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THE FINITE TIME RUIN PROBABILITY OF THE COMPOUND POISSON MODEL WITH CONSTANT INTEREST FORCE

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Abstract

In this paper we establish a simple asymptotic formula with respect to large initial surplus for the finite time ruin probability of the compound Poisson model with constant interest force and subexponential claims. The formula is consistent with known results for the ultimate ruin probability and, in particular, it is uniform for all time horizons when the claim size distribution is regularly varying tailed.

Keywords: Asymptotics, finite time ruin probability, Poisson process, regular variation, subexponentiality, uniform convergence.

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1 The compound Poisson model

Consider the compound Poisson model, in which the claim sizes X_k, $k = 1, 2, \ldots$, form a sequence of independent, identically distributed (i.i.d.), and nonnegative random variables with common distribution B, while the arrival times σ_k, $k = 1, 2, \ldots$, constitute a homogenous Poisson process

$$N(t) = \#\{k = 1, 2, \ldots : \sigma_k \leq t\}, \qquad t \geq 0,$$

with intensity $\lambda > 0$. Let $\{C(t)\}_{t\geq 0}$ with $C(0) = 0$ be a nondecreasing and right continuous stochastic process, denoting the total amount of premiums accumulated up to time t, let $r > 0$ be the constant interest force (that is, after time t one dollar becomes into e^{rt} dollars), and let $x \geq 0$ be the initial surplus. Then, the total surplus up to time t, denoted by $S_r(t)$, satisfies the equation

$$S_r(t) = x\mathrm{e}^{rt} + \int_0^t \mathrm{e}^{r(t-s)} C(\mathrm{d}s) - \sum_{k=1}^{N(t)} X_k \mathrm{e}^{r(t-\sigma_k)}, \qquad t \geq 0, \tag{1.1}$$

where, by convention, a summation over an empty set of index is 0.

We define, as usual, the time to ruin of this model as

$$\tau(x) = \inf \{ t > 0 : S_r(t) < 0 \mid S_r(0) = x \}, \tag{1.2}$$

where $\inf \phi = \infty$ by convention. Hence, the probability of ruin within a finite time $T > 0$ is defined by

$$\psi_r(x, T) = \Pr(\tau(x) \leq T), \tag{1.3}$$

while the probability of ultimate ruin is defined by

$$\psi_r(x) = \psi_r(x, \infty) = \lim_{T \to \infty} \psi_r(x, T) = \Pr(\tau(x) < \infty).$$

In this paper we investigate the asymptotic behavior of the finite time ruin probability $\psi_r(x, T)$ under the assumption that the claim size distribution B is heavy tailed.

The remaining part of this paper consists of three sections. After briefly reviewing some related recent works in Section 2, we present two main results in Section 3, and prove them in Section 4 after recalling several lemmas.

2 A brief review on related results

Throughout, all limit relationships are for $x \to \infty$ unless stated otherwise; for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$.

We shall restrict ourselves to the case of heavy-tailed claim size distributions. The most important class of heavy-tailed distributions is the subexponential class $\mathcal{S}$. By definition, a distribution F on $[0, \infty)$ is subexponential, denoted by $F \in \mathcal{S}$, if $\overline{F}(x) = 1 - F(x) > 0$ holds for all $x \geq 0$ and the relation

$$\lim_{x \to \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n \tag{2.1}$$

holds for some (hence for all) $n = 2, 3, \ldots$, where F^{*n} denotes the n-fold convolution of F; see Embrechts et al. (1979). It is well known that each subexponential distribution F is long tailed, denoted by $F \in \mathcal{L}$, in the sense that the relation

$$\lim_{x \to \infty} \frac{\overline{F}(x + y)}{\overline{F}(x)} = 1 \tag{2.2}$$

holds for each $y > 0$. A useful subclass of subexponential distributions is $\mathcal{R}$, the class of distributions with regular variations. By definition, a distribution F on $[0, \infty)$ belongs to the class $\mathcal{R}$ if $\overline{F}(x) > 0$ holds for all $x \geq 0$ and there exists some $\alpha > 0$ such that the relation

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \tag{2.3}$$

holds for each $y > 0$. We denote by $F \in \mathcal{R}_{-\alpha}$ the regularity property in (2.3). The last statement of Theorem 1.5.2 of Bingham et al. (1987) tells us that the convergence of relation (2.3) is uniform for $y \in [1, \infty)$. That is,

$$\lim_{x\to\infty} \sup_{1\leq y<\infty} \left| \frac{\overline{F}(xy)}{\overline{F}(x)} - y^{-\alpha} \right| = 0.$$

For more details of heavy-tailed distributions and their applications to insurance and finance, the reader is referred to Embrechts et al. (1997).

The asymptotic behavior of the ultimate ruin probability $\psi_r(x)$ of the risk model introduced in Section 1 with $C(\cdot)$ a deterministic linear function, B heavy tailed, and $\{X_k\}_{k=1}^{\infty}$ and $\{N(t)\}_{t\geq 0}$ mutually independent, has been investigated in the recent literature. Under the condition $B \in \mathcal{R}_{-\alpha}$ for some $\alpha > 1$, starting from an integral equation of Sundt and Teugels (1995), Klüppelberg and Stadtmüller (1998) developed a sophisticated L_p transform technique in proving the result

$$\psi_r(x) \sim \frac{\lambda}{\alpha r}\overline{B}(x); \tag{2.4}$$

see their Corollary 2.4. Asmussen (1998, Corollary 4.1(ii)) and Asmussen et al. (2002) proved a more general result that the relation

$$\psi_r(x) \sim \frac{\lambda}{r}\int_x^{\infty} \frac{\overline{B}(y)}{y}\mathrm{d}y \tag{2.5}$$

holds under the condition $B \in \mathcal{S}^*$, where the class $\mathcal{S}^*$ was introduced by Klüppelberg (1988) and is characterized by the relation

$$\int_0^x \overline{B}(x-y)\overline{B}(y)\mathrm{d}y \sim 2\mu\overline{B}(x)$$

with $\mu = \int_0^{\infty} \overline{B}(y)\mathrm{d}y \in (0, \infty)$. About the class $\mathcal{S}^*$, Klüppelberg (1988, Theorem 3.2) pointed out that if $B \in \mathcal{S}^*$ then both B itself and its integrated tail distribution B_I, which is defined by

$$B_I(x) = \frac{1}{\mu}\int_0^x \overline{B}(y)\mathrm{d}y, \qquad x \geq 0,$$

are subexponential. Lately, also starting from the work of Sundt and Teugels (1995) but using a simpler treatment, Kalashnikov and Konstantinides (2000) and Konstantinides et al. (2002) rebuilt relation (2.5) under a different condition that the integrated tail distribution B_I is an element of the class $\mathcal{A}$. That is, B_I is subexponential and satisfies

$$\limsup_{x\to\infty} \frac{\overline{B_I}(xy)}{\overline{B_I}(x)} < 1 \qquad \text{for some } y > 1.$$

To our knowledge, whether or not the condition $B_I \in \mathcal{S}$ is sufficient for relation (2.5) remains unknown.

It is also worth mentioning that $B \in \mathcal{S}^*$ neither implies nor is implied by $B_I \in \mathcal{A}$. A simple illustration for the assertion "$B \in \mathcal{S}^* \nRightarrow B_I \in \mathcal{A}$" is the distribution with a tail satisfying

$$\overline{B}(x) \sim x^{-1} \ln^{-2} x.$$

To see the other assertion "$B_I \in \mathcal{A} \nRightarrow B \in \mathcal{S}^*$", let us look at the random variable

$$Z = a^{\pi}, \tag{2.6}$$

where π is geometric with probability function $\Pr(\pi = k) = (1-p)p^k$ for $0 < p < 1$, $k = 0, 1, \ldots$, and a is arbitrarily fixed satisfying $1 < a < 1/p$. Clearly, the random variable Z has a finite mean and its distribution B satisfies

$$\lim_{x\to\infty} \frac{\overline{B}(ax)}{\overline{B}(x)} = p < \infty.$$

Based on this, it is easy to see that $B_I \in \mathcal{S}$ (see Theorem 1 of Embrechts and Omey (1984) or Proposition 1.4.4 of Embrechts et al. (1997)), that $B \notin \mathcal{L}$ (hence $B \notin \mathcal{S}^*$), and that

$$\lim_{x\to\infty} \frac{\overline{B_I}(ax)}{\overline{B_I}(x)} = ap < 1.$$

Therefore, $B_I \in \mathcal{A}$.

By the way, for an arbitrarily large number $v > 0$, by suitably choosing the parameters a and p in (2.6) such that $a^v p < 1$, we have $\mathrm{E}Z^v < \infty$. This means that the condition $B_I \in \mathcal{A}$ allows for some distributions that are not so "heavy-tailed" and are not in the class $\mathcal{L}$ (hence are not in the class $\mathcal{S}^*$).

Recently, Tang (2004) extended the work of Konstantinides et al. (2002) to the discrete time model while Tang (2005) extended the work of Klüppelberg and Stadtmüller (1998) to the ordinary renewal model.

3 Main results

In this paper we use a different method to establish a similar formula for the finite time ruin probability with B ranging over the whole class $\mathcal{S}$. Our first main result is given below:

Theorem 3.1. *Consider the compound Poisson model introduced in Section 1, in which all sources of randomness, $\{X_k\}_{k=1}^{\infty}$, $\{N(t)\}_{t\geq 0}$, and $\{C(t)\}_{t\geq 0}$, are mutually independent. If $B \in \mathcal{S}$, then for each $T > 0$,*

$$\psi_r(x, T) \sim \frac{\lambda}{r} \int_x^{x e^{rT}} \frac{\overline{B}(y)}{y} \mathrm{d}y. \tag{3.1}$$

Apparently, relation (3.1) is consistent with relation (2.5). In particular, if $B \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, by the uniformity of relation (2.3) we have

$$\begin{aligned}\int_x^{x\mathrm{e}^{rT}} \frac{\overline{B}(y)}{y}\mathrm{d}y &= \overline{B}(x)\int_x^{x\mathrm{e}^{rT}} \frac{\overline{B}(y)}{\overline{B}(x)}\frac{1}{y}\mathrm{d}y \\ &\sim \overline{B}(x)\int_x^{x\mathrm{e}^{rT}} \left(\frac{y}{x}\right)^{-\alpha}\frac{1}{y}\mathrm{d}y \\ &= \frac{1}{\alpha}\overline{B}(x)\left(1-\mathrm{e}^{-\alpha rT}\right).\end{aligned}$$

Hence in this case, it follows from (3.1) that for each $T > 0$,

$$\psi_r(x,T) \sim \frac{\lambda}{\alpha r}\overline{B}(x)\left(1-\mathrm{e}^{-\alpha rT}\right), \tag{3.2}$$

which is consistent with relation (2.4).

For each $T \in (0,\infty]$, denote by $\widetilde{C}(T)$ the total discounted amount of premiums accumulated up to time T. That is,

$$\widetilde{C}(T) = \int_0^T \mathrm{e}^{-rt}C(\mathrm{d}t) \qquad \text{for } T \in (0,\infty]. \tag{3.3}$$

The following result makes the statement of relation (3.2) somewhat stronger:

Theorem 3.2. *Consider the compound Poisson model introduced in Section 1 with $B \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ and $\widetilde{C}(\infty)$ in (3.3) finite almost surely. Then, relation (3.2) holds uniformly for $T \in (0,\infty]$, that is,*

$$\lim_{x\to\infty}\sup_{0<T\leq\infty}\left|\frac{\psi_r(x,T)}{\frac{\lambda}{\alpha r}\overline{B}(x)\left(1-\mathrm{e}^{-\alpha rT}\right)}-1\right| = 0, \tag{3.4}$$

if one of the following two assumptions is valid:

1. *$\{X_k\}_{k=1}^\infty$, $\{N(t)\}_{t\geq 0}$, and $\{C(t)\}_{t\geq 0}$ are mutually independent;*

2. *$\{X_k\}_{k=1}^\infty$ and $\{N(t)\}_{t\geq 0}$ are mutually independent and $\widetilde{C}(\infty)$ satisfies*

$$\Pr\left(\widetilde{C}(\infty) > x\right) = o\left(\overline{B}(x)\right). \tag{3.5}$$

As pointed out by Tang (2005), allowing dependence between the premium process and the claim process is not only of purely academic interest since very often the premium rate depends on the history of the surplus process.

Admittedly, there are a lot of advantages in knowing the uniformity of an asymptotic relation. Below are some direct applications of the uniformity described by Theorem 3.2:

1. The relation

$$\psi_r(x,T(x)) \sim \frac{\lambda}{\alpha r}\overline{B}(x)\left(1-\mathrm{e}^{-\alpha r T(x)}\right)$$

holds for every function $T(\cdot) \in (0,\infty]$. Moreover, if $T(x) \to \infty$ then the relation above is reduced to

$$\psi_r(x,T(x)) \sim \frac{\lambda}{\alpha r}\overline{B}(x) \sim \psi_r(x).$$

2. For a random variable $\mathcal{T}$, which is independent of the risk system and has a distribution H with $\overline{H}(0) > 0$, denote by $\psi_r(x,\mathcal{T})$ the probability of "ruin within the random horizon $\mathcal{T}$". We have

$$\begin{aligned}\psi_r(x,\mathcal{T}) &= \int_0^\infty \psi_r(x,T)H(\mathrm{d}T)\\ &\sim \int_0^\infty \frac{\lambda}{\alpha r}\overline{B}(x)\left(1-\mathrm{e}^{-\alpha r T}\right)H(\mathrm{d}T)\\ &= \frac{\lambda}{\alpha r}\overline{B}(x)\mathrm{E}\left(1-\mathrm{e}^{-\alpha r\mathcal{T}}\right)1_{(\mathcal{T}>0)},\end{aligned} \tag{3.6}$$

where 1_A denotes the indicator function of an event A.

3. Relation (3.6) further enables us to derive an asymptotic estimate for the Laplace-Stieltjes transform of the ruin time $\tau(x)$. For this purpose, we identify the $\mathcal{T}$ in (3.6) as an exponentially distributed random variable with mean $1/\kappa$. On the one hand, recalling relation (1.3) and using Fubini's theorem we have

$$\psi_r(x,\mathcal{T}) = \int_0^\infty \mathrm{E}1_{(\tau(x)\le T)}H(\mathrm{d}T) = \mathrm{E}\exp\{-\kappa\tau(x)\}1_{(\tau(x)<\infty)} = \mathrm{E}\exp\{-\kappa\tau(x)\};$$

on the other hand, relation (3.6) gives that

$$\psi_r(x,\mathcal{T}) \sim \frac{\lambda}{\alpha r}\overline{B}(x)\mathrm{E}\left(1-\mathrm{e}^{-\alpha r\mathcal{T}}\right) = \frac{\lambda}{\alpha r+\kappa}\overline{B}(x).$$

It follows that

$$\mathrm{E}\exp\{-\kappa\tau(x)\} \sim \frac{\lambda}{\alpha r+\kappa}\overline{B}(x).$$

4 Proofs of Theorems 3.1 and 3.2

4.1 Lemmas

Before giving the proofs we need recall some preliminaries.

Lemma 4.1. *If F is subexponential, then for each $\varepsilon > 0$, there exists some constant $C_\varepsilon > 0$ such that the inequality*

$$\overline{F^{*n}}(x) \le C_\varepsilon(1+\varepsilon)^n\overline{F}(x)$$

holds for all $n = 1, 2, \ldots$ and $x \ge 0$.

Proof. This inequality is classical and it was established by Chistyakov (1964) and Athreya and Ney (1972); see also Embrechts et al. (1997, Lemma 1.3.5). □

Lemma 4.2. *Let X and Y be two independent and nonnegative random variables. If X is subexponentially distributed while Y is bounded and nondegenerate at 0, then the product XY is subexponentially distributed.*

Proof. See Corollary 2.5 of Cline and Samorodnitsky (1994). □

Lemma 4.3. *Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with arrival times σ_k, $k = 1, 2, \ldots$. Given $N(T) = n$ for arbitrarily fixed $T > 0$ and $n = 1, 2, \ldots$, the random vector $(\sigma_1, \cdots, \sigma_n)$ is equal in distribution to the random vector $(TU_{(1,n)}, \cdots, TU_{(n,n)})$ with $U_{(1,n)}, \ldots, U_{(n,n)}$ being the order statistics of n i.i.d. $(0, 1)$ uniformly distributed random variables $U_1, \ldots, U_n$.*

Proof. This result is well known; see, for example, Theorem 2.3.1 of Ross (1983). □

Lemma 4.4. *If a sequence of distributions $\{F_t\}_{t\geq 0}$ converges to a continuous distribution F as $t \to \infty$, then the convergence is uniform in the sense that*

$$\lim_{t\to\infty} \sup_{-\infty\leq x\leq\infty} |F_t(x) - F(x)| = 0.$$

Proof. See Theorem 1.11 of Petrov (1995), though the sequence under his discussion is $\{F_n\}_{n=1}^{\infty}$ instead of $\{F_t\}_{t\geq 0}$. □

4.2 Proof of Theorem 3.1

It follows from (1.3) and (1.2) that

$$\psi_r(x, T) = \Pr\left(\mathrm{e}^{-rt} S_r(t) < 0 \text{ for some } t \in (0, T] \mid S_r(0) = x\right). \tag{4.1}$$

Furthermore, for each $t \in (0, T]$ it follows from (1.1) that

$$x - \sum_{k=1}^{N(T)} X_k \mathrm{e}^{-r\sigma_k} \leq \mathrm{e}^{-rt} S_r(t) \leq x + \widetilde{C}(T) - \sum_{k=1}^{N(t)} X_k \mathrm{e}^{-r\sigma_k}, \tag{4.2}$$

where $\widetilde{C}(T)$ is defined in (3.3). For notational convenience, we write

$$\widetilde{X}(t) = \sum_{k=1}^{N(t)} X_k \mathrm{e}^{-r\sigma_k}$$

as the total discounted amount of claims accumulated up to time $t > 0$. Clearly, equality (4.1) and the first inequality in (4.2) imply that

$$\psi_r(x, T) \leq \Pr\left(\widetilde{X}(T) > x\right), \tag{4.3}$$

while equality (4.1) and the second inequality in (4.2) imply that

$$\psi_r(x,T) \geq \Pr\left(\widetilde{X}(t) > x + \widetilde{C}(T) \text{ for some } t \in (0,T]\right) = \Pr\left(\widetilde{X}(T) > x + \widetilde{C}(T)\right). \quad (4.4)$$

Hence, if we prove that

$$\Pr\left(\widetilde{X}(T) > x + \widetilde{C}(T)\right) \sim \Pr\left(\widetilde{X}(T) > x\right) \sim \lambda \int_0^T \Pr\left(X_1 \mathrm{e}^{-ru} > x\right) \mathrm{d}u, \quad (4.5)$$

then it follows that

$$\psi_r(x,T) \sim \lambda \int_0^T \Pr\left(X_1 \mathrm{e}^{-ru} > x\right) \mathrm{d}u,$$

which, upon a trivial substitution, implies the announced result (3.1).

Let us successively prove the two asymptotic relations in (4.5). By Lemma 4.3 we have

$$\begin{aligned}\Pr\left(\widetilde{X}(T) > x\right) &= \sum_{n=1}^{\infty} \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\sigma_k} > x \;\middle|\; N(T) = n\right) \Pr(N(T) = n) \\ &= \sum_{n=1}^{\infty} \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-rTU_{(k,n)}} > x\right) \Pr(N(T) = n),\end{aligned}$$

where $U_{(k,n)}$ for $k = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$ come from Lemma 4.3 and are independent of $\{X_k\}_{k=1}^{\infty}$. Therefore,

$$\Pr\left(\widetilde{X}(T) > x\right) = \sum_{n=1}^{\infty} \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-rTU_k} > x\right) \Pr(N(T) = n). \quad (4.6)$$

By Lemma 4.2 we know that the i.i.d. products $X_k \mathrm{e}^{-rTU_k}$, $k = 1, 2, \ldots$, are subexponentially distributed; by Lemma 4.1 we also know that for an arbitrarily fixed $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that the inequality

$$\Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-rTU_k} > x\right) \leq C_\varepsilon (1+\varepsilon)^n \Pr\left(X_1 \mathrm{e}^{-rTU_1} > x\right)$$

holds for all $n = 1, 2, \ldots$ and $x \geq 0$. Since $\mathrm{E}(1+\varepsilon)^{N(T)} < \infty$, applying the definition in (2.1) of the subexponentiality and the dominated convergence theorem, we obtain from (4.6) that

$$\begin{aligned}\Pr\left(\widetilde{X}(T) > x\right) &\sim \Pr\left(X_1 \mathrm{e}^{-rTU_1} > x\right) \sum_{n=1}^{\infty} n \Pr(N(T) = n) \\ &= \lambda \int_0^T \Pr\left(X_1 \mathrm{e}^{-ru} > x\right) \mathrm{d}u. \quad (4.7)\end{aligned}$$

This proves the second relation in (4.5).

Using (4.7), it is not difficult to prove the first asymptotic relation in (4.5). Actually, since the product $X_1 \mathrm{e}^{-rTU_1}$ is subexponentially distributed, by (4.7) it is easy to see that the sum $\widetilde{X}(T)$ is long tailed. Using the dominated convergence theorem and the property in (2.2) of long-tailed distributions, we obtain that

$$\lim_{x\to\infty} \frac{\Pr\left(\widetilde{X}(T) > x + \widetilde{C}(T)\right)}{\Pr\left(\widetilde{X}(T) > x\right)} = \int_0^\infty \lim_{x\to\infty} \frac{\Pr\left(\widetilde{X}(T) > x + y\right)}{\Pr\left(\widetilde{X}(T) > x\right)} \Pr\left(\widetilde{C}(T) \in \mathrm{d}y\right) = 1.$$

This ends the proof of Theorem 3.1.

4.3 Proof of Theorem 3.2

First, we prove that relation (3.2) holds for each $T \in (0, \infty]$. In view that for both cases the relation

$$\psi_r(x) \sim \frac{\lambda}{\alpha r} \overline{B}(x) \tag{4.8}$$

is a direct consequence of Theorem 1 of Tang (2005) and that under assumption 1 relation (3.2) with $T \in (0, \infty)$ has been proved by Theorem 3.1, we only prove relation (3.2) for each $T \in (0, \infty)$ under assumption 2. In this case, following the proof of Theorem 3.1, inequalities (4.3) and (4.4) remain valid and, moreover,

$$\Pr\left(\widetilde{X}(T) > x\right) \sim \lambda \int_0^T \Pr\left(X_1 \mathrm{e}^{-ru} > x\right) \mathrm{d}u \sim \frac{\lambda}{\alpha r} \overline{B}(x) \left(1 - \mathrm{e}^{-\alpha r T}\right). \tag{4.9}$$

Hence, it suffices to prove that

$$\liminf_{x\to\infty} \frac{\Pr\left(\widetilde{X}(T) > x + \widetilde{C}(T)\right)}{\Pr\left(\widetilde{X}(T) > x\right)} \geq 1. \tag{4.10}$$

To this end, note that relation (4.9) indicates that the distribution of $\widetilde{X}(T)$ belongs to the class $\mathcal{R}_{-\alpha}$. For an arbitrarily fixed number $l > 0$, applying (4.9) and (3.5) we obtain that

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{\Pr\left(\widetilde{X}(T) > x + \widetilde{C}(T)\right)}{\Pr\left(\widetilde{X}(T) > x\right)} \\
&\geq \liminf_{x\to\infty} \frac{\Pr\left(\widetilde{X}(T) > (1+l)x\right) - \Pr\left(\widetilde{C}(\infty) > lx\right)}{\Pr\left(\widetilde{X}(T) > x\right)} \\
&\geq \liminf_{x\to\infty} \frac{\Pr\left(\widetilde{X}(T) > (1+l)x\right)}{\Pr\left(\widetilde{X}(T) > x\right)} - \limsup_{x\to\infty} \frac{\Pr\left(\widetilde{C}(\infty) > lx\right)}{\overline{B}(lx)} \frac{\overline{B}(lx)}{\overline{B}(x)} \frac{\overline{B}(x)}{\frac{\lambda}{\alpha r}\overline{B}(x)\left(1 - \mathrm{e}^{-\alpha r T}\right)} \\
&= (1+l)^{-\alpha}.
\end{aligned}$$

Hence, relation (4.10) follows since the number l above can be arbitrarily close to 0.

Then, we prove the uniformity of relation (3.2) with respect to $T \in (0, \infty]$. Write $\Pr^{(x)}(\cdot) = \Pr(\cdot \mid \tau(x) < \infty)$ for $x \geq 0$. Recall the definition in (1.3). From relations (3.2) and (4.8) we obtain that for each $T \in (0, \infty]$,

$$\lim_{x \to \infty} \Pr^{(x)}(\tau(x) \leq T) = \lim_{x \to \infty} \frac{\psi_r(x, T)}{\psi_r(x)} = 1 - \mathrm{e}^{-\alpha r T}. \tag{4.11}$$

This means that in $\Pr^{(x)}$, the limit distribution of the ruin time $\tau(x)$ is exponential with mean $1/(\alpha r)$. Applying Lemma 4.4 we know that the convergence in (4.11) is uniform with respect to $T \in (0, \infty]$. That is,

$$\lim_{x \to \infty} \sup_{0 < T \leq \infty} \left| \frac{\psi_r(x, T)}{\psi_r(x)} - \left(1 - \mathrm{e}^{-\alpha r T}\right) \right| = 0. \tag{4.12}$$

By (4.8), it is easy to see that relation (4.12) is equivalent to relation (3.4). This ends the proof of Theorem 3.2.

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