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The Finite Toda Lattices

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Abstract. Connection is established between one-dimensional Toda lattices, constructed on the basis of the systems of simple roots of classical and affine Lie algebras, and other integrable systems of interacting particles. That connection allows us to find new lattices differing from the known ones by the interaction of particles near the ends. Some of the new lattices admit non-Abelian generalizations.

1. Introduction

The study of the propagation of waves in infinite one-dimensional lattices made it possible to derive exact analytic solutions to an infinite system of nonlinear differential equations describing the exponential interaction between the nearest neighbouring particles [1]. Once Henon, Flaschka [2] and Moser [3] have shown complete integrability of finite-dimensional Hamiltonian systems of that type (nonperiodic and periodic Toda lattices), a number of papers have appeared devoted to the investigation of their properties. Kostant [6], Olshanetsky and Perelomov [7] have established the connection between nonperiodic lattices and classical Lie algebras. Hamiltonians of those lattices may be constructed with the use of systems of simple roots $\{p\}$ of classical Lie algebras $\{\mathfrak{G}\}$,

$$H=\sum_{j=1}^{n}\frac{p_j^2}{2}+V_{\mathfrak{G}}, \qquad U_{\mathfrak{G}}=\sum_{\alpha\in\{p\}}\exp(\alpha q), \tag{1}$$

where α are root vectors, p_j and q_j are respectively momenta and coordinates of particles. For the algebras $\mathscr{A}_n$, $\mathscr{B}_n$, $\mathscr{C}_n$, and $\mathscr{D}_n$ the potentials $V_{\mathfrak{G}}$ are of the form

$$\begin{aligned} V_{\mathscr{A}_n}&=\sum_{j=1}^{n-1}\exp(q_j-q_{j+1}), \qquad U_{\mathscr{B}_n}=V_{\mathscr{A}_n}+\exp(q_n), \\ V_{\mathscr{C}_n}&=V_{\mathscr{A}_n}+\exp(2q_n), \qquad V_{\mathscr{D}_n}=U_{\mathscr{A}_n}+\exp(q_{n-1}+q_n). \end{aligned} \tag{2}$$

For the exceptional algebras systems like (1) have no simple mechanical interpretation and will not be considered here. Bogoyavlensky [4] has pointed out a method for constructing generalized periodic lattices that, as first shown in [9], corresponds to the use in (1) of a system of simple roots of affine algebras. Potentials V for loop algebras $\mathscr{A}_n^{(1)}$, $\mathscr{B}_n^{(1)}$, $\mathscr{C}_n^{(1)}$, and $\mathscr{D}_n^{(1)}$ can be written in the form

$$\begin{aligned} V_{\mathscr{A}_n^{(1)}} &= V_{\mathscr{A}_n} + \exp(q_n - q_1), \quad & V_{\mathscr{B}_n^{(1)}} &= V_{\mathscr{B}_n} + \exp(-q_1 - q_2), \\ V_{\mathscr{C}_n^{(1)}} &= V_{\mathscr{C}_n} + \exp(-2q_1), \quad & V_{\mathscr{D}_n^{(1)}} &= V_{\mathscr{D}_n} + \exp(-q_1 - q_2). \end{aligned} \tag{3}$$

To the twisted loop algebras there correspond the potentials [18]

$$\begin{aligned} V_{\mathscr{A}_{2n}^{(2)}} &= V_{\mathscr{A}_n} + \exp(q_n) + \exp(-2q_1), \\ V_{\mathscr{A}_{2n+1}^{(2)}} &= V_{\mathscr{A}_n} + \exp(-q_1 - q_2) + \exp(2q_n), \\ V_{\mathscr{D}_{n+1}^{(2)}} &= V_{\mathscr{A}_n} + \exp(q_n) + \exp(-q_1). \end{aligned} \tag{4}$$

The Lax representations for generalized periodic lattices (3) and (4) depend on the spectral parameter, and the corresponding Hamiltonian flows are linearized on the Jacobi varieties of the algebraic curves associated with the L-matrix spectrum [7, 8]. The list (2–4) covers all the potentials corresponding to infinite series of the Kac-Moody algebras. The existence of other, different from (2–4), integrable systems with exponential potentials like $V = \sum_{j=1}^{n+1} \exp\left(\sum_{k=1}^{n} N_{jk} q_k\right)$ has been questioned by Adler and van Moerbeke [10]. They have shown that when the rank of N equals n, the Kowalewski-Painlevé property pertains only to the trajectories of systems (3, 4). Matrices of a smaller rank were not analysed.

The main purpose of this paper is as follows. It is known [13–15], that multiparticle systems in which the structure of an interaction potential is defined by systems of all positive, and not just simple roots of algebras $\mathfrak{G}$, are also completely integrable. I shall show that all the above-listed Toda lattices are particular cases of these systems corresponding to certain limit situations. And what is more, taking advantage of that correspondence, I will obtain new integrable lattices. This does not contradict the results of [10] as they may be interpreted as systems with matrices N_{jk} of a rank smaller than the number of degrees of freedom. Also, lattices will be found for which the interaction of particles at the ends is not exponential.

In Sect. II, I describe the systems given in [13–15] and the limit transitions via the lattices (2–4) are obtained. In Sect. III, explicit expressions are presented for the Lax matrices of new lattices and non-Abelian generalizations of some of them. These matrices depend on the spectral parameter, and the number of poles of the L matrix is twice that for the earlier known lattices (3–4). Some unsolved problems are also pointed out.

2. Toda Lattices as Limit Cases of Integrable Systems with a Non-Exponential Interaction

Moser [13], Olshanetsky and Perelomov [14] have reported a class of integrable Hamiltonian systems of type (1) with a potential $\mathscr{U}(q)$ constructed with the use of systems of positive roots $\{\Delta_+\}$ of classical algebras,

$$\mathscr{U}(q)=\sum_{\alpha\in\Delta_+} g_\alpha^2 V(\alpha q)\,. \tag{5}$$

The constants g_α may depend on the length of the root vector but not on its direction. Systems of type (5) generally describe the motion of $2n$ particles of unit mass interacting via the potential $V(\xi)$ and with an external field under the symmetric initial conditions $p_j=-p_{j+n}$, $q_j=-q_{j+n}$, $1\leqq j\leqq n$. We have proposed further generalization of (5) in [15] and [19], assuming that the form of the function $V(\alpha q)$ may change with changing the length of the vector α.

Thus, it suffices to consider the root systems $\mathscr{A}_n$ and $\mathscr{BC}_n$ to which there correspond the potentials

$$\mathscr{U}(q)=\mathscr{U}_1(q)+\varepsilon[\mathscr{U}_2(q)+\mathscr{U}_3(q)]\,,\qquad \varepsilon=0,1\,, \tag{6}$$

$$\mathscr{U}_1(q)=g^2\sum_{j>k}\mathscr{P}(q_j-q_k)\,,\qquad \mathscr{U}_2(q)=g^2\sum_{j>k}\mathscr{P}(q_j+q_k)\,, \tag{7}$$

$$\mathscr{U}_3(q)=\sum_{j=1}^{n}\left(g_1^2\mathscr{P}(q_j)+g_2^2\mathscr{P}\left(g_j+\frac{\omega_1}{2}\right)+g_3^2\mathscr{P}\left(q_j+\frac{\omega_2}{2}\right)+g_4^2\mathscr{P}\left(q_j+\frac{\omega_1+\omega_2}{2}\right)\right) \tag{8}$$

Here $\mathscr{P}(\xi)$ is doubly periodic Weierstrass function,

$$\mathscr{P}(\xi)=\frac{1}{\xi^2}+\sum_{\substack{m1,m_2\in\mathbb{Z}\\ m_1^2+m_2^2>0}}\left[\frac{1}{(\xi+m_1\omega_1+m_2\omega_2)^2}-\frac{1}{(m_1\omega_1+m_2\omega_2)^2}\right].$$

The Lax representation for the systems (6) at $\varepsilon=0$ has been found by Moser [13]; in ref. [14] the integrability has been established in two cases: $g_1=g_2=g_3=g_4$ and $g_2=g_3=g_4$, $g_1^2=g_2^2+2g^2\pm2\sqrt{2}gg_2$, i.e. when the sum in brackets in (8) can be written as $g_1^2\mathscr{P}(q_j)+g_2^2\mathscr{P}(2q_j)$. In ref. [15] we have determined the Lax matrices for (6–8) for the constants $\{g_\gamma\}$ obeying the only condition $\left(\sum_{\gamma=1}^{4}g_\gamma^4-\sum_{\beta\neq\gamma}^{4}g_\beta^2g_\gamma^2\right)^2=64\prod_{\gamma=1}^{4}g_\gamma^2$. And finally, in [19] also this constraint has been removed, and the Lax representation obtained here is valid for arbitrary values of g, $\{g_\gamma\}$ and contains the spectral parameter defined on a complex torus, the factor $\mathbb{C}$ on the lattice of periods of some elliptic Jacobi functions.

Let us now show how from (6–8) one can obtain potentials of the Toda lattices (2–4). To start with, note that the Weierstrass function is real-valued if and only if one of the periods, for instance ω_1 is purely imaginary (in what follows we shall set $\omega_1=2\pi i$, $i^2=-1$); and the other is real-valued. In the degenerated case, $\omega_2\to\infty$, $\mathscr{P}(\xi)$ is a trigonometric function, $\mathscr{P}(\xi)=\frac{1}{4}\left(\frac{1}{3}+\frac{1}{\mathrm{sh}^2\xi/2}\right)$, and the potentials (7, 8) assume the form

$$\mathscr{U}_1(q)=g^2\sum_{j>k}^{n}\mathrm{sh}^{-2}\left(\frac{q_j-q_k}{2}\right),\qquad \mathscr{U}_2(q)=g^2\sum_{j>k}^{n}\mathrm{sh}^{-2}\left(\frac{q_j+q_k}{2}\right), \tag{7a}$$

$$\mathscr{U}_3(q)=\sum_{j=1}^{n}\left[\frac{g_1^2}{\mathrm{sh}^2\frac{q_j}{2}}+\frac{g_2^2}{\mathrm{sh}^2 q_j}+g_3^2\,\mathrm{ch}\,q_j+g_4^2\,\mathrm{ch}\,2q_j\right]. \tag{8a}$$

All the constants here are arbitrary, the corresponding Lax representation with the spectral parameter has already been found in [15]. Introduce now variables x_j by the relations

$$q_j = x_j + (j-1)\Delta, \quad \Delta > 0. \tag{9}$$

Since g, $\{g_\gamma\}$ can be chosen arbitrary, we set $g = \frac{g_0}{2}\exp\left(\frac{\Delta}{2}\right)$, and in (7a) take the limit $\Delta \to \infty$ with q_j given by (9). As at large Δ one may expand $\left[\operatorname{sh}\left(\frac{q_j - q_k}{2}\right)\right]^{-2}$,

$$\left[\operatorname{sh}\left(\frac{q_j - q_k}{2}\right)\right]^{-2} \simeq 4\exp(-\Delta(j-k))\exp(x_k - x_j) + O(e^{-\Delta(j-k)}), \quad j > k \tag{10}$$

in the potential $\mathscr{U}_1(q)$ when $\Delta \to \infty$ there remain only terms corresponding to the interaction of nearest-neighbour particles:

$$\tilde{\mathscr{U}}_1(x) = \lim_{\Delta \to \infty} \mathscr{U}_1(q) = g_0 V_{\mathscr{A}_n}(x). \tag{11}$$

Thus, at $\varepsilon = 0$ we obtained the potential of nonperiodic Toda lattice (2) from the Moser potential. Let us clarify the matter with other terms in (7, 8a). It is easy to see that $\mathscr{U}_2$ now contains only one term,

$$\tilde{\mathscr{U}}_2(x) = \lim_{\Delta \to \infty} \mathscr{U}_2(q) = g_0 \exp(-x_1 - x_2). \tag{12}$$

Inserting (9) into (8a) and using (10) we observe that the limit when $\Delta \to \infty$ is finite only when g_1 and g_2 do not depend on Δ, whereas g_3 and g_4 are exponentially decreasing, $g_3^2 \sim \mathscr{C} e^{-\Delta(n-1)}$ and $g_4^2 \sim \mathscr{D} e^{-2\Delta(n-1)}$,

$$\tilde{\mathscr{U}}_3(x) = \lim_{\Delta \to \infty} \mathscr{U}_3(q) = g_0\left[\frac{\tilde{\mathscr{A}}}{\operatorname{sh}^2 \frac{x_1}{2}} + \frac{\tilde{\mathscr{B}}}{\operatorname{sh}^2 x_1} + \mathscr{C} e^{x_n} + \mathscr{D} e^{2x_n}\right].$$

Here as in (8a) the constants $\tilde{\mathscr{A}}$, $\tilde{\mathscr{B}}$, $\mathscr{C}$, and $\mathscr{D}$ are arbitrary. Thus, we have got the system with the potential

$$\begin{aligned} V_1(x) &= \tilde{\mathscr{U}}_1(x) + \varepsilon[\tilde{\mathscr{U}}_2(x) + \tilde{\mathscr{U}}_3(x)] = g_0 V_{\mathscr{A}_n}(x) \\ &\quad + \varepsilon g_0\left[e^{-x_1 - x_2} + \frac{\tilde{\mathscr{A}}}{\operatorname{sh}^2 \frac{x_1}{2}} + \frac{\tilde{\mathscr{B}}}{\operatorname{sh}^2 x_1} + \mathscr{C} e^{x_n} + \mathscr{D} e^{2n_n}\right]. \end{aligned} \tag{13}$$

When $\tilde{\mathscr{A}} = \tilde{\mathscr{B}} = 0$ and $\varepsilon = 1$, (13) represents the potential of the lattice with exponential interaction

$$V_2(x) = g_0[V_{\mathscr{A}_n}(x) + e^{-x_1 - x_2} + \mathscr{C} e^{x_n} + \mathscr{D} e^{2x_n}]. \tag{14}$$

The cases $\mathscr{D}_n$, $B_n^{(1)}$ and $\mathscr{A}_{2n+1}^{(1)}$ (2–4) follow from (14) with the choice of constants $\mathscr{C} = \mathscr{D} = 0$, $\mathscr{D} = 0$ and $\mathscr{C} = 0$, respectively. Any of nonzero constants in (14) can be made equal to 1 by a finite shift of all the coordinates.

Shifting the whole system to the right, $x_j \to x_j + \Delta_1$, $\Delta_1 > 0$, $1 \leqq j \leqq n$, and performing "renormalization" of the constants $\tilde{\mathscr{A}} \to \frac{\mathscr{A} e^{\Delta_1}}{4}$, $\tilde{\mathscr{B}} \to \frac{\mathscr{B} e^{2\Delta}}{4}$, $\mathscr{C} \to \mathscr{C} e^{-\Delta_1}$, and $\mathscr{D} \to \mathscr{D} e^{-2\Delta_1}$, we obtain from (13), when $\Delta_1 \to \infty$, the potential

$$V_3(x) = g_0 [V_{\mathscr{A}_n}(x) + \mathscr{A} e^{-x_1} + \mathscr{B} e^{-2x_1} + \mathscr{C} e^{x_n} + \mathscr{D} e^{2x_n}]. \tag{15}$$

It describes the motion of the Toda system whose extreme particles interact with an external field, the interaction potentials being different for the first and last particles. When part of the constants vanishes in (15), we obtain the known lattices of the type (2–4):

$$\begin{gathered} V_{\mathscr{B}_n}, \mathscr{C} = 1, \mathscr{A} = \mathscr{B} = \mathscr{D} = 0, \qquad V_{\mathscr{A}^{(2)}_{2n}}, \mathscr{C} = \mathscr{B} = 1, \mathscr{A} = \mathscr{D} = 0, \\ V_{\mathscr{C}_n}, \mathscr{D} = 1, \mathscr{A} = \mathscr{B} = \mathscr{C} = 0, \qquad V_{\mathscr{D}^{(2)}_{h+1}}, \mathscr{A} = \mathscr{C} = 1, \mathscr{B} = \mathscr{D} = 0, \\ V_{\mathscr{C}^{(1)}_h}, \mathscr{B} = \mathscr{D} = 1, \mathscr{A} = \mathscr{C} = 0. \end{gathered} \tag{16}$$

By finite shifts of the type $x_j \to x_j + \delta_1 + j\delta_2$ and change of the time scale one of the nonzero constants can be made equal to 1 in each of the pairs $(\mathscr{A}, \mathscr{B})$, $(\mathscr{C}, \mathscr{D})$ so that the potential actually contains only two arbitrary constants.

Thus, from the trigonometric degenerations of (7, 8) we have obtained the most part of the known lattices with exponential interactions (2–4) and some new ones (13–15). There are, however, two cases: $\mathscr{A}_n^{(1)}$ (the periodic lattice) and $\mathscr{D}_n^{(1)}$ that cannot be derived from (7a, 8a) in any limit. Nevertheless, they may also be included into the general potential (7, 8). To demonstrate this, note that we passed from (7, 8) to (13–15) via two stages: first we tended the real-valued period of the Weierstrass function, ω_2, to infinity, and then carried out shifts of the coordinates of particles without correlation with ω_2. Now let us put

$$q_j = x_j + (j-1)\omega_2 \tau, \qquad 0 < \tau < \frac{1}{h-1}, \tag{17}$$

and use for $\mathscr{P}(\xi)$ the representation ($\omega_1 = 2\pi i$):

$$\mathscr{P}(\xi) = \frac{1}{4} \sum_{m=-\infty}^{\infty} \left[\operatorname{sh}\left(\frac{\xi - m\omega_2}{2} \right) \right]^{-2} + C, \quad C = \frac{1}{12} - \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{\operatorname{sh}^2 \frac{m\omega_2}{2}}. \tag{18}$$

In the limit of large values of the argument and real-valued period, from (18) we derive the expansion

$$\mathscr{P}(\xi + \omega_2 \delta) = C + \exp\{-\xi(\delta - \tfrac{1}{2})|\delta - \tfrac{1}{2}|^{-1} - \omega_2(\tfrac{1}{2} - |\delta - \tfrac{1}{2}|)\} + O(e^{-\omega_2(\frac{1}{2} - |\delta - \frac{1}{2}|)}), \tag{19}$$

where $0 < \delta < 1$, $\delta \neq 1/2$.

We put $g^2 = e^{\omega_2 \tau}$ in (7) and calculate the limits of the potentials $\mathscr{U}_1(q)$ and $\mathscr{U}_2(q)$ with the coordinates (17) when $\omega_2 \to \infty$. From (19) it follows that the limit of $\mathscr{U}_1(q)$ may contain an additional to (11) term only for the choice $\tau = \frac{1}{n}$ when there are valid the expansions $\mathscr{P}(g_{j+1} - q_j) \sim C + \exp(x_j - x_{j+1}) \times \exp\left(-\frac{\omega_2}{n}\right)$, $\mathscr{P}(q_n - q_1) \sim C + \exp(x_n - x_1) \exp\left(-\frac{\omega_2}{n}\right)$,

$$\lim_{\omega_2 \to \infty} \mathscr{U}_1(q) = V_{\mathscr{A}_n}(x) + e^{x_n - x_1} = V_{\mathscr{A}^{(1)}_h}(x). \tag{20}$$

Here and in what follows we shall, in calculating the limits, omit the constant C that can be made zero by subtracting the constant $n(n+3)C$, inessential for the equations of motion, from the potential (6). The potential $\mathcal{U}_2(q)$, when $\omega_2 \to \infty$, and with a given choice of q, τ for $n>2$ diverges as the numbers $k<j\leqq n$ may always include such for which $j+k=n+2$, $q_j+q_k=x_j+x_k+\omega_2$, and $\mathcal{P}(q_j+q_k)-C$ remains constant, while g^2 increases indefinitely. Consequently, for $n>2$ we should put in (6) $\varepsilon=0$, and then we arrive at a periodic Toda lattice. For $n=2$ and $\tau=1/2$ the expansion (19) does not hold valid ($\delta=1/2$) and it is to be changed to the asymptotic representations

$$\mathcal{P}(\xi)\sim\frac{1}{4\,\mathrm{sh}^2\dfrac{\xi}{2}},\qquad \mathcal{P}\left(\xi+\frac{\omega_2}{2}\right)\sim e^{-\omega_2/2}\,\mathrm{ch}\,\xi,$$

$$\mathcal{P}\left(\xi+\frac{\omega_1}{2}\right)\sim-\frac{1}{4\,\mathrm{ch}^2\dfrac{\xi}{2}},\qquad \mathcal{P}\left(\xi+\frac{\omega_1+\omega_2}{2}\right)\sim-e^{-\omega_2/2}\,\mathrm{ch}\,\xi.$$

By this means we obtain a lattice with the potential

$$V^{(2)}(x)=e^{x_1-x_2}+e^{x_2-x_1}+e^{x_1+x_2}+e^{-x_1-x_2}+\frac{\mathcal{A}}{\mathrm{sh}^2\dfrac{x_1}{2}}+\frac{\mathcal{B}}{\mathrm{sh}^2 x_1}+\frac{\mathcal{C}}{\mathrm{sh}^2\dfrac{x_2}{2}}+\frac{\mathcal{D}}{\mathrm{sh}^2 x_2}. \tag{21}$$

When $n>2$, there is one more possible choice in (17) not leading to divergences of $\mathcal{U}_2(q)$. It is realized under the condition

$$\tau\cdot\min(j+k-2)=1-\tau\cdot\max(j+k-2),\qquad n\geqq j>k\geqq 1. \tag{22}$$

From (22) it follows that $\tau=\dfrac{1}{2(n-1)}$, and at $g^2=e^{\omega_2\tau}$ the limits of all the terms in (6), according to (19), are finite:

$$\lim_{\omega_2\to\infty}\mathcal{U}_1(q)=V_{\mathcal{A}_n}(x),\qquad \lim_{\omega_2\to\infty}\mathcal{U}_2(q)=e^{-x_1-x_2}+e^{x_{n-1}+x_n},$$
$$\lim_{\omega_2\to\infty}\mathcal{U}_3(q)=\frac{\tilde{\mathcal{A}}}{\mathrm{sh}^2\dfrac{x_1}{2}}+\frac{\tilde{\mathcal{B}}}{\mathrm{ch}^2\dfrac{x_1}{2}}+\frac{\tilde{\mathcal{C}}}{\mathrm{sh}^2\dfrac{x_n}{2}}+\frac{\tilde{\mathcal{D}}}{\mathrm{ch}^2\dfrac{x_n}{2}}, \tag{23}$$

which corresponds to a lattice with the potential

$$V(x)=V_{\mathcal{A}_n}(x)+e^{-x_1-x_2}+e^{x_{n-1}+x_n}+\frac{\mathcal{A}}{\mathrm{sh}^2\dfrac{x_1}{2}}+\frac{\mathcal{B}}{\mathrm{sh}^2 x_1}+\frac{\mathcal{C}}{\mathrm{sh}^2\dfrac{x_n}{2}}+\frac{\mathcal{D}}{\mathrm{sh}^2 x_n}. \tag{24}$$

The constants $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are arbitrary and connected with constants in (23) by linear relations. It is easy to see that (21) differs from the potential (24) only by the term $e^{x_2-x_1}$ if we put $n=2$ there. The case of $\mathcal{D}_n^{(1)}$ in (3) corresponds to vanishing all four arbitrary constants in (24) of terms describing nonexponential interaction of the first and last particles of the lattice with external fields.

3. The Lax Representations. Non-Abelian Lattices

Above we have considered the limits that allow us to derive the Hamiltonians of the Toda lattices from (6). Integrability of new lattices requires the corresponding limits to exist for the Lax matrices, i.e. the divergences to be absent in the constants of motion of higher orders in momenta. For the lattices (13–15) it can easily be proved with the Lax representation [15] for the trigonometric degeneration (7a, 8a) of the potentials (7, 8). For the lattice (24) immediately obtained from (7), (8) it is simpler to construct the Lax matrices anew, as in the representation found in [19] use has been made of matrices of a higher dimension than in [14] and [15]. Here we shall cite the results for the lattices (14), (15), and (24). In all the cases the matrices L and M obeying the Lax equation $\frac{dL}{dt}=[L,M]$ have the dimension $2n \times 2n$ and the structure

$$L=\begin{pmatrix} l & \psi \\ \psi^* & -l \end{pmatrix}, \qquad M=\begin{pmatrix} m & \chi \\ -\chi^* & m \end{pmatrix}, \tag{25}$$

where l, m, ψ, and χ are $n\times n$ matrices, l and m being the same for almost all the lattices:

$$l_{jk}=\delta_{jk}p_j+i\left[\delta_{j,k-1}\exp\left(\frac{q_j-q_{j+1}}{2}\right)+\delta_{j-1,k}\exp\left(\frac{q_k-q_{k+1}}{2}\right)\right],$$

$$m_{jk}=\delta_{jk}\mu_j+i\left[\delta_{j,k-1}\exp\left(\frac{q_j-q_{j+1}}{2}\right)-\delta_{j-1,k}\exp\left(\frac{q_k-q_{k+1}}{2}\right)\right].$$

The matrices ψ, ψ^*, χ, and χ^* are meromorphic functions of the spectral parameter h and for the lattices (14), (15), (24) are of the following form (for simplicity we set $\mathscr{D}=1$ in (14), $\mathscr{B}=\mathscr{D}=1$ in (15)):

$$\psi_{jk}=\lambda_{jk}+\sqrt{2}\left(\frac{Ch}{h^2-1}+h\delta_{jn}e^{q_n}\right)\delta_{jk},$$

$$\psi^*_{jk}=-\lambda_{jk}+\sqrt{2}\left(\frac{-Ch}{h^2-1}+\delta_{jn}\frac{e^{q_n}}{h}\right)\delta_{jk},$$

$$\chi_{jk}=-\frac{1}{2}\lambda_{jk}+\frac{\sqrt{2}}{2}h\delta_{jk}\delta_{jn}e^{q_n}, \qquad \chi^*_{jk}=\frac{\lambda_{jk}}{2}+\frac{\sqrt{2}}{2}\delta_{jk}\delta_{jn}\frac{e^{q_n}}{h},$$

$$\mu_j=0, \qquad \lambda_{jk}=i\,e^{-\frac{q_1+q_2}{2}}(\delta_{j1}\delta_{k2}+\delta_{j2}\delta_{k1}), \tag{14a}$$

$$\psi_{jk}=\sqrt{2}\,\delta_{jk}\left(\frac{h(\mathscr{A}h-\mathscr{C})}{h^2-1}+\delta_{j1}e^{-q_1}+h\delta_{jn}e^{q_n}\right),$$

$$\psi^*_{jk}=\sqrt{2}\,\delta_{jk}\left(\frac{\mathscr{C}h-\mathscr{A}}{h^2-1}+\delta_{j1}e^{-q_1}+h^{-1}\delta_{jn}e^{q_n}\right),$$

$$\chi_{jk}=\frac{\sqrt{2}}{2}\delta_{jk}(-\delta_{j1}e^{-q_1}+h\delta_{jn}e^{q_n}), \qquad \chi^*_{jk}=\frac{\sqrt{2}}{2}\delta_{jk}\left(-\delta_{j1}e^{-q_1}+\frac{\delta_{jh}}{h}e^{q_n}\right),$$

$$\mu_j=0. \tag{15a}$$

For the lattice (24) with four arbitrary constants let us introduce the following notation:

$$a=\frac{\sqrt{\mathscr{B}}+\sqrt{4\mathscr{A}+\mathscr{B}}}{\sqrt{2}},\qquad \lambda=\frac{-\sqrt{\mathscr{B}}+\sqrt{4\mathscr{A}+\mathscr{B}}}{\sqrt{2}},$$

$$b=\frac{\sqrt{\mathscr{D}}+\sqrt{4\mathscr{C}+\mathscr{D}}}{\sqrt{2}},\qquad \tau=\frac{-\sqrt{\mathscr{D}}+\sqrt{4\mathscr{C}+\mathscr{D}}}{\sqrt{2}},$$

$$\varrho_{jk}=(\delta_{j1}\delta_{k2}+\delta_{j2}\delta_{k1})\exp\left(-\frac{q_1+q_2}{2}\right),\; v_{jk}=(\delta_{j,n-1}\delta_{kn}+\delta_{k,n-1}\delta_{jn})\exp\left(\frac{q_{n-1}+q_n}{2}\right).$$

The matrices ψ, ψ^*, χ, χ^*, m in this notation can be represented in a relatively simple form

$$\begin{aligned}
\psi_{jk}=i\Bigg\{\delta_{jk}\Bigg[&\frac{2h(\lambda h-\tau)}{h^2-1}+\left(\lambda(\operatorname{cth}q_1-1)+\frac{a}{\operatorname{sh}q_1}\right)\delta_{j1}\\
&-\left(\tau(\operatorname{cth}q_h-1)+\frac{b}{\operatorname{sh}q_n}\right)h\delta_{jn}\Bigg]+\varrho_{jk}+hv_{jk}\Bigg\},\\
\psi^*_{jk}=-i\Bigg\{\delta_{jk}\Bigg[&\frac{2(\tau h-\lambda}{h^2-1}+\left(\lambda(\operatorname{cth}q_1-1)+\frac{a}{\operatorname{sh}q_1}\right)\delta_{j1}\\
&-\left(\tau(\operatorname{cth}q_h-1)+\frac{b}{\operatorname{sh}q_h}\right)h^{-1}\delta_{jh}\Bigg]+\varrho_{jk}+\frac{v_{jk}}{h}\Bigg\},\\
\chi_{jk}=\frac{i}{2}\Bigg\{\delta_{jk}\Bigg[&-\frac{\lambda+a\operatorname{ch}q_1}{\operatorname{sh}^2q_1}\delta_{j1}\\
&+\frac{\tau+b\operatorname{ch}q_n}{\operatorname{sh}^2q_n}\delta_{jn}h\Bigg]-\varrho_{jk}+hv_{jk}\Bigg\},\\
\chi^*_{jk}=-\frac{i}{2}\Bigg\{\delta_{jk}\Bigg[&-\frac{\lambda+a\operatorname{ch}q_1}{\operatorname{sh}^2q_1}\delta_{j1}\\
&+\frac{\tau+b\operatorname{ch}q_h}{\operatorname{sh}^2q_n}\delta_{jn}h^{-1}\Bigg]-\varrho_{jk}+h^{-1}v_{jk}\Bigg\},\\
\mu_j=-i[(a+&\lambda\operatorname{ch}q_1)(\operatorname{sh}q_1)^{-2}\delta_{j1}\\
&+(\tau\operatorname{ch}q_n+b)(\operatorname{sh}q_n)^{-2}\delta_{jn}].
\end{aligned}\tag{24a}$$

The matrix L (25) as a function of h has extra poles at the points $h=\pm 1$. This phenomenon is characteristic of almost all the trigonometric degenerations of the potential (7, 8) including those describing the motion of systems of interacting particles of the Sutherland type in an external field with the potential $W(\xi)=\mathscr{A}\operatorname{ch}(2\xi)+\mathscr{B}\operatorname{ch}(\xi+\gamma)$ when $\mathscr{A}\mathscr{B}\neq 0$ [16]. For the rational degenerations of the potential (7, 8a) the matrix L has one pole at $h=0$ and a pole common with M at infinity. On each of $2n$ sheets of the spectral curves $\det(L(h)-Iz)=0$ the eigenvectors of L possess essential singularities at the poles of the matrix M, which allows us to construct the vector Baker-Akhiezer functions and to obtain explicit expressions for particle trajectories in the systems (14, 15, 24) as combinations of multidimensional theta functions. Results of these calculations for the Hamiltonians with the potentials (7a), (8a) and all their degenerations, trigonometrical and rational, listed in [15] and [16] will be published elsewhere.

The above consideration may also be applied to relativistic analogs of the potentials (6–8) whose integrability at $\varepsilon=0$ has been established by Ruijsenaars [20] and at $\varepsilon=1$, $n=2$ by the author [21].

Note also that the lattices (15) possess non-Abelian generalizations: the systems of nonlinear matrix equations for matrices $g_j\in gl(s,\mathbb{R})$, $1\leqq j\leqq n$,

$$\frac{d}{dt}\left(\frac{dg_i}{dt}g_j^{-1}\right)=g_{j-1}g_j^{-1}-g_jg_{j+1}^{-1}+\delta_{j1}(g_1^{-2}+\alpha g_1^{-1})-\delta_{jn}(g_n^2+\beta g_n) \qquad (26)$$

admit the Lax representation with the structure of L and M (25) where l, m, ψ, and χ are block $(ns\times ns)$-matrices:

$$l_{jk}=\delta_{jk}\frac{dg_j}{dk}g_j^{-1}+\delta_{j,k-1}g_jg_k^{-1}+\delta_{j-1,k},$$

$$m_{jk}=\frac{1}{2}\left(-\delta_{jk}\frac{dg_j}{dt}g_j^{-1}+\delta_{j,k-1}g_jg_k^{-1}-\delta_{j-1,k}\right),$$

$$\psi_{jk}=\frac{\delta_{jk}}{\sqrt{2}}\left(\frac{2(\alpha h-\beta)h}{h^2-1}+g_1^{-1}\delta_{j1}+hg_n\delta_{jk}\right),$$

$$\psi^*_{jk}=\frac{\delta_{jk}}{\sqrt{2}}\left(\frac{2(\beta h-\alpha)}{h^2-1}+g_1^{-1}\delta_{j1}+h^{-1}g_n\delta_{jn}\right),$$

$$\chi_{jk}=\frac{1}{2\sqrt{2}}\delta_{jk}[-g_1^{-1}\delta_{j1}+hg_n\delta_{jn}],$$

$$\chi^*_{jk}=\frac{1}{2\sqrt{2}}\delta_{jk}[-g_1^{-1}\delta_{j1}+g_nh^{-1}\delta_{jn}].$$

The systems (26) represent generalizations of the non-Abelian lattices given in [17] and analogous of Abelian systems of the type $\mathscr{C}_n^{(1)}$ and $\mathscr{D}_{n+1}^{(2)}$. Two-dimensional analogs existing for all earlier known lattices (2–4) are absent for the systems (14) and (15) which cannot be included into the Zakharov-Shabat scheme of construction of two-dimensional nonlinear evolution equations.

To conclude, it is to be noted that of extreme interest seems to be the group-theoretical interpretation of the systems (14, 15, 24). For the lattices (2–4) it has been obtained in refs. [6, 9, 11, 12, and 17] where a detailed study has been made for the connection of the Lax matrices with orbits of a coadjoint representation of the Kac-Moody algebras and solution of equations of motion has been reduced to the problem of factorization in the corresponding infinite-dimensional groups. For the systems (14) and (15) with the exponential interaction the Lax matrices in the gauge $\tilde{L}=\Omega L\Omega^{-1}$, $\tilde{M}=\Omega M\Omega^{-1}$,

$$\Omega=\begin{pmatrix}E & 0\\ 0 & \mathscr{I}\end{pmatrix},\quad E_{jk}=\delta_{jk},\ \mathscr{I}_{jk}=\delta_{j,n-k+1}$$

can also be realised as orbits of the Kac-Moody algebras. As for the systems (24), it is necessary first to obtain the algebraic interpretation of the potentials (7), (8) containing the Weierstrass function. At present there is no such interpretation even for the simplest case of $\varepsilon=0$. Solution of this problem would permit finding the methods of studying quantum systems with potentials (6–8) and their nontrivial trigonometric and rational degenerations [15, 16].

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