial interpretations can be given to sequences that satisfy higher-order recurrences. That is, if we define $a_j = 0$ for $j < 0$ and $a_0 = 1$, then for $n \geq 1$, $a_n = c_1a_{n-1} + \cdots + c_ka_{n-k}$ counts the number of ways to tile a board of length n with colored tiles of length at most k, where each tile of length i has c_i choices of color. Again, this interpretation can be extended to handle complex values of c_i and arbitrary initial conditions. See Chapter 3 of [**4**]. Of course, the identities tend to be prettier for the two-term recurrences, and are usually prettiest for the traditional Fibonacci and Lucas numbers.

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REFERENCES

1. A. T. Benjamin, C. R. H. Hanusa, and F. E. Su, Linear recurrences and Markov chains, *Utilitas Mathematica* **64** (2003), 3–17.
2. A. T. Benjamin, G. M. Levin, K. Mahlburg, and J. J. Quinn, Random approaches to Fibonacci identities, *Amer. Math. Monthly* **107**:6 (2000), 511–516.
3. A. T. Benjamin and J. J. Quinn, Recounting Fibonacci and Lucas Identities, *College Math. J.* **30**:5 (1999), 359–366.
4. ———, *Proofs That Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, Washington, D.C., 2003.
5. D. Kalman and R. Mena, The Fibonacci Numbers—Exposed, this MAGAZINE **76**:3 (2003), 167–181.
6. T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, NY, 2001.
7. S. Vajda, *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications*, John Wiley and Sons, New York, 1989.