The finite dimensional basis problem with an appendix on nets of Grassmann manifolds

by

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1. Introduction and the main results

It is well-known that if B is a normed space, then for every n-dimensional (or n-codimensional) subspace F there is a projection from B onto F of norm not exceeding $\sqrt{n}$. But it has so far been an open question whether for every B of dimension greater than n there is some n-dimensional subspace F onto which there is a projection of norm $o(\sqrt{n})$ or even K, with K an absolute constant ([2]). In this paper we will construct a $2n$-dimensional space in which every projection of rank n in B has norm greater than $C\sqrt{n}$ where C is an absolute positive constant. This example, which is in an obvious sense the best—or the worst, depending on the point of view—possible, also settles in the negative "the finite-dimensional basis problem", i.e. the question whether there is an absolute constant K such that every finite-dimensional space has a basis with basis constant $\leqslant K$. We recall that the (Schauder) basis constant of a basis (x_j) of B is the smallest number K such that

$$\left\|\sum_{j=1}^{k} t_j x_j\right\|_B \leqslant K \left\|\sum_{j=1}^{n} t_j x_j\right\|_B$$

for all scalars $t_1, t_2, \ldots, t_n$ and every $k \leqslant n = \dim B$. When the sum on the left is allowed to run over any subset of $\{1, \ldots, n\}$, we get the unconditional basis constant. The

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([2]) After this paper was submitted to publication, G. Pisier ([16]) constructed an example of an infinite dimensional Banach space with many surprising properties, in particular the norm of any rank n projection on it is of order $\sqrt{n}$. In spite of a very similar formulation, this result neither implies our Theorem 1.1 nor follows from it (the construction is strictly infinite dimensional).

problem of the basis constant was known as early as in the '30's to mathematicians of the Lvov school. The first step towards a solution was taken by F. Bohnenblust in 1941 (see [1]). The problem of the unconditional basis constant was settled by Y. Gordon and D. Lewis (see [7]) who showed that some well-known n-dimensional spaces of operators have unconditional basis constants of order $n^{1/4}$. Later T. Figiel, S. Kwapien and A. Pelczynski (see [4]) gave examples of n-dimensional spaces for which the unconditional basis constants were of order $\sqrt{n}$, the same as we obtained for the basis constants.

Let us state our main result again.

THEOREM 1.1. *There is a constant $C>0$ such that, for every n, there exists a $2n$-dimensional normed space B such that, for every projection P on B of rank n,*

$$\|P\colon B\to B\| > C\sqrt{n}.$$

As an immediate consequence of Theorem 1.1 we get

COROLLARY 1.2. *For every n there exists a normed space B, $\dim B=n$, such that the basis constant of every basis of B exceeds $C'\sqrt{n}$, where C' is an absolute constant.*

Recently (independently of this paper and a month or two before the work on it was completed) E. Gluskin has proved the following result (see [6]), which also settles the finite dimensional basis problem: for each n there exists an n-dimensional normed space such that, for every projection P on it with $\operatorname{rank} P<n/3$, $\|P\|\geq cn^{-1/2}(1+\log n)^{-1/2}\operatorname{rank} P$. In particular his result, while being slightly weaker than Theorems 1.1 and 1.4 if rank $P=O(n)$, yields nontrivial estimates if rank P is of order n^s, $s>1/2$—better than our Theorem 1.6.

The construction in the proof of Theorem 1.1 is of a "random" nature: We define a whole class of spaces and show that, in a certain sense, "most of" them satisfy the assertion of Theorem 1.1. The nature of the example is close to one constructed by Gluskin in [5] to obtain two n-dimensional normed spaces, whose Banach-Mazur distance is of order n (cf. Remark 4.6). Random spaces of essentially the same kind (but with different distribution) were considered in [4] as having the "worst" possible order of constants of local unconditional structure. The construction in [4] was based on some phenomenon discovered in [9] (cf. Remark 4.5).

A common feature of all these examples is that they (or their duals) are quotients of "small" l^1_m-spaces or, in other words, the number of extreme points of their unit balls is comparable with their dimension. A consequence of this, crucial in both [5] and here, is that such a unit ball, while containing sufficiently many points "far away" from the origin to keep the Banach-Mazur distance between the space and l^2 (and, in random, even l^1) of the corresponding dimension large, has volume very close to that of the largest Euclidean ball contained in it. This phenomenon was observed in the case of l_n^1 in [17], named "small volume ratio" and investigated more systematically in [20] and finally noticed in the case of spaces considered here in [10] and [18].

We feel that the methods of the proofs of Theorem 1.1 and its generalisations Theorem 1.4 and Theorem 1.6 below may be of use for some other problems, which we will now mention.

The infinite dimensional version of the basis problem was solved, also in the negative, by P. Enflo in 1972 (see [3]). However, his proof was based on a completely different, infinite dimensional considerations, yielding in fact an example of a space failing to satisfy much weaker property than having a Schauder basis. In particular the following problem still remains open.

Problem 1.3. Does there exist a normed space without Schauder basis, but with bounded approximation property (i.e. the identity operator is a pointwise limit of operators of finite rank)? A finite dimensional Schauder decomposition?

Corollary 1.2 indicates that the answer may be positive.

Our proof of Theorem 1.1 yields actually a significantly stronger result.

THEOREM 1.4. *Given $\delta>0$ there exists $C=C(\delta)$ such that, for every n, there is a normed space B, $\dim B=n$, satisfying*

$$||T\colon B\to B|| \geq C\sqrt{n}$$

for every operator T on B with $\operatorname{rank} T\leq(1-\delta)n$, $\operatorname{rank}(I-T)\leq(1-\delta)n$.

Theorem 1.4 may be seen as a step toward a solution of the following well-known problem (or at least its finite dimensional version).

Problem 1.5. Does there exist an infinite dimensional Banach space such that every continuous operator on it is of the form $\lambda I+K$, where K is compact?

Modifying the proof of Theorem 1.1 a little bit one can generalize it to

THEOREM 1.6. *There is a constant $b>0$, such that, for every n and $m\leq n/2$, there exists an n-dimensional normed space B satisfying*

$$||P\colon B\to B|| \geq m^{1/2-(bn^2/m^2\ \ln n)}$$

for every projection P on B with $m\leq \operatorname{rank} P\leq n-m$.

Observe that the above estimate is nontrivial (and of type m^{α}, $\alpha>0$) if and only if $m>\sqrt{2b}\; n/\sqrt{\ln n}$. If $m\geq\delta n$, $\delta>0$, we get the estimate $C(\delta)\sqrt{n}$ (as in Theorem 1.4).

Theorem 1.1 and the above remark suggests the following questions.

Problem 1.7. Does there exist a sequence $K_m\nearrow\infty$ such that, for every n, one can find an n-dimensional normed space B satisfying: if P is a projection on B, then

$$||P\colon B\to B||\geq K_m,$$

where $m=\min\{\text{rank } P,\ n-\text{rank } P\}$? Can we take $c\sqrt{m}$ as K_m?

Problem 1.8. Does there exist a constant K and a sequence of integers $k_n\nearrow\infty$ such that, for every n-dimensional normed space B, there is a projection P on B with $||P||\leq K$ and $k_n\leq \text{rank}\, P\leq n/2$?

Both problems were probably asked many times. Of course positive answer to one of them implies negative answer to the other.

All spaces considered in this paper are real. The complex case does not follow formally from the real one (as it is in the case of positive statements). However, one can construct an analogous example in the complex case. The first seven sections can be copied almost word by word: we can treat $\mathbf{C}^{4n}$ as $\mathbf{R}^{8n}$, complex l^1_{4n} as real l^1_{8n} (the ratio of the corresponding norms is always between 1 and $\sqrt{2}$) etc. In section 8 we must replace $O(m)$ by the unitary group $U(m)$ etc.—all inequalities remain true in the complex case because of their algebraic origin and the proof is somewhat simpler (however, less intuitive geometrically).

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2. Organization of the paper

Section 3 establishes notation.

Section 4 describes construction of a space, whose existence is asserted in Theorem 1.1. It contains Proposition 4.1 (which is essentially a specification of Theorem 1.1), its immediate corollaries and some remarks.

Section 5 reduces Proposition 4.1 to two technical statements: Fact 5.1 and Fact 5.2 (the first of them is essentially contained in [5]).

Finally, sections 6, 7, and 8 are devoted to the proof of Fact 5.2. Section 6 reduces it to Proposition 6.8, which is a purely Hilbert space problem.

Proposition 6.8 is proved in section 7.

Section 8 is an appendix. It contains proof of Lemma 7.3 and lists other (precise) estimates for the minimal cardinalities of ε-nets of orthogonal groups and Grassmann manifolds (with respect to unitary ideal norms and their analogues respectively)—mostly without proofs. More systematic exposition of that subject is contained in [19].

Finally, section 9 indicates changes in the argument necessary to prove Theorems 1.4 and 1.6.

3. Notation

Our notation generally follows [13]. Some of frequently used (but less standard) conventions are listed below.

$(e_1, \ldots, e_m)$ will always mean the standard unit vector basis of $\mathbf{R}^m$.

The Lebesgue measure of a measurable subset S of $\mathbf{R}^m$ will be denoted by m-vol(S) or shortly vol S.

For a normed space X we shall denote by $B(X)$ its unit ball, the ball with radius r and center at the origin by $B_r(X)$ or just B_r. We shall also denote $B(l_m^p)$ shortly by B_m^p.

If A, B are normed spaces, we shall denote the space of bounded linear operators from A to B, equipped with operator norm, by $L(A, B)$ or, in the case $A=B$, just by $L(A)$. The operator norm will be denoted by $\| \cdot : A \to B\|$, $\| \cdot \|_{L(A,B)}$ or $L(A, B)$-norm etc.

Moreover, we shall identify the spaces A, B with their unit balls. In particular we may write $\| \cdot \|_B$ for the norm generated by an absolutely convex body $B \subset \mathbf{R}^m$, $L(A, B)$ for the space of operators between the spaces generated by absolutely convex bodies A, B etc.

As usually, the group of orthogonal operators on $\mathbf{R}^m$ will be denoted by $O(m)$, the normalized Haar measure on it by $\mu = \mu_m$.

As it is well-known, every operator $T: H_1 \to H_2$ (H_1, H_2, finite dimensional Hilbert spaces) can be written as

$$T = \sum_j \lambda_j \langle h_j, \cdot \rangle h_j',$$

where (h_j) and (h_j') are orthonormal systems in H_1 and H_2 respectively. We shall always assume that (λ_j) is a non-increasing sequence (of length $\dim H_1$; add zeros if necessary) of nonnegative numbers and refer to such representation as "the polar decomposition" (the sequence (λ_j) is then uniquely determined by T; (h_i) and (h_i') not always).

For a Hilbert space H and its subspace F we shall denote by P_F the orthogonal projection of H onto F. If $\dim H = m$ and $|\cdot|$ is the Hilbert norm, then by "*a Gaussian variable with distribution* $N(0, 1, H)$" we shall mean an H-valued random variable g with density

$$\varrho_g(x) = \left(\frac{m}{2\pi}\right)^{m/2} e^{-m|x|^2/2}$$

against the Lebesgue measure. g can be also represented as $(1/\sqrt{m})\ \Sigma_{j=1}^m \gamma_j h_j$, where the γ_j's are independent *real* Gaussian variables with distribution $N(0, 1)$ and (h_j) is an orthonormal basis of H. Observe that

(3.1) $\mathbf{E}|g|^2 = 1$ ($\mathbf{E}$ stands for expectation).

(3.2) $\mathcal{P}(\{1/2 \le |g| \le 2\}) \ge 1 - e^{-cm}$, c absolute.

(3.3) If E is another Hilbert space, $V: H \to E$ an isometry onto and E_0 a k-dimensional subspace of E, then

$$\sqrt{\frac{m}{k}}\, P_{E_0} V g$$

is a Gaussian variable with distribution $N(0, 1, E_0)$.

In (3.2) and throughout the paper $\mathcal{P}$ stands for a probability measure, which may be different in different places.

The letters C, c, c', c or C with a subscript etc. will stand for universal constants, in particular not depending on dimensions of considered spaces, unless otherwise explicitly stated. The same symbols may correspond to different numerical values in different sections.

4. Construction of the space

Consider $\mathbf{R}^{4n}$ with the standard unit vector basis $e_1, e_2, \ldots, e_{4n}$. Let $F = \text{span}\,\{e_1, \ldots, e_n\}$. Let, for $j = 1, \ldots, n$, $f_j \colon \Omega_0 \to F$ be independent Gaussian variables with distribution $N(0, 1, F)$, where $(\Omega_0, \mathcal{P}_0)$ is the corresponding probability space. Consider another probability space $(\Omega, \mathcal{P}) \overset{df}{=} (O(4n) \times \Omega_0,\ \mu \otimes \mathcal{P}_0)$, where $\mu = \mu_{4n}$ is the normalized Haar measure on $O(4n)$.

Now, for $j = 1, \ldots, n$, define the random variables g_j on Ω by

$$g_j = g_j(U, \omega) \overset{df}{=} U f_j(\omega),$$

and finally let (ac = absolute convex hull)

$$B \overset{df}{=} \text{ac}\,\{e_1, \ldots, e_{4n}, g_1, \ldots, g_n\}.$$

Of course $g_1, \ldots, g_n$, and hence B, depend on U and ω. Thus, identifying a normed space with its unit ball, one can consider B as a random variable on Ω, whose values are $4n$-dimensional normal spaces. Denote by $\mathscr{B}$ its range, i.e. the set of all spaces obtained in the above way. One can consider $\mathscr{P}$ as a probability measure on $\mathscr{B}$.

Now we are able to state

PROPOSITION 4.1. *There exist absolute constants $c_1, c_2 > 0$ so that*
$$\mathscr{P}(\{B: \|P: B \to B\| \leq c_1 \sqrt{n} \text{ for some rank } 2n \text{ projection } P\}) \leq e^{-c_2 n}.$$

Observe that to prove Theorem 1.1 it is enough to show that the probability above is strictly smaller than 1.

The following two facts are immediate consequences of Proposition 4.1.

COROLLARY 4.2. *For every $p \geq 2$ and k there is a $2k$-dimensional subspace E of an L^p-space such that no projection of rank k on E has norm not exceeding $c_3 k^{1/2-1/p}$, where c_3 is an absolute constant. Moreover, we can take l^p_{3k} as L^p-space above.*

Notice that since, by [12], $d(E, l^2_{2k}) \leq (2k)^{1/2-1/p}$ (d is the Banach-Mazur distance), every k-dimensional subspace of E is a range of projection (on E) of norm not exceeding $(2k)^{1/2-1/p}$, Corollary 4.2 is, in a sense, optimal.

Proof. Clearly it is enough to consider the case of even k; say, $k=2n$. Observe that every $B \in \mathscr{B}$ is isometically a quotient of l^1_{5n}. Hence B^*, which satisfies the assertion of Theorem 1.1 simultaneously with B, is isometrically a subspace of l^∞_{5n}. Now it is enough to take B^*, considered as a linear subspace of $\mathbf{R}^{5n}$ equipped with l^p_{5n}-norm, as E and the conclusion will follow immediately from the inequalities

$$\|x\|_\infty \leq \|x\|_p \leq (5n)^{1/p} \|x\|_\infty$$

valid for $x \in \mathbf{R}^{5n}$. Q.E.D.

COROLLARY 4.3. *For every n there is an n-dimensional subspace of l^1_{4n}, which is not contained in the range of any projection P of rank $2n$ with $\|P: l^1_{4n} \to l^1_{4n}\| \leq c_1 \sqrt{n}$.*

Proof. Consider any $B \in \mathscr{B}$, for which the condition in the brackets in Proposition 4.1 is not satisfied. Now it is enough to take $\text{span}\{g_1, \ldots, g_n\}$ and look at it as a subspace of l^1_{4n}. Q.E.D.

Remark 4.4. Since, by (3.2),

$$\mathcal{P}(\{1/2\leq\|g_j\|_2\leq 2 \text{ for } j=1,\dots,n\}) \geq 1-ne^{-cn}\geq 1-e^{-c'n},$$

one can choose B of the form $B=\mathrm{ac}\{e_1,\dots,e_{4n}, x_1,\dots,x_n\}$, $\|x_j\|_2=1$ for $j=1,\dots,n$, still failing to satisfy the condition from Proposition 4.1 (with possibly a slightly "worse" constant c_1).

Remark 4.5. By somewhat more subtle considerations one can show that B, still failing to satisfy the condition from Proposition 4.1, can be chosen in such a way that its cotype 2 constant does not exceed C_1 (a universal constant) and in fact it happens for "most of" choices of B. In order to show this it is enough to prove that if q: $l_{5n}^1\to B$ is the quotient map, then $\ker q$ is "usually" nearly Euclidean, i.e. $d(\ker q, l_n^2)\leq C_2$ for some universal constant C_2 (by [11], [14] or [15], cf. [4]). To show this identify $\mathbf{R}^{5n}$ with $\mathbf{R}^{4n}\oplus\mathbf{R}^n$ and observe that $\ker q=\{(-\tilde{G}x,x)\colon x\in R_n\}$, where $\tilde{G}$ is the random $4n\times n$ matrix, whose jth column is g_j. Clearly it is enough to prove that, for "most of" $\tilde{G}s$,

$$\|(-\tilde{G}x,x)\|_2\leq\|\tilde{G}x\|_2+\|x\|_2\leq C_2/\sqrt{5n}\,\|(-\tilde{G}x,x)\|_1 \tag{4.1}$$

for all $x\in\mathbf{R}^n$. Estimating $\|\tilde{G}x\|_2$ is easy, since, by our construction, im $\tilde{G}$ is a random subspace of $\mathbf{R}^{4n}$ and hence $\|\cdot\|_1$ and $\|\cdot\|_2$ are roughly proportional on it (see [9] or [17]). To estimate $\|x\|_2$ we may copy the argument from [17]. First we show that, for every $x\in\mathbf{R}^n$ and $r>0$,

$$\mathcal{P}(\{\|\tilde{G}x\|_1\leq r\sqrt{n}\,\|x\|_2\})=\mathcal{P}(\{\|g_1\|_1\leq r\sqrt{n}\})\leq(C_3 r)^n$$

by Lemma 6.5 from this paper and considerations similar to the proof of Claim 6.1. Then, combining the above with Lemma 1 from [17] we conclude that, for "most of" $\tilde{G}s$,

$$\mu(\{x\in S_{n-1}:\|x\|_1\leq r\sqrt{n} \text{ and } \|\tilde{G}x\|_1\leq r\sqrt{n}\})\leq(C_4 r)^{2n},$$

where μ is the normalized Haar measure on S_{n-1}. This in turn, by Lemma 3 from [17], implies (4.1) (we must only choose $r<(1/2)\,C_4^{-2}$).

Remark 4.6. Since some refinement of the argument from [5] can be used to our class $\mathcal{B}$ and its variants discussed in section 9 (we use Lemma 6.5 from this paper instead of Lemma 1 from [5]), the following statement is true:

"Given n there exist n-dimensional normed spaces B_1 and B_2 such that, for some universal constant C,

(1) *the cotype 2 constants of* B_1 *and* B_2 *do not exceed* C,

(2) $d(B_1, b_2) \geqslant C^{-1}n$,

(3) *for* $i=1,2$, $\|P: B_i \to B_i\| \geqslant C^{-1}\sqrt{n}$ *for projections* P *such that* $0.01n \leqslant \operatorname{rank} P \leqslant 0.99n$".

5. Strategy of the proof of Proposition 4.1 and preliminary reduction

From this point on we fix n. Proposition 4.1 will follow from the following two facts.

Fact 5.1. *Let* $Z \subset B(L(l_m^1))$. *Then, for each* $\varepsilon \in (0,1)$, *there is an* $\varepsilon/\sqrt{m}$*-net—in the* $L(l_m^2)$*-norm—of* Z, *whose cardinality does not exceed* $(C_0/\varepsilon)^{m^2}$.

In the sequel we shall frequently denote, for fixed $U \in O(4n)$,

$$G = UF \ (F \overset{df}{=} \operatorname{span}\{e_1, \dots, e_n\}) \quad \text{and} \quad Q = P_{G^\perp} (= I - P_G).$$

In particular, for given $(U_0, \omega_0) \in \Omega$, we shall mean by G and Q the objects related to U_0 by the above convention. One can visualise G as $\operatorname{span}\{g_1, \dots, g_n\}$ (they coincide with probability 1).

Fact 5.2. *Let* T *be an operator on* $\mathbf{R}^{4n}$ *such that*

(i) rank $T \leqslant 3n$ *and*

(ii) rank$(I-T) \leqslant 3n$.

Then, for every $K \in (0, \sqrt{n})$,

$$\mathscr{P}(\{B: QTg_j \in 2KQB_{4n}^1 \ \ for \ \ j=1,\dots,n\}) \leqslant \left(C_1 \frac{K}{\sqrt{n}}\right)^{C_2 n^2}.$$

Fact 5.1 is essentially contained in Gluskin's paper [5] (for the sake of completeness we sketch the proof of it below). Also the general strategy of the proof of Proposition 4.1 is similar to that of [5]. Namely, in order to show that, for some $B \in \mathscr{B}$, *no* operator on B, which is a projection of rank $2n$, has small norm, we prove two statements:

(1) a given projection (which corresponds somehow to T from Fact 5.2) is "good" for a "very few" $B \in \mathscr{B}$ (Fact 5.2),

(2) there are not that many projections (or operators from $B(L(l_{4n}^1))$ associated with them) we must consider (Fact 5.1).

Hence for "most of" $B \in \mathscr{B}$ no projection is "good".

Sketch of the proof of Fact 5.1. It is enough to show that

(a) $1/\sqrt{m}\, B(L(l^2_m)) \subset B(L(l^1_m))$

(b) $\mathrm{vol}\,(B(L(l^1_m))) \leq \left(\frac{C_0}{3}\right)^{m^2} \mathrm{vol}\,(1/\sqrt{m}\, B(L(l^2_m)))$

and then the same construction which yields an ε-net of a unit ball of a k-dimensional normed space of cardinality not exceeding $(3/\varepsilon)^k$, leads to the desired conclusion.

Now (a) is obvious. The left hand side of (b) may be computed directly and shown to be of the same order as $\mathrm{vol}\, B^2_{m^2}$ (up to factor of type C^{m^2}). Estimating the right hand side of (b) is precisely the content of the proposition from [5]. The following argument is due to S. Kwapień and N. Tomczak-Jaegermann and has been in circulation for some time. We have

$$\frac{\mathrm{vol}\, B(L(l^2_m))}{\mathrm{vol}\, B^2_{m^2}} = \int \frac{dx}{\|(x_{ij})\|^{m^2}} \geq \left(\int \frac{dx}{\|(x_{ij})\|^2}\right)^{m^2/2} \geq \left(\int \|(x_{ij})\|^2\right)^{-m^2/2},$$

where $\|(x_{ij})\|$ is the $L(l^2_m)$-norm of an $m\times m$ matrix (x_{ij}) and the integration is with respect to the normalized measure on S_{m^2-1} and $l^2_{m^2}$ being identified with the space of $n\times n$ matrices equipped with the (Hilbert-Schmidt) norm $(\Sigma_{i,j}|x_{ij}|^2)^{1/2}$. By the standard argument, we have

$$\int \|(x_{ij})\|^2\, dx = \frac{1}{m^2}\int \|(g_{ij})\|^2\, d\mathcal{P},$$

where g_{ij}, $1 \leq i,j \leq m$, are independent real Gaussian variables with distribution $N(0, 1)$. In turn, by Lemma 3.1(2), [2] or by a much more elementary direct argument,

$$\int \|(g_{ij})\|^2\, d\mathcal{P} \leq 8m.$$

Combining the last three formulas we obtain the desired estimate. Q.E.D

We postpone the proof of Fact 5.2 to sections 6–8.

Proof of Proposition 4.1. We shall prove Proposition 4.1 with

$$c_1 = [C_1(2\cdot(2C_0)^{16})^{1/C_2}]^{-1} \tag{5.6}$$

where C_0, C_1 and C_2 come from Facts 5.1 and 5.2.

To this end set $K=c_1\sqrt{n}$, c_1 as above. Let Z' be the set of all linear operators T on $\mathbf{R}^{4n}$ satisfying the conditions (i), (ii) from Fact 5.2 such that $\|T\|_{L(l^1_{4n})}\leq K$. Applying Fact 5.1 with $Z=K^{-1}Z'$, $m=4n$ and $\varepsilon=1/2$ we see that there exists a finite subset $\mathcal{N}$ of Z, $\#\mathcal{N}\leq(2C_0)^{16n^2}$, which is a $K/(2\sqrt{4n})$-net of Z' in the $L(l^2_{4n})$-norm. For each $T\in\mathcal{N}$ let $\mathcal{B}_T=\{B\in\mathcal{B}: QTg_j\in 2KQB^1_{4n} \text{ for } j=1,\dots,n\}$.

Then, by Fact 5.2,

$$\mathcal{P}(\mathcal{B}_T)\leq\left(C_1\frac{K}{\sqrt{n}}\right)^{C_2n^2}$$

and hence

$$\mathcal{P}\left(\bigcup_{T\in\mathcal{N}}\mathcal{B}_T\right)\leq\#\mathcal{N}\cdot\left(C_1\frac{K}{\sqrt{n}}\right)^{C_2n^2}\leq(2C_0)^{16n^2}\left(C_1\frac{K}{\sqrt{n}}\right)^{C_2n^2}$$

which in turn, by (5.6) and the choice of K, equals $(1/2)^{n^2}$. Hence

$$\mathcal{P}(\mathcal{B}\setminus\bigcup_{T\in\mathcal{N}}\mathcal{B}_T)\geq 1-(1/2)^{n^2}$$

Let $\mathcal{B}_0=\{B\in\mathcal{B}: B\not\subset 2B^2_{4n}\}=\{B:\|g_j\|_2>2 \text{ for some } j\leq n\}$. Then by (3.2),

$$\mathcal{P}(\mathcal{B}_0)\leq ne^{-c\cdot 4n}\leq e^{-c'n},\quad c'>0,$$

and hence

$$\mathcal{P}(\mathcal{B}\setminus\bigcup_{T\in\mathcal{N}}\mathcal{B}_T\setminus\mathcal{B}_0)\geq 1-(1/2)^{n^2}-e^{-c'n}.$$

Fix $B\in\mathcal{B}\setminus\bigcup_{T\in\mathcal{N}}\mathcal{B}_T\setminus\mathcal{B}_0$. We shall show that the norm of every rank $2n$ projection on B is greater than K $(=c_1\sqrt{n})$—this, by the estimate above, will prove Proposition 4.1.

Suppose not. Let P, rank $P=2n$, be a projection on $\mathbf{R}^{4n}$ such that

$$\|P: B\to B\|\leq K.$$

In particular

$$Pg_j\in KB\quad\text{for } j=1,\dots,n,$$

and, as a consequence,

$$QPg_j\in KQB\quad\text{for } j=1,\dots,n.$$

Now $B=\mathrm{ac}(B^1_{4n} \cup \{g_1, \ldots, g_n\})$ and $Qg_j=0$ for $j=2, \ldots, n$ show that $QB=QB^1_{4n}$, so we have

$$QP\,g_j \in KQB^1_{4n} \quad \text{for } j=1, \ldots, n. \tag{5.7}$$

Similarly we obtain $QP\,e_i \in KQB^1_{4n}$ for $j=1, \ldots, 4n$, or, in other words,

$$\|QP{:}B^1_{4n} \to QB^1_{4n}\| \leq K.$$

In turn, the above can be written as

$$\|qP{:}\, l^1_{4n} \to l^1_{4n}/G\| \leq K$$

where q: $l^1_{4n} \to l^1_{4n}/G$ is a quotient map, because l^1_{4n}/G can be realized as $Q\mathbf{R}^{4n}=G^\perp$ with the norm generated by QB^1_{4n}.

Now, by lifting, we can find an operator T_0 on $\mathbf{R}^{4n}$ such that

$$qP = qT_0 \tag{5.8}$$

$$\left|\left|T_0{:}\, l^1_{4n} \to l^1_{4n}\right|\right| \leq K. \tag{5.9}$$

Clearly (5.8) may be written as

$$QP = QT_0. \tag{5.10}$$

Observe that, since $T_0=QT_0+P_G T_0=QP+P_G T_0$, $\mathrm{rank}T_0 \leq \mathrm{rank}QP+\mathrm{rank}P_G \leq \mathrm{rank}P+\dim G=2n+n=3n$. On the other hand, since $Q(I-T_0)=Q-QT_0=Q-QP=Q(I-P)$ and $\mathrm{rank}(I-P)=2n$, the same argument yields that $\mathrm{rank}(I-T_0) \leq 3n$. This, together with (5.9), shows that $T_0 \in Z'$ and hence there exists $T_1 \in \mathcal{N}$ such that

$$\left|\left|T_1-T_0\right|\right|_{L(l^2_{4n})} \leq \frac{K}{2\sqrt{4n}}. \tag{5.11}$$

By the choice of B, $B \notin \mathcal{B}_{T_1}$. Hence

$$QT_1\, g_i \notin 2KQB^1_{4n} \quad \text{for some } i \leq n. \tag{5.12}$$

On the other hand we have, for every $j \leq n$,

$$\begin{aligned} \|QT_1\, g_j - QPg_j\|_2 &= \|QT_1\, g_j - QT_0\, g_j\|_2 \leq \|T_1\, g_j - T_0\, g_j\|_2 \\ &\leq \|T_1 - T_0\|_{L(l^2_{4n})} \|g_j\|_2 \leq \frac{K}{2\sqrt{4n}} \cdot 2 = \frac{K}{\sqrt{4n}} \end{aligned}$$

(we used (5.10), (5.11) and the fact that $B\notin\mathscr{B}_0$ consequtively), which is the same as

$$(QT_1 g_j - QPg_j) \in \frac{K}{\sqrt{4n}} B^2_{4n} \cap G^\perp \quad \text{for every } j\leq n. \tag{5.13}$$

Now, since $(1/\sqrt{4n})B^2_{4n} \subset B^1_{4n}$, we have

$$\frac{K}{\sqrt{4n}} B^2_{4n} \cap G^\perp \subset KB^1_{4n} \cap G^\perp \subset KQB^1_{4n},$$

which combined with (5.13) gives

$$(QT_1 g_j - QPg_j) \in KQB^1_{4n} \quad \text{for } j=1, \dots, n.$$

Now the above and (5.7) contradict (5.12). This shows that $\|P\colon B\to B\| > K = c_1\sqrt{n}$, ending the proof of Proposition 4.1. Q.E.D.

It remains to prove Fact 5.2.

6. Proof of Fact 5.2 : 1st reduction

The purpose of this section is to reduce Fact 5.2 to Proposition 6.8.

By definition of Ω, the probability in the assertion of Fact 5.2 equals

$$\int_{O(4n)} \mathscr{P}_0(\{QTg_j \in 2KQB^1_{4n} \quad \text{for } j = 1, \dots, n\})\mu(dU). \tag{6.1}$$

Now, for fixed $U\in O(4n)$ (and hence fixed $G=UF$ and $Q=P_{G^\perp}$), $g_j = Uf_j$ are independent Gaussian variables with distribution $N(0, 1, G)$. So (6.1) may be written as

$$\int_{O(4n)} [\mathscr{P}(\{QTg \in 2KQB^1_{4n}\})]^n \, \mu(dU), \tag{6.2}$$

where, for each U, $g=g_U$ is a Gaussian variable with distribution $N(0, 1, G)$.

Let us introduce some notation. Given a linear operator $W\colon G\to H$ (G, H Hilbert spaces) and k, $\varepsilon\geq 0$ we shall say that W is (k,ε)-thick (resp. (k,ε)-thin) if, for some subspace G_1 of G, $\dim G_1\geq k$, $|Wx|\geq\varepsilon|x|$ for $x\in G_1$ (resp. $|Wx|<\varepsilon|x|$ for $x\in G_1\setminus\{0\}$), where $|\cdot|$ denotes the corresponding Hilbert norm.

The following simple and well-known lemma explains the meaning of these notions.

LEMMA 6.1. *Let $W\colon G\to H$, k and ε be as above.*
Let $\Sigma_j \lambda_j \langle u_j, \cdot\, \rangle v_j$ be the polar decomposition of W. Then

(a) *W is (k, ε)-thick iff $\#\{j\colon \lambda_j\geq\varepsilon\}\geq k$*

(b) *W is (k, ε)-thin iff $\#\{j\colon \lambda_j<\varepsilon\}\geq k$.*

Fact 5.2 will immediately follow from the following two facts.

Claim 6.2. *Let ε, $K'>0$. Let W: $G\to H$, $\dim G=n$, be $(n/3,\varepsilon)$-thick. Let $A\subset H$ be of the form $A=\mathrm{ac}\{y_1,\dots,y_{4n}\}$, $|y_j|\leq K'$ for $j\leq 4n$. Finally, let g be a Gaussian variable with distribution $N(0,1,G)$. Then*

$$\mathcal{P}(\{Wg\in A\})\leq\left(\frac{c_1K'}{\varepsilon\sqrt{n}}\right)^{n/3}.$$

Claim 6.3. *Let $\varepsilon\geq 0$; $T, G=UF$ and Q as in Fact* 5.2. *Then*

$$\mu(\{U\in O(4n)\colon QT|_G \text{ is not } (n/3,\varepsilon)\text{-thick}\})\leq(c_2\varepsilon)^{n^2/9}.$$

Indeed, suppose we have proved Claims 6.2 and 6.3. Then, applying Claim 6.2 with $K'=2K$ and $A=QB^1_{4n}$ we see that if, for some G and $\varepsilon>0$, $QT|_G$ is $(n/3,\varepsilon)$-thick, then

$$\mathcal{P}(\{QTg\in 2KQB^1_{4n}\})\leq\left(\frac{2c_1K}{\varepsilon\sqrt{n}}\right)^{n/3}.$$

Combining this with Claim 6.3 we conclude that, for every ε, (6.2) is not bigger than

$$\left(\frac{2c_1K}{\varepsilon\sqrt{n}}\right)^{n^2/3}+(c_2\varepsilon)^{n^2/9}.$$

Now the choice $\varepsilon=(2c_1K/c_2\sqrt{n})^{1/2}$ yields the desired estimate. Q.E.D

Remark 6.4. There is nothing special about the number 1/3 in Claims 6.2 and 6.3. One can prove their analogues with any $a\in(0,1)$ instead of 1/3 and conclude the proof in essentially the same way.

For the proof of Claim 6.1 we need two further lemmas.

LEMMA 6.5. *Let $A=\mathrm{ac}\{x_1,x_2,\dots,x_M\}\subset\mathbf{R}^m$ with $\|x_j\|_2\leq 1$ for $j=1,2,\dots,m$. Then*

$$\frac{\mathrm{vol}\,A}{\mathrm{vol}\,B^2_m}\leq\left(\frac{c_3M}{m^{3/2}}\right)^m.$$

Proof. Standard and well-known (see e.g. [10] or [18]). We have (cf. proof of (5.1))

$$\mathrm{vol}\,A\leq\binom{M}{m}\mathrm{vol}\,B^1_m\leq\binom{M}{m}\left(\frac{2e}{\pi}\right)^{m/2}\mathrm{vol}\,B^2_m$$

and it is enough to observe that

$$\binom{M}{m} \leqslant \left(\frac{Me}{m}\right)^m$$

to obtain Lemma 6.5 with $c_3=\sqrt{2e^3/\pi}$. Q.E.D.

LEMMA 6.6. *Let E be a Hilbert space,* $\dim E=m$, *and let g_0 be a Gaussian variable with distribution $N(0, 1, E)$. Let $A \subset E$ be a measurable set and let L be an invertible operator on E. Then*

$$\mathcal{P}(\{Lg_0 \in A\}) \leqslant c_4^m \frac{\operatorname{vol} A}{\operatorname{vol} B_m^2} (\det L)^{-1}.$$

Proof. $\mathcal{P}(\{Lg_0 \in A\}) = \mathbf{E}\chi_{\{Lg_0 \in A\}} = \int_{\{Lx \in A\}} \left(\frac{m}{2\pi}\right)^{m/2} e^{-\frac{m|x|^2}{2}} dx$

$$= \int_A \left(\frac{m}{2\pi}\right)^{m/2} e^{-(m|L^{-1}y|^2)/2} (\det L)^{-1} dy$$

$$\leqslant \left(\frac{m}{2\pi}\right)^{m/2} \operatorname{vol} A (\det L)^{-1} \leqslant c_4^m \frac{\operatorname{vol} A}{\operatorname{vol} B_m^2} (\det L)^{-1},$$

where $|\cdot|$ is the corresponding Hilbert norm, integration is with respect to m-dimensional Lebesgue measure and

$$\operatorname{vol} B_m^2 = \pi^{m/2} / \Gamma\left(\frac{m}{2}+1\right).$$ Q.E.D.

Proof of Claim 6.2. Fix ε, K, W and A satisfying the assumptions of Claim 6.2. Let $W=\Sigma_{j=1}^n \lambda_j \langle u_j, \cdot \rangle v_j$ be the polar decomposition. Let $m=\#\{j: \lambda_j > \varepsilon\}$. By Lemma 6.1, $m \geqslant n/3$. Denote $E=\operatorname{span}\{v_1, \dots, v_m\}$.

Observe that

(1) if $Wg \in A$, then also $P_E Wg \in P_E A$,

(2) since $P_E W=\Sigma_{j=1}^m \lambda_j \langle u_j, \cdot \rangle v_j$, we can write $P_E W=\Lambda P_E V$, where $\Lambda \overset{df}{=} \Sigma_{j=1}^m \lambda_j \langle v_j, \cdot \rangle v_j$, and V: $G \to H$ is the isometry defined by $Vu_j=v_j$, $j=1, \dots, n$. Hence, by (3.3), the distribution of $P_E W$ is the same as the distribution of $\sqrt{m/n}\,\Lambda g_0$, where g_0 is a Gaussian variable with distribution $N(0, 1, E)$.

In view of (1) and (2) it is enough to estimate

$$\mathcal{P}(\{\sqrt{m/n}\,\Lambda g_0 \in P_E A\}),$$

which, by Lemma 6.6 applied with $L=\sqrt{m/n}\,\Lambda$ and $P_E A$ instead of A, does not exceed

$$c_4^m(\sqrt{m/n})^{-m}(\det\Lambda)^{-1}\,\frac{\operatorname{vol}(P_E A)}{\operatorname{vol} B_m^2}.$$

Now, by our assumptions, $\det\Lambda\geqslant\varepsilon^{-m}$ (because $\lambda_j\geqslant\varepsilon$ for $j=1,\dots,m$) and $m\geqslant n/3$. So the above, and hence also the probability from Claim 6.2, is not greater than

$$\left(\frac{c_4\sqrt{3}}{\varepsilon}\right)^m\frac{\operatorname{vol}(P_E A)}{\operatorname{vol} B_m^2}.$$

Notice that $P_E A=\operatorname{ac}\{P_E y_j,\ j=1,2,\dots,4n\}$ and $|y_j|\leqslant K'$ for $j=1,\dots,4n$. This, by Lemma 6.5, shows that

$$\frac{\operatorname{vol}(P_E A)}{\operatorname{vol} B_m^2}\leqslant\left(\frac{c_3\cdot 4n\cdot K'}{m^{3/2}}\right)^m.$$

Inserting this estimate to the previous one we see that that probability from Claim 6.2 is at most

$$\left(\frac{4\sqrt{3}\,c_3c_4K'n}{\varepsilon\, m^{3/2}}\right)^m\leqslant\left(\frac{36\,c_3c_4K'}{\varepsilon\sqrt{n}}\right)^m$$

which proves Claim 6.2 with $c_1=36c_3c_4$ (remember that $m\geqslant n/3$). Q.E.D.

Proof of Claim 6.3. Denote $S=I-T$, $H_1=(\operatorname{im}S)^\perp$ and $H_2=(\operatorname{im}T)^\perp$. Observe that

$$\dim H_i\geqslant n \quad \text{for } i=1,2. \tag{6.3}$$

Indeed, $\dim H_1=4n-\operatorname{rank}S\geqslant 4n-3n$ (by assumption (ii), Fact 5.2). The same argument applies to H_2.

Our present goal is to prove (as a step towards showing Claim 6.3)

Claim 6.7. *If, for some G and ε, $QT|_G$ is not $(n/3,\ \varepsilon)$-thick, then either $P_{H_1}|_G$ or $P_{H_2}|_G$ is $(n/3,\ 2\varepsilon)$-thin.*

Proof of Claim 6.7. Suppose not; let for some G and ε neither $P_{H_1}|_G$ nor $P_{H_2}|_G$ be $(n/3,\ 2\varepsilon)$-thin.

Let $\Sigma_1^n\alpha_j\langle x_j,\ \cdot\ \rangle z_j$ be the polar decomposition of $P_{H_1}|_G$. Then, by Lemma 6.1, the fact that $P_{H_1}|_G$ is not $(n/3,\ 2\varepsilon)$-thin implies

$$k\overset{\mathrm{df}}{=}\#\{j\colon\alpha_j\geqslant 2\varepsilon\}>\frac{2n}{3}. \tag{6.4}$$

Let $G'=\operatorname{span}\{x_1,\dots,x_k\}$. Clearly $\|P_{H_1}x\|_2\geq 2\varepsilon\|x\|_2$ if $x\in G$. Moreover, if $x\in G'$ and $y\in G$ with $y\perp x$, then

$$\|P_{H_1}(x+y)\|_2\geq 2\varepsilon\|x\|_2. \tag{6.5}$$

To show this, denote $y'=P_{G'}y$ and $H_1'=\operatorname{span}\{z_1,\dots,z_k\}\subset H_1$. Clearly $\langle x,y'\rangle=\langle x,y\rangle=0$ and hence

$$\|P_{H_1}(x+y)\|_2\geq\|P_{H_1'}(x+y)\|_2=\|P_{H_1}(x+y')\|_2$$
$$\geq 2\varepsilon\|x+y'\|_2\geq 2\varepsilon\|x\|_2$$

as required. Similarly we show that the assumption $P_{H_2}|_G$ is not $(n/3,\,2\varepsilon)$-thin implies

$$\|P_{H_2}(x+y)\|_2\geq 2\varepsilon\|x\|_2. \tag{6.6}$$

for $x\in G''$ and $y\in G$ with $y\perp x$, where G'' is some subspace of G, $\dim G''\overset{df}{=}p>2n/3$.

We prove now that if $x\in G'\cap G''$ with $\|x\|_2=1$, then

$$\|QTx\|_2\geq\varepsilon. \tag{6.7}$$

Since $\dim G'\cap G''\geq k+p-n>2n/3+2n/3-n=n/3$, this contradicts the assumption that $QT|_G$ is not $(n/3,\,\varepsilon)$-thick, thus proving Claim 6.7.

Once again suppose (6.7) does not hold. Then for some $x\in G'\cap G''$, $\|x\|_2=1$ and $\|QTx\|_2<\varepsilon$. In other words, for some real t and $y\in G$, $y\perp x$,

$$\varepsilon>\|Tx+tx+y\|_2=\|(1+t)\,Tx+tSx+y\|_2$$

(because $S+T=I$). Assume first that $|t|\leq 1/2$. Then $1+t\geq 1/2$ and

$$\varepsilon>\|(1+t)\,Tx+tSx+y\|_2\geq\|P_{H_1}[(1+t)\,Tx+tSx+y]\|_2$$
$$=\|P_{H_1}[(1+t)\,x+y]\|_2=(1+t)\,\|P_{H_1}(x+(1+t)^{-1}\,y)\|_2\geq\frac{1}{2}\|P_{H_1}(x+(1+t)^{-1}\,y)\|_2$$

which, by (6.5), contradicts $x\in G'$ (we used the identities $P_{H_1}S=P_{(\operatorname{im}S)^\perp}S=0$ and $P_{H_1}T=P_{H_1}$). Similarly, assuming $|t|\geq 1/2$ and using P_{H_2} and (6.6) instead of P_{H_1} and (6.5) we obtain a contradiction to $x\in G''$. This shows (6.7) and concludes the proof of Claim 6.7. Q.E.D.

Let us return to the proof of Claim 6.3. Claim 6.7 shows that it is enough to prove that

$$\mu(\{U\in O(4n)\colon P_{H_i}|_G \text{ is } (n/3,\, 2\varepsilon)\text{-thin}\}) \leqslant (C_2'\,\varepsilon)^{n^2/9}$$

for $i=1, 2$. Arguments in both cases are the same, one uses only (6.3). In other words, everything reduces to the following:

PROPOSITION 6.8. *Let $E\subset\mathbf{R}^{4n}$, $\dim E\geqslant n$ and let $\delta>0$. Then*

$$\mu(\{U\in O(4n)\colon \left\|P_E|_{G_0}\right\| \leqslant \delta \text{ for some } G_0\subset G=UF,\ \dim G_0 \geqslant n/3\})\leqslant(c_2''\delta)^{n^2/9}$$

where $\|\cdot\|$ is the operator norm with respect to the l_{4n}^2-norm.

7. Proof of Proposition 6.8

We shall prove the slightly more general

PROPOSITION 7.1. *Let $F=\operatorname{span}\{e_1,\dots,e_n\}\subset\mathbf{R}^m$, let $E\subset\mathbf{R}^m$, $\dim E=p$, and let $\delta\in(0,1)$. Then, for every positive integer q,*

$$\mu\left(\left\{\begin{matrix} U\in O(m)\colon \text{there exists } G_0\subset UF, \\ \dim G_0\geqslant q, \text{ such that} \left\|P_E|_{G_0}\right\| \leqslant\delta\end{matrix}\right\}\right)\leqslant C^{m^2}\delta^{q(p-n-q)}.$$

Clearly Proposition 6.8 is a special case of Proposition 7.1 with $q=n/3$ and $m=4n$. Before passing to the proof of Proposition 7.1 we must introduce some notation (for more details see section 8).

Let $G_{k,m}$ be the Grassmann manifold (i.e. the set of k-dimensional subspaces of $\mathbf{R}^m$), ν the normalized Haar measure on it, induced by the action of $O(m)$, and let ϱ be the metric on $G_{k,m}$ defined by

$$\varrho(H_1, H_2) = \inf_{V\in O(m)\colon VH_1=H_2} \|I-V\|.$$

Here and throughout this section $\|\cdot\|$ will always denote the operator norm with respect to l_m^2-norm or restrictions thereof.

It is easy to observe that $\varrho(H_1, H_2)$ is the same as the Hausdorff distance between $S_{m-1}\cap H_1$ and $S_{m-1}\cap H_2$. In particular it depends only on the position of H_1 in relation to H_2 and not on a "superspace" containing both of them. In the sequel we shall not distinguish between metrics obtained by considering different such "super-spaces".

For the proof of Proposition 7.1 we need two further lemmas.

LEMMA 7.2. *For every* $\eta \in [0, \sqrt{2}]$ *and* $H_0 \in G_{n,m}$ *we have*

$$C_1^{m^2}\,\eta^{n(m-n)} \leq \nu(\{H\colon \varrho(H,H_0) \leq \eta\}) \leq C_2^{m^2}\,\eta^{n(m-n)}.$$

Observe that:

(1) $n(m-n) = \dim G_{n,m}$,

(2) by invariance of ν and ϱ with respect to the action of $O(m)$, the measure in the assertion does not depend on H_0,

(3) $\sqrt{2} = \operatorname{diam} G_{n,m}$.

LEMMA 7.3. *Let, for some* $\eta \in (0, \sqrt{2})$, $\mathcal{N}$ *be an* η*-net of* $G_{n,m}$ *(with respect to* ϱ*) of minimal cardinality. Then*

$$C_1^{m^2}\,\eta^{-n(m-n)} \leq \#\mathcal{N} \leq C_2^{m^2}\,\eta^{-n(m-n)}.$$

Since, as one can easily show, if $\mathcal{N}$ satisfies the assumptions of Lemma 7.3, then

$$\nu(\{H\colon \varrho(H,H_0) \leq \eta\})^{-1} \leq \#\mathcal{N} \leq \nu(\{H\colon \varrho(H,H_0) \leq \eta/2\})^{-1},$$

it is enough to prove one of the two lemmas above and the other one will automatically follow. We shall prove Lemma 7.3 (in section 8).

Proof of Proposition 7.1. Let us identify $G_{q,n}$ with the set of all q-dimensional subspaces of $F \overset{df}{=} \operatorname{span}\{e_1, \dots, e_n\}$. Then, by Lemma 7.3, this set admits a δ-net $\mathcal{N}_1$ with

$$\#\mathcal{N}_1 \leq C_2^{n^2}\,\delta^{-q(n-q)}. \tag{7.1}$$

Now let $U \in O(m)$ and $G = UF \in G_{n,m}$ satisfy the condition from Proposition 7.1, i.e.

$$\|P_E|_{G_0}\| \leq \delta \text{ for some } G_0 \subset G, \dim G_0 = q.$$

Since $U\mathcal{N}_1$ is clearly a δ-net in the set of q-dimensional subspaces of G, we can find $H \in \mathcal{N}_1$ and an isometry V on G, $\|V-I\| \leq \delta$, such that $VG_0 = UH$. Hence

$$\begin{aligned}\|P_E|_{UH}\| &= \|P_E|_{VG_0}\| = \|P_E V|_{G_0}\| \leq \|P_E|_{G_0}\| + \|P_E(I-V)|_{G_0}\| \\ &\leq \delta + \delta = 2\delta.\end{aligned}$$

This shows that the measure in the assertion of Proposition 7.1 does not exceed

$$\mu\left(\bigcup_{H\in\mathcal{N}_1}\{U:\|P_E|_{UH}\|\leq 2\delta\}\right)\leq\sum_{H\in\mathcal{N}_1}\mu(\{U:\|P_E|_{UH}\|\leq 2\delta\}).$$

The measure after the summation sign clearly does not depend on H, so the sum equals $\#\mathcal{N}_1$ times the single term. Besides, for fixed H, $\|P_E|_{UH}\|$ depends only on UH and not on the particular choice of $U\in O\,(m)$.

Hence the sum above equals

$$\#\mathcal{N}_1\cdot\nu(\{H\in G_{q,m}:\|P_E|_H\|\leq 2\delta\}). \tag{7.2}$$

We need the following elementary

LEMMA 7.4. *Let E and H_1 be subspaces of $\mathbf{R}^m$ and $\dim H_1+\dim E\leq m$. Then there exists $H_2\subset E^\perp$, $\dim H_2=\dim H_1$, such that $\varrho(H_1,H_2)\leq\sqrt{2}\,\|P_E|_{H_1}\|$.*

Proof. Denote $d=\|P_E|_{H_1}\|$. We can assume that $0<d<1$ (otherwise the assertion is satisfied trivially). Let $H_2=P_{E^\perp}H_1$, then $\dim H_2=\dim H_1$. Choose $x\in H_1$, $\|x\|_2=1$, such that $\operatorname{dist}(x,S_{m-1}\cap H_2)=\varrho(H_1,H_2)$. Then we have

$$d\geq\|P_E x\|_2=\|x-P_{E^\perp}x\|_2\overset{df}{=}d_1.$$

Now let $y=P_{E^\perp}x/\|P_{E^\perp}x\|_2$. Of course $y\in H_2$, $\|y\|_2=1$ and

$$\varrho(H_1,H_2)\leq\|x-y\|_2=(d_1^2+[1-(1-d_1^2)^{1/2}]^2)^{1/2}=\sqrt{2}\,(1-(1-d_1^2)^{1/2})^{1/2}\leq\sqrt{2}\,d_1\leq\sqrt{2}\,d.$$

This proves Lemma 7.4. Q.E.D.

Now, to estimate (7.2), consider $H\in G_{q,m}$ such that $\|P_E|_H\|\leq 2\delta$. Clearly we can assume that $\dim E+\dim H=p+q\leq m$ (otherwise $\|P_E|_{G_0}\|=1$ for every G_0 with $\dim G_0=q$ and Proposition 7.1 holds trivially with $C=1$). Then, by Lemma 7.4, there exists a q-dimensional subspace H' of $E^\perp$ such that $\varrho(H,H')\leq 2\sqrt{2}\,\delta<3\delta$. By Lemma 7.3, there exists a δ-net $\mathcal{N}_2$ of the set of all q-dimensional subspaces of $E^\perp$ (identified with $G_{q,m-p}$) with

$$\#\mathcal{N}_2\leq C_2^{m^2}\,\delta^{-q(m-p-q)}. \tag{7.3}$$

Choose $H''\in\mathcal{N}_2$ so that $\varrho(H',H'')\leq\delta$ and hence $\varrho(H,H'')\leq 3\delta+\delta=4\delta$.

Then we have

$$\{H \in G_{q,m} : \|P_E|_H\| \leq 2\delta\} \subset \bigcup_{H'' \in \mathcal{N}_2} \{H \in G_{q,m} : \varrho(H, H'') \leq 4\delta\}. \tag{7.4}$$

By Lemma 7.2, the measure of the latter set doesn't exceed

$$\#\mathcal{N}_2 \cdot C_2^{m^2} (4\delta)^{q(m-q)} \leq \#\mathcal{N}_2 \cdot (4C_2)^{m^2} \delta^{q(m-q)}. \tag{7.5}$$

Combining (7.4) and (7.5) we see that (7.2), and hence the measure from the assertion of Proposition 7.1, is not bigger than

$$\#\mathcal{N}_1 \cdot \#\mathcal{N}_2 \cdot (4C_2)^{m^2} \delta^{q(m-q)}.$$

By (7.1) and (7.3), the above is at most

$$c_2^{m^2} \delta^{-q(n-q)} c_2^{m^2} \delta^{-q(m-p-q)} (4C_2)^{m^2} \delta^{q(m-q)} = (4c_2^2 C_2)^{m^2} \delta^{q(p-n+q)}$$

which proves Proposition 7.1 with $C = 4c_2^2 C_2$. Q.E.D.

It remains to prove Lemma 7.3.

8. Appendix on Grassmann manifolds

The main objective of this section is to prove (for notation see below)

LEMMA 7.3. *Let, for some positive integers m, n and $\eta \in (0, \sqrt{2})$, $\mathcal{N}$ be an η-net of $G_{n,m}$ (with respect to ϱ) of minimal cardinality. Then*

$$c_1^{m^2} \eta^{-n(m-n)} \leq \#\mathcal{N} \leq c_2^{m^2} \eta^{-n(m-n)},$$

where c_1 and c_2 are constants independent of m, n and η.

Remark 8.1. Independence of c_1, c_2 of m and n is the main point in Lemma 7.3. A similar statement with c_1, c_2 depending on m and n, but independent of η, is very easy to prove.

At the end of the section we list (without proofs) a number of strengthenings and generalizations of Lemma 7.3 and their analogues in the case of orthogonal group (Remarks 8.4–8.6). More specifically, we give exact order of minimal cardinalities of ε-nets of $O(m)$ (resp. $G_{n,m}$) with respect to unitary ideal norms (resp. their quotients). The proofs and a more systematic exposition of the subject may be found in [19].

Let $G_{n,m}$ be the Grassmann manifold, i.e. the set of all n-dimensional subspaces of $\mathbf{R}^m$. To be more strict, denote $F=\text{span}\{e_1,\dots,e_n\}\subset\mathbf{R}^m$ and let $O(n;m)\overset{df}{=}\{V\in O(m): VF=F\}$. Clearly $O(n;m)$ is a subgroup of $O(m)$, isomorphic to $O(n)\times O(m-n)$. Now let us identify $G_{n,m}$ with $O(m)/O(n;m)$, the set of left cosets of $O(n;m)$, via $VF\sim VO(n;m)$. Denote by ν the Haar measure on $G_{n,m}$ resulting from this identification. Let $q:O(m)\to O(m)/O(n;m)=G_{n,m}$ be the quotient map and let ϱ be the quotient metric on $G_{n,m}$ induced by the operator norm on $O(m)$. More explicitly, for $H_1,H_2\in G_{n,m}$,

$$\varrho(H_1,H_2)=\inf\{\|I-V\|: V\in O(m),\ VH_1=H_2\}.$$

where we can consider H_1 and H_2 either as subspaces of $\mathbf{R}^m$ or cosets of $O(n;m)$ in $O(m)$ and $\|\cdot\|$ is the $L(l_n^2)$-norm (operator norm).

If K is a subset of a metric space and d the corresponding metric, we shall denote, for $\varepsilon>0$,

$$N(K,d,\varepsilon)=\inf\{\#\mathcal{N}:\mathcal{N}\text{ an }\varepsilon\text{-net of }K\text{ with respect to }d\}.$$

As a rule, we shall consider only $\varepsilon\in(0,\operatorname{diam}K]$. If d is induced by a norm $\|\cdot\|$, we may write $N(K,\|\cdot\|,\varepsilon)$ instead of $N(K,d,\varepsilon)$.

In the notation introduced above, Lemma 7.3 can be written as

LEMMA 7.3 A. *If $0<\eta\leq\sqrt{2}$, then*

$$c_1^{m^2}\,\eta^{-n(m-n)}\leq N(G_{n,m},\varrho,\eta)\leq c_2^{m^2}\,\eta^{-n(m-n)},$$

where c_1 and c_2 are independent of n,m and η.

We shall need the following simple facts about the function $N(\cdot,\cdot,\cdot)$.

LEMMA 8.2. (a) *Let (K_1,d_1) and (K_2,d_2) be metric spaces, $(K_1\times K_2,d_1\times d_2)$ their product with $d_1\times d_2((x_1,x_2),(y_1,y_2))\overset{df}{=}\max\{d_1(x_1,y_1),d_2(x_2,y_2)\}$. Then, for every $\varepsilon>0$,*

$$N(K_1,d_1,2\varepsilon)\,N(K_2,d_2,2\varepsilon)\leq N(K_1\times K_2,d_1\times d_2,\varepsilon)\leq N(K_1,d_1,\varepsilon)\,N(K_2,d_2,\varepsilon).$$

(b) *Let G be a compact group, H its subgroup, d an invariant metric on G and d' the corresponding quotient metric on G/H. Then, for every $\varepsilon>0$,*

$$\frac{N(G,d,2\varepsilon)}{N(H,d,\varepsilon)}\leq N(G/H,d',\varepsilon)\leq\frac{N(G,d,\varepsilon/2)}{N(H,d,\varepsilon)}.$$

(c) *Let* $(X, \|\cdot\|)$ *be a normed space,* $\dim X = D$, *and* (M, d) *a metric space. Let* $\Phi: X \to M$ *be a map satisfying, for some* $r, R, l, L > 0$,

(i) $\Phi(B_R) = M$

(ii) $\varrho(\Phi(x), \Phi(y)) \le L\|x-y\|$ *for* $x, y \in B_R$

(iii) $\varrho(\Phi(x), \Phi(y)) \ge l\|x-y\|$ *for* $x, y \in B_r$

where B_s *stands for the ball of radius s in X. Then, for every* $\varepsilon \in (0, \operatorname{diam} M]$,

$$(c'/\varepsilon)^D \le N(M, d, \varepsilon) \le (c''/\varepsilon)^D,$$

where $c' = rl/2$, $c'' = 3RL$.

The proofs of all parts of Lemma 8.2 are standard. Let us just mention that (c) follows almost immediately from the very well-known fact that, for any $\varepsilon, s, 0 < \varepsilon \le 2s$,

$$(s/\varepsilon)^D \le N(B_s, \|\cdot\|, \varepsilon) \le (3s/\varepsilon)^D.$$

For the rest of this section $\|\cdot\|$ will always denote the l_m^2 operator norm (the $L(l_m^2)$-norm).

Lemma 7.3 A will be deduced from

PROPOSITION 8.3. *There exist universal constants* c_3, c_4 *such that, for all* $\varepsilon \in (0,2]$ *and all m,*

$$c_3^{m^2} \varepsilon^{-D} \le N(O(m), \|\cdot\|, \varepsilon) \le c_4^{m^2} \varepsilon^{-D},$$

where $D = D(m) = m(m-1)/2 = \dim O(m)$.

Proof. Consider the special orthogonal group $SO(m) \overset{df}{=} \{U \in O(m): \det U = 1\}$. Since geometrically $O(m)$ is a disjoint union of two copies of $SO(m)$, it is clearly enough to prove Proposition 8.3 with $SO(m)$ instead of $O(m)$.

We shall apply Lemma 8.2(c) with $(M, d) = (SO(m), \|\cdot\|)$. As X we shall choose $A(m)$, the subspace of $L(l_m^2)$ consisting of all skewsymmetric operators (i.e. $A \in A(m)$ iff $A^* = -A$), as Φ the exponential map defined, as usually, by $\exp A = \Sigma_{k=0}^{\infty} A^k/k!$.

It is a well-known fact from differential geometry that $\exp(A(m)) = SO(m)$. The main step is to show that in fact

$$\exp(B_\pi(A(m))) = SO(m), \tag{8.1}$$

i.e. the condition (i) of Lemma 8.2(c) holds with $R=\pi$. To show this consider $U\in SO(m)$ and let $A_0\in A(m)$ be such that $\exp A_0=U$. Then A_0 can be written in the form

$$\begin{bmatrix} 0 & \lambda_1 & & & & & & & \\ -\lambda_1 & 0 & & & & 0 & & & \\ & & 0 & \lambda_2 & & & & & \\ & & -\lambda_2 & 0 & & & & & \\ 0 & & & & \ddots & & & & \\ & & & & & 0 & \lambda_k & & \\ & & & & & -\lambda_k & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{bmatrix} \tag{8.2}$$

with respect to some orthonormal basis of $\mathbf{R}^m$, where $k\leq m/2$ and $\lambda_1,\dots,\lambda_k$ are reals.

Let, for any real λ, $\tilde{\lambda}$ be the (unique) number satisfying

(α) $(\lambda-\tilde{\lambda})/2\pi$ is an integer

(β) $-\pi<\tilde{\lambda}\leq\pi$.

Define $A_1\in A(m)$ to be the matrix obtained from (8.2) by replacing λ_j's by $\tilde{\lambda}_j$'s. Then, in view of (α) and (β),

(α)′ $\exp A_1=\exp A_0=U$

(β)′ $\|A_1\|=\max_{j\leq k}|\tilde{\lambda}_j|\leq\pi$,

which proves (8.1).

To conclude the proof of Proposition 8.3 observe that the definition of exp yields immediately

$$(2-e^a)\,\|A-B\|\leq\|\exp A-\exp B\|\leq e^a\|A-B\|$$

for any $a\geq 0$ and $A,B\in L(l_m^2),\|A\|,\|B\|\leq a$. Hence the conditions (ii) and (iii) from Lemma 8.2(c) hold with $R=\pi$, $L=e^{\pi}$, $r=0.4$ and $l=0.5$ $(<2-e^{0.4})$. This shows Proposition 8.3 (e.g. with $c_3=0.3$ and $c_4=15$). Q.E.D.

Proof of Lemma 7.3A. We identify $G_{n,m}$ with $O(m)/O(n;m)$. As was observed at the beginning of this section, $O(n;m)$ is naturally isomorphic to $O(n)\times O(m-n)$ (via

$U \sim (U|_F, U|_{F^\perp}))$. Moreover, the product metric, as defined in Lemma 8.2(a), coincides with the operator metric inherited from $O(m)$. Hence, by Lemma 8.2(a), for any $\varepsilon>0$,

$$N(O(n), \|\cdot\|, 2\varepsilon)\, N(O(m-n), \|\cdot\|, 2\varepsilon) \leqslant N(O(n;m), \|\cdot\|, \varepsilon) \leqslant N(O(n), \|\cdot\|, \varepsilon)\, N(O(m-n), \|\cdot\|, \varepsilon).$$

Combining this with Lemma 8.2(b) we get, for any $\varepsilon>0$,

$$\frac{N(O(m), \|\cdot\|, 2\varepsilon)}{N(O(n), \|\cdot\|, \varepsilon)\, N(O(m-n), \|\cdot\|, \varepsilon)} \leqslant N(G_{n,m}, \varrho, \varepsilon) \leqslant \frac{N(O(m), \|\cdot\|, \varepsilon/2)}{N(O(n), \|\cdot\|, 2\varepsilon)\, N(O(m-n), \|\cdot\|, 2\varepsilon)}.$$

Now, applying the corresponding estimate from Proposition 8.3 to each of the quantities of type $N(O(k), \|\cdot\|, \eta)$ appearing above, we get Lemma 7.3A, and hence also Lemma 7.3. Q.E.D.

Remark 8.4. For n (or $m-n$) significantly smaller than m Lemma 7.3 (or 7.3A) is not sharp. In fact one can show that, under the same assumptions,

$$(c_1'/\eta)^{n(m-n)} \leqslant N(G_{n,m}, \varrho, \eta) \leqslant (c_2'/\eta)^{n(m-n)}$$

with universal constants c_1', c_2'. On the other hand, Proposition 8.3 is basically sharp.

Remark 8.5. Let α be any unitary ideal norm on the space of operators on $\mathbf{R}^m$ (i.e. $\alpha(T)=\alpha(UTV)$ for $U, V \in O(m)$ and any $T, \alpha(S)=\|S\|$ if $\operatorname{rank} S=1$). Then for every $\eta \in (0, 2\alpha(I)]$, $(c_3\,\alpha(I)/\eta)^{\dim O(m)} \leqslant N(O(m), \alpha, \eta) \leqslant (c_4\,\alpha(I)/\eta)^{\dim O(m)}$, where c_3 and c_4 are positive constants indpendent of α, m and η.

Remark 8.6. In the notation of Remark 8.5, let, for some $n<m, \varrho=\varrho(\alpha)$ be the corresponding quotient metric on $G_{n,m}$. Denote $d=\operatorname{diam} G_{n,m}=\sqrt{2}\,\alpha(P_{\operatorname{span}\{e_1,\dots,e_k\}})$, where $k=2\min\{n, m-n\}$. Then, for any $\eta \in (0, d]$.

$$(c_5\, d/\eta)^{n(m-n)} \leqslant N(G_{n,m}, \varrho(\alpha), \eta) \leqslant (c_6\, d/\eta)^{n(m-n)}$$

with $c_5, c_6>0$ independent of α, m, n and η.

The estimates given in Remarks 8.4–8.6 are "isomorphically" sharp. The proofs can be found in [19].

9. Sketches of the proofs of Theorems 1.4 and 1.6

The case of Theorem 1.4. In the proof of Theorem 1.1 we added n additional extreme points (corresponding to $g_1, \ldots, g_n$) to the unit ball of l^1_{4n}. In this case we add $k=[\delta n/2]$ points to the unit ball of l^1_n and the argument is the same. The crucial observation is that if T is an operator satisfying the assumption of Theorem 1.4 and T_0 the corresponding lifting satisfying analogues of (5.8) and (5.9) (with T instead of P), then $\operatorname{rank} T_0 \leq \operatorname{rank} T+[\delta n/2] \leq (1-\delta/2)n$ and the same holds for rank $(I-T_0)$. Q.E.D.

The case of Theorem 1.6. Here we add $k=[m/2]$ g_j's. Then, similarly as above, the corresponding lifting T_0 of P satisfies

$$\operatorname{rank} T_0 \leq n-\frac{m}{2}, \quad \operatorname{rank}(I-T_0) \leq n-\frac{m}{2}.$$

Applying the same line of argument as in the proof of Theorem 1.1 we are able to show the following analogue of Fact 5.2

"If T_0 satisfies the conditions above, then

$$\mathcal{P}(\{QT_0 g_j \in 2KQB^1_n \quad \text{for } j=1, \ldots, k\}) \leq \left(\frac{CKn^{3/2}}{k^2}\right)^{ck^2}\text{''}.$$

Combining this with Fact 5.1 we see that in our present setting

$$\mathcal{P}(\{B: \|P: B \to B\| \leq K\}) \leq (2C_0)^{n^2}\left(\frac{CKn^{3/2}}{k^2}\right)^{ck^2} + e^{-c'k}$$
$$\leq C_1^{n^2}\left(\frac{C_2 Kn^{3/2}}{m^2}\right)^{c_1 m^2} + e^{-c_2 m},$$

which turns out to be strictly smaller than 1 provided

$$K < m^{1/2 - bn^2/m^2 \ln n},$$

proving Theorem 1.6. Q.E.D.

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