The finite-element modeling of three-dimensional time-domain electromagnetic fields in strongly inhomogeneous media

Gerrit Mur
Faculty of Electrical Engineering
Delft University of Technology
P.O. Box 5031, 2600GA Delft,The Netherlands

Abstract-**An efficient and accurate finite-element method is presented for computing transient electromagnetic fields in three-dimensional configurations containing arbitrarily inhomogeneous media that may be anisotropic. To obtain accurate results with an optimum computational efficiency, both edge and Cartesian elements are used for approximating the spatial distribution of the field. The efficiency and the storage requirements of the method are further optimized by choosing an irreducible implicit formulation, by solving the resulting system of algebraic equations in terms of the time-dependent expansion coefficients iteratively and by using an incomplete LU-decomposition for preconditioning. A method is described for imposing the divergence condition in a weighted sense.**

I. INTRODUCTION

In earlier papers [1,2] finite-element methods have been described for computing time-harmonic electromagnetic fields in three-dimensional configurations containing strongly inhomogeneous media. In [2] it was shown that using a combination of edge elements and Cartesian elements for the expansion of the electric and/or the magnetic field strength yields optimum computational results. By adding the divergence condition [3], the latter formulation was further improved, both as regards its computational efficiency (storage and time) and as regards its accuracy. In [4] it was shown that the numerically optimum combination of edge and Cartesian elements mentioned above can also be used for approximating the spatial distribution of the field in a mixed formulation of the three-dimensional time-domain form of Maxwell's equations. In the latter paper, an explicit method was used for the integration of the system of coupled differential equations along the time axis. In the present paper, the mixed formulation presented in [4] is replaced by an irreducible one in terms of the electric field strength only, the media may be anisotropic. The irreducible formulation was chosen both to make it easier to implement implicit methods for carrying out the integration along the time axis and to reduce the storage requirements.

Comparing our approach with other valid methods for solving time-domain electromagnetic-field problems in three-dimensional inhomogeneous configurations, such as the methods using potentials [5] and methods using edge elements that are divergence-free [6], we note that our approach has a higher order of accuracy since it employs a consistently linear approximation of the electric field strength. Potentials require a numerical differentiation because of which they yield only piecewise constant approximations for the electric field strength (assuming linear expansions for the potentials). Divergence-free edge elements yield approximations that are constants along the edges they refer to, because of this they also yield poor convergence properties [1,7]. In comparison with many methods that use potentials, our approach has the additional advantages that multiply connected domains do not cause any difficulties, that (in)homogeneous Dirichlet as well as (in)homogeneous Neumann boundary conditions can be implemented straightforwardly and that, finally, no gauge is required for ensuring the uniqueness of the method [5].

Finally we mention that the linear edge elements we use are, apart from a constant factor, identical to the "pyramid vector fields" used by McMahon [8] for computing lower bounds for the electrostatic capacity of a cube, McMahon's paper seems to be the oldest reference using edge elements.

II. THE CHOICE OF THE EXPANSION FUNCTIONS

For topological reasons [9], the geometrical domain $\mathcal{D}$ in which the finite-element method is applied, is subdivided into a number of adjoining tetrahedra (simplices in $\mathbb{R}^3$). This subdivision can either be done exactly, when $\mathcal{D}$ is a polyhedron, or approximately if $\mathcal{D}$ is not a polyhedron. In each tetrahedron $\mathcal{T}$ the set of local (i.e. belonging to a particular tetrahedron) Cartesian (nodal) expansion functions $\{\boldsymbol{W}_{i,j}^{(C)}(\boldsymbol{x})\}$ is given by $\boldsymbol{W}_{i,j}^{(C)}(\boldsymbol{x}) = \phi_i(\boldsymbol{x})\boldsymbol{i}_j$ $(i = 0,...,3,\ j = 1,2,3)$, where $\boldsymbol{i}_j$ are the base vectors with respect to the (background) Cartesian reference frame and where $\phi_i(\boldsymbol{x})$ are the barycentric coordinates [4]. The set of local edge expansion functions $\{\boldsymbol{W}_{i,j}^{(E)}(\boldsymbol{x})\}$ is given by $\boldsymbol{W}_{i,j}^{(E)}(\boldsymbol{x}) = a_{i,j}\phi_i(\boldsymbol{x})\nabla\phi_j(\boldsymbol{x})$, $(i,j = 0,...,3,\ i \neq j)$ where $a_{i,j} = |\boldsymbol{x}_i - \boldsymbol{x}_j|$ denotes the length of the edge joining the vertices $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. The factor $a_{i,j}$ is introduced for making $\boldsymbol{W}_{i,j}^{(E)}(\boldsymbol{x})$ dimensionless and for normalizing it for improving the condition of the system of algebraic equations that constitute the system matrix. When $\boldsymbol{x} \in \mathcal{T}$, the electric field strength $\boldsymbol{E}(\boldsymbol{x},t)$ is expanded as

$$\boldsymbol{E}(\boldsymbol{x},t) = \sum_{i=0}^{3}\sum_{j} e_{i,j}(t)\boldsymbol{W}_{i,j}^{(C,E)}(\boldsymbol{x}), \tag{1}$$

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where $\{e_{i,j}(t)\}$ denotes the local set of unknown time-dependent expansion coefficients. The local expansion functions are taken from the set $\{\boldsymbol{W}_{i,j}^{(C,E)}(\boldsymbol{x})\} = \{\boldsymbol{W}_{i,j}^{(C)}(\boldsymbol{x}), \boldsymbol{W}_{i,j}^{(E)}(\boldsymbol{x})\}$ since, depending on the local degree of inhomogeneity [2], both edge and Cartesian expansion functions may be used in each tetrahedron. The summation index j in (1) runs over values that depend on the type of expansion used near node i.

III. THE SYSTEM OF DIFFERENTIAL EQUATIONS

Eliminating the magnetic field strength $\boldsymbol{H}$ from Maxwell's equations we obtain

$$\partial_t^2 \boldsymbol{\epsilon} \cdot \boldsymbol{E} + \partial_t \boldsymbol{\sigma} \cdot \boldsymbol{E} + \nabla \times (\boldsymbol{\mu}^{-1} \cdot \nabla \times \boldsymbol{E}) = -\partial_t \boldsymbol{J}^{\text{ext}} - \nabla \times (\boldsymbol{\mu}^{-1} \cdot \boldsymbol{K}^{\text{ext}}). \tag{2}$$

where $\boldsymbol{J}^{ext}$ and $\boldsymbol{K}^{ext}$ denote external sources of electrical and magnetic current, and where $\boldsymbol{\epsilon}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ denote the permittivity, the conductivity and the permeability tensor, respectively. After substituting the expansion (1) for the electric field strength in (2), a system of equations in the expansion coefficients is obtained by applying the method of weighted residuals. The set of weighting functions $\{\boldsymbol{W}_{p,q}^{(C,E)}(\boldsymbol{x})\}$ that is used is the same as the set of expansion functions. Using an integration by parts and adding the resulting equations over all tetrahedra, we obtain a system of coupled ordinary differential equations for $\{e_{i,j}\}$ that can be written as

$$\begin{aligned}
&\sum_{i,j} \partial_t^2 e_{i,j} \int_{\mathcal{D}} \boldsymbol{W}_{p,q} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{W}_{i,j} \mathrm{d}V \\
&+ \sum_{i,j} \partial_t e_{i,j} \int_{\mathcal{D}} \boldsymbol{W}_{p,q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{W}_{i,j} \mathrm{d}V \\
&+ \sum_{i,j} e_{i,j} \int_{\mathcal{D}} (\nabla \times \boldsymbol{W}_{p,q}) \cdot \boldsymbol{\mu}^{-1} \cdot (\nabla \times \boldsymbol{W}_{i,j}) \mathrm{d}V = \\
&\qquad \partial_t \int_{\partial\mathcal{D}} \boldsymbol{W}_{p,q} \cdot (\boldsymbol{n} \times \boldsymbol{H}) \mathrm{d}A \\
&\qquad - \int_{\mathcal{D}} \boldsymbol{W}_{p,q} \cdot \boldsymbol{J}^{\text{ext}} \mathrm{d}V \\
&\qquad - \int_{\mathcal{D}} (\nabla \times \boldsymbol{W}_{p,q}) \cdot \boldsymbol{\mu}^{-1} \cdot \boldsymbol{K}^{\text{ext}} \mathrm{d}V \;\; \forall p, q,
\end{aligned} \tag{3}$$

where $\partial\mathcal{D}$ denotes the outer boundary of the domain of computation $\mathcal{D}$ and $\boldsymbol{n}$ the unit vector along the outward normal to $\partial\mathcal{D}$. Note that, contrary to the summation in (1), the summation indices i and j in (3) refer to the global numbering of the expansion functions and that p and q refer to the global numbering of the weighting functions. For deriving (3) we have used the continuity of the tangential components of the magnetic field strength over all internal interfaces.

Assuming the configuration to be free of electric charges, it follows from Maxwell's equations that the electric flux density $\boldsymbol{D} = \boldsymbol{\epsilon} \cdot \boldsymbol{E}$ should be free of divergence. The latter condition is not automatically enforced by either (2) or (3). To impose this freedom of divergence, the system of differential equations (3) is modified by adding the conditions

$$\sum_{i,j} e_{i,j} \int_{\mathcal{T}} (\nabla \cdot \boldsymbol{W}_{p,q}) |\boldsymbol{\epsilon} \cdot \boldsymbol{\mu}|^{-1} \nabla \cdot (\boldsymbol{\epsilon} \cdot \boldsymbol{W}_{i,j}) \mathrm{d}V = 0 \;\; \forall p, q \tag{4}$$

to it for all tetrahedra. Note that the validity of (4) follows from the divergence condition and that it has the same dimension as the terms in (3). The norm in (4) has to be taken such that it reduces to $\varepsilon\mu$ for isotropic media. Using the above procedure, the divergence condition is imposed in a weighted sense. A numerical solution that is obtained using (4) will be free of divergence in the numerical sense, i.e. the divergence condition is satisfied to the same degree of accuracy as (2) is. In comparison with methods using divergence-free (mixed) finite elements [10], our method uses a consistently linear approximation of the field because of which it has superior convergence properties [1,7]. To summarize, using (4) is a numerically optimum method of imposing the divergence condition.

Assuming the medium properties $\boldsymbol{\epsilon}, \boldsymbol{\sigma}$ and $\boldsymbol{\mu}^{-1}$ as well as the external magnetic field strength $\boldsymbol{H}$ and the external source distributions $\boldsymbol{J}^{\text{ext}}$ and $\boldsymbol{K}^{\text{ext}}$ to be constant, or to vary linearly, over each tetrahedron or triangle, all integrations implied in (3) and (4) can be carried out analytically [11].

The system of coupled differential equations can, in matrix form, be written as

$$\boldsymbol{M}_{\varepsilon} \partial_t^2 \boldsymbol{e} + \boldsymbol{M}_{\sigma} \partial_t \boldsymbol{e} + \boldsymbol{M}_{\mu} \boldsymbol{e} = \partial_t \boldsymbol{h} - \boldsymbol{j}^{\text{ext}} - \boldsymbol{k}^{\text{ext}}. \tag{5}$$

Together with the appropriate initial and boundary conditions (5) constitutes a system of ordinary diffential equations from which the evolution in time of the expansion coefficients can be obtained. When the problem to be solved contains Dirichlet boundary conditions, some of the elements of the vector $\boldsymbol{e}$ will have prescribed, time-dependent, values, those elements should be eliminated from the vector of unknowns $\boldsymbol{e}$ and (5) has to be rewritten such that their contribution becomes a part of the right-hand side vector.

IV. THE INTEGRATION OF THE SYSTEM OF DIFFERENTIAL EQUATIONS

For the integration of the system of coupled differential equations (5) along the time axis the most obvious chioces are single and two-step time marching schemes [12,13]. We have used both methods with a number of different weighting functions in time. As regards computational efficiency, two-step methods proved to be slightly more efficient for solving our type of problems than single-step methods. Single-step methods, however, have the advantage that with them the step-size can be varied more easily. When the solution vector varies only slowly in time for certain intervals along the time axis, this may be used to improve the efficiency of the method. All results to be presented in the present paper have been obtained using the two-step scheme with the unconditionally stable "average acceleration" weighting function over the relevant time interval [13].

Using an incomplete LU-decomposition [14] of the system matrix for preconditioning, the iterative (conjugate-gradient) solution of the system of linear algebraic equations that has to be determined for each time-step has proved to be an extremely fast method, requiring only a few, usually 2 to 5 (see Table 1), iteration steps for each time step.

As regards explicit methods, such as the one reported in [4], we note that they have the disadvantage of being only conditionally stable, which causes explicit finite-element methods for solving electromagnetic-field problems in inhomogeneous media to be relatively inefficient.

V. NUMERICAL RESULTS

To demonstrate the accuracy and efficiency of our method, we have applied the FEMAXT code, in which the present theory was implemented, to a test problem for which the solution is known analytically. This test problem was solved under the condition that edge expansion functions are used when the relative contrast in the numerical value of ε and/or σ in two adjacent tetrahedra exceeds 10%; Cartesian expansion functions are used for lower contrasts. The configuration used for testing is the cubic source-free region $\mathcal{D}$ $\{-0.5 < x_1 < 0.5\mathrm{m}, 0 < x_2 < 1\mathrm{m}, 0 < x_3 < 1\mathrm{m}\}$ consisting of two homogeneous parts with different medium properties viz. $\mathcal{D}_1$ $\{-0.5 < x_1 < 0\mathrm{m}, 0 < x_2 < 1\mathrm{m}, 0 < x_3 < 1\mathrm{m}\}$ with the properties of a vacuum $\{\varepsilon_1 = \varepsilon_0, \mu_1 = \mu_0, \sigma_1 = 0\}$ and $\mathcal{D}_2 = \mathcal{D} \setminus \mathcal{D}_1$ with the medium properties $\{\varepsilon_1 = 10\varepsilon_0, \mu_1 = \mu_0, \sigma_1 = 0.01\mathrm{S/m}\}$ (this choice applies to the realistic case of a plane interface between a vacuum and a lossy dielectric). The magnetic field strength is chosen as the time-harmonic plane wave in $\mathcal{D}_1$ polarized along the x_3-axis. Because of the discontinuity at $x_1 = 0$ we also have a reflected wave in $\mathcal{D}_1$ and a transmitted wave in $\mathcal{D}_2$. Using complex arithmetic the field can be written as

$$\boldsymbol{H} = \Re(\exp(j\omega t - \boldsymbol{\gamma}^i \cdot \boldsymbol{x}) + R\exp(j\omega t + \boldsymbol{\gamma}^r \cdot \boldsymbol{x}))\boldsymbol{i}_3 \qquad (6)$$

when $x_1 < 0$ and

$$\boldsymbol{H} = \Re(T\exp(j\omega t - \boldsymbol{\gamma}^t \cdot \boldsymbol{x}))\boldsymbol{i}_3 \qquad (7)$$

when $x_1 \geq 0$. Writing $\boldsymbol{\gamma}^i = \gamma^i \boldsymbol{s}^i$, where $\boldsymbol{s}^i$ denotes the unit vector in the direction of propagation of the incident wave, we have chosen $s_1^i = s_2^i = 2^{-1/2}$ and $s_3^i = 0$. With this choice, (6,7) represents a plane wave in $\mathcal{D}_1$ travelling in the (x_1, x_2) plane and having an angle of incidence of 45°, together with a reflected wave and a transmitted wave in $\mathcal{D}_2$. Expressions for the unknown constants in (6) and (7) and for the electric field strength can be easily obtained. Because of the discontinuity in the properties of the medium, E_1 will be discontinuous at $x_1 = 0$, the remaining field components being continuous throughout the domain of computation, $E_3 \equiv 0$. The field has a frequency $f = 10^8$ Hz ($\omega = 2\pi f$). In all computations $\mathcal{D}_1$ consists of $18 \times 18 \times 6$ bricks of equal size and $\mathcal{D}_2$ consists of $18 \times 18 \times 12$ bricks of equal size, all bricks being subdivided into 6 tetrahedra. The computations are carried out for the time interval $t_0 = 0 \leq t \leq 10^{-7}\mathrm{s} = t_{\mathrm{end}}$, using 200 steps in time ($\Delta t = 5 * 10^{-10}\mathrm{s}$). The initial conditions at $t = -\Delta t$ and $t = 0$ as well as the boundary conditions are taken from the analytical expressions that are given in, or derived from, (6) or (7). The boundary conditions have been chosen as follows, at the parts of the outer boundary in the planes $x_3 = 0$ and $x_3 = 1$ the tangential component of magnetic field strength is specified, the tangential component of the electric field strength is specified along the remaining part of the outer boundary. With these boundary conditions we have a total number of 24248 degrees of freedom, 2956 degrees of freedom being prescribed because of the boundary conditions.

Table 1: Relative accuracies and average number of iterations NIT per time step as a function of the frequency.

Frequency Hz	RMS error in % E_1	E_2	E_3	NIT per time step
$4 * 10^8$	24.0	25.7	8.1	3.41
$2 * 10^8$	8.23	13.4	11.3	2.96
10^8	0.77	0.67	0.65	2.00
$4 * 10^7$	0.094	0.104	0.058	2.00
$2 * 10^7$	0.036	0.035	0.023	2.00

In Fig. 1 a contour plot is given of the local relative error in the solution for $E_1(x_1, x_2, x_3 = 0.5, t = t_{\mathrm{end}})$. The local relative error is defined as the value of the error at a given point and at a certain moment of time divided by the maximum absolute value of the exact solution, anywhere in the configuration and at the same instant of time. In Fig. 2 the numerical value of $E_2(x_1 = 0.01, x_2 = 0.49, x_3 = 0.49, t)$ as a function of time ($0.75 * 10^{-7} \leq t \leq 10^{-7}$) is plotted together with behaviour in time of the exact solution.

To obtain insight into the convergence properties of the method, the RMS-error in the solution is given, in Table 1, as a function of the frequency. All results are given for the same discretization of the domain of computation, for the same step-size in time and for the same medium properties. Comparing the RMS errors for different frequencies, it can be concluded that the error in the results is approximately $\mathrm{O}((h/\lambda)^2)$, where h denotes the maximum diameter of a tetrahedron and where λ denotes the wavelength of the incident field. This behaviour is in accordance with the expectations for linear expansion functions [15].

We have also solved the present time-harmonic problems and a number of other time-harmonic problems with all initial values set to zero, letting the solution "develop" to its approximate time-harmonic value and filtering out all components not having the frequency f. In this way, highly accurate solutions to time-harmonic problems can be obtained with a computational effort that is much less than the computational effort required for solving the complex system of equations one obtains, solving the problem in the frequency domain.

All computations have been carried out at a VAXstation 3100 M76, requiring about 1 minute of CPU time for each step in time and about 10Mbytes for storing the matrices. The SEPRAN finite-element package [16] is used for carrying out a number of tasks like generating the mesh and assembling the system matrices from the element matrices generated by FEMAXT.

VI. CONCLUSION

The theory discussed in the present paper was implemented in the FEMAXT code that was, apart from the time-dependent aspects of it, developed along the same lines as its counterpart for time-harmonic problems, the FEMAX3 package [17]. We have shown that our approach yields an efficient and very accurate method for computing transient electromagnetic fields in strongly inhomogeneous media. The present time-domain formulation can also be used for the efficient and accurate solution in the time-domain of problems involving time-harmonic fields and as such it is a very attractive alternative to a formulation in the frequency domain.

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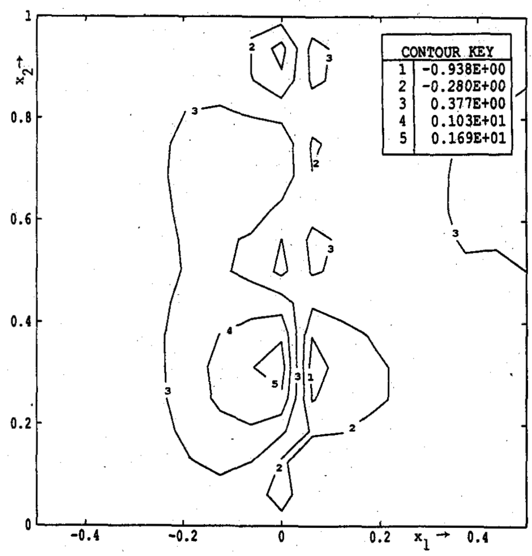

Fig. 1. Plot of the local relative error in $E_1(x_1, x_2, 0.5, t_{end})$ in %.

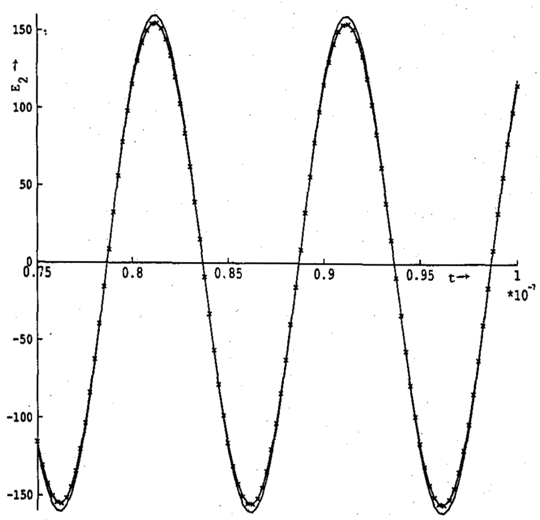

Fig. 2. Plot of $E_2(0.01, 0.49, 0.49, t)$, x = numerical result, the second line gives analytical results.