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THE FINITE MODEL PROPERTY FOR MIPQ AND SOME CONSEQUENCES

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The proof given in [1] that **MIPQ** has the finite model property is shown to be erroneous. Through the use of slightly different techniques we prove that this property nonetheless does hold. We then prove a representation theorem for finite Monadic Heyting algebras which, together with the finite model property for **MIPQ**, yields an algebraic proof of a theorem concerning a translation map from **MIPQ** to a bimodal calculus (S4, S5)-**C** (see [3]).

1 **MIPQ** *has the finite model property* Following a suggestion of Prior, R. A. Bull considers in [1] a modal calculus (**MIPQ**) which contains as its base the intuitionist propositional calculus (**IC**) instead of the usual classical logic. For reasons that will become clear in section **2**, from here on this calculus will be called S5-**IC**. Hence S5-**IC** contains **IC** and the following rules:

R1 $\dfrac{\alpha \rightarrow \beta}{L\alpha \rightarrow \beta}$

R2 $\dfrac{\alpha \rightarrow \beta}{\alpha \rightarrow M\beta}$

R3 $\dfrac{\alpha \rightarrow \beta}{\alpha \rightarrow L\beta}$, *if* α *is fully modalized*

R4 $\dfrac{\alpha \rightarrow \beta}{M\alpha \rightarrow \beta}$, *if* β *is fully modalized.*

For this modal extension of **IC**, R. A. Bull shows completeness with respect to canonical models. We recall that in a canonical model $\langle A, B, \cup, \cap, \rightarrow, 0, 1, \mathsf{K}, \mathsf{I}\rangle$, A is a Heyting algebra, B is a relatively complete subalgebra of A and K, I are two operators on A such that $\mathsf{K}x = \mathrm{Min}\{y \in B: x \leq y\}$ and $\mathsf{I}x = \mathrm{Max}\{y \in B: y \leq x\}$. In [1] it is proven that S5-**IC** is characterized by finite models, but it appears that the proof in question is wrong. For, starting from a canonical model, the author uses a construction which, as opposed to what is maintained, does not yield a finite canonical model. Let us consider the argument in [1]: given a canonical model $\langle A, B, \cup, \cap, \rightarrow, 0, 1, \mathsf{K}, \mathsf{I}\rangle$

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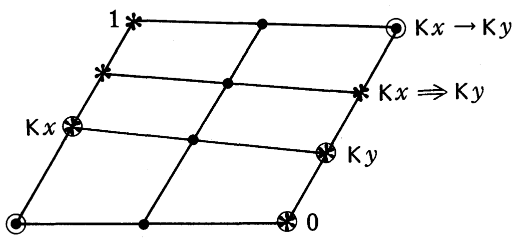
and a sequence $a_1, \ldots, a_n \in A$, take the finite sublattice generated by $0, 1, a_1, \ldots, a_n$. Call it A' and define

$$a \Rightarrow b = \mathrm{Max}\,\{c \in A' : c \leqslant a \rightarrow b\}, \qquad a, b \in A'$$

It is known that $\langle A', \cup, \cap, \Rightarrow, 1\rangle$ is a Heyting algebra. Now if the finite sublattice $B' = B \cap A'$, could be shown to be closed with respect to $\Rightarrow$, then B' would be a relatively complete subalgebra of A' determining K' and I' and thus concluding the construction of a finite canonical model from an arbitrary one. Unfortunately the proof that B' is a subalgebra of A' is erroneous. To be more precise, given the above notations, it can be shown that in general B' is not a Heting subalgebra of A'. Consider, for instance, the following example:



The twelve elements lattice represented in this figure is a Heyting algebra A; a sublattice A' is given by the starred elements and a relatively complete subalgebra B of A is represented by the circled elements. Setting $B' = A' \cap B$, it is immediate that B' is not a subalgebra of A' for it does not contain $\mathsf{K}x \Rightarrow \mathsf{K}y$. Note that the proof in [1] could not be corrected by simply taking the subalgebra B_0 generated by B' in A'. For in order to prove the finite model property we want a subalgebra B_0 of A' such that if $\mathsf{K}_0, \mathsf{I}_0$ are the operators with range B_0, then $\mathsf{K}_0 a = \mathsf{K}a$ and $\mathsf{I}_0 a = \mathsf{I}a$ for each a such that $a, \mathsf{K}a, \mathsf{I}a \in A'$. B_0 does not in general satisfy to this condition as can readily be seen, if we consider the example above and take $B_0 = A'$ and $a = \mathsf{K}x \Rightarrow \mathsf{K}y$.

Yet as we shall see below, S5-**IC** does have in fact the finite model property. To show this, we first recall the concept of Monadic Heyting Algebras. We say that $\langle A; \cup, \cap, \rightarrow, \mathsf{K}, \mathsf{I}\rangle$ is a *Monadic Heyting algebra* (**HM**) if $\langle A, \cup, \cap, \rightarrow\rangle$ is a Heyting algebra and $\mathsf{K}, \mathsf{I}: A \rightarrow A$ (called associated quantifiers on A) are such that

(i) $\mathsf{K}[A]$, the range of K in A, is a subalgebra of A
(ii) $\mathsf{K}[A] = \mathsf{I}[A]$
(iii) (a) $x \leqslant y$ iff $\mathsf{K}x \leqslant y$, $\qquad x \in A, y \in \mathsf{K}[A]$
(b) $x \leqslant y$ iff $x \leqslant \mathsf{I}y$, $\qquad x \in \mathsf{K}[A], y \in A$.

In [4] Monadic Heyting algebras were shown to be equivalent to canonical models and since the former structures prove to be more convenient for the problems we want to deal with, we shall from here on think of the **HM**'s as the structures which give completeness for S5-**IC**. We then prove

Theorem 1 *Let* $\mathfrak{G} = \langle A; \cup, \cap, \rightarrow, \mathsf{K}, \mathsf{I}\rangle$ *be a Monadic Heyting algebra and let* $A_0 \subseteq A$ *be a finite subset of* A. *Then there exists a finite* **HM**, $\mathfrak{G}' = \langle A'; \cup, \cap, \Rightarrow, \mathsf{K}, \mathsf{I}'\rangle$ *such that*

(i) $A_0 \cup \{1_A\} \cup \{0_A\} \subseteq A'$,
(ii) $\langle A', \cup, \cap, \mathsf{K}\rangle$ *is a (finite) existential sublattice of* $\mathfrak{G}$,[1]
(iii) *if* $x, y, x \rightarrow y \in A'$, *then* $x \rightarrow y = x \Rightarrow y$,
(iv) *if* $x, \mathsf{I}x \in A'$, *then* $\mathsf{I}x = \mathsf{I}'x$.

Proof: Let $\langle A', \cup, \cap, \mathsf{K}\rangle$ be the existential sublattice generated by $A_0 \cup \{1, 0\}$. It is known that the structure thus obtained is finite ([8]). Letting I_0 be the interior operator determined by A', (i.e., $\mathsf{I}_0[A] = A'$), and defining

$$x \Rightarrow y = \mathsf{I}_0(x \rightarrow y), \qquad x, y \in A'$$
$$\mathsf{I}'x = \mathsf{I}_0\mathsf{I}x, \qquad x \in A'$$

we show that $\mathfrak{G}' = \langle A'; \cup, \cap, \Rightarrow, \mathsf{K}, \mathsf{I}'\rangle$ is an **HM**. It is straightforward to check that $\langle A', \cup, \cap, \Rightarrow\rangle$ is a Heyting algebra for which condition (iii) holds. We prove first that the sublattice $\mathsf{K}[A']$ is a subalgebra of $\langle A', \cup, \cap, \Rightarrow\rangle$, i.e., that it is closed with respect to $\Rightarrow$. Note that $\mathsf{K}[A]$ is a Heyting subalgebra of $\mathfrak{G}$, hence for $x, y \in A'$, $\mathsf{K}x \rightarrow \mathsf{K}y = \mathsf{K}z$ for some $z \in A$. Thus $\mathsf{K}x \Rightarrow \mathsf{K}y \leqslant \mathsf{K}x \rightarrow \mathsf{K}y = \mathsf{K}z$ which by condition (iii,a) on **HM**'s, yields $\mathsf{K}(\mathsf{K}x \Rightarrow \mathsf{K}y) \leqslant \mathsf{K}z$. But A' is closed with respect to K, so $\mathsf{K}(\mathsf{K}x \Rightarrow \mathsf{K}y) = \mathsf{I}_0\mathsf{K}(\mathsf{K}x \Rightarrow \mathsf{K}y) \leqslant \mathsf{I}_0\mathsf{K}z = \mathsf{K}x \Rightarrow \mathsf{K}y$. Hence $\mathsf{K}x \Rightarrow \mathsf{K}y = \mathsf{K}(\mathsf{K}x \Rightarrow \mathsf{K}y) \in \mathsf{K}[A']$. To prove that $\mathsf{K}[A'] = \mathsf{I}'[A']$, it is sufficient to show that for all $x \in A'$, $\mathsf{I}'\mathsf{K}x = \mathsf{K}x$ and $\mathsf{K}\mathsf{I}'x \leqslant \mathsf{I}'x$. Since $\mathsf{K}x = \mathsf{I}\mathsf{K}x$ (see [4]) we have $\mathsf{K}x = \mathsf{I}_0\mathsf{K}x = \mathsf{I}_0\mathsf{I}\mathsf{K}x = \mathsf{I}'\mathsf{K}x$. Second from $\mathsf{I}'x \leqslant \mathsf{I}x$, it follows $\mathsf{K}\mathsf{I}'x \leqslant \mathsf{K}\mathsf{I}x = \mathsf{I}x$ and therefore $\mathsf{K}\mathsf{I}'x = \mathsf{I}_0\mathsf{K}\mathsf{I}'x \leqslant \mathsf{I}_0\mathsf{I}x = \mathsf{I}'x$. That K satisfies (iii,a) of the definition of **HM**'s is immediate since it is the original operator restricted to A' and $\mathsf{K}[A'] \subseteq \mathsf{K}[A]$. In order to prove that I' satisfies (iii,b) we suppose that $x \in \mathsf{K}[A']$ and $x \leqslant y$. Then for some $z \in A'$, $\mathsf{K}z = x \leqslant y$ and thus $\mathsf{K}z \leqslant \mathsf{I}y$. But $\mathsf{K}z = \mathsf{I}_0\mathsf{K}z \leqslant \mathsf{I}_0\mathsf{I}y = \mathsf{I}'y$. The second half of (iii,b) is staightforward. Last note that if $x, \mathsf{I}x \in A'$, then $\mathsf{I}x = \mathsf{I}_0\mathsf{I}x = \mathsf{I}'x$. Consequently $\mathfrak{G}'$ is an **HM** having the required properties.

Theorem 2 S5-**IC** *has the finite model property*, i.e., *for each non-theorem* α *of* S5-**IC**, *there is a finite* **HM** *which falsifies* α.[2]

Proof: Suppose that α is a non-theorem of S5-**IC** with n subformulas. Then there is an **HM**$\mathfrak{G}$ and a homomorphism $v: \mathfrak{F} \rightarrow \mathfrak{G}$ such that $v(\alpha) \neq 1_A$. Let $A_0 = \{a_1, \ldots, a_n\}$ be the set of elements that v assigns to the subformulas of α. Then by Theorem 1, there is a finite **HM**$\mathfrak{G}'$ satisfying conditions (i) through (iv). Clearly these conditions ensure the existence of a homomorphism $v': \mathfrak{F} \rightarrow \mathfrak{G}'$ such that $v'(\alpha) = v(\alpha)$. For by (i) it is possible to

1. We recall that an existential lattice is a pair $\langle A, \mathsf{K}\rangle$ where A is a distributive lattice with 0, 1 and K is an operator on A such that $x \leqslant \mathsf{K}x$, $\mathsf{K}0 = 0$, $\mathsf{K}(x \cup y) = \mathsf{K}x \cup \mathsf{K}y$, $\mathsf{K}(x \cap \mathsf{K}y) = \mathsf{K}x \cap \mathsf{K}y$.

2. A formula $\alpha \in \mathfrak{F}$ (where $\mathfrak{F}$ is the algebra of formulas of S5-**IC**) is true in an **HM**$\mathfrak{G}$ if for every homomorphism $v: \mathfrak{F} \rightarrow \mathfrak{G}$, $v(\alpha) = 1_{\mathfrak{G}}$.

choose v' in such a way that v' coincides with v on the propositional variables of α. Hence by (i) through (iv), $v(\beta) = v'(\beta)$ for every subformula β of α and hence $v'(\alpha) = v(\alpha) \neq 1_A = 1_{A'}$, i.e., α is falsified in $\mathfrak{G}'$.

2 *A consequence of the finite model property* In a sense which was explained in [1] and [2], S5-**IC** turns out to be an intuitionistic analogue of Lewis' S5. In [3] we proposed a general criterion by means of which it is possible to find the intuitionist counterpart of many other classical modal systems. This criterion is based upon the existence of a theorem-preserving translation T, from S5-**IC** to a bimodal calculus (S4, S5)-**C**. While the theorem concerning the existence of translation T in [3] makes use of both algebraic semantics and syntactic methods, we want to show in this paper that once the finite model property for S5-**IC** has been established, this same theorem can be proven using algebraic techniques only.[3]

In order to make this paper more readable, I recall a few concepts that can be found in [3]. Let (S4, S5)-**C** be a "bimodal calculus" having as primitive connectives ($\vee$, $\neg$, L_1, L_2) with the following axioms and rules:

(b_1) all classical propositional tautologies with the usual definitions for $\wedge$, $\rightarrow$.

(b_2) S4 axioms and rules on L_1:

$$L_1\alpha \rightarrow \alpha$$
$$L_1(\alpha \rightarrow \beta) \rightarrow L_1\alpha \rightarrow L_1\beta$$
$$L_1\alpha \rightarrow L_1L_1\alpha$$
$$\frac{\vdash \alpha}{\vdash L_1\alpha}$$

(b_3) S5 axioms and rules on L_2:

$$L_2\alpha \rightarrow \alpha$$
$$L_2(\alpha \rightarrow \beta) \rightarrow L_2\alpha \rightarrow L_2\beta$$
$$M_2\alpha \rightarrow L_2M_2\alpha, \text{ where } M_2 = \neg L_2 \neg$$
$$\frac{\vdash \alpha}{\vdash L_2\alpha}$$

(b_4) $M_2L_1\alpha \rightarrow L_1M_2\alpha$.

Now let T be the function from S5-**IC** formulas to (S4, S5)-**C** formulas such that for each propositional variable p,

$$\mathrm{T}(p) = L_1p$$

and if α, β are formulas of S5-**IC**, then

$$\begin{aligned} \mathrm{T}(\alpha \vee \beta) &= \mathrm{T}\alpha \vee \mathrm{T}\beta \\ \mathrm{T}(\alpha \wedge \beta) &= \mathrm{T}\alpha \wedge \mathrm{T}\beta \\ \mathrm{T}(\alpha \rightarrow \beta) &= L_1(\mathrm{T}\alpha \rightarrow \mathrm{T}\beta) \\ \mathrm{T}(\neg\alpha) &= L_1\neg\mathrm{T}\alpha \\ \mathrm{T}(M\alpha) &= M_2\mathrm{T}\alpha \\ \mathrm{T}(L\alpha) &= L_1L_2\mathrm{T}\alpha \end{aligned}$$

In [3], we proved

3. Then, this paper settles a question mentioned in [3].

Theorem 3 *If* $\vdash_{\overline{\text{S5-IC}}} \alpha$, *then* $\vdash_{\overline{\text{(S4, S5)-C}}} \mathsf{T}\alpha$,

using first, completeness results for S5-**IC** and (S4, S5)-**C** in the class of all **HM**'s and all bimodal algebras[4] respectively, and second the following

Lemma 1 *Let* $\mathfrak{B} = \langle B, \mathsf{I}_1, \mathsf{K}_2\rangle$ *be a bimodal algebra. If we define* $\mathsf{G}(B) = \{x \in B: x = \mathsf{I}_1 x\}$ *and*

$$\mathsf{K} = \mathsf{K}_2 \upharpoonright \mathsf{G}(B)$$
$$\mathsf{I} = \mathsf{I}_1\mathsf{I}_2 \upharpoonright \mathsf{G}(B), \textit{ where } \mathsf{I}_2 = \sim \mathsf{K}_2 \sim$$

then $\mathsf{G}(\mathfrak{B}) = \langle \mathsf{G}(B), \mathsf{K}, \mathsf{I}\rangle$ *is an* **HM** *to be called the* **HM** *associated with the bimodal algebra* $\mathfrak{B}$.

Our aim is to give an algebraic proof of the converse of Theorem 3. We shall be able to do so, once we establish that every finite **HM** is, up to isomorphism, associated with a bimodal algebra and such will be the topic of section **3**.

3 *A representation theorem for finite* **HM***'s* From here on $\Rightarrow, \rightarrow$ will stand for the Heyting and Boolean relative pseudo complement respectively. Similarly - will denote the Heyting pseudo complement while $\sim$ will represent Boolean complementation. Now let $\langle A; \mathsf{K}, \mathsf{I}\rangle$ be a finite **HM**. It is well known (see [6]) that there is a topological Boolean algebra $\langle B; \mathsf{I}_1\rangle$ such that $A = \mathsf{G}(B) = \mathsf{I}_1[B]$ with $x \Rightarrow y = \mathsf{I}_1(x \rightarrow y)$ and $-x = \mathsf{I}_1(\sim x)$ for all $x, y \in \mathsf{G}(B)$. Then $\mathsf{K}[A] = \mathsf{I}[A] \subseteq A \subseteq B$. Let B' be the Boolean subalgebra of B generated by $\mathsf{K}[A]$. Bearing these facts in mind we can prove the following two lemmas

Lemma 2 *There is a (Boolean) quantifier* K_2 *on* B *such that* $\mathsf{K}_2[B] = B'$.

Proof: B' is a finite, hence relatively complete, subalgebra of B.

Lemma 3 *For every* $x \in B'$, $\mathsf{I}_1 x \in \mathsf{K}[A] = \mathsf{I}[A]$.

Proof: If $x \in B'$, then $x = \bigcap_{i=1}^{m} (a_i \rightarrow b_i)$ with $a_i, b_i \in \mathsf{K}[A]$ (see [6]). So:

$$\mathsf{I}_1 x = \mathsf{I}_1 \bigcap_{i=1}^{m} (a_i \rightarrow b_i) = \bigcap_{i=1}^{m} \mathsf{I}_1(a_i \rightarrow b_i) = \bigcap_{i=1}^{m} (a_i \Rightarrow b_i) \in \mathsf{K}[A].$$

And now for the representation theorem for finite **HM**'s:

Theorem 4 *For every finite* **HM**, $\mathfrak{G} = \langle A; \mathsf{K}, \mathsf{I}\rangle$, *there exists a finite bimodal algebra* $\mathfrak{B} = \langle B, \mathsf{I}_1, \mathsf{K}_2\rangle$ *such that* $\mathfrak{G}$ *is (up to isomorphism) the* **HM** *associated with* $\mathfrak{B}$.

Proof: Let $\langle B; \mathsf{I}_1\rangle$ be as in the comments preceding Lemma 2. Let K_2 be as in Lemma 2 and put $\mathsf{K}_1 = \sim \mathsf{I}_1 \sim$, $\mathsf{I}_2 = \sim \mathsf{K}_2 \sim$. We prove that $\mathfrak{B} = \langle B, \mathsf{I}_1, \mathsf{K}_2\rangle$ is

4. A bimodal algebra (see [3]) is a triple $\langle B, \mathsf{I}_1, \mathsf{K}_2\rangle$ where B is a boolean algebra, K_2 a boolean quantifier and I_1 an interior operator having the following property: for all $x \in B$, $\mathsf{K}_2\mathsf{I}_1 x \leqslant \mathsf{I}_1\mathsf{K}_2 x$ or equivalently $\mathsf{K}_2\mathsf{I}_1 x = \mathsf{I}_1\mathsf{K}_2\mathsf{I}_1 x$.

a bimodal algebra with $\mathfrak{G} = \mathsf{G}(\mathfrak{B})$. We show first that $\mathsf{K} = \mathsf{K}_2 \upharpoonright A$. For every $x \in A$, $x \leqslant \mathsf{K}x \in B' = \mathsf{K}_2[B]$ (Lemma 2) thus $\mathsf{K}_2 x \leqslant \mathsf{K}x$. On the other hand $\mathsf{K}_2 x \in B'$ so by Lemma 3, $\mathsf{I}_1\mathsf{K}_2 x \in \mathsf{K}[A]$. But $x = \mathsf{I}_1 x \leqslant \mathsf{I}_1\mathsf{K}_2 x$ and hence $\mathsf{K}x \leqslant \mathsf{I}_1\mathsf{K}_2 x \leqslant \mathsf{K}_2 x$. We can now prove that $\mathfrak{B}$ is a bimodal algebra, i.e., that $\mathsf{K}_2\mathsf{I}_1 x \leqslant \mathsf{I}_1\mathsf{K}_2 x$ for every $x \in B$. Since $A = \mathsf{I}_1[B]$ and K and K_2 coincide on elements of A, we have $\mathsf{K}_2\mathsf{I}_1 x = \mathsf{K}\mathsf{I}_1 x = \mathsf{I}_1\mathsf{K}\mathsf{I}_1 x = \mathsf{I}_1\mathsf{K}_2\mathsf{I}_1 x \leqslant \mathsf{I}_1\mathsf{K}_2 x$. Last we prove $\mathsf{I} = \mathsf{I}_1\mathsf{I}_2 \upharpoonright A$. Let $x \in A$; from $x \geqslant \mathsf{I}x \in B'$ we get $\mathsf{I}_2 x \geqslant \mathsf{I}x$ and hence $\mathsf{I}_1\mathsf{I}_2 x \geqslant \mathsf{I}_1\mathsf{I}x = \mathsf{I}x$. Vice versa, from Lemma 3 we have $\mathsf{I}_1 x \geqslant \mathsf{I}_1\mathsf{I}_2 x \in \mathsf{K}[A]$, thus $\mathsf{I}\mathsf{I}_1 x \geqslant \mathsf{I}_1\mathsf{I}_2 x$. It follows $\mathsf{I}x \geqslant \mathsf{I}_1\mathsf{I}_2 x$.

4 *A theorem on translation* T We can now give an algebraic proof of the converse of Theorem 3. First we recall that in [3], we proved

Lemma 4 *Let* $\mathfrak{B} = \langle B; \mathsf{I}_1, \mathsf{K}_2\rangle$ *be a bimodal algebra. For every formula* α *of* S5-**IC**, $\mathsf{T}\alpha$ *is true in* $\mathfrak{B}$ *iff* α *is true in* $\mathsf{G}(\mathfrak{B})$, *the* **HM** *associated with* $\mathfrak{B}$.

We then prove

Theorem 5 *If* $\vdash_{\overline{(\mathrm{S4},\, \mathrm{S5})\text{-}\mathbf{C}}} \mathsf{T}\alpha$, *then* $\vdash_{\overline{\mathrm{S5}\text{-}\mathbf{IC}}} \alpha$.

Proof: Suppose that α is a non-theorem of S5-**IC**, then by Theorem 2, there exists a finite **HM**, $\mathfrak{G}$ which falsifies α. By Theorem 4, there is a bimodal algebra $\mathfrak{B}$ such that $\mathfrak{G} = \mathsf{G}(\mathfrak{B})$ is the **HM** associated with $\mathfrak{B}$. Then by Lemma 4, $\mathsf{T}\alpha$ is not true in $\mathfrak{B}$, hence by completeness $\mathsf{T}\alpha$ is not a theorem of (S4, S5)-**C**.

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