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THE FINITE ELEMENT SOLUTION OF PARABOLIC EQUATIONS

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In papers on solution of parabolic differential equations by the finite element method, error bounds are derived either in the case that the union of finite elements (straight or curved) matches exactly the given domain (e.g. in Zlámal's papers) or in the case of curved elements which do not cover, in general, the given domain (e.g. in Ciarlet-Raviart's papers). In the former case the error bounds are given for fully (i.e. both in space and time) discretized approximate solutions. In the latter case the numerical integration is taken into account; however, the error bounds are given only for semidiscrete (not discretized in time) approximate solutions. Error bounds introduced in this paper are given for fully discretized approximate solutions and for arbitrary curved domains. Simplicial curved elements in n -dimensional space are applied. Degrees of accuracy of quadrature formulas are determined so that numerical integration does not worsen the optimal order of convergence in L_2 -norm of the method.

1. NOTATION. THE CONSTRUCTION OF FINITE ELEMENT SPACE.
ISOPARAMETRIC INTEGRATION

The norm and the scalar product in the space $L^2(A)$ is denoted by $\|\cdot\|_{0,A}$ and $(\cdot, \cdot)_{0,A}$ respectively.

$H^m(A) \equiv W_2^{(m)}(A)$, $m = 0, 1, \dots$. Here $W_2^{(m)}(A)$ is a Sobolev space with the norm

$$\|v\|_{W_2^{(m)}(A)} \equiv \sqrt{\left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_{0,A}^2\right)}, \text{ where } D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

We denote

$$|v|_{W_2^{(m)}(A)} = \sqrt{\left(\sum_{|\alpha|=m} \|D^\alpha v\|_{0,A}^2\right)}, \quad \|v\|_{m,A} = \|v\|_{W_2^{(m)}(A)}, \quad |v|_{m,A} = |v|_{W_2^{(m)}(A)}.$$

$H_0^1(A)$ is the closure of the set $C_0^\infty(A)$ (i.e. the set of infinitely differentiable functions with compact support in A) in the norm $\|\cdot\|_{1,A}$. $H^{-1}(A)$ is the space dual to $H_0^1(A)$ (with dual norm).

$L^\infty(H^m(A))$ is the space of all functions $v(x, t)$, $x = (x_1, \dots, x_n) \in A$, $t \in [0, T]$ such that $v(x, t) \in H^m(A)$, $\forall t \in [0, T]$ and the function $\|v(x, t)\|_{m,A}$ is bounded for almost all $t \in [0, T]$. We denote

$$\|v\|_{L^\infty(H^m)} = \text{vrai sup}_{t \in [0, T]} \|v\|_{m,A}.$$

In the same way as in [1] we define the k -regular family $\{e\}_k$ of simplicial isoparametric finite elements e in the following manner:

We are given

- (a) A set $\hat{\Sigma} = \bigcup_{i=1}^N \{\hat{a}_i\}$ of N distinct points from R^n such that its closed convex hull \hat{T} is a unit n -simplex.
- (b) A finite dimensional space \hat{P} of functions defined on \hat{T} with $\dim \hat{P} = N$ such that $\hat{\Sigma}$ is \hat{P} -unisolvent, i.e. the Lagrange interpolation problem¹⁾: "Find $\hat{p} \in \hat{P}$ such that $\hat{p}(\hat{a}_i) = \alpha_i$, $1 \leq i \leq N$ " has a unique solution for any real numbers α_i . We suppose $\hat{P} \in C^{k+1}(\hat{T})$, $\hat{P} \supset \hat{P}(1)$. Here for any integer $r \geq 0$, $\hat{P}(r)$ is the space or restrictions to \hat{T} of all polynomials of degree $\leq r$ in n variables $\hat{x}_1, \dots, \hat{x}_n$.
- (c) A set $\Sigma = \bigcup_{i=1}^N \{a_i\}$ of N distinct points from R^n .

Then the simplicial finite element $e \in \{e\}_h$ is the image (i.e. $e = F_e(\hat{T})$) of the set \hat{T} through the unique mapping $F_e : \hat{T} \rightarrow R^n$ which satisfies

$$F_e \in \hat{P}^n, \quad F_e(\hat{a}_i) = a_i, \quad 1 \leq i \leq N.$$

We suppose

- (d) For all h , the mapping F_e is a C^{k+1} -diffeomorphism and there exist constants c_l , $0 \leq l \leq k+1$, independent of h , such that for all h :

$$(1) \quad \sup_{\hat{x} \in \hat{T}} \max_{|x|=l} |D^x F_e(\hat{x})| \leq c_l h^l, \quad 1 \leq l \leq k+1$$

and

$$(2) \quad 0 < \frac{1}{c_0} h^n \leq |J_e(\hat{x})| \leq c_0 h^n,$$

where $J_e(\hat{x})$ is the Jacobian of the mapping F_e at the point $\hat{x} \in \hat{T}$.

With every element e we associate the finite dimensional space P_e (with $\dim = N$) of functions

$$(3) \quad P_e = \{p_e : e \rightarrow R; p_e = \hat{p}(F_e^{-1}), \forall \hat{p} \in \hat{P}\}.$$

The e -interpolate $\pi_e u$ of a given function $u : e \rightarrow R$ is the unique function which satisfies

$$(4) \quad \pi_e u \in P_e, \quad \pi_e u(a_i) = u(a_i), \quad 1 \leq i \leq N.$$

¹⁾ The analogous analysis can be given for Hermite interpolation.

For a k -regular family $\{e\}_h$ of finite elements the following interpolation theorem is true (see [1], Theorem 2, p. 429):

Interpolation theorem.

Let a k -regular family $\{e\}_h$ of simplicial finite elements such that $\hat{P}(k) \subset \hat{P}$ be given. Let

$$(5) \quad k > \frac{n}{2} - 1$$

Then for any integer m such that $0 \leq m \leq k + 1$, there exists a constant c independent of h such that for any $e \in \{e\}_h$ and for any function $u \in H^{k+1}(e)$ we have

$$(6) \quad |u - \pi_e u|_{m,e} \leq ch^{k+1-m} \|u\|_{k+1,e}$$

We define now a k -regular triangulation \mathcal{C}_h of a set Ω :

Let Ω be a bounded domain. Let Ω_h be the union of a finite number of simplicial finite elements $e = F_e(\hat{T})$. Every element e is determined by N points $a_{i,e}$. We suppose that all points $a_{i,e}$ belong to $\bar{\Omega}$. The family of elements constructed in this way is called a triangulation of Ω (or of Ω_h) and is denoted by \mathcal{C}_h . We say that a triangulation \mathcal{C}_h of Ω is k -regular if the family of all elements from which the triangulation is formed is k -regular and if for the boundary elements (i.e. for elements e such that $e \not\subset \Omega$) of the triangulation \mathcal{C}_h we have

$$(7) \quad \max_{y' \in \mathcal{A}} |\psi_h(y') - \psi(y')| \leq ch^{k+1}$$

where c is a constant independent of h and the notation is that of Fig. 1.

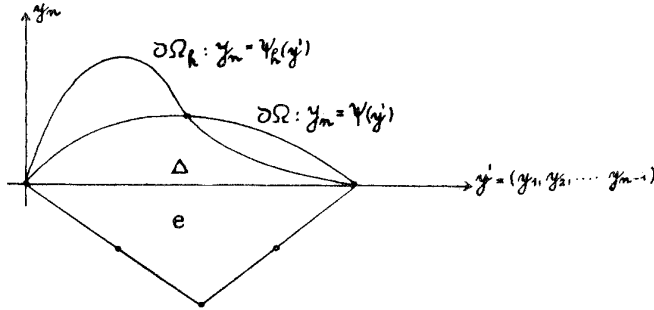


Fig. 1.

With a given k -regular triangulation \mathcal{C}_h , we associate the finite dimensional space V_h of functions v defined by

$$(8) \quad V_h = \{v \in C^0(\bar{\Omega}_h); v_e \in P_e \text{ for all } e \in \mathcal{C}_h, v = 0 \text{ on } \partial\Omega_h\},$$

where v_e is the restriction of the function v to the set e .

In our paper we suppose that $\hat{P} = \hat{P}(k)$. This restriction is not essential. It enables us to give simpler proofs.

Let $\Phi(x)$ be any function defined on the element e . Then the function $\Phi(F_e(\hat{x}))$ is defined on \hat{T} . In the sequel we will denote it by $\Phi^*(\hat{x})$.

Let us suppose that we have at our disposal a quadrature formula of a degree d over the reference set \hat{T} . In other words,

$$(9) \quad \int_{\hat{T}} \Phi^*(\hat{x}) \, d\hat{x} \quad \text{is approximated by} \quad \sum_r \hat{\omega}_r \Phi^*(\hat{b}_r)$$

for some specified points $\hat{b}_r \in \hat{T}$ and weights $\hat{\omega}_r$, which will be assumed once and for all to satisfy

$$(10) \quad \hat{\omega}_r > 0^2).$$

Concerning \hat{b}_r , we suppose that for every r , \hat{b}_r lies either inside \hat{T} or it coincides with some of the points \hat{a}_i .

With the quadrature scheme (9) we associate the error

$$(11) \quad \hat{E}(\Phi^*) = \int_{\hat{T}} \Phi^*(\hat{x}) \, d\hat{x} - \sum_r \hat{\omega}_r \Phi^*(\hat{b}_r).$$

Using the standard formula for the change of variables in multiple integrals, we find that

$$(12) \quad \int_e \Phi(x) \, dx \quad \text{is approximated by} \quad \sum_r \omega_{r,e} \Phi(b_{r,e}),$$

where

$$(13) \quad \omega_{r,e} = \hat{\omega}_r J_e(\hat{b}_r)^3, \quad b_{r,e} = F_e(\hat{b}_r).$$

We see that the quadrature scheme (9) over the reference set \hat{T} induces the quadrature scheme (12) over the finite element e , a circumstance which we call “*isoparametric*” numerical integration. With the scheme (12), we associate the error

$$(14) \quad E_e(\Phi) = \int_e \Phi(x) \, dx - \sum_r \omega_{r,e} \Phi(b_{r,e})$$

so that we have

$$(15) \quad E_e(\Phi) = \hat{E}(\Phi^* J_e) \quad \text{and} \quad \hat{E}(\Phi^*) = E(\Phi J_e^{-1}).$$

²⁾ This assumption is by no means necessary but it yields simpler proofs.

³⁾ We may, and will, assume that the Jacobian $J_e(\hat{x}) > 0$ for all $\hat{x} \in \hat{T}$.

2. FORMULATION OF THE PROBLEM

Let $x = (x_1, \dots, x_n) \in R^n$. Let Ω be a bounded domain in R^n . Let functions $g(x)$, $g_{ij}(x)$, $i, j = 1, \dots, n$ defined on $\bar{\Omega}$ and a function $f(x, t)$ defined on $\Omega \times [0, T]$ be smooth enough. Let

$$(16) \quad g_{ij}(x) = g_{ji}(x), \quad g(x) \geq g_0 (= \text{const}) > 0, \quad \forall x \in \bar{\Omega}$$

and let the differential operator

$$(17) \quad L = \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(g_{ij}(x) \frac{\partial}{\partial x_i} \right)$$

be strongly elliptic, i.e. there exists a constant $g_1 > 0$ such that

$$(18) \quad \sum_{i,j=1}^n g_{ij}(x) \xi_i \xi_j \geq g_1 \sum_{i=1}^n \xi_i^2 \quad \text{for all } x \in \bar{\Omega} \quad \text{and for all } (\xi_1, \dots, \xi_n) \in R^n.$$

Let $a(u, v)$ be the bilinear form corresponding to the operator L , i.e.

$$(19) \quad a(u, v) = \int_{\Omega} \sum_{i,j=1}^n g_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx.$$

We study the following problem:

Find a function $u(x, t)$ such that

$$(20) \quad u \in L^\infty(H_0^1(\Omega)), \quad \frac{\partial u}{\partial t} \in L^\infty(H^{-1}(\Omega)),$$

$$\left(g \frac{\partial u}{\partial t}, v \right)_{0,\Omega} + a(u, v) = (f, v)_{0,\Omega_h}, \quad \forall v \in H_0^1(\Omega) \quad \text{and } t \in (0, T],$$

$$u(x, 0) = u_0(x) \in L^2(\Omega).$$

First, we discretize this problem by the finite element method with respect to x . Let \mathcal{C}_h be a k -regular triangulation of the set Ω and let V_h be the corresponding finite element space. The union of the elements e from \mathcal{C}_h forms a set Ω_h which, in general, differs from Ω . We extend the functions $g(x)$, $g_{ij}(x)$, $u_0(x)$ to a greater set $\tilde{\Omega} \supset \bar{\Omega}$ so that the conditions (16) and (18) are satisfied (with positive constants \tilde{g}_0, \tilde{g}_1). In this way we obtain functions $\tilde{g}(x)$, $\tilde{g}_{ij}(x)$, $\tilde{u}_0(x)$. Obviously, for sufficiently small h

$$(21) \quad \Omega_h \subset \tilde{\Omega}.$$

The solution u of the problem (20) is supposed to satisfy

$$(22) \quad u, \frac{\partial u}{\partial t} \in L^\infty(H^{k+3}(\Omega)).$$

The validity of the assumption (22) may be assured by a sufficient smoothness of the given functions and the boundary $\partial\Omega$ of the set Ω . By the Calderon theorem, for every $t \in (0, T]$ there exist extensions $\tilde{u}(x, t)$, \tilde{u}_t ⁴) of the functions u , $\partial u/\partial t$ such that

$$(23) \quad \|\tilde{u}\|_{k+3, \tilde{\Omega}} \leq c \|u\|_{k+3, \Omega}, \quad \|\tilde{u}_t\|_{k+3, \tilde{\Omega}} \leq c \left\| \frac{\partial u}{\partial t} \right\|_{k+3, \Omega},$$

where c is a constant independent of h and of t (it depends on Ω only).

Let us denote

$$(24) \quad \tilde{f}(x, t) = \tilde{g}(x) \frac{\partial \tilde{u}}{\partial t} - \tilde{L}\tilde{u},$$

where

$$(25) \quad \tilde{L} = \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(\tilde{g}_{ij}(x) \frac{\partial}{\partial x_i} \right).$$

According to (20) we define now the following semidiscrete problem (see [2]):

Find a function $u_s(x, t)$ such that

$$(26) \quad u_s, \frac{\partial u_s}{\partial t} \in L^\infty(V_h(\Omega_h)),$$

$$\left(\tilde{g}(x) \frac{\partial u_s}{\partial t}, v \right)_{0, \Omega_h} + \tilde{a}(u_s, v) = (\tilde{f}, v)_{0, \Omega_h}, \quad \forall v \in V_h, \quad t \in (0, T],$$

$$u_s(x, 0) = u_0 \in V_h$$

where u_0 is an approximate of $\tilde{u}_0(x)$ and $\tilde{a}(u, v)$ is the bilinear form

$$(27) \quad \tilde{a}(u, v) = \int_{\Omega_h} \sum_{i,j=1}^n \tilde{g}_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx.$$

We named the problem (26) semidiscrete because it is discretized with respect to x only. It is obvious that (26) is a system of ordinary differential equations with an unknown vector function of parameter t . This suggests the way how to discretize the problem with respect to t . We solve the system of ordinary differential equations by ν -step A-stabil method (for $\nu = 1, 2$) of order q . We divide the time interval $[0, T]$ into a finite number of mutually equal parts Δt . We introduce the notation

$$(28) \quad \Phi^m = \Phi^m(x) = \Phi(x, m \Delta t), \quad m = 0, 1, \dots$$

for any function $\Phi(x, t)$.

⁴) The identity $\tilde{u}_t = \partial \tilde{u} / \partial t$ is not supposed to be true.

According to (26) and to the described way of the time discretization we define the following discrete problem:

Find a function $u_d(x, t)$ such that

$$(29) \quad \begin{aligned} u_d &= V_h \quad \text{for any } t = 0, \Delta t, 2\Delta t, \dots, T, \\ (\tilde{g}(x) \sum_{j=0}^v \alpha_j u_d^{m+j}, v)_{0, \Omega_h} + \Delta t \tilde{a}(\sum_{j=0}^v \beta_j u_d^{m+j}, v) &= \\ &= \Delta t (\sum_{j=0}^v \beta_j \tilde{f}^{m+j}, v)_{0, \Omega_h}, \quad \forall v \in V_h, \quad m = 0, 1, \dots, \\ u_d^0 &= u_0 \in V_h. \end{aligned}$$

It is easy to prove that the problem (29) has one and only one solution. This solution can be considered an approximate of the function $\tilde{u}(x, t)$.

Since it is either too costly or simply impossible to evaluate exactly the integrals $(\cdot, \cdot)_{0, \Omega_h}$, $\tilde{a}(\cdot, \cdot)$, we must now take into account the fact that approximate integration is used for their computation. For this purpose we use the isoparametric numerical integration, i.e. with agreement with (12) we replace

$$(30) \quad (w, z)_{0, \Omega_h} \approx (w, z)_h, \quad \tilde{a}(w, z) \approx a_h(w, z)$$

where

$$(31) \quad (w, z)_h = \sum_{e \in \mathcal{E}_h} \sum_r \omega_{r,e} w(b_{r,e}) z(b_{r,e}),$$

$$(32) \quad a_h(w, z) = \sum_{e \in \mathcal{E}_h} \sum_r \omega_{r,e} \left[\sum_{i,j=1}^n g_{ij}(b_{r,e}) \frac{\partial w(b_{r,e})}{\partial x_i} \frac{\partial z(b_{r,e})}{\partial x_j} \right]$$

(in (32) g_{ij} is written instead of \tilde{g}_{ij} because $b_{r,e} \in \bar{\Omega}$ for sufficiently small h).

According to (29) and (30) we define the following problem:

Find a function $u_h(x, t)$ such that

$$(33) \quad \begin{aligned} u_h &\in V_h \quad \text{for } t = 0, \Delta t, \dots, T, \\ (g(x) \sum_{j=0}^v \alpha_j u_h^{m+j}, v)_h + \Delta t a_h(\sum_{j=0}^v \beta_j u_h^{m+j}, v) &= \\ &= \Delta t (\sum_{j=0}^v \beta_j f^{m+j}, v)_h, \quad \forall v \in V_h, \quad m = 0, 1, \dots, \\ u_h^0 &= u_0 \in V_h. \end{aligned}$$

The aim of the paper is to derive error bounds for

$$(34) \quad \lambda = u(x, t) - u_h(x, t) \quad \text{for } t = 0, \Delta t, \dots, T$$

in the norm $\|\cdot\|_{0, \Omega \cap \Omega_h}$.

3. THE THEOREM ON RITZ APPROXIMATION

A function $\eta(x, t) \in V_h(V_h \subset H_0^1(\Omega_h))$, $\forall t \in (0, T]$ such that

$$(35) \quad (\tilde{g}(x) \tilde{u}_t, v)_{0, \Omega_h} + \tilde{a}(\eta(x, t), v) = (\tilde{f}(x, t), v)_{0, \Omega_h}, \quad \forall v \in V_h$$

is called the *Ritz approximation* of the function $\tilde{u}(x, t)$.

Let us prove that the function $\eta(x, t)$ is an orthogonal projection onto V_h of the function $\tilde{u}(x, t)$ in the energetic norm given by the bilinear form $\tilde{a}(\cdot, \cdot)$, i.e. it satisfies

$$(36) \quad \tilde{a}(\tilde{u} - \eta, v) = 0, \quad \forall v \in V_h.$$

Really, (35) implies

$$\tilde{a}(\eta, v) = (\tilde{f} - \tilde{g}\tilde{u}_t, v)_{0, \Omega_h}.$$

The Green theorem together with (24) yield

$$\tilde{a}(\tilde{u}, v) = -(\tilde{L}\tilde{u}, v)_{0, \Omega_h} = (\tilde{f} - \tilde{g}\tilde{u}_t, v)_{0, \Omega_h}.$$

The aim of this chapter is to estimate the norm

$$(37) \quad \|\tilde{u}(x, t) - \eta(x, t)\|_{i, \Omega_h}, \quad i = 0, 1.$$

In the sequel the constants independent of h and t will be denoted by c . The notation is generic, i.e., c will not denote the same constant in any two places.

From (27) and (18) it follows that

$$(38) \quad \tilde{a}(v, v) \geq \tilde{g}_1 |v|_{1, \Omega_h}^2, \quad \forall v \in H^1(\Omega_h).$$

The continuity assumption and the Cauchy-Schwarz inequality imply

$$(39) \quad |\tilde{a}(z, v)| \leq c |z|_{1, \Omega_h} |v|_{1, \Omega_h}, \quad \forall z, v \in H^1(\Omega_h).$$

Let $v \in H^{k+1}(\Omega_h)$ and let \mathcal{C}_h be a k -regular triangulation of Ω_h . Let $\pi_h v$ be the function which is equal to $\pi_e v$ on every element $e \in \mathcal{C}_h$. Here $\pi_e v$ is an e -interpolate of v (see (4)). The interpolation theorem (see (6)) implies

$$|v - \pi_e v|_{m, e}^2 \leq ch^{2(k+1-m)} \|v\|_{k+1, e}^2, \quad 0 \leq m \leq k+1.$$

Hence

$$(40) \quad \|v - \pi_h v\|_{1, \Omega_h} \leq ch^k \|v\|_{k+1, \Omega_h}.$$

Obviously $\pi_h \tilde{u} = \pi_h u = 0$ on $\partial\Omega_h$. Hence

$$(41) \quad \pi_h \tilde{u} \in V_h \quad \text{for any } t \in (0, T].$$

From (38), (36), (41) and (39) we obtain

$$|\tilde{u} - \eta|_{1, \Omega_h}^2 \leq c \tilde{a}(\tilde{u} - \eta, \tilde{u} - \eta) \leq c \tilde{a}(\tilde{u} - \pi_h \tilde{u}, \tilde{u} - \pi_h \tilde{u}) \leq c |\tilde{u} - \pi_h \tilde{u}|_{1, \Omega_h}^2.$$

From here and from (40) it follows

$$(42) \quad |\tilde{u} - \eta|_{1, \Omega_h} \leq ch^k \|\tilde{u}\|_{k+1, \Omega_h}, \quad \forall t \in (0, T].$$

Let us denote

$$(43) \quad \psi(x, t) = \begin{cases} \tilde{u}(x, t) - \eta(x, t) & \text{for } x \in \bar{\Omega}_h \\ 0 & \text{for } x \in \tilde{\Omega} - \bar{\Omega}_h. \end{cases}$$

For any $t \in (0, T]$ we solve the homogeneous Dirichlet problem

$$(44) \quad -L\Phi(x, t) = \psi(x, t) \quad \text{in } \Omega, \quad \Phi(x, t) = 0 \quad \text{on } \partial\Omega.$$

If $\partial\Omega$ is smooth enough then

$$(45) \quad \begin{aligned} \Phi &\in H_0^1(\Omega) \cap H^2(\Omega), \\ \|\Phi\|_{2, \Omega} &\leq c\|\psi\|_{0, \Omega} \leq c\|\psi\|_{0, \tilde{\Omega}} = c\|\psi\|_{0, \Omega_h}, \quad \text{i.e.} \\ \|\Phi\|_{2, \Omega} &\leq c\|\tilde{u} - \eta\|_{0, \Omega_h}, \quad \forall t \in (0, T]. \end{aligned}$$

Using the Calderon theorem we extend the function Φ from Ω onto $\tilde{\Omega}$. In this way we obtain a function $\tilde{\Phi} \in H^2(\tilde{\Omega})$ such that

$$\|\tilde{\Phi}\|_{2, \tilde{\Omega}} \leq c\|\Phi\|_{2, \Omega}.$$

From here and from (45) it follows

$$(46) \quad \|\Phi\|_{2, \Omega_h} \leq c\|\tilde{u} - \eta\|_{0, \Omega_h}.$$

Let $\zeta(x, t)$ be the orthogonal projection of $\Phi(x, t)$ onto the space $V_h(\Omega_h)$ in the energetic norm. Then in the same way as in (42) we get

$$|\tilde{\Phi} - \zeta|_{1, \Omega_h} \leq ch\|\tilde{\Phi}\|_{2, \Omega_h}.$$

From here and from (46) it follows

$$(47) \quad |\tilde{\Phi} - \zeta|_{1, \Omega_h} \leq ch\|\tilde{u} - \eta\|_{0, \Omega_h}.$$

It is easy to verify that $(\psi + L\Phi = \psi + \tilde{L}\tilde{\Phi} \text{ on } \Omega)$

$$(48) \quad \|\tilde{u} - \eta\|_{0, \Omega_h}^2 = \int_{\Omega_h - \Omega} (\tilde{u} - \eta)(\psi + \tilde{L}\tilde{\Phi}) \, dx - \int_{\Omega_h} (\tilde{u} - \eta) \tilde{L}\tilde{\Phi} \, dx.$$

The Green theorem ($\eta = 0$ on $\partial\Omega_h$) yields

$$(49) \quad - \int_{\Omega_h} (\tilde{u} - \eta) \tilde{L}\tilde{\Phi} \, dx = \tilde{a}(\tilde{u} - \eta, \tilde{\Phi}) - \int_{\partial\Omega_h} \tilde{u} \frac{\partial \tilde{\Phi}}{\partial \nu} \, dS.$$

Since $\zeta \in V_h$, we get from (36) $\tilde{a}(\tilde{u} - \eta, \zeta) = 0$, i.e. $\tilde{a}(\tilde{u} - \eta, \tilde{\Phi}) = \tilde{a}(\tilde{u} - \eta, \tilde{\Phi} - \zeta)$. From here and from (49) and (48) it follows

$$(50) \quad \|\tilde{u} - \eta\|_{0, \Omega_h}^2 \leq \left| \int_{\Omega_h - \Omega} (\tilde{u} - \eta) (\psi + \tilde{L}\tilde{\Phi}) dx \right| + |\tilde{a}(\tilde{u} - \eta, \tilde{\Phi} - \zeta)| + \left| \int_{\partial\Omega_h} \tilde{u} \frac{\partial \tilde{\Phi}}{\partial \nu} dS \right|.$$

Before estimating the expressions on the right hand side of (50) we introduce some lemmas and notes.

Lemma 1. Let \mathcal{C}_h be a k -regular triangulation of Ω_h . Let $v(x, t) \in H^1(\tilde{\Omega})$, $\forall t \in (0, T]$ and let

$$(51) \quad v(y', y_n, t) = 0 \quad \text{on} \quad \partial\Omega$$

(for notation see Fig. 1). Then there exist a constant c such that

$$(52) \quad \|v\|_{0, \Omega_h - \Omega} \leq ch^{k+1} |v|_{1, \Omega_h - \Omega}.$$

Proof. From (51), from the Schwarz inequality and from (7) it follows

$$(53) \quad |v(y', y_n, t)|^2 = \left| \int_{\psi(y')}^{y_n} \frac{\partial v(y', \tau, t)}{\partial y_n} d\tau \right|^2 \leq |y_n - \psi(y')| \left| \int_{\psi(y')}^{y_n} \left(\frac{\partial v(y', \tau, t)}{\partial y_n} \right)^2 d\tau \right| \leq ch^{k+1} \left| \int_{\psi(y')}^{y_n} \left(\frac{\partial v(y', \tau, t)}{\partial y_n} \right)^2 d\tau \right|.$$

By integrating (53) and summing over all the boundary elements $e \in \mathcal{C}_h$ we obtain (52).

Remark 1. The proof implies immediately that the assumption (51) may be replaced by the assumption

$$(51') \quad v(y', y_n, t) = 0 \quad \text{on} \quad \partial\Omega_h.$$

Remark 2. (53) yields immediately the inequality

$$(54) \quad |v(x, t)| \leq ch^{k+1} |v|_{1, \infty, \tilde{\Omega}}$$

valid for every $x \in \partial\Omega_h$ (under the assumption $v = 0$ on $\partial\Omega$).

Lemma 2.

$$(55) \quad \|v\|_{0, \partial\Omega_h} \leq c \|v\|_{1, \Omega_h}, \quad \forall v \in H^1(\tilde{\Omega}).$$

Proof. Lemma is a consequence of the proof of Theorem 1.2, p. 15 of [3] and the inclusion $\Omega_h \subset \Omega$ for all h .

We estimate now the expressions on the right hand side of (50).

The Schwarz inequality gives

$$(56) \quad \left| \int_{\Omega_h - \Omega} (\tilde{u} - \eta)(\psi + \tilde{L}\tilde{\Phi}) dx \right| \leq \|\tilde{u} - \eta\|_{0, \Omega_h - \Omega} \|\psi + \tilde{L}\tilde{\Phi}\|_{0, \Omega_h - \Omega}.$$

From (46) we get

$$\begin{aligned} \|\psi + \tilde{L}\tilde{\Phi}\|_{0, \Omega_h - \Omega} &\leq \|\psi + \tilde{L}\tilde{\Phi}\|_{0, \Omega_h} \leq \|\psi\|_{0, \Omega_h} + \|\tilde{L}\tilde{\Phi}\|_{0, \Omega_h} \leq \\ &\leq \|\psi\|_{0, \Omega_h} + c\|\tilde{\Phi}\|_{2, \Omega_h} \leq \|\psi\|_{0, \Omega_h} + c\|\tilde{u} - \eta\|_{0, \Omega_h}. \end{aligned}$$

From here and from (43) it follows that

$$(57) \quad \|\psi + \tilde{L}\tilde{\Phi}\|_{0, \Omega_h - \Omega} \leq c\|\tilde{u} - \eta\|_{0, \Omega_h}.$$

From (38), (36) and (39) we get

$$|\eta|_{1, \Omega_h}^2 \leq c\tilde{a}(\eta, \eta) = c\tilde{a}(\tilde{u}, \eta) \leq c|\tilde{u}|_{1, \Omega_h} |\eta|_{1, \Omega_h}.$$

Hence

$$|\eta|_{1, \Omega_h} \leq c|\tilde{u}|_{1, \Omega_h}.$$

From here, from (52) and from the Remark 2 (we remember $\tilde{u} = 0$ on $\partial\Omega$ and $\eta = 0$ on $\partial\Omega_h$) we obtain

$$\begin{aligned} \|\tilde{u} - \eta\|_{0, \Omega_h - \Omega} &\leq \|\tilde{u}\|_{0, \Omega_h - \Omega} + \|\eta\|_{0, \Omega_h - \Omega} \leq \\ &\leq ch^{k+1}\{|\tilde{u}|_{1, \Omega_h - \Omega} + |\eta|_{1, \Omega_h - \Omega}\} \leq ch^{k+1}|\tilde{u}|_{1, \Omega_h}. \end{aligned}$$

Substituting from here and from (57) into (56) we get

$$(58) \quad \left| \int_{\Omega_h - \Omega} (\tilde{u} - \eta)(\psi + \tilde{L}\tilde{\Phi}) dx \right| \leq ch^{k+1}|\tilde{u}|_{1, \Omega_h} \|\tilde{u} - \eta\|_{0, \Omega_h}.$$

From (39), (42) and (47) we get

$$(59) \quad |\tilde{a}(\tilde{u} - \eta, \tilde{\Phi} - \zeta)| \leq ch^{k+1}\|\tilde{u}\|_{k+1, \Omega_h} \|\tilde{u} - \eta\|_{0, \Omega_h}.$$

The Schwarz inequality implies

$$(60) \quad \left| \int_{\partial\Omega_h} \tilde{u} \frac{\partial\tilde{\Phi}}{\partial\nu} dS \right| \leq \|\tilde{u}\|_{0, \partial\Omega_h} \left\| \frac{\partial\tilde{\Phi}}{\partial\nu} \right\|_{0, \partial\Omega_h}.$$

From (55) and (46) it follows that

$$(61) \quad \begin{aligned} \left\| \frac{\partial\tilde{\Phi}}{\partial\nu} \right\|_{0, \partial\Omega_h} &\leq c \sum_{i=1}^n \left\| \frac{\partial\tilde{\Phi}}{\partial x_i} \right\|_{0, \partial\Omega_h} \leq c \sum_{i=1}^n \left\| \frac{\partial\tilde{\Phi}}{\partial x_i} \right\|_{1, \Omega_h} \leq \\ &\leq c\|\tilde{\Phi}\|_{2, \Omega_h} \leq c\|\tilde{u} - \eta\|_{0, \Omega_h}. \end{aligned}$$

Using (54), (55), (21) and the first Sobolev theorem (see for example [4]) we get (we suppose (5) to be satisfied)

$$\begin{aligned} \|\tilde{u}\|_{0,\partial\Omega_h} &= \sqrt{(\int_{\partial\Omega_h} \tilde{u}^2 ds)} \leq ch^{k+1} \|\tilde{u}\|_{1,\infty,\tilde{\Omega}} \|0,\partial\Omega_h\| \leq ch^{k+1} \|\tilde{u}\|_{1,\infty,\tilde{\Omega}} \|1,\Omega_h\| \leq \\ &\leq ch^{k+1} \text{mes } \Omega_h \|\tilde{u}\|_{1,\infty,\tilde{\Omega}} \leq ch^{k+1} \|\tilde{u}\|_{k+2,\tilde{\Omega}}. \end{aligned}$$

Substituting from here and from (61) into (60) we get

$$(62) \quad \left| \int_{\partial\Omega_h} \tilde{u} \frac{\partial\tilde{\Phi}}{\partial\nu} ds \right| \leq ch^{k+1} \|\tilde{u}\|_{k+2,\tilde{\Omega}} \|\tilde{u} - \eta\|_{0,\Omega_h}.$$

(50), (58), (59) and (60) implies

$$(63) \quad \|\tilde{u} - \eta\|_{0,\Omega_h} \leq ch^{k+1} \|\tilde{u}\|_{k+2,\tilde{\Omega}}.$$

From here and from (42) it follows that

$$\|\tilde{u} - \eta\|_{1,\Omega_h} \leq \|\tilde{u} - \eta\|_{0,\Omega_h} + |\tilde{u} - \eta|_{1,\Omega_h} \leq ch^{k+1} \|\tilde{u}\|_{k+2,\tilde{\Omega}} + ch^k \|\tilde{u}\|_{k+1,\tilde{\Omega}}.$$

Hence

$$(64) \quad \|\tilde{u} - \eta\|_{1,\Omega_h} \leq ch^k \|\tilde{u}\|_{k+2,\tilde{\Omega}}.$$

Now we can summarize the results into the following

Theorem 1 (*Theorem on the Ritz approximaton*). Let $u(x, t)$ be a solution of the problem (20) such that

$$(65) \quad u(x, t) \in H^{k+3}(\Omega), \quad \forall t \in (0, T].$$

Let a triangulation \mathcal{C}_h of the set Ω be k -regular. Let $\Omega_h \subset \tilde{\Omega}$ and let

$$(66) \quad k > \frac{n}{2} - 1.$$

Let $\tilde{u}(x, t) \in H^{k+3}(\tilde{\Omega})$, $\forall t \in (0, T]$ be an extension of $u(x, t)$ from Ω onto $\tilde{\Omega}(\Omega \subset \tilde{\Omega})$ such that

$$(67) \quad \|\tilde{u}\|_{k+3,\tilde{\Omega}} \leq c_1 \|u\|_{k+3,\Omega}$$

where c_1 is a constant (independent of h and t).

Let $\eta(x, t) \in V_h(\Omega_h)$, $\forall t \in (0, T]$ be the Ritz approximation of the function $\tilde{u}(x, t)$.

Then there exists a constant c independent of h and t such that

$$(68) \quad \|\tilde{u} - \eta\|_{i,\Omega_h} \leq ch^{k+1-i} \|u\|_{k+3,\Omega}, \quad i = 0, 1.$$

Proof. The inequality (68) follows immediately from (63), (64) and (67).

Note 3. In Theorem 1 the number $k + 3$ can be replaced by the number $k + 2$ in all places. Nevertheless in what follows we shall need the above formulation.

4. ISOPARAMETRIC INTEGRATION

Now we derive some estimates of errors due to isoparametric integration which was defined in (10)–(15). First, we give some technical lemmas.

Lemma 3. *For any functions φ and ψ from the class $C^{(\alpha_1, \dots, \alpha_n)}$ the following inequalities are true*

$$(69) \quad |D^{\alpha_1, \dots, \alpha_n}(\varphi\psi)| \leq c_1 \sum_{\substack{\beta_1=0, \dots, \alpha_1 \\ \beta_n=0, \dots, \alpha_n}} |D^{(\beta_1, \dots, \beta_n)} \varphi| |D^{(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)} \psi|,$$

$$(70) \quad |\varphi\psi|_{i, \mathcal{T}}^2 \leq c_2 \sum_{j=0}^i \{|\varphi|_{j, \mathcal{T}}^2 \cdot \sup_{\mathcal{T}} \max_{|\alpha|=i-j} (D^\alpha \psi)^2\},$$

where c_1 and c_2 are constants independent of h and t .

Proof. The proof of inequality (69) is trivial. The inequality (70) follows from (69) by simple calculation.

Lemma 4. *For polynomials r, s on the reference set \hat{T} the following inequalities are true*

$$(71) \quad \max_{\hat{T}} |D^\alpha r| \leq c_1 |r|_{|\alpha|, \hat{T}},$$

$$(72) \quad |r|_{j, \hat{T}}^2 \leq c_2 |r|_{i, \hat{T}}^2 \quad \text{for } j \geq i \geq 0,$$

$$(73) \quad |rs|_{i, \hat{T}}^2 \leq c_3 \sum_{j=0}^i |r|_{j, \hat{T}}^2 |s|_{i-j, \hat{T}}^2.$$

Proof. For the proof of (71) see [7], p. 356. The inequality (72) is an immediate consequence of (70).

Lemma 5. *Let \mathcal{C}_h be a k -regular triangulation of Ω_h . Let J_e be the Jacobian of the transformation $e = F_e(\hat{T})$, $e \in \mathcal{C}_h$. Let $J_e^{(l, m)}$ be a cofactor of J_e . Then*

$$(74) \quad D^\alpha J_e = 0(h^{|\alpha|+n}),$$

$$(75) \quad D^\alpha J_e^{(l, m)} = 0(h^{|\alpha|+n-1}),$$

$$(76) \quad D^\alpha \left(\frac{1}{J_e} \right) = 0(h^{|\alpha|-n}).$$

Proof. Using the mathematical induction the following assertion can be proved:

Let $D^\beta \varphi_s = 0(h^{|\beta|+\kappa_s})$ for $s = 1, \dots, r$, $|\beta| = 0, 1, \dots, |\alpha|$. Then

$$(77) \quad D^\alpha(\varphi_1 \varphi_2 \dots \varphi_r) = 0(h^{|\alpha|+\kappa_1+\dots+\kappa_r}).$$

The relations (74) and (75) follow immediately from (77). The relation (76) can be easily proved by the mathematical induction.

Lemma 6. Let $\tau^*(\hat{x}, t) \in H^{k+1}(\hat{T})$, $\tau(x, t) \in H^{k+1}(e)$, $\forall t \in (0, T]$, $e \in \mathcal{C}_h$, let \mathcal{C}_h be a k -regular triangulation. Then there exists a constant c such that

$$(78) \quad \|\tau^*\|_{k+1, \hat{T}} \leq ch^{-n/2+k+1} \|\tau\|_{k+1, e}.$$

Proof. Lemma is an immediate consequence of Lemma 1 from [1], p. 427.

We introduce the notation

$$(79) \quad E(\Phi) = \sum_{e \in \mathcal{C}_h} E_e(\Phi) \quad (\text{for the definition of } E_e(\Phi) \text{ see (14)}),$$

$$(80) \quad [\Phi]_{s, \mathcal{T}} = \sqrt{\left(\sum_{i=0}^s h^{-2i} |\Phi|_{i, \mathcal{T}}^2\right)}.$$

Lemma 7. Let

$$(81) \quad \psi(\hat{x}) \in H^{k+2}(\hat{T}),$$

$$(82) \quad \tau(\hat{x}) \text{ be a polynomial of degree } \leq k,$$

$$\delta(\hat{x}) \in C^{k+1}(\hat{T}) \text{ be a function such that ,}$$

$$(83) \quad D^\alpha \delta = 0 (h^{|\alpha|+\varepsilon}) \text{ for } 0 \leq |\alpha| \leq k+1, \quad \alpha \dots \text{ integer .}$$

Let d be the order of a quadrature formula on the reference set \hat{T} . Then there exist constants c_1 and c_2 such that

$$(84) \quad \begin{aligned} |\hat{E}(\psi\tau\delta)| &\leq c_1 h^\alpha \{h^{k+1} \|\tau\|_{0, \mathcal{T}} (h^{-(k+1)} |\psi|_{k+1, \mathcal{T}} + \|\psi\|_{k+1, \mathcal{T}}) + \\ &\quad + h^{d+1} [\tau]_{k, \mathcal{T}} (h^{-k} |\psi|_{k+1, \mathcal{T}} + [\psi]_{k, \mathcal{T}})\}, \end{aligned}$$

$$(85) \quad \begin{aligned} \left| \hat{E} \left(\frac{\partial \psi}{\partial \hat{x}_i} \frac{\partial \tau}{\partial \hat{x}_j} \delta \right) \right| &\leq c_2 h^\alpha \{h^{k+1} |\tau|_{1, \mathcal{T}} (h^{-(k+1)} |\psi|_{k+2, \mathcal{T}} + \sum_{j=0}^{k+1} |\psi|_{j+1, \mathcal{T}} + \\ &\quad + h^{d+2-k} |\tau|_{1, \mathcal{T}} (h^{-k} |\psi|_{k+2, \mathcal{T}} + h [\psi]_{k+1, \mathcal{T}})\}. \end{aligned}$$

Provided that

$$(81') \quad \psi(\hat{x}) \text{ is a polynomial of degree } \leq k,$$

the inequalities (84) and (85) reduce to

$$(84') \quad |\hat{E}(\psi\tau\delta)| \leq c_1 h^\alpha \{h^{k+1} \|\tau\|_{0, \mathcal{T}} \|\psi\|_{0, \mathcal{T}} + h^{d+1} [\tau]_{k, \mathcal{T}} [\psi]_{k, \mathcal{T}}\},$$

$$(85') \quad \left| \hat{E} \left(\frac{\partial \psi}{\partial \hat{x}_i} \frac{\partial \tau}{\partial \hat{x}_j} \delta \right) \right| \leq c_2 h^\alpha \{h^{k+1} |\tau|_{1, \mathcal{T}} |\psi|_{1, \mathcal{T}} + h^{d+3-k} |\tau|_{1, \mathcal{T}} [\psi]_{k, \mathcal{T}}\}.$$

Proof. Obviously

$$(86) \quad |\hat{E}(\psi\tau\delta)| \leq |\hat{E}(\psi\tau(\delta - \hat{\pi}\delta))| + |\hat{E}(\psi\tau\hat{\pi}\delta)|,$$

where $\hat{\pi}\delta$ is the \hat{T} -interpolate of the function $\delta(\hat{x})$ (i.e. $\hat{\pi}\delta \in \hat{P}(k)$, $\hat{\pi}\delta(\hat{a}_i) = \delta(\hat{a}_i)$). The first Sobolev theorem (we suppose (66) to be true), the Bramble-Hilbert lemma (shortly the B.-H. lemma) and (83) imply

$$(87) \quad |\hat{E}(\psi\tau(\delta - \hat{\pi}\delta))| = \left| \int_{\hat{T}} \psi\tau(\delta - \hat{\pi}\delta) d\hat{x} - \sum_r \hat{\omega}_r [\psi(\hat{b}_r) \tau(\hat{b}_r) (\delta(\hat{b}_r) - \hat{\pi}\delta(\hat{b}_r))] \right| \leq \\ \leq \|\psi\|_{0,\hat{T}} \|\delta - \hat{\pi}\delta\|_{0,\hat{T}} \max_{\hat{T}} |\tau| + c \max_{\hat{T}} |\tau| \sup_{\hat{T}} |\psi| \sup_{\hat{T}} |\delta - \hat{\pi}\delta| \leq \\ \leq c \|\tau\|_{0,\hat{T}} \|\psi\|_{k+1,\hat{T}} \|\delta - \hat{\pi}\delta\|_{k+1,\hat{T}} \leq c \|\tau\|_{0,\hat{T}} \|\psi\|_{k+1,\hat{T}} |\delta|_{k+1,\hat{T}} \leq \\ \leq ch^{\alpha+k+1} \|\tau\|_{0,\hat{T}} \|\psi\|_{k+1,\hat{T}}.$$

Further,

$$(88) \quad |\hat{E}(\psi\tau\hat{\pi}\delta)| \leq |\hat{E}(\tau\hat{\pi}\delta(\psi - \hat{\pi}\psi))| + |\hat{E}(\tau\hat{\pi}\delta\hat{\pi}\psi)|.$$

In the same way as in (87) we get

$$(89) \quad |\hat{E}(\tau\hat{\pi}\delta(\psi - \hat{\pi}\psi))| \leq c \|\tau\|_{0,\hat{T}} \|\hat{\pi}\delta\|_{k+1,\hat{T}} |\psi|_{k+1,\hat{T}}.$$

From the B.-H. lemma and from (83) it follows that

$$|\hat{\pi}\delta|_{j,\hat{T}} \leq |\hat{\pi}\delta - \delta|_{j,\hat{T}} + |\delta|_{j,\hat{T}} \leq c|\delta|_{k+1,\hat{T}} \leq c\{\alpha+k+1 + h^{\alpha+j}\}, \quad 0 \leq j \leq k+1.$$

Hence

$$(90) \quad |\hat{\pi}\delta|_{j,\hat{T}} \leq ch^{\alpha+j}.$$

Substituting (90) into (89) we get

$$(91) \quad |\hat{E}(\tau\hat{\pi}\delta(\psi - \hat{\pi}\psi))| \leq ch^{\alpha} \|\tau\|_{0,\hat{T}} |\psi|_{k+1,\hat{T}}.$$

From the B.-H. lemma, from (73), (90) and (70) we conclude

$$(92) \quad |\hat{E}(\tau\hat{\pi}\delta\hat{\pi}\psi)|^2 \leq c |\tau\hat{\pi}\delta\hat{\pi}\psi|_{d+1,\hat{T}}^2 \leq \\ \leq c \sum_{j=0}^{d+1} |\tau\hat{\pi}\psi|_{j,\hat{T}}^2 |\hat{\pi}\delta|_{d+1-j,\hat{T}}^2 \leq ch^{2(\alpha+d+1)} \sum_{j=0}^{2k} h^{-2j} \sum_{i=0}^j |\tau|_{i,\hat{T}}^2 |\hat{\pi}\psi|_{j-i,\hat{T}}^2.$$

It is easy to verify (we use the B.-H. lemma and the notation introduced in (80)) that

$$\sum_{j=0}^{2k} h^{-2j} \sum_{i=0}^j |\tau|_{i,\hat{T}}^2 |\hat{\pi}\psi|_{j-i,\hat{T}}^2 = \left(\sum_{j=0}^k h^{-2i} |\tau|_{i,\hat{T}}^2 \right) \left(\sum_{j=0}^k h^{-2j} |\hat{\pi}\psi|_{j,\hat{T}}^2 \right) \leq \\ \leq c [\tau]_{k,\hat{T}}^2 \left\{ \sum_{j=0}^k h^{-2j} (|\psi|_{k+1,\hat{T}}^2 + |\psi|_{j,\hat{T}}^2) \right\} \\ \leq c [\tau]_{k,\hat{T}}^2 \{ h^{-2k} |\psi|_{k+1,\hat{T}}^2 + [\psi]_{k,\hat{T}}^2 \}.$$

Substituting this inequality into (92) we get

$$(93) \quad |\hat{E}(\tau \hat{\pi} \delta \hat{\pi} \psi)| \leq ch^{s+d+1} [\tau]_{k, \mathcal{T}} \{h^{-k} |\psi|_{k+1, \mathcal{T}} + [\psi]_{k, \mathcal{T}}\}.$$

From (86), (87), (88), (91) and (93) we obtain (84). The inequality (84') is an immediate consequence of the inequalities (84) and (72). We prove now the inequality (85). From (84) it follows that

$$(94) \quad \left| \hat{E} \left(\frac{\partial \psi}{\partial \hat{x}_i} \frac{\partial \tau}{\partial \hat{x}_j} \delta \right) \right| \leq ch^s \left\{ h^{k+1} \left\| \frac{\partial \tau}{\partial \hat{x}_j} \right\|_{0, \mathcal{T}} \left(h^{-(k+1)} \left| \frac{\partial \psi}{\partial \hat{x}_i} \right|_{k+1, \mathcal{T}} + \left\| \frac{\partial \psi}{\partial \hat{x}_i} \right\|_{k+1, \mathcal{T}} \right) + h^{d+1} \left[\frac{\partial \tau}{\partial \hat{x}_j} \right]_{k, \mathcal{T}} \left(h^{-k} \left| \frac{\partial \psi}{\partial \hat{x}_i} \right|_{k+1, \mathcal{T}} + \left[\frac{\partial \psi}{\partial \hat{x}_i} \right]_{k, \mathcal{T}} \right) \right\},$$

while (80), (82) and (72) imply

$$\begin{aligned} \left[\frac{\partial \tau}{\partial \hat{x}_j} \right]_{k, \mathcal{T}}^2 &= \sum_{i=0}^k h^{-2i} \left| \frac{\partial \tau}{\partial \hat{x}_j} \right|_{i, \mathcal{T}}^2 \leq \sum_{i=0}^k h^{-2i} |\tau|_{i+1, \mathcal{T}}^2 = \\ &= \sum_{i=0}^{k-1} h^{2i} |\tau|_{i+1, \mathcal{T}}^2 \leq c \sum_{i=0}^{k-1} h^{-2i} |\tau|_{1, \mathcal{T}}^2. \end{aligned}$$

Hence

$$(95) \quad \left[\frac{\partial \tau}{\partial \hat{x}_j} \right]_{k, \mathcal{T}} \leq ch^{-k+1} |\tau|_{1, \mathcal{T}}.$$

In the same way we get

$$(96) \quad \left[\frac{\partial \psi}{\partial \hat{x}_i} \right]_{k, \mathcal{T}} \leq h [\psi]_{k+1, \mathcal{T}}.$$

Substituting from here and from (95) into (94) we get (85). The inequality (85') is an immediate consequence of (85).

We can now formulate the results concerning the isoparametric integration. We remember that \mathcal{C}_h is supposed to be a k -regular triangulation of Ω_h . In the sequel we assume that (66) is satisfied.

Theorem 2. Let $w(x, t) \in H^{k+1}(\Omega_h)$, $v(x, t) \in V_h(\Omega_h)$, $\forall t \in (0, T]$. Let the quadrature formula given on the reference set \hat{T} be of a degree

$$(97) \quad d \geq 2k - 1.$$

Then there exists a constant c such that

$$(98) \quad |E(wv)| \leq ch^{k+1} \|w\|_{k+1, \Omega_h} \|v\|_{1, \Omega_h}.$$

Proof. (79) and (15) imply

$$(99) \quad E(wv) = \sum_{e \in \mathcal{E}_h} \hat{E}(v^* w^* J_e).$$

It is easy to verify that all the assumptions of Lemma 7 are satisfied for $\varkappa = n$ (this follows from (74)). Hence, the inequality (84) may be applied. When substituting the inequalities (which can be easily derived from (78) and from the relation $[v^*]_{k,\mathcal{T}}^2 = |v^*|_{0,\mathcal{T}}^2 + \sum_{i=1}^k h^{-2i} |v^*|_{i,\mathcal{T}}^2$):

$$(100) \quad \|v^*\|_{0,\mathcal{T}} \leq ch^{-n/2} \|v\|_{0,e},$$

$$(101) \quad |w^*|_{k+1,\mathcal{T}} \leq ch^{-n/2+k+1} \|w\|_{k+1,e},$$

$$(102) \quad \|w^*\|_{k+1,\mathcal{T}} \leq ch^{-n/2} \|w\|_{k+1,e},$$

$$(103) \quad [v^*]_{k,\mathcal{T}} \leq ch^{-n/2-k+1} \|v\|_{1,e},$$

$$(104) \quad [w^*]_{k,\mathcal{T}} \leq ch^{-n/2} \|w\|_{k,e}$$

into (84) and applying (97) then we conclude by simple calculation

$$|\hat{E}(w^* v^* J_e)| \leq ch^{k+1} \|v\|_{1,e} \|w\|_{k+1,e}.$$

From here and from the Schwarz inequality (98) follows.

Theorem 3. Let $b(x) \in C^{k+1}(\Omega_h)$, let $u(x, t) \in H^{k+3}(\Omega) \forall t \in (0, T]$ be a solution of the problem (20) and $\tilde{u}(x, t)$ be an extension of the function $u(x, t)$ satisfying the condition (23). Let $\eta(x, t) \in V_h(\Omega_h)$ be the Ritz approximation of $\tilde{u}(x, t)$. Let $v \in V_h$ and let the quadrature formula satisfy (97).

Then there exists a constant c such that

$$(105) \quad |E(b\eta v)| \leq ch^{k+1} \|u\|_{k+3,\Omega} \|v\|_{1,\Omega_h}.$$

Proof. In the same way as in (99) we get

$$(106) \quad E(b\eta v) = \sum_{e \in \mathcal{E}_h} \hat{E}(\eta^* v^* b^* J_e).$$

Evidently $D^\alpha b^* = 0(h^{|\alpha|})$ and $D^\alpha J_e = 0(h^{n+|\alpha|})$. (77) implies $D^\alpha(b^* J_e) = 0(h^{n+|\alpha|})$. Hence, all the assumptions of Lemma 7 for $\varkappa = n$ are satisfied. From (84') it follows that

$$(107) \quad |\hat{E}(b^* \eta^* v^* J_e)| \leq ch^2 \{h^{k+1} \|v^*\|_{0,\mathcal{T}} \|\eta^*\|_{0,\mathcal{T}} + h^{d+1} [v^*]_{k,\mathcal{T}} [\eta^*]_{k,\mathcal{T}}\}$$

while (78) implies

$$(108) \quad \|v^*\|_{0,\mathcal{T}} \leq ch^{-n/2} \|v\|_{0,e}, \quad \|\eta^*\|_{0,\mathcal{T}} \leq ch^{-n/2} \{\|\eta - \tilde{u}\|_{0,e} + \|\tilde{u}\|_{0,e}\}.$$

It can be easily verified that

$$(109) \quad [\eta^*]_{k,T} \leq c\{[\eta^* - (\pi_e \tilde{u})^*]_{k,T} + [(\pi_e \tilde{u})^* - \tilde{u}^*]_{k,T} + [\tilde{u}^*]_{k,T}\}.$$

From (80), (72) and from the interpolation theorem (see (6)) we get

$$\begin{aligned} [\eta^* - (\pi_e \tilde{u})^*]_{k,T}^2 &\leq ch^{-2k} \|\eta^* - (\pi_e \tilde{u})^*\|_{0,T}^2 \leq \\ &\leq ch^{-2k} \{ \|\eta^* - \tilde{u}^*\|_{0,T}^2 + \|\tilde{u}^* - (\pi_e \tilde{u})^*\|_{0,T}^2 \} \leq \\ &\leq ch^{-2k-n} \{ \|\eta - \tilde{u}\|_{0,e}^2 + h^{2(k+1)} \|\tilde{u}\|_{k+1,e}^2 \}. \end{aligned}$$

Hence

$$(110) \quad [\eta^* - (\pi_e \tilde{u})^*]_{k,T} \leq ch^{-n/2-k} \{ \|\eta - \tilde{u}\|_{0,e} + h^{k+1} \|\tilde{u}\|_{k+1,e} \}.$$

Analogously we get

$$(111) \quad [(\pi_e \tilde{u})^* - \tilde{u}^*]_{k,T} \leq c\sqrt{\sum_{i=0}^k h^{-n} \|\pi_e \tilde{u} - \tilde{u}\|_{i,e}^2} \leq ch^{-n/2+1} \|\tilde{u}\|_{k+1,e},$$

$$(112) \quad [\tilde{u}]_{k,T} \leq ch^{-n/2} \|\tilde{u}\|_{k,e}.$$

If we substitute from (110), (111) and (112) into (109) and use elementary calculation then we obtain

$$(113) \quad [\eta^*]_{k,T} \leq ch^{-n/2} \{ h^{-k} \|\eta - \tilde{u}\|_{0,e} + \|\tilde{u}\|_{k+1,e} \}.$$

Substituting from (108), (103) and (113) into (107) and using (97) we get

$$(114) \quad |\hat{E}(\eta^* v^* b^* J_e)| \leq ch^{k+1} \|v\|_{1,e} \{ h^{-k} \|\eta - \tilde{u}\|_{0,e} + \|\tilde{u}\|_{k+1,e} \},$$

which together with (106), the Schwarz inequality, Theorem on Ritz approximation (see (68)) and (67) yields

$$\begin{aligned} |E(b\eta v)| &\leq ch^{k+1} \|v\|_{1,\Omega_h} \{ h^{-k} \|\eta - \tilde{u}\|_{0,\Omega_h} + \|\tilde{u}\|_{k+1,\Omega_h} \} \leq \\ &\leq ch^{k+1} \|v\|_{1,\Omega_h} \{ h \|u\|_{k+3,\Omega} + \|u\|_{k+3,\Omega} \} \end{aligned}$$

and the inequality (105) is proved.

Theorem 4. *Let all the assumptions of Theorem 3 be satisfied. Then there exists a constant c such that*

$$(115) \quad \left| E \left(b \frac{\partial \eta}{\partial x_i} \frac{\partial v}{\partial x_j} \right) \right| \leq ch^{k+1} \|u\|_{k+3,\Omega} \|v\|_{1,\Omega_h}.$$

Proof. Analogously as in (106) the following relation is true

$$(116) \quad E \left(b \frac{\partial \eta}{\partial x_i} \frac{\partial v}{\partial x_j} \right) = \sum_{e \in \mathcal{C}_h} \hat{E} \left(b^* \left(\frac{\partial \eta}{\partial x_i} \right)^* \left(\frac{\partial v}{\partial x_j} \right)^* J_e \right).$$

From the rule on the composed function derivative it follows that

$$(117) \quad \hat{E} \left(b^* \left(\frac{\partial \eta}{\partial x_i} \right)^* \left(\frac{\partial v}{\partial x_j} \right)^* J_e \right) = \sum_{i,m=1}^n \hat{E} \left(\frac{\partial \eta^*}{\partial \hat{x}_i} \frac{\partial v^*}{\partial \hat{x}_m} b^* \frac{J_e^{(i,i)} J_e^{(m,j)}}{J_e} \right)$$

(75), (76) and (77) implies

$$D^2(b^*(J_e^{(i,i)} J_e^{(m,j)})/J_e) = O(h^{|z|+n-2}).$$

We may apply Lemma 7 (with $\kappa = n - 2$) to the expressions on the right hand side of (116). From (85') we obtain in the same way as in (114)

$$(118) \quad \left| \hat{E} \left(\frac{\partial \eta^*}{\partial \hat{x}_i} \frac{\partial v^*}{\partial \hat{x}_m} b \frac{J_e^{(i,i)} J_e^{(m,j)}}{J_e} \right) \right| \leq ch^{k+1} \|v\|_{1,e} \{h^{-k} \|\eta - \tilde{u}\|_{1,e} + \|\tilde{u}\|_{k+1,e}\}.$$

From here, from (117), (116), from the Schwarz inequality, from Theorem on Ritz approximation and from (67) the inequality (115) follows.

Before proving the next theorem, we formulate Lemma 8 which is an obvious consequence of Theorem 3 from [1], p. 436.

Lemma 8. *If the quadrature formula on the reference set \hat{T} is of a degree $d \geq \geq 2k - 2$ then there exists a constant c such that*

$$(119) \quad \sum_r \omega_{r,e} \sum_{i=1}^n \left(\frac{\partial p}{\partial x_i} (b_{r,e}) \right)^2 \geq c |p|_{1,e}^2, \quad \forall p \in P_e, \quad e \in \mathcal{C}_h$$

(P_e is defined in (3)).

We introduce the notation

$$(120) \quad |v|_h^2 = (g(x) v, v)_h, \quad \|v\|_h^2 = a_h(v, v)$$

where the forms $(\cdot, \cdot)_h$, $a_h(\cdot, \cdot)$ are defined in (31) and (32).

Theorem 5. *There exist constants c_1 and c_2 such that*

$$(121) \quad \text{a) } c_1 \|v\|_{0,\Omega_h} \leq |v|_h, \quad \forall v \in V_h$$

provided the quadrature formula on the reference set \hat{T} is of a degree $d \geq 2k$.

$$(122) \quad \text{b) } c_2 |v|_{1,\Omega_h} \leq \|v\|_h, \quad \forall v \in V_h$$

provided the quadrature formula on the reference set \hat{T} is of a degree $d \geq 2k - 2$.

Proof. From (31), (16), (12) and from the fact that δ^2 is a polynomial of degree $\leq 2k$ on \hat{T} we obtain

$$|v|_h^2 \geq g_0 \sum_{e \in \mathcal{C}_h} \min_{\hat{T}} J_e \sum_r \omega_r (v^*)^2 (\hat{b}_r) = g_0 \sum_{e \in \mathcal{C}_h} \min_{\hat{T}} J_e \int_{\hat{T}} (v^*)^2 d\hat{x} \geq$$

$$\geq g_0 \sum_{e \in \mathcal{E}_h} \frac{\min J_e}{\max J_e} \int_e (v^*)^2 J_e \, d\hat{x} = g_0 \sum_{e \in \mathcal{E}_h} \frac{\min J_e}{\max J_e} \int_e v^2 \, dx.$$

From (2) it follows $\min_{\hat{T}} J_e / \max_{\hat{T}} J_e \geq 1/c_0^2$. Hence

$$|v|_h^2 \geq \frac{g_0}{c_0^2} \sum_{e \in \mathcal{E}_h} \int_e v^2 \, dx = \frac{g_0}{c_0^2} \|v\|_{0,\Omega_h}^2$$

and (121) is proved.

From (32), (18) and (12) it follows that

$$\|v\|_h^2 \geq g_1 \sum_{e \in \mathcal{E}_h} \sum_r \omega_{r,e} \sum_{i=1}^n \left[\frac{\partial v}{\partial x_i} (b^s, e) \right]^2.$$

This together with (119) proves the inequality (122).

5. ERROR ESTIMATES IN ONE AND TWO-STEPS A-STABLE METHODS

Let us recall the notation introduced in (28). As we said in (34) the aim of our paper is to derive error bounds for

$$(123) \quad \|\lambda\|_{0,\Omega \cap \Omega_h} = \|u^j - u_h^j\|_{0,\Omega \cap \Omega_h}, \quad 1 \leq j \leq \frac{T}{\Delta t},$$

where u and u_h are solutions of the problems (20) and (33) respectively.

Let

$$(124) \quad \tilde{u}^j = \eta^j + \xi^j$$

where $\eta^j = \eta(x, j \Delta t)$ is the Ritz approximation of the function $\tilde{u}^j = \tilde{u}(x, j \Delta t)$ (we remind that \tilde{u} is an extension of u satisfying the inequality (23)).

From Theorem on Ritz approximation we get

$$(125) \quad \|\xi^j\|_{0,\Omega_h} = \|\tilde{u}^j - \eta^j\|_{0,\Omega_h} \leq ch^k \|u\|_{k+3,\Omega}.$$

It is evident that

$$(126) \quad \|u^j - u_h^j\|_{0,\Omega \cap \Omega_h} \leq \|\tilde{u}^j - u_h^j\|_{0,\Omega_h} \leq \|\xi^j\|_{0,\Omega_h} + \|\eta^j - u_h^j\|_{0,\Omega_h}.$$

Hence it is sufficient to estimate error bounds for

$$(127) \quad \varepsilon^j = \eta^j - u_h^j.$$

From (35) we get

$$(128) \quad \tilde{a}(\eta^m, v) = (\tilde{f}^m, v)_{0,\Omega_h} - (\tilde{g}\tilde{u}_r^m, v)_{0,\Omega_h}, \quad \forall v \in V_h.$$

Hence

$$\Delta t \tilde{a}(\sum_{j=0}^{\nu} \beta_j \eta^{m+j}, v) = \Delta t (\sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}, v)_{0, \Omega_h} - \Delta t (\tilde{g} \sum_{j=0}^{\nu} \beta_j \tilde{u}_t^{m+j}, v)_{0, \Omega_h}.$$

If we add $(\tilde{g} \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}, v)_{0, \Omega_h}$ to both sides of this identity and apply (124) then we get

$$(129) \quad \begin{aligned} & (\tilde{g} \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}, v)_{0, \Omega_h} + \Delta t \tilde{a}(\sum_{j=0}^{\nu} \beta_j \eta^{m+j}, v) = \\ & = \Delta t (\sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}, v)_{0, \Omega_h} + (\pi_v^m - \omega_v^m, v)_{0, \Omega_h} \end{aligned}$$

where

$$(130) \quad \pi_v^m = \tilde{g} \sum_{j=0}^{\nu} (\alpha_j \tilde{u}^{m+j} - \Delta t \beta_j \tilde{u}_t^{m+j}), \quad \omega_v^m = \tilde{g} \sum_{j=0}^{\nu} \alpha_j \xi^{m+j}.$$

From (31), (32) and (79) it follows that

$$(131) \quad (z, v)_{0, \Omega_h} - (z, v)_h = E(zv), \quad \tilde{a}(z, v) - a_h(z, v) = E\left(\sum_{i,j=1}^n \tilde{g}_{ij} \frac{\partial z}{\partial x_i} \frac{\partial v}{\partial x_j}\right).$$

From here and from (129) we get

$$(132) \quad \begin{aligned} & (g \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}, v)_h + E(\tilde{g}v \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}) + \Delta t a_h(\sum_{j=0}^{\nu} \beta_j \eta^{m+j}, v) + \\ & + \Delta t E\left(\sum_{i,j=1}^n \tilde{g}_{ij} \frac{\partial v}{\partial x_j} \left(\sum_{l=0}^{\nu} \beta_l \frac{\partial \eta^{m+l}}{\partial x_i}\right)\right) = \\ & = \Delta t (\sum_{j=0}^{\nu} \beta_j f^{m+j}, v)_h + \Delta t E(v \sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}) + (\pi_v^m - \omega_v^m, v)_{0, \Omega_h}. \end{aligned}$$

If we subtract (33) from (132) and take (127) into account then we come to the identity

$$(133) \quad \begin{aligned} & (g \sum_{j=0}^{\nu} \alpha_j \varepsilon^{m+j}, v)_h + \Delta t a_h(\sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}, v) = \\ & = (\pi_v^m - \omega_v^m, v)_{0, \Omega_h} + \Delta t E(v \sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}) - E(\tilde{g}v \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}) - \\ & - \Delta t E\left(\sum_{i,j=1}^n \tilde{g}_{ij} \frac{\partial v}{\partial x_j} \sum_{l=0}^{\nu} \beta_l \frac{\partial \eta^{m+l}}{\partial x_i}\right), \quad \forall v \in V_h. \end{aligned}$$

Let us denote

$$(134) \quad A_v^m = (g \sum_{j=0}^{\nu} \alpha_j \varepsilon^{m+j}, \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j})_h,$$

$$(135) \quad B_v^m = a_h \left(\sum_{j=0}^v \beta_j \varepsilon^{m+j}, \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right),$$

$$(136) \quad D_v^m = \left(\pi_v^m - \omega_v^m, \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right)_{0, \Omega_h},$$

$$(137) \quad F_v^m = E \left(\sum_{j=0}^v \beta_j \varepsilon^{m+j}, \sum_{j=0}^v \beta_j \tilde{J}^{m+j} \right),$$

$$(138) \quad G_v^m = E \left(\tilde{g} \sum_{j=0}^v \beta_j \varepsilon^{m+j}, \sum_{j=0}^v \alpha_j \eta^{m+j} \right),$$

$$(139) \quad H_v^m = E \left(\sum_{i,j=1}^n \tilde{g}_{ij} \left[\sum_{l=0}^v \beta_l \frac{\partial \varepsilon^{m+l}}{\partial x_j} \sum_{l=0}^v \beta_l \frac{\partial \eta^{m+l}}{\partial x_i} \right] \right),$$

$$(140) \quad Q_v^m = \Delta t F_v^m - G_v^m - \Delta t H_v^m.$$

The identity (133) is true for all $v \in V_h$. Hence it is also true for

$$(141) \quad v = \sum_{j=0}^v \beta_j \varepsilon^{m+j}.$$

From this and from (134)–(140) we get for any s such that $s \Delta t \leq T$, $s \geq v$ the following basic identity:

$$(142) \quad \sum_{m=0}^{s-v} A_v^m + \Delta t \sum_{m=0}^{s-v} B_v^m = \sum_{m=0}^{s-v} D_v^m + \sum_{m=0}^{s-v} Q_v^m.$$

Now, we estimate the expressions in (142) when one and two-step A -stable methods are used. From [11] and [12] it follows:

a) for one-step A -stable methods

$$(143) \quad v = 1, \quad \alpha_1 = 1, \quad \alpha_0 = -1, \quad \beta_1 = 1 - \theta, \quad \beta_0 = \theta, \quad \theta \leq \frac{1}{2} \text{ is any real number.}$$

If $\theta = \frac{1}{2}$ then the method is of the order $q = 2$, in all other cases the method is of the order $q = 1$.

b) for two-step A -stable methods

$$(144) \quad v = 2, \quad \alpha_2 = \theta, \quad \alpha_1 = 1 - 2\theta, \quad \alpha_0 = -1 + \theta, \quad \beta_2 = \frac{1}{2}\theta + \delta, \\ \beta_1 = \frac{1}{2} - 2\delta, \quad \beta_0 = \frac{1}{2} - \frac{1}{2}\theta + \delta, \quad \theta \geq \frac{1}{2}, \quad \delta > 0.$$

From (134), (143) and (120) it follows that

$$A_1^m = (g(\varepsilon^{m+1} - \varepsilon^m), (1 - \theta) \varepsilon^{m+1} + \theta \varepsilon^m)_h = \\ = (1 - \theta) |\varepsilon^{m+1}|_h^2 - (1 - 2\theta) (g \varepsilon^{m+1}, \varepsilon^m)_h - \theta |\varepsilon^m|_h^2 \geq \\ \geq (1 - \theta) |\varepsilon^{m+1}|_h^2 - \frac{1}{2} (1 - 2\theta) |\varepsilon^{m+1}|_h^2 - \frac{1}{2} (1 - 2\theta) |\varepsilon^m|_h^2 - \theta |\varepsilon^m|_h^2 = \\ = \frac{1}{2} |\varepsilon^{m+1}|_h^2 - \frac{1}{2} |\varepsilon^m|_h^2.$$

Hence

$$\sum_{m=0}^{s-1} A_1^m \geq \frac{1}{2} |e^s|_h^2 - \frac{1}{2} |e^0|_h^2.$$

From here and from (121) we have

$$(145) \quad \sum_{m=0}^{s-1} A_1^m \geq c \|e^s\|_{0, \Omega_h}^2 - \frac{1}{2} |e^0|_h^2.$$

For $v = 2$ we find out by simple calculation that (see [11])

$$\begin{aligned} A_2^m &= (g(\theta e^{m+2} + (1 - 2\theta) e^{m+1} + (-1 + \theta) e^m), (\frac{1}{2}\theta + \delta) e^{m+2} + \\ &\quad + (\frac{1}{2} - 2\delta) e^{m+1} + (\frac{1}{2} - \frac{1}{2}\theta + \delta) e^m)_h = \\ &= \frac{1}{2}(\theta^2 + \delta) |e^{m+2}|_h^2 - (\theta - \frac{1}{2}) |e^{m+1}|_h^2 - \frac{1}{2}[(\theta - 1)^2 + \delta] |e^m|_h^2 - \\ &\quad - [\theta(\theta - 1) + \delta] [(g e^{m+2}, e^{m+1})_h - (g e^{m+1}, e^m)_h] + \\ &\quad + \delta(\theta - \frac{1}{2}) |e^{m+2} - 2e^{m+1} + e^m|_h^2. \end{aligned}$$

Hence

$$\begin{aligned} (146) \quad \sum_{m=0}^{s-2} A_2^m &\geq \sum_{m=0}^{s-2} \{ \frac{1}{2}(\theta^2 + \delta) |e^{m+2}|_h^2 - (\theta - \frac{1}{2}) |e^{m+1}|_h^2 - \\ &\quad - \frac{1}{2}[(\theta - 1)^2 + \delta] |e^m|_h^2 - \\ &\quad - [\theta(\theta - 1) + \delta] (g e^{m+2}, e^{m+1})_h + [\theta(\theta - 1) + \delta] (g e^{m+1}, e^m)_h \} = \\ &= \sum_{m=2}^{s-2} \{ \frac{1}{2}(\theta^2 + \delta) - (\theta - \frac{1}{2}) - \frac{1}{2}[(\theta - 1)^2 + \delta] \} |e^m|_h^2 - \\ &\quad - \sum_{m=1}^{s-2} \{ [\theta(\theta - 1) + \delta] - [\theta(\theta - 1) + \delta] \} (g e^{m+1}, e^m)_h + \\ &\quad + \frac{1}{2}(\theta^2 + \delta) [|e^{s-1}|_h^2 + |e^s|_h^2] - (\theta - \frac{1}{2}) [|e^{s-1}|_h^2 + |e^1|_h^2] - \\ &\quad - \frac{1}{2}[(\theta - 1)^2 + \delta] [|e^0|_h^2 + |e^1|_h^2] - [\theta(\theta - 1) + \delta] (g e^s, e^{s-1})_h + \\ &\quad + [\theta(\theta - 1) + \delta] (g e^1, e^0)_h \geq M - c_1 [|e^0|_h^2 + |e^1|_h^2] \end{aligned}$$

where

$$(147) \quad \begin{aligned} M &= \frac{1}{2}(\theta^2 + \delta) |e^s|_h^2 + \frac{1}{2}[(\theta - 1)^2 + \delta] |e^{s-1}|_h^2 - \\ &\quad - |(\sqrt{g} e^s, \sqrt{g} [\theta(\theta - 1) + \delta] e^{s-1})_h|. \end{aligned}$$

Let us suppose first $\theta(\theta - 1) + \delta = 0$. Then (147) implies

$$(148) \quad M \geq \frac{1}{2}(\theta^2 + \delta) |e^s|_h^2.$$

Let $\theta(\theta - 1) + \delta \neq 0$ now. Then using the inequality $|ab| \leq \frac{1}{2}\tau a^2 + b^2/2\tau$ for $\tau = [\theta(\theta - 1) + \delta]^2/[(\theta - 1)^2 + \delta]$ we get

$$\begin{aligned} M &\geq \frac{1}{2} \left\{ \theta^2 + \delta - \frac{[\theta(\theta - 1) + \delta]^2}{(\theta - 1)^2 + \delta} \right\} |\varepsilon^s|_h^2 = \\ &= \frac{1}{2} (\theta^2 + \delta) \left\{ 1 - \frac{[\theta(\theta - 1) + \delta]^2}{[\theta(\theta - 1) + \delta]^2 + \delta} \right\} |\varepsilon^s|_h^2. \end{aligned}$$

Hence the inequality (148) is again true.

From (148), (146) and (121) we get

$$(149) \quad \sum_{m=0}^{s-2} A_2^m \geq c_2 \|\varepsilon^s\|_{0,\Omega_h}^2 - c_1 [|\varepsilon^0|_h^2 + |\varepsilon^1|_h^2].$$

Both the inequalities (145) and (149) can be written in the same way as

$$(150) \quad \sum_{m=0}^{s-v} A_v^m \geq c_2 \|\varepsilon^s\|_{0,\Omega_h}^2 - c_1 [|\varepsilon^0|_h^2 + |\varepsilon^{v-1}|_h^2], \quad v = 1, 2.$$

From (135) and (120) it follows that

$$B_v^m = \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_h^2.$$

From here and from (122) we get

$$(151) \quad \sum_{m=0}^{s-v} B_v^m \geq c_3 \sum_{m=0}^{s-v} \left| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right|_{1,\Omega_h}^2, \quad v = 1, 2.$$

Substituting (150) and (151) into (142) we get

$$(152) \quad \begin{aligned} &c_2 \|\varepsilon^s\|_{0,\Omega_h}^2 + c_3 \Delta t \sum_{m=0}^{s-v} \left| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right|_{1,\Omega_h}^2 \leq \\ &\leq \sum_{m=0}^{s-v} |D_v^m| + \sum_{m=0}^{s-v} |Q_v^m| + c_1 [|\varepsilon^0|_h^2 + |\varepsilon^{v-1}|_h^2], \quad v = 1, 2. \end{aligned}$$

From (136) and (130) it follows that

$$(153) \quad \begin{aligned} |D_1^m| &\leq \|\pi_1^m - \omega_1^m\|_{0,\Omega_h} \left\| \sum_{j=0}^1 \beta_j \varepsilon^{m+j} \right\|_{0,\Omega_h} \leq \\ &\leq [\|\pi_1^m\|_{0,\Omega_h} + \|\omega_1^m\|_{0,\Omega_h}] \left\| \sum_{j=0}^1 \beta_j \varepsilon^{m+j} \right\|_{0,\Omega_h} \end{aligned}$$

where

$$(154) \quad \begin{aligned} \pi_1^m &= \tilde{g}[\tilde{u}^{m+1} - \tilde{u}^m - \Delta t((1-\theta)\tilde{u}_t^{m+1} + \theta\tilde{u}_t^m)], \\ \omega_1^m &= \tilde{g}(\xi^{m+1} - \xi^m). \end{aligned}$$

In the sequel we suppose additionally that

$$(155) \quad \frac{\partial^l u(x, t)}{\partial t^l} \in L^\infty(H^{k+3}(\Omega)), \quad l = 1, \dots, q + 1$$

where q is the order of the method and $u(x, t)$ is a solution of (20). From the Calderon theorem and from (154) we get

$$(156) \quad \|\pi_1^m\|_{0, \Omega_h} \leq c \left\| u^{m+1} - u^m - \Delta t \left((1 - \theta) \frac{\partial u^{m+1}}{\partial t} + \theta \frac{\partial u^m}{\partial t} \right) \right\|_{k+3, \Omega}.$$

a) Let $q = 1$ (i.e. $\theta = \frac{1}{2}$). Then the Taylor theorem yields

$$u^{m+1} - u^m - \Delta t \left((1 - \theta) \frac{\partial u^{m+1}}{\partial t} + \theta \frac{\partial u^m}{\partial t} \right) = \frac{1}{2} \Delta t^2 \frac{\partial^2 u^{x_1}}{\partial t^2} - (1 - \theta) \Delta t^2 \frac{\partial^2 u^{x_2}}{\partial t^2}.$$

Hence

$$(157) \quad \|\pi_1^m\|_{0, \Omega_h} \leq c \Delta t^2.$$

b) Let $q = 2$ (i.e. $\theta = \frac{1}{2}$). Then the Taylor theorem yields

$$u^{m+1} - u^m - \Delta t \left((1 - \theta) \frac{\partial u^{m+1}}{\partial t} + \theta \frac{\partial u^m}{\partial t} \right) = \frac{1}{6} \Delta t^3 \frac{\partial^3 u^{x_3}}{\partial t^3} - \frac{1}{4} \Delta t^3 \frac{\partial^3 u^{x_4}}{\partial t^3}.$$

Hence

$$(158) \quad \|\pi_1^m\|_{0, \Omega_h} \leq c \Delta t^3.$$

From (157) and (158) we conclude

$$(159) \quad \|\pi_1^m\|_{0, \Omega_h} \leq c \Delta t^{q+1}.$$

From (154) and (124) it follows that

$$\omega_1^m = \tilde{g}[\tilde{u}^{m+1} - \tilde{u}^m - (\eta^{m+1} - \eta)].$$

Obviously the function $\eta^{m+1} - \eta^m$ is the Ritz approximation of $\tilde{u}^{m+1} - \tilde{u}^m$. Theorem on Ritz approximation yields

$$(160) \quad \|\omega_1^m\|_{0, \Omega_h} \leq ch^{k+1} \|u^{m+1} - u^m\|_{k+3, \Omega} = ch^{k+1} \Delta t \left\| \frac{\partial u^{x_2}}{\partial t} \right\|_{k+3, \Omega}.$$

Hence

$$(161) \quad \|\omega_1^m\|_{0, \Omega_h} \leq ch^{k+1} \Delta t.$$

(153), (159) and (161) imply

$$|D_1^m| \leq c \Delta t (\Delta t^q + h^{k+1}) \left\| \sum_{j=0}^1 \beta_j \varepsilon^{m+j} \right\|_{0, \Omega_h}.$$

In the same way an analogous inequality may be derived for $v = 2$. Hence

$$(162) \quad \sum_{m=0}^{s-v} |D_v^m| \leq c \Delta t (\Delta t^q + h^{k+1}) \sum_{m=0}^{s-v} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{0, \Omega_h}, \quad v = 1, 2.$$

From (140) we have

$$(163) \quad \sum_{m=0}^{s-v} |Q_v^m| \leq \Delta t \sum_{m=0}^{s-v} |F_v^m| + \sum_{m=0}^{s-v} |G_v^m| + \Delta t \sum_{m=0}^{s-v} |H_v^m|.$$

From (137) and (98) (supposing (97)) we obtain

$$|F_v^m| \leq ch^{k+1} \left\| \sum_{j=0}^v \beta_j \tilde{f}^{m+1} \right\|_{k+1, \Omega_h} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{1, \Omega_h}.$$

From here, from (24), (23) and (22) we get

$$(164) \quad \sum_{m=0}^{s-v} |F_v^m| \leq ch^{k+1} \sum_{m=0}^{s-v} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{1, \Omega_h}.$$

(138) and (105) (under the assumption (97)) imply

$$(165) \quad |G_v^m| \leq ch^{k+1} \left\| \sum_{j=0}^v \alpha_j u^{m+j} \right\|_{k+3, \Omega} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{1, \Omega_h}.$$

We estimate now $\left\| \sum_{j=0}^v \alpha_j u^{m+j} \right\|_{k+3, \Omega}$ for $v = 1$. From (143) and (22) it follows that

$$(166) \quad \left\| \sum_{j=0}^1 \alpha_j u^{m+j} \right\|_{k+3, \Omega} = \|u^{m+1} - u\|_{k+3, \Omega} = \Delta t \left\| \frac{\partial u^{\alpha_6}}{\partial t} \right\|_{k+3, \Omega} \leq c \Delta t.$$

In the same way for $v = 2$ it follows from (144)

$$(167) \quad \begin{aligned} \left\| \sum_{j=0}^2 \alpha_j u^{m+j} \right\|_{k+3, \Omega} &= \|\theta u^{m+2} + (1 - 2\theta) u^{m+1} + (-1 + \theta) u^m\|_{k+3, \Omega} \leq \\ &\leq \theta \|u^{m+2} - u^{m+1}\|_{k+3, \Omega} + |1 - \theta| \|u^{m+1} - u^m\|_{k+3, \Omega} \leq c \Delta t. \end{aligned}$$

Substituting (166) and (167) into (165) we get

$$|G_v^m| \leq c \Delta t h^{k+1} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{1, \Omega_h}, \quad v = 1, 2.$$

Hence

$$(168) \quad \sum_{m=0}^{s-v} |G_v^m| \leq c \Delta t h^{k+1} \sum_{m=0}^{s-v} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{1, \Omega_h}, \quad v = 1, 2.$$

From (139) and (115) it follows that

$$|H_v^m| \leq ch^{k+1} \left\| \sum_{j=0}^v \beta_j u^{m+j} \right\|_{k+3, \Omega} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{1, \Omega_h}.$$

Hence

$$(169) \quad \sum_{m=0}^{s-v} |H_v^m| \leq c h^{k+1} \sum_{m=0}^{s-v} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{1, \Omega_h}.$$

Substituting (164), (168) and (169) into (163) we get

$$(170) \quad \sum_{m=0}^{s-v} |Q_v^m| \leq c \Delta t h^{k+1} \sum_{m=0}^{s-v} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+1} \right\|_{1, \Omega_h}.$$

From (152), (162) and (170) it follows that

$$(171) \quad \begin{aligned} \|\varepsilon^s\|_{0, \Omega_h}^2 + \Delta t \sum_{m=0}^{s-v} \left| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right|_{1, \Omega_h}^2 &\leq \\ &\leq c \{ \Delta t (\Delta t^q + h^{k+1}) \sum_{m=0}^{s-v} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{0, \Omega_h} + \\ &+ \Delta t h^{k+1} \sum_{m=0}^{s-v} \left\| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right\|_{1, \Omega_h} + |\varepsilon^0|_h^2 + |\varepsilon^{v-1}|_h^2 \} \leq \\ &\leq c \{ \Delta t (\Delta t^q + h^{k+1}) \sum_{m=0}^s \|\varepsilon^m\|_{0, \Omega_h} + \\ &+ \Delta t h^{k+1} \sum_{m=0}^{s-v} \left| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right|_{1, \Omega_h} + |\varepsilon^0|_h^2 + |\varepsilon^{v+1}|_h^2 \}. \end{aligned}$$

Using the inequality

$$(172) \quad |ab| \leq \frac{1}{2} \tau a^2 + \frac{1}{2\tau} b^2$$

for $1/2\tau = 1/c$ we get

$$\begin{aligned} c \Delta t h^{k+1} \sum_{m=0}^{s-v} \left| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right|_{1, \Omega_h} &\leq c \sum_{m=0}^{s-v} \left[\frac{1}{2} \tau \Delta t h^{2(k+1)} + \frac{1}{2\tau} \Delta t \left| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right|_{1, \Omega_h}^2 \right] \leq \\ &\leq \frac{c^2 T}{4} h^{2(k+1)} + \Delta t \sum_{m=0}^{s-v} \left| \sum_{j=0}^v \beta_j \varepsilon^{m+j} \right|_{1, \Omega_h}^2. \end{aligned}$$

If we substitute this inequality into (171) then we obtain

$$(173) \quad \begin{aligned} \|\varepsilon^s\|_{0, \Omega_h}^2 &\leq c \{ \Delta t (\Delta t^q + h^{k+1}) [\|\varepsilon^s\|_{0, \Omega_h} + \sum_{m=0}^{s-1} \|\varepsilon^m\|_{0, \Omega_h}] + \\ &+ h^{2(k+1)} + |\varepsilon^0|_h^2 + |\varepsilon^{v-1}|_h^2 \}. \end{aligned}$$

Using the inequality (172) for $\tau = c$ we get

$$c \Delta t (\Delta t^q + h^{k+1}) \|\varepsilon^s\|_{0, \Omega_h} \leq \frac{c^2}{2} \Delta t^2 (\Delta t^q + h^{k+1})^2 + \frac{1}{2} \|\varepsilon^s\|_{0, \Omega_h}^2.$$

If we substitute this inequality into (173) then we obtain

$$(174) \quad \|\varepsilon^s\|_{0,\Omega_h}^2 \leq c \left\{ \Delta t (\Delta t^q + h^{k+1}) \sum_{m=0}^{s-1} \|\varepsilon^m\|_{0,\Omega_h} + \Delta t^2 (\Delta t^q + h^{k+1})^2 + h^{2(k+1)} + |\varepsilon^0|_h^2 + |\varepsilon^{v-1}|_h^2 \right\}.$$

When using the inequality (172) for $\tau = c/2$ we get

$$\begin{aligned} c \Delta t (\Delta t^q + h^{k+1}) \sum_{m=0}^{s-1} \|\varepsilon^m\|_{0,\Omega_h} &\leq c \sum_{m=0}^{s-1} \left\{ \frac{1}{2} \Delta t (\Delta t^q + h^{k+1})^2 + \frac{1}{2\tau} \Delta t \|\varepsilon^m\|_{0,\Omega_h}^2 \right\} \leq \\ &\leq \frac{c^2 T}{4} (\Delta t^q + h^{k+1})^2 + \Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|_{0,\Omega_h}^2. \end{aligned}$$

From here and from (174) it follows that

$$(175) \quad \|\varepsilon^s\|_{0,\Omega_h}^2 \leq c (\Delta t^{2q} + h^{2(k+1)} + |\varepsilon^0|_h^2 + |\varepsilon^{v-1}|_h^2) + \Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|_{0,\Omega_h}^2.$$

In [9] (see Lemma 2.1, p. 396) the following lemma is formulated:

Lemma 9. *Let Φ, ϕ, χ be nonnegative functions defined for $t = j \Delta t, j = 0, 1, \dots, M$ and let χ be nondecreasing. If*

$$(176) \quad \Phi^s + \phi^s \leq \chi^s + c \Delta t \sum_{m=0}^{s-1} \Phi^m, \quad s = 0, 1, \dots, M,$$

where c is a positive constant then

$$(177) \quad \Phi^s + \phi^s \leq \chi^s e^{cs\Delta t}, \quad s = 0, 1, \dots, M.$$

For the proof see [10].

Applying (176) onto (175) we get

$$\|\varepsilon^s\|_{0,\Omega_h}^2 \leq c (\Delta t^{2q} + h^{2(k+1)} + |\varepsilon^0|_h^2 + |\varepsilon^{v-1}|_h^2) e^{s\Delta t}.$$

Hence

$$(178) \quad \|\varepsilon^s\|_{0,\Omega_h} \leq c (\Delta t^q + h^{k+1} + |\varepsilon^0|_h + |\varepsilon^{v-1}|_h).$$

Now we are able to formulate the result

Theorem 6. *Let $u(x, t)$ be a solution of the problem (20) such that $u, \partial^l u / \partial t^l \in L^\infty(H^{k+3}(\Omega))$, $l = 1, \dots, q$. Let \mathcal{C}_h be a k -regular triangulation of the set Ω_h where k is a positive integer such that $k > n/2 - 1$. Let a quadrature formula on the reference set \hat{T} for calculation of the forms $(\cdot, \cdot)_{0,\Omega_h}$ and $\tilde{a}(\cdot, \cdot)$ be of degrees $d \geq 2k$ and $d \geq 2k - 1$, respectively. Let a given v -step time discretization method be A -stable and of an order q . Let $v = 1$ or 2 .*

Then the discrete problem (33) has one and only one solution $u_h(x, t)$ and there exists a constant c independent of t and h such that

$$(179) \quad \|u^s - u_h^s\|_{0, \Omega \cap \Omega_h} \leq c(\Delta t^q + h^{k+1} + |\varepsilon^0|_h + |\varepsilon^{v-1}|_h).$$

Proof. The existence and uniqueness of the solution u_h is a consequence of A -stability and Theorem 5. The inequality (179) is a consequence of the inequalities (126), (125), (123) and (178).

Remark 1. From (179) we see that L_2 -norm of the error is of a magnitude of the order Δt^q ($q = 1, 2$) with respect to Δt and of the order h^{k+1} with respect to h .

Remark 2. According to our result, for 1-regular triangulation (i.e. for linear isoparametric elements) the quadrature formula on the reference set \hat{T} for calculation of the forms $(\cdot, \cdot)_{0, \Omega}$ and $a(\cdot, \cdot)$ must be, in general, of degree 2 and 1, respectively. It can be proved that using the quadrature formula

$$(180) \quad \int_{\hat{T}} \varphi(\hat{x}) \, d\hat{x} \approx \frac{\text{mes } \hat{T}}{n} [\varphi(0, 0, \dots, 0) + \varphi(0, 1, \dots, 0) + \varphi(0, 0, \dots, 1)]$$

(which is of degree 1) for calculation of the form $(\cdot, \cdot)_{0, \Omega}$ we obtain the same estimate as in (179). In this case the mass matrix is diagonal. In the engineering literature this effect is called the *mass lumping*.

Remark 3. For the three-dimensional space the simplicial curved elements have no practical use. For such case the theory using quadrilateral elements must be developed. We are working on this problem now.

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Souhrn

ŘEŠENÍ PARABOLICKÝCH ROVNIC METODOU KONEČNÝCH PRVKŮ

JOSEF NEDOMA

V dosud publikovaných pracích o řešení parabolických rovnic metodou konečných prvků jsou odvozeny odhady chyb buďto pro případ kdy sjednocení konečných elementů (rovných nebo křivých) přesně pokrývá danou oblast (tak je tomu např. ve Zlámalových pracích) nebo pro případ kdy sjednocení konečných elementů (křivých) pokrývá danou oblast jen přibližně (tak je tomu např. v Ciarlet-Raviartových pracích). V prvním případě jsou odhady odvozeny pro plně diskretizovaná (tj. v prostoru i čase) přibližná řešení. Není však brána v úvahu chyba způsobená numerickou integrací. Ve druhém případě je sice uvažována i chyba způsobená numerickou integrací, odhady jsou však odvozeny pouze pro semidiskretní (nediskretizovaná v čase) přibližná řešení.

V této práci jsou odvozeny odhady chyb pro plně diskretizovaná řešení a pro libovolné křivé oblasti. Jsou odvozeny požadavky na stupeň přesnosti kvadraturních formulí tak, aby numerická integrace nezhoršovala optimální řád konvergence v L_2 normě daný metodou konečných prvků. Obdržený výsledek je následující: Jestliže metoda konečných prvků je řádu $k + 1$, potom k tomu, aby se numerickou integrací řád nesnížil, je třeba použít pro výpočet integrálů, ve kterých vystupují samotné funkce, kvadraturních formulí stupně přesnosti aspoň $2k$ a pro výpočet integrálů, ve kterých vystupují derivace funkcí kvadraturních formulí stupně přesnosti aspoň $2k - 1$.

Práce je rozdělena do pěti částí. Vprvé části je konstruován prostor konečných elementů. Vychází se zde z isoparametrické třídy simplicialních elementů definované Ciarletem a Raviartem. Důvodem k tomu je, že bylo tak možné beze změny převzít interpolační teorém odvozený těmito autory. Obdržené výsledky lze odvodit i pro jiné elementy, pokud ovšem zůstane zachována platnost interpolačního teorému ve tvaru uvedeném v (6).

Ve druhé části je formulován problém a cíl celé práce. Základní parabolický problém je definován ve (20) a to přímo ve variačním tvaru. Předpokládá se, že čtenáři je dostatečně známo, jak tento variační parabolický model souvisí s modelem klasickým. Z metodických důvodů je v (26) formulován příslušný semidiskretní model, tj. model diskretizovaný metodou konečných prvků vzhledem k prostorovým proměnným. Ve (33) je formulován plně diskretní model, tj. model, který vznikne ze semidiskretního modelu diskretizací vzhledem k časové proměnné. Pro tuto diskretizaci je použito v -krokových lineárních metod.

Ve třetí části je odvozena věta, kterou jsme nazvali větou o Ritzově aproximaci. Ritzova aproximace se ukázala být velmi silným prostředkem v dalších důkazech a proto je jí věnována tak velká pozornost.

Ve čtvrté části jsou odvozeny odhady chyb při použití isoparametrické integrace. Přesto, že výsledky jsou formulovány s přihlédnutím k jejich dalšímu využití pro naše účely, mají obecnější charakter a platnost.

Konečně v páté, závěrečné části jsou provedeny odhady chyb v případě, kdy pro časovou diskretizaci jsou použity A -stabilní jedno nebo dvoukrokové metody. Podmínka A -stability je zde podstatná. Finální výsledek celé práce je pak formulován v závěru páté části ve větě 6.

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