THE FINITE VERTEX-PRIMITIVE AND VERTEX-BIPRIMITIVE s-TRANSITIVE GRAPHS FOR $s \geq 4$

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ABSTRACT. A complete classification is given for finite vertex-primitive and vertex-biprimitive s-transitive graphs for $s \geq 4$. The classification involves the construction of new 4-transitive graphs, namely a graph of valency 14 admitting the Monster simple group M, and an infinite family of graphs of valency 5 admitting projective symplectic groups $\mathrm{PSp}(4, p)$ with p prime and $p \equiv \pm 1 \pmod 8$. As a corollary of this classification, a conjecture of Biggs and Hoare (1983) is proved.

1. INTRODUCTION

Denote by Γ a finite connected graph with vertex set $V\Gamma$ and edge set $E\Gamma$. For a positive integer s, an s-*arc* of Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $\{v_{i-1}, v_i\} \in E\Gamma$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. Let Aut Γ denote the full automorphism group of Γ. If $G \leq$ Aut Γ and G is transitive on $V\Gamma$ and on the set of s-arcs of Γ, then Γ is called a (G, s)-*arc transitive* graph; while if in addition G is not transitive on the set of $(s+1)$-arcs of Γ, then Γ is called a (G, s)-*transitive graph.* In particular, if $G =$ Aut Γ, then a (G, s)-arc transitive or (G, s)-transitive graph is simply called an s-*arc transitive graph* or an s-*transitive graph*, respectively. (We remark that for any positive integer s a cycle is $(s+1)$-arc transitive so that it is not s-transitive, so the valency of an s-transitive graph is greater than 2.) The main purpose of this paper is to classify finite vertex-primitive and vertex-biprimitive s-transitive graphs for $s \geq 4$.

Interest in s-transitive graphs stems from a beautiful result of Tutte (1947) who proved that there exist no finite s-transitive cubic graphs for $s \geq 6$. Tutte's Theorem was generalized by Weiss (1981) who proved that there exist no finite s-transitive graphs for $s = 6$ and $s \geq 8$. Since then, s-transitive graphs have received considerable attention in the literature (see, for example, [11, 13, 24, 34]), and investigating the following problem has become one of the central topics in algebraic graph theory:

Problem. Construct and characterize s-transitive graphs for $s \geq 2$, and in particular, for $s = 4, 5$ and 7.

It is known that for each value of $s \in \{1, 2, 3, 4, 5, 7\}$, there exist s-transitive graphs (see Section 2). However, it is widely believed that s-transitive graphs for

large values of s, that is $s \in \{4, 5, 7\}$, are rare and hard to construct. An s-transitive graph is said to be *basic* if it is not a *normal cover* of a smaller s-transitive graph (see [26]). It was shown by Conway that any s-transitive graph has a nontrivial s-transitive normal cover (see [3, Section 19]). This suggests that the study of s-transitive graphs should be divided into two steps: studying basic s-transitive graphs and studying their normal covers (see [26]). Using such an approach, it is shown in [18, 19] that the order of a finite s-transitive graph for $s \geq 4$ is neither a prime-power nor an odd integer.

Let s be an integer in $\{4, 5, 7\}$. Up to now, the only known basic s-transitive graphs consist of the following graphs (see Sections 2 and 3): several infinite families of s-transitive graphs of valency 3 and valency 4, constructed in [4, 6, 7, 39]; the classical generalized $(s-1)$-gon graphs which are bipartite; and three graphs from sporadic simple groups: the Janko group J_3, the Rudvalis group Ru and the Thompson group Th, which were constructed in [37] and [28]. The J_3-graph has valency 5; the Ru-graph and the Th-graph have valency 6. These three graphs were the only known non-bipartite s-transitive graphs for $s \geq 4$ of valency greater than 4 until now. Here we construct a 4-transitive graph of valency 14 from the Monster sporadic simple group, and an infinite family of 4-transitive graphs of valency 5 from the projective symplectic groups $\mathrm{PSp}(4, p)$ with p prime.

Theorem 1.1. *There exists exactly one 4-transitive graph Γ such that Γ is vertex-primitive and $\mathrm{Aut}\,\Gamma = \mathrm{M}$, the Monster simple group. Further, the graph Γ has valency 14.*

The Monster-graph is the only known non-bipartite basic s-transitive graph of valency greater than 6, for $s \geq 4$.

Theorem 1.2. *For each prime $p \equiv \pm 1 \pmod 8$, there exists exactly one 4-transitive graph Γ such that Γ is vertex-primitive and $\mathrm{Aut}\,\Gamma = \mathrm{PSp}(4, p)$. Further, Γ has valency 5.*

The $\mathrm{PSp}(4, p)$-graphs form the only known infinite family of basic s-transitive graphs of valency greater than 4, for $s \geq 4$.

Moreover, it will be shown that the Monster-graph and the $\mathrm{PSp}(4, p)$-graphs, together with some other known graphs, form the class of vertex-primitive or vertex-biprimitive s-transitive graphs for $s \geq 4$. (A finite graph Γ is said to be *vertex-primitive* if $\mathrm{Aut}\,\Gamma$ acts primitively on $V\Gamma$; while a bipartite graph Γ with bi-parts Δ_1 and Δ_2 is said to be *vertex-biprimitive* if the setwise stabilizer $(\mathrm{Aut}\,\Gamma)_{\Delta_i}$ is primitive on Δ_i.)

For two groups G and F, let $G.F$ denote an extension of G by F, and $G \rtimes F$ denote a semidirect product of G by F. For a positive integer n, as in the Atlas [8], denote by $[n]$ a group of order n throughout the paper. Given a finite group G, a core-free subgroup H of G and a 2-element $g \in G \setminus H$ with $g^2 \in H$, the *coset graph* $\Gamma = \Gamma(G, H, HgH)$ is the graph with vertex set $[G : H]$, the set of right cosets of H in G, such that Hx, Hy are adjacent if and only if $yx^{-1} \in HgH$ (see Section 2 for more detailed discussions). The next theorem gives a complete classification of finite vertex-primitive s-transitive graphs for $s \in \{4, 5, 7\}$.

Theorem 1.3. *Let k, s be positive integers, and assume that $s \geq 4$. A finite graph Γ of valency k is vertex-primitive and s-transitive if and only if $\Gamma \cong \Gamma(G, H, HgH)$, where G, H, k, s are as listed in Table 1. Further, such a graph Γ is uniquely determined by its order $|V\Gamma|$.*

TABLE 1.

Aut $\Gamma = G$	H	k	s	reference
PSL$(2,p)$ $p \equiv \pm1 \pmod{16}$ prime	S$_4$	3	4	Wong (1967)
PSL$(3,3).\mathbb{Z}_2$	S$_4 \times \mathbb{Z}_2$	3	5	Wong (1967)
PSp$(4,p)$ $p \equiv \pm1 \pmod{8}$ prime	$\mathbb{Z}_2^4 \rtimes \mathrm{GL}(2,4)$	5	4	Theorem 1.2
J$_3$	$\mathbb{Z}_2^4 \rtimes \mathrm{GL}(2,4)$	5	4	Weiss (1986)
Ru	$\mathbb{Z}_5^2 \rtimes \mathrm{GL}(2,5)$	6	4	Stroth & Weiss (1990)
Th	$\mathbb{Z}_5^2 \rtimes \mathrm{GL}(2,5)$	6	4	Stroth & Weiss (1990)
M	$\mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4.\mathrm{PGL}(2,13))$	14	4	Theorem 1.1

TABLE 2.

$T = \mathrm{soc}(G)$	G	$H \cap T$	k	s	reference
PSL$(3,q)$	$G \leq \mathrm{Aut}(T)$	$[q^2] \rtimes (\mathbb{Z}_l.\mathrm{PGL}(2,q))$ where $l = \frac{q-1}{(q-1,3)}$	$q+1$	4	Section 2
Sp$(4,2^m)$	$G \leq \mathrm{Aut}(T)$	$[2^{3m}] \rtimes \mathrm{GL}(2,2^m)$	2^m+1	5	Section 2
G$_2(3^m)$	$G \leq \mathrm{Aut}(T)$	$[3^{5m}] \rtimes \mathrm{GL}(2,3^m)$	3^m+1	7	Section 2
PSL$(2,p)$ $p \equiv \pm1 \pmod{8}$	PGL$(2,p)$	$H = $ S$_4$	3	4	Biggs & Hoare [4]
A$_6$	PΓL$(2,9)$	$H = $ S$_4 \times \mathbb{Z}_2$	3	5	Tutte's 8-Cage
M$_{12}$	M$_{12}.\mathbb{Z}_2$	$H = \mathbb{Z}_3^2 \rtimes \mathrm{GL}(2,3)$	4	4	Weiss (1985)

Remark. The J$_3$-graph was used to characterize the Janko simple group J$_3$ by Weiss [37]; the Ru-graph was used to characterize the Rudvalis simple group Ru by Stroth and Weiss [28].

Some vertex-biprimitive *s*-transitive graphs are covers of vertex-primitive *s*-transitive graphs. For a (G,s)-transitive graph $\Sigma = \Gamma(G,H,HgH)$, let $F = G \times \langle z \rangle$ where z is an involution, and let $\Gamma = \Gamma(F,H,HgzH)$. Then Γ is an (F,s)-transitive bipartite graph, and is called the *standard double cover* of Σ. The following theorem gives a complete classification of finite vertex-biprimitive *s*-transitive graphs for $s \in \{4,5,7\}$.

Theorem 1.4. *Let k,s be positive integers, and assume that $s \geq 4$. A finite graph Γ of valency k is vertex-biprimitive and s-transitive if and only if either*

  (i) *Γ is the standard double cover of a vertex-primitive s-transitive graph (classified in Theorem 1.3); or*
  (ii) *$\Gamma \cong \Gamma(G,H,HgH)$ such that G, $\mathrm{soc}(G) \cap H$, k,s are as listed in Table 2, where q is a prime-power, and p is a prime.*

Further, such a graph Γ is uniquely determined by its order $|V\Gamma|$.

Remark. The M$_{12}$-graph was used to characterize the Mathieu group M$_{12}$ by Weiss [36].

This result does not help to describe all *s*-transitive graphs for $s \geq 4$ as there is no reduction to the primitive and biprimitive case. The major problem remaining is to handle the quasiprimitive and bi-quasiprimitive cases (see [20]).

It follows from these theorems that there exist no finite vertex-primitive 7-transitive graphs, and that all finite vertex-biprimitive 7-transitive graphs of valency k admit $\mathrm{G}_2(3^m)$ and have $k = 3^m + 1$, for some m.

Wong [39] constructed the $\mathrm{PSL}(3,3).\mathbb{Z}_2$-graph (see Example 3.3) and proved that it is the unique vertex-primitive cubic 5-transitive graph. Biggs and Hoare [4] constructed an infinite family of bipartite 5-transitive cubic graphs from $\mathrm{P\Gamma L}(2, p^2)$ (see Example 3.6). Motivated by Wong's classification, they further conjectured that each 5-transitive cubic biprimitive graph is either one of these graphs, or is a standard double cover of the $\mathrm{PSL}(3,3).\mathbb{Z}_2$-graph given in Table 1. It is an immediate consequence of Theorems 1.3 and 1.4 that this conjecture is true.

Corollary 1.5. *The only vertex-biprimitive 5-transitive graphs of valency 3 are the $\mathrm{P\Gamma L}(2,9)$-graph, which is Tutte's 8-Cage, and the standard double cover of the $\mathrm{PSL}(3,3).\mathbb{Z}_2$-graph.*

In Section 2, we describe generalized polygon graphs in terms of coset graphs, and state the results of Weiss [33] and Gardiner [10], which provide a method for constructing s-transitive graphs for $s \geq 4$. Then in Section 3, we reproduce most of the known s-transitive graphs for $s \in \{4, 5, 7\}$ and construct some new 4-transitive graphs in a uniform manner. Finally, we complete the proofs of Theorems 1.1–1.4 in Section 4.

2. Amalgams of s-transitive graphs for $s \in \{4, 5, 7\}$

In this section, we study a method of constructing s-transitive graphs in terms of coset graphs. Let G be a finite group, and let H be a subgroup of G. The subgroup H is said to be *core-free* if H contains no nontrivial normal subgroups of G. Denote by $[G : H]$ the set of right cosets of H in G, and for a subset S of G, denote by $\underline{S}$ the set of right cosets of H contained in HS, that is, $\{Hs \mid s \in S\}$. Recall that for an element $g \in G \setminus H$ such that $g^2 \in H$, the coset graph $\Gamma(G, H, HgH)$ is the graph with vertex set $[G : H]$ and with Hx, Hy adjacent if and only if $yx^{-1} \in HgH$. By definition, the set of vertices Hx of $\Gamma(G, H, HgH)$ which are adjacent to the vertex H is equal to $\underline{gH} = \{Hgh \mid h \in H\}$, denoted by $\Gamma(H)$ as usual. It follows that the group G in its coset action by right multiplication on $[G : H]$ is transitive on the vertex set of $\Gamma(G, H, HgH)$, and in this G-action the subgroup H is the stabilizer of the vertex H in G and acts transitively on $\Gamma(H)$. Thus G acts transitively on the vertex set and the arc set of $\Gamma(G, H, HgH)$, that is, $\Gamma(G, H, HgH)$ is a G-arc-transitive graph. The valency of $\Gamma(G, H, HgH)$ equals the cardinality of $\Gamma(H)$, that is, $|[HgH : H]|$. Clearly, $|[HgH : H]| = |[H : H \cap H^g]|$, and by definition, it is also clear that $\Gamma(G, H, HgH)$ is connected if and only if $\langle H, g \rangle = G$.

Let α and β denote the vertex H and the vertex Hg, respectively. Then, as observed before, $G_\alpha = H$, and further, $G_\beta = H^g$. Since $g^2 \in H$, g interchanges α and β, and so interchanges G_α and G_β. Now $G_{\alpha\beta}$, the subgroup of G which fixes both α and β, equals $H \cap H^g$; while $G_{\{\alpha,\beta\}}$, the subgroup of G which fixes the edge $\{\alpha, \beta\}$, equals $\langle H \cap H^g, g \rangle$.

As we have noticed, the subgroup H acts transitively on $\Gamma(H)$ and has point-stabilizer $H \cap H^g$. Thus the H-action on $\Gamma(H)$ is equivalent to the coset action of H on $[H : H \cap H^g]$. For convenience, we summarize the above observations in the following lemma (see [27]).

Lemma 2.1. *Let G be a finite group, and let $\Gamma = \Gamma(G, H, HgH)$ be a coset graph of G. Denote by α, β the vertices H, Hg, respectively, of Γ. Then the following are true:*

(1) *Γ is G-arc-transitive and has valency $|[H : H \cap H^g]|$;*
(2) *Γ is connected if and only if $\langle H, g \rangle = G$;*
(3) *$G_\alpha = H$, $G_\beta = H^g$, $G_{\alpha\beta} = H \cap H^g$ and $G_{\{\alpha,\beta\}} = \langle H \cap H^g, g \rangle$.*

These simple properties suggest some method for constructing arc-transitive graphs with certain extra conditions.

Typical examples of s-transitive graphs for $s \geq 4$ come from the parabolic geometries of the rank 2 Lie type groups $A_2(q) \cong \mathrm{PSL}(3, q)$, $B_2(2^m) \cong \mathrm{Sp}(4, 2^m)$ and $G_2(3^m)$, corresponding to $s = 4, 5$ and 7, respectively. These graphs are the so-called *classical generalized polygon graphs* (see [31] or [13, p. 423]). In this section, we first describe these graphs in terms of coset graphs, and then state a result of Weiss about automorphism groups of s-transitive graphs, that is the classification of the amalgams for s-transitive graphs for $s \in \{4, 5, 7\}$. Recall that for a positive integer n, $[n]$ denotes a group of order n.

A 4-transitive graph from $\mathrm{PSL}(3, q)$. Let $T = \mathrm{PSL}(3, q)$, where $q = p^m$, p is a prime and $m \geq 1$. Then T has two conjugacy classes of maximal parabolic subgroups, and has a graph automorphism g, which is of order 2; further, g fuses the two conjugacy classes of maximal parabolic subgroups. Let P be a maximal parabolic subgroup of T. Then $P = [q^2] \rtimes (\mathbb{Z}_{\frac{q-1}{(q-1,3)}} . \mathrm{PGL}(2, q))$ and $P \cap P^g = [q^3] \rtimes (\mathbb{Z}_{q-1} \times \mathbb{Z}_{\frac{q-1}{(q-1,3)}})$. Now $g^2 = 1 \in P$, $\langle P, g \rangle = \langle T, g \rangle$, and $|[P : P \cap P^g]| = q + 1$. Thus the coset graph

$$\Gamma = \Gamma(\langle T, g \rangle, P, PgP)$$

is a bipartite $(\langle T, g \rangle, 4)$-transitive graph of valency $q + 1$ (see Theorem 2.2). Moreover, $\mathrm{Aut}\,\Gamma = \mathrm{Aut}(\mathrm{PSL}(3, q))$, and thus Γ is $(G, 4)$-transitive for each G with $\langle T, g \rangle \leq G \leq \mathrm{Aut}(\mathrm{PSL}(3, q))$; in particular, Γ is 4-transitive. Let d, f be a diagonal or a field automorphism, respectively, of T. Then $o(d) = (q - 1, 3)$, $o(f) = m$, and d, f, P can be chosen so that d, f normalize P. Let R be a subgroup of $\langle d, f \rangle$, and let

$$G = T.R.\langle g \rangle.$$

Then R normalizes T, P and $P \cap P^g$. Thus $T.R$ has a subgroup $H := P.R$ such that $L := H \cap H^g = (P \cap P^g).R$. Then we have

$$H = ([q^2] \rtimes (\mathbb{Z}_{\frac{q-1}{(q-1,3)}} . \mathrm{PGL}(2, q))).R, \quad L = ([q^3] \rtimes (\mathbb{Z}_{q-1} \times \mathbb{Z}_{\frac{q-1}{(q-1,3)}})).R.$$

Finally, we note that $\mathbf{N}_G(L) = L.\mathbb{Z}_2$ (see [16]). $\qquad\qquad\square$

A 5-transitive graph from $\mathrm{Sp}(4, 2^m)$. Let $T = \mathrm{Sp}(4, q)$, where $q = 2^m$ for some $m \geq 1$. Then T has two conjugacy classes of maximal parabolic subgroups, which are fused in $\mathrm{Aut}(T)$. Let f be a field automorphism, and let $g \in \mathrm{Aut}(T) \backslash \mathrm{Inn}(T).\langle f \rangle$. (Note that we are identifying T with $\mathrm{Inn}(T)$.) Then $o(f) \,\big|\, m$, $\mathrm{Aut}(T)/\langle T, f \rangle \cong \mathbb{Z}_2$ and $g^2 \in \langle f \rangle$. Let P be a maximal parabolic subgroup of T. Then g, f, P may be chosen so that f normalizes P but g does not, and $P = [q^3] \rtimes \mathrm{GL}(2, q)$. Let $R \leq \langle f \rangle$ contain g^2, and set

$$G = T.\langle R, g \rangle.$$

TABLE 3.

$k = q + 1$	s	$G_\alpha = H$	$G_{\alpha\beta} = L$
$q + 1$	4	$([q^2] \rtimes (\mathbb{Z}_{\frac{q-1}{(q-1,3)}} . \mathrm{PGL}(2,q))).R$	$([q^3] \rtimes (\mathbb{Z}_{q-1} \times \mathbb{Z}_{\frac{q-1}{(q-1,3)}})).R$
$2^m + 1$	5	$([q^3] \rtimes \mathrm{GL}(2,q)).R$	$([q^4] \rtimes \mathbb{Z}_{q-1}^2).R$
$3^m + 1$	7	$([q^5] \rtimes \mathrm{GL}(2,q)).R$	$([q^6] \rtimes \mathbb{Z}_{q-1}^2).R$

Then $T.R$ has a subgroup $H := P.R$ such that $g^2 \in H$ and $\langle H, g \rangle = G$. Thus the
coset graph $\Gamma = \Gamma(G, H, HgH)$ is a $(G, 5)$-transitive graph of valency $q + 1$ (see
Theorem 2.2). Moreover, Aut $\Gamma = \mathrm{Aut}(T)$, and Γ is 5-transitive. The subgroups
$H, L < G$ have the form

$$H = ([q^3] \rtimes \mathrm{GL}(2,q)).R, \quad L = ([q^4] \rtimes \mathbb{Z}_{q-1}^2).R.$$

We notice that $\mathbf{N}_G(L) = L.\mathbb{Z}_2$ (see [16]).   □

A 7-transitive graph from $\mathrm{G}_2(3^m)$. Let $T = \mathrm{G}_2(q)$, where $q = 3^m$ for some
$m \geq 1$. Then T has two conjugacy classes of maximal parabolic subgroups, and has
a graph automorphism g, which is of order 2; further, g fuses the two conjugacy
classes of parabolic subgroups. Let P be a parabolic subgroup of T. Then $P =
[q^5] \rtimes \mathrm{GL}(2,q)$, and P, g may be chosen so that $P \cap P^g = [q^6] \rtimes \mathbb{Z}_{q-1}^2$. Now
$g^2 = 1 \in P$, $\langle P, g \rangle = \langle T, g \rangle$, and $|[P : P \cap P^g]| = q + 1$. Thus the coset graph

$$\Gamma = \Gamma(\langle T, g \rangle, P, PgP)$$

is a bipartite $(\langle T, g \rangle, 7)$-transitive graph of valency $q + 1$ (see Theorem 2.2). More-
over, Aut $\Gamma = \mathrm{Aut}(\mathrm{G}_2(q))$, and thus Γ is $(G, 7)$-transitive for each G with $\langle T, g \rangle \leq
G \leq \mathrm{Aut}(\mathrm{G}_2(q))$; in particular, Γ is 7-transitive. Choose a field automorphism f of
T such that f normalizes P. Let R be a subgroup of $\langle f \rangle$, and let

$$G = T.R.\langle g \rangle.$$

Then R normalizes T, P and $P \cap P^g$. Thus $T.R$ has a subgroup $H := P.R$ such
that $L := H \cap H^g = (P \cap P^g).R$. Then we have

$$H = ([q^5] \rtimes \mathrm{GL}(2,q)).R, \quad L = ([q^6] \rtimes \mathbb{Z}_{q-1}^2).R.$$

Finally, we note that, by [15, Theorem B], $\mathbf{N}_G(L) = L.\mathbb{Z}_2$.   □

These three classes of graphs are important because of the following result (see
[10] and [33, 35], also refer to [13, Theorem 2.1.5]).

Theorem 2.2. *Let Γ be a connected graph of valency k, and let $G \leq \mathrm{Aut}\,\Gamma$ be such
that Γ is transitive on $V\Gamma$. Let $\alpha, \beta \in V\Gamma$ be a pair of adjacent vertices.*

   (1) *(Weiss [33, 35]) Assume that $s \geq 4$ and that Γ is (G, s)-transitive. Then
       $s \in \{4, 5, 7\}$, $k = q + 1$ where q is a prime power, and the stabilizers G_α and
       $G_{\alpha\beta}$ are completely known, as in Table 3. Further, either Γ is s-transitive, or
       $k = 3$, $s = 4$, and Γ is 5-transitive.*
   (2) *(Gardiner [10, Theorem 3.1]) Assume that H is a subgroup of G which satisfies
       line 1, 2 or 3 of Table 3. If there exists an arc-transitive graph $\Gamma(G, H, HgH)$
       of valency k, then $\Gamma(G, H, HgH)$ is s-transitive for $s = 4$, 5 or 7, respectively.*

Remark on Theorem 2.2. The valency of the graph Γ and the arc-stabilizer L are
uniquely determined by the point-stabilizer H. Actually, L is the normalizer of a
Sylow p-subgroup of H, where q is a p-power, and the valency is equal to $|H|/|L|$.

Hence, in particular, all subgroups of H isomorphic to L are conjugate (in H) to L.

To end this section, we describe a possible (and often efficient) method for constructing s-arc transitive graphs for $s \geq 4$. Let $\Gamma(G, H, HgH)$ be a coset graph, and let $L = H \cap H^g$ and $K = \langle L, g \rangle$. Then we have three subgroups H, L, K of G, which have the properties:

$H \cap K = L$, $|[K : L]| = 2$, and $|[H : L]|$ equals the valency of $\Gamma(G, H, HgH)$.

A triple (H, L, K) with these properties is called an *amalgam* (see [11, 13]). Thus, constructing arc-transitive graphs with certain extra property is equivalent to seeking suitable groups G and suitable pairs $\{H, g\}$; the latter is in turn equivalent to seeking suitable amalgams (H, L, K). A procedure for constructing s-transitive graphs may be stated as follows:

(A) find a suitable group G which has a subgroup H as listed in Table 3;

(B) find a subgroup L of H such that H acting on $[H : L]$ is of degree $q + 1$;

(C) analyze $\mathbf{N}_G(L)$ to decide whether there exists a 2-element $g \in \mathbf{N}_G(L) \setminus L$ such that $g^2 \in H$ and $\langle H, g \rangle = G$.

We note that there may be different 2-elements $g \in \mathbf{N}_G(L) \setminus L$ such that $g^2 \in H$ and $\langle H, g \rangle = G$. Thus for a pair (G, H), there may be different arc-transitive graphs $\Gamma(G, H, HgH)$, which in general are not isomorphic. However, for certain s-transitive graphs for $s \geq 4$, we have the uniqueness of such graphs as in Lemma 2.3.

Lemma 2.3. *Let $\Gamma = \Gamma(G, H, HgH)$ be a connected (G, s)-transitive graph, where $s \in \{4, 5, 7\}$, and let $L = H \cap H^g$. Assume that $\mathbf{N}_G(L) = K = L.\mathbb{Z}_2$. For any 2-element $f \in G$ with $f^2 \in H$, if $\Gamma(G, H, HfH)$ is connected and s-transitive, then*

$$\Gamma(G, H, HfH) = \Gamma(G, H, HgH).$$

Proof. By Theorem 2.2, $H \cap H^f \cong L$. It then follows from the remark on Theorem 2.2 that $H \cap H^f$ is conjugate in H to L. Hence

$$H \cap H^{h^{-1}fh} = (H \cap H^f)^h = L$$

for some $h \in H$. Since $(h^{-1}fh)^2 \in H$, we have that

$$L^{h^{-1}fh} = (H \cap H^{h^{-1}fh})^{h^{-1}fh} = H^{h^{-1}fh} \cap H = L,$$

that is, $h^{-1}fh \in \mathbf{N}_G(L)$. Since $|\mathbf{N}_G(L)/L| = 2$ and $g \in \mathbf{N}_G(L) \setminus L$, we have $\mathbf{N}_G(L) = L \cup Lg$. Hence $h^{-1}fh \in \mathbf{N}_G(L) \setminus L = Lg$, and so $h^{-1}fh = h'g$ for some $h' \in L$. Therefore, $HfH = Hh^{-1}fhH = Hh'gH = HgH$, and thus we have that $\Gamma(G, H, HfH) = \Gamma(G, H, HgH)$. $\square$

An immediate consequence of this lemma is

Corollary 2.4. *The classical generalized polygon graphs (constructed above) are uniquely determined by their automorphism groups.*

3. Constructing s-arc transitive graphs

In this section, we will first describe the constructions of most known examples of s-arc transitive graphs for $s \in \{4, 5, 7\}$ in a uniform way, apart from the polygonal graphs described in Section 2, and then construct some new 4-transitive graphs: one from the Monster simple group M; an infinite family of graphs from $\text{PSp}(4, p)$ with p a prime. (As usual, for positive integers m and n, by the notation $m \mid n$ we

mean m divides n, and by $m \,\|\, n$ we mean m strictly divides n, that is, $m \,|\, n$ and m is coprime to n/m.)

In the following, for a finite group G and a prime $p \,|\, |G|$, denote by G_p a Sylow p-subgroup of G. The first lemma provides a method to construct certain arc-transitive graphs.

Lemma 3.1. *Let G be a finite group. Assume that G has a maximal subgroup H which is core-free, and assume further that H contains a subgroup P such that $|\mathbf{N}_G(P)|/|\mathbf{N}_H(P)|$ is even. Then there exists $g \in G \setminus H$ such that $\Gamma := \Gamma(G, H, HgH)$ is a connected G-arc transitive graph of valency $|[H : H \cap H^g]|$.*

Proof. By the assumption, there exists a 2-element $g \in \mathbf{N}_G(P) \setminus \mathbf{N}_H(P)$ such that $g^2 \in \mathbf{N}_H(P)$. Thus we have $g^2 \in H$, and $g \notin H$. Since H is a maximal subgroup of G, $G = \langle H, g \rangle$. By Lemma 2.1(1), the coset graph $\Gamma(G, H, HgH)$ is a connected G-arc transitive graph of valency $|[H : H \cap H^g]|$. $\qquad\square$

Lemma 3.2. *Let G be a finite group, and let K be a 2-subgroup of G. Assume that $|G|/|K|$ is even. Then $\mathbf{N}_G(K)/K$ is of even order.*

Proof. Let G_2 be a Sylow 2-subgroup of G which contains K. Then $G_2 > K$, and by [30, p. 88], $\mathbf{N}_{G_2}(K) > K$, so the conclusion of the lemma follows. $\qquad\square$

In the rest of this section, we describe some constructions of s-transitive graphs for $s \in \{4, 5, 7\}$. All of them involve nonabelian simple groups.

3.1. **Graphs of valency 3 and 4.** Note that $\mathrm{GL}(2,2) \cong \mathrm{S}_3$, $\mathbb{Z}_2^2 \rtimes \mathrm{GL}(2,2) \cong \mathrm{S}_4$, and $\mathbb{Z}_2^3 \rtimes \mathrm{GL}(2,2) \cong \mathrm{S}_4 \times \mathbb{Z}_2$. Thus, for cubic s-transitive graphs, the amalgams have been often written as special forms, that is, either

$$s = 4 \text{ and } (H, L) = (\mathrm{S}_4, \mathrm{D}_8), \text{ or}$$
$$s = 5 \text{ and } (H, L) = (\mathrm{S}_4 \times \mathbb{Z}_2, \mathrm{D}_8 \times \mathbb{Z}_2).$$

In particular, in either case L is a 2-group. Assume that G is a finite group which has subgroups H and L satisfying these conditions. By Lemma 3.2, if $|G|/|L|$ is even, then $\mathbf{N}_G(L)/L$ is of even order, and thus there exists a 2-element $g \in \mathbf{N}_G(L) \setminus H$ such that $g^2 \in H$. Assume further that H is a maximal subgroup. Then $\langle H, g \rangle = G$. Therefore, we have the following examples of s-transitive graphs for $s = 4$ or 5. These were first constructed by Wong (1967).

Example 3.3. Let

$$G = \mathrm{Aut}(\mathrm{PSL}(3,3)) = \mathrm{PSL}(3,3).\mathbb{Z}_2.$$

By the Atlas [8], G has a maximal subgroup

$$H \cong \mathrm{S}_4 \times \mathbb{Z}_2,$$

and $|G|/|H|$ is even. Thus there exists a vertex-primitive $(G, 5)$-transitive cubic graph $\Gamma = \Gamma(G, H, HgH)$ such that $\mathrm{Aut}\,\Gamma = G$ and

$$L = H \cap H^g \cong \mathrm{D}_8 \times \mathbb{Z}_2.$$

Noting that $2^5 \,\|\, |G|$ (see the Atlas [8]), it follows from Lemma 3.2 that $\mathbf{N}_G(L) = [2^5]$, and so $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 5)$-transitive cubic graph.

Example 3.4. Let

$$G = \mathrm{PSL}(2, p) \text{ where } p \equiv \pm 1 \pmod{16} \text{ is a prime.}$$

Then by [30, p. 417], G has a maximal subgroup

$$H \cong \mathrm{S}_4,$$

and thus $|G|/|H$ is even. So there exists a vertex-primitive $(G, 4)$-transitive cubic graph $\Gamma = \Gamma(G, H, HgH)$ such that $\mathrm{Aut}\,\Gamma = G$ and

$$L = H \cap H^g \cong \mathrm{D}_8.$$

Checking subgroups of G which contain L (see [30, p. 417]), we conclude that $\mathbf{N}_G(L) \cong \mathrm{D}_{16}$ and so $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 4)$-transitive cubic graph.

Assume now that $|G|/|L|$ is odd but $|\mathrm{Aut}(G)|/|L|$ is even. Then in some circumstances there exists a 2-element $g \in \mathbf{N}_{\langle G,g \rangle}(L) \setminus H$ such that $g^2 \in H$. If, in addition, H is a maximal subgroup of G, then $\langle H, g \rangle = \langle G, g \rangle$. Thus we have the next examples of bipartite *s*-transitive graphs for $s = 4$ or 5. These were first constructed by Biggs and Hoare [4] (also see [3, 18b]).

Example 3.5. Let $T = \mathrm{PSL}(2, p)$ where $p \equiv \pm 1 \pmod{8}$. Then T has two conjugacy classes of maximal subgroups S_4, which are fused in

$$G = \mathrm{PGL}(2, p),$$

see [30, p. 417]. Let H be such that

$$\mathrm{S}_4 \cong H < T.$$

Then $|T|/|H|$ is odd but $|G|/|H|$ is even. Thus there exists a 2-element $g \in G \setminus H$ such that $g^2 \in H$, $\langle H, g \rangle = G$ and $H \cap H^g = L$. Thus there exists a bipartite $(G, 4)$-transitive cubic graph $\Gamma = \Gamma(G, H, HgH)$ such that $\mathrm{Aut}\,\Gamma = G.\mathbb{Z}_2 = \mathrm{PGL}(2, p)$ and

$$L = H \cap H^g \cong \mathrm{D}_8.$$

Checking subgroups of G which contain L (see [30, p. 417]), we may conclude that $\mathbf{N}_G(L)$ is a Sylow 2-subgroup of G, which is of order 16. Thus $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 4)$-transitive cubic graph.

The 5-transitive cubic graphs given in the next example were also constructed by Biggs and Hoare [4].

Example 3.6. Let $T = \mathrm{PSL}(2, p^2)$ where p is a prime and $p^2 \equiv \pm 1 \pmod{8}$. Then $\mathrm{Out}(T) \cong \mathbb{Z}_2^2$ and

$$G := \mathrm{Aut}(T) = \mathrm{P\Gamma L}(2, p^2).$$

By [30, p. 417], T has two conjugacy classes of subgroups which are isomorphic to S_4, which are fused in G. It follows that the group $\mathrm{P\Sigma L}(2, p^2)$ has two conjugacy classes of subgroups that are isomorphic to $\mathrm{S}_4 \times \mathbb{Z}_2$, which are fused in G. Let H be a subgroup of $\mathrm{P\Sigma L}(2, p^2)$ such that

$$H \cong \mathrm{S}_4 \times \mathbb{Z}_2.$$

Then a Sylow 2-subgroup L of H is isomorphic to $\mathrm{D}_8 \times \mathbb{Z}_2$, and is a proper subgroup of a Sylow 2-subgroup S of G. Therefore, there exists a 2-element $g \in S \setminus L$ normalizing L, and so

$$H \cap H^g = L \cong \mathrm{D}_8 \times \mathbb{Z}_2.$$

Now it is easily shown that $\langle H, g \rangle$ is contained in no proper subgroup of G, so $\langle H, g \rangle = G$. Therefore, by Lemma 3.1 and Theorem 2.2, the coset graph $\Gamma = \Gamma(G, H, HgH)$ is a 5-transitive cubic graph, which is bipartite. Analyzing subgroups of G which contain L (see [30, p. 417]), we may obtain that $\mathbf{N}_G(L)$ is a Sylow 2-subgroup of G, which has order 32. Hence $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 5)$-transitive cubic graph. Since H is maximal in $\mathrm{P\Sigma L}(2, p^2)$ if and only if $p = 3$, Γ is biprimitive if and only if $p = 3$. We note that $\mathrm{PSL}(2, 9) \cong \mathrm{A}_6$, and the corresponding graph is Tutte's 8-Cage.

Next we consider 4-valent graphs. Note that $\mathrm{GL}(2, 3) \cong \mathrm{Q}_8 \rtimes \mathrm{S}_3$. Thus, for s-transitive graphs of valency 4 with $s \in \{4, 5, 7\}$, the amalgams also have a special forms, that is, either

$$s = 4 \text{ and } (H, L) = (\mathbb{Z}_3^2 \rtimes (\mathrm{Q}_8 \rtimes \mathrm{S}_3), [3^3] \rtimes \mathbb{Z}_2^2),$$

or

$$s = 7 \text{ and } (H, L) = ([3^5] \rtimes (\mathrm{Q}_8 \rtimes \mathrm{S}_3), [3^6] \rtimes \mathbb{Z}_2^2).$$

We note that in this case $L = \mathbf{N}_H(L_3)$, where L_3 is a Sylow 3-subgroup of H, and thus $\mathbf{N}_G(L) \le \mathbf{N}_G(L_3)$. The 4-transitive graph given in the next lemma was constructed and used to characterize the group M_{12} by Weiss [36].

Lemma 3.7. *Let $T = \mathrm{M}_{12}$, the Mathieu simple group of degree 12, and let $G = \mathrm{Aut}(T) = T.\mathbb{Z}_2$. Then there exists a unique connected 4-transitive graph Γ of valency 4 such that Γ is biprimitive and $\mathrm{Aut}\,\Gamma = G$.*

Proof. By the Atlas [8], T has two conjugacy classes of maximal subgroups isomorphic to $\mathbb{Z}_3^2 \rtimes (\mathbb{Z}_2.\mathrm{S}_4)$, which are fused in G into maximal subgroups of G isomorphic to $3_+^{1+2} \rtimes \mathrm{D}_8$. Let H be a subgroup of T such that

$$H \cong \mathbb{Z}_3^2 \rtimes (\mathbb{Z}_2.\mathrm{S}_4).$$

(We note that in this case, H is isomorphic to $\mathbb{Z}_3^2 \rtimes \mathrm{GL}(2, 3)$.) Let H_3 be a Sylow 3-subgroup of H and let $L = \mathbf{N}_H(H_3)$. Then $H_3 \cong 3_+^{1+4}$, and

$$L \cong 3_+^{1+4} \rtimes [2^2].$$

It follows from the Atlas [8] that $\mathbf{N}_G(L) = 3_+^{1+4} \rtimes \mathrm{D}_8$. Thus there exists a 2-element $g \in \mathbf{N}_G(L) \setminus H$ such that $L = H \cap H^g$. Since H is a maximal subgroup of T, it follows that $\langle H, g \rangle = G$. Thus, by Lemma 2.1, the coset graph $\Gamma = \Gamma(G, H, HgH)$ is a connected arc-transitive graph of valency 4, and further by Lemma 3.1 and Theorem 2.2, Γ is 4-transitive. Since $g \notin T$, T is not transitive on $V\Gamma$, and thus Γ is bipartite, say with parts Δ_1 and Δ_2. As H is a maximal subgroup of T, Γ is biprimitive. It follows from Theorem 2.2 that $\mathrm{Aut}\,\Gamma = G$. Since $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$, by Lemma 2.3, Γ is the unique $(G, 4)$-transitive graph of valency 4. $\qquad\square$

An infinite family of non-bipartite 5-transitive cubic graphs was constructed by Conder (1988) from the alternating groups A_n and the symmetric groups S_n. Very recently, an infinite family of non-bipartite 7-transitive graphs of valency 4 was constructed by Conder and Walker (1998), also from the alternating groups A_n and the symmetric groups S_n. These graphs are neither vertex-primitive nor vertex-biprimitive.

3.2. **Graphs of valency 5.** Weiss (1986) constructed a 4-transitive graph of valency 5 from the Janko simple group J_3, and used the graph to characterize the simple group. Here we first give a description for the Weiss graph, and then construct an infinite family of 4-transitive graphs of valency 5 from a class of symplectic simple groups. We notice that for a $(G, 4)$-transitive graph of valency 5,

$$\mathbb{Z}_2^4 \rtimes \mathrm{GL}(2,4) \leq H \leq (\mathbb{Z}_2^4 \rtimes \mathrm{GL}(2,4)).\mathbb{Z}_2,$$

and

$$[2^6].\mathbb{Z}_3^2 \leq L \leq [2^6].\mathbb{Z}_3^2.\mathbb{Z}_2;$$

in particular, L is a Hall $\{2,3\}$-subgroup of H.

Lemma 3.8. *Let T be the Janko simple group J_3. Then there exists a unique 4-transitive graph Γ of valency 5 such that $\mathrm{Aut}\,\Gamma = \mathrm{Aut}(T) = T.\mathbb{Z}_2$ and Γ is vertex-primitive.*

Proof. Let $G = \mathrm{Aut}(T)$. By the Atlas [8], G has one conjugacy class of maximal subgroups isomorphic to $\mathbb{Z}_2^4 \rtimes (\mathbb{Z}_3 \times A_5).\mathbb{Z}_2 \cong \mathbb{Z}_2^4 \rtimes \mathrm{GL}(2,4).\mathbb{Z}_2$. Let $H < G$ be such that

$$H \cong \mathbb{Z}_2^4 \rtimes \mathrm{GL}(2,4).\mathbb{Z}_2,$$

and let L be a Hall $\{2,3\}$-subgroup of H. Then

$$L \cong [2^6].\mathbb{Z}_3^2.\mathbb{Z}_2,$$

and by the Atlas [8], $\mathbf{N}_G(L) = 2^{2+4} \rtimes (S_3 \times S_3) = L.\mathbb{Z}_2$. Thus there exists a 2-element g in $\mathbf{N}_G(L) \setminus H$ such that $\langle H, g \rangle = G$, $g^2 \in H$, $L = H \cap H^g$ and $|H : L| = 5$. Therefore, by Lemma 2.1 and Theorem 2.2, the coset graph $\Gamma = \Gamma(G, H, HgH)$ is a 4-transitive graph of valency 5, which is the Weiss graph given in [37]. It follows from Theorem 2.2 that $\mathrm{Aut}\,\Gamma = G$. Finally, by Lemma 2.3, Γ is a unique $(G, 4)$-transitive graph of valency 5. $\qquad\square$

Next we construct an infinite family of 4-transitive graphs of valency 5.

Lemma 3.9. *Let $G = \mathrm{PSp}(4, p)$, where p is a prime and $2^4.3 \,\|\, (p^2 - 1)$. Then there exists a unique 4-transitive graph Γ of valency 5 such that $G_\alpha \cong \mathbb{Z}_2^4 \rtimes S_5$, $\mathrm{Aut}\,\Gamma = G$ and Γ is vertex-primitive.*

Proof. By [16, Proposition 4.6.9], G has one conjugacy class of maximal subgroups isomorphic to $\mathbb{Z}_2^4.(\Omega_4^-(2).\mathbb{Z}_2) \cong \mathbb{Z}_2^4.\mathrm{PGL}(2,4) \cong \mathbb{Z}_2^4.S_5$. Let H be a maximal subgroup of G such that

$$H \cong \mathbb{Z}_2^4.S_5,$$

and let L be a subgroup of H such that

$$L \cong \mathbb{Z}_2^4.S_4\,.$$

Write $H = N_1 . \mathrm{S}_5$, and $L = N_1 . (N_2 . \mathrm{S}_3)$, where $N_1 \cong \mathbb{Z}_2^4$ and $N_2 \cong \mathbb{Z}_2^2$. Let L_2 be a Sylow 2-subgroup of L, and let G_2 be a Sylow 2-subgroup of G containing L_2. Then $|G_2| = 2^8$, $|L_2| = 2^7$, and so $G_2 > L_2$. Now $M := \mathbf{O}_2(L) = N_1 . N_2$. Since $\mathbf{C}_H(N_1) = N_1$, we have that M is not abelian, and hence M has exponent 4. Let $b \in M$ be of order 4. Then $\overline{b} = bN_1 \in (N_1 . \mathrm{A}_4)/N_1$ is an involution. It is clear that each involution of A_4 is a square of an element of S_4 of order 4, that is, there exists $\overline{c} \in L/N_1 = \mathrm{S}_4$ such that $\overline{c}^2 = \overline{b}$. Let c be a preimage of $\overline{c}$ in L. Then the order of c is divisible by 8, and so L_2 is of exponent 8.

Since $G_2 > L_2$, there exists $g \in \mathbf{N}_G(L_2) \setminus L_2$ such that $g^2 \in L_2$. Suppose that $M^g \neq M$. Then, since M, M^g are subgroups of L_2 of index 2, $\langle M, M^g \rangle = M \circ M^g = L_2$. However, as M is of exponent 4, every element of $M \circ M^g$ is of order at most 4, which is a contradiction since $L_2 = M \circ M^g$ is of exponent 8. Hence $M^g = M$, that is g normalizes M. In particular, M is a normal subgroup of $\langle L, g \rangle$. It is now easy to obtain that $\langle L, g \rangle / M$ is of order 12 and thus g normalizes L, so that $L = H \cap H^g$. Hence the coset graph $\Gamma = \Gamma(G, H, HgH)$ is arc-transitive and of valency 5. It then follows from Theorem 2.2 that Γ is 4-transitive and $\mathrm{Aut}\,\Gamma = G$. Analyzing maximal subgroups of G which contain L (see [16]), we may conclude that $\mathbf{N}_G(L) = \langle L, g \rangle \cong L.\mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 4)$-transitive graph of valency 5. $\qquad\blacksquare$

3.3. **Graphs of valency greater than 5.** Here we will construct two 4-transitive graphs of valency 6, and a 4-transitive graph of valency 14. We observe that if $k = p + 1 \geq 6$ for some prime p, then a $(G, 4)$-arc transitive graph of valency k is 4-transitive;

$$H = \mathbb{Z}_p^2 \rtimes \mathbb{Z}_{\frac{p-1}{(p-1,3)}} . \mathrm{PGL}(2, p),$$

and

$$L = [p^3] \rtimes (\mathbb{Z}_{p-1} \times \mathbb{Z}_{\frac{p-1}{(p-1,3)}}).$$

In particular, $L = \mathbf{N}_H(L_p)$ for a Sylow p-subgroup L_p. Thus $\mathbf{N}_G(L) \leq \mathbf{N}_G(L_p)$. The construction of the graphs of valency 6 in the next lemma was first obtained by Stroth and Weiss [28].

Lemma 3.10. *Let T be one of the sporadic simple groups* Ru, Th *or* M. *Then there exists a unique 4-transitive graph Γ such that* $\mathrm{Aut}\,\Gamma = T$, *and further, Γ has valency 6 in the cases where $T = $ Ru or Th, while Γ has valency 14 when $T = $ M.*

Proof. Since the construction for the Ru-graph is similar to the construction for the Th-graph, we only describe the latter.

Let T be the Thompson group Th. By the Atlas [8], T has one conjugacy class of maximal subgroups that are isomorphic to $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_4 . \mathrm{S}_5 \cong \mathbb{Z}_5^2 \rtimes \mathrm{GL}(2, 5)$. Let H be a subgroup of T such that

$$H \cong \mathbb{Z}_5^2 \rtimes \mathrm{GL}(2, 5),$$

and let H_5 be a Sylow 5-subgroup of H. Then

$$L := \mathbf{N}_H(H_5) \cong 5_+^{1+2} . (\mathbb{Z}_4 . \mathbb{Z}_4),$$

and however, by the Atlas [8], $\mathbf{N}_T(H_5) \cong 5_+^{1+2} . (\mathbb{Z}_4 . \mathrm{S}_4)$. Let M be a Hall $\{2, 5\}$-subgroup of $\mathbf{N}_T(H_5)$ which contains L. Then L has index 2 in M, and thus there exists a 2-element $g \in M \setminus L$ such that $g^2 \in L < H$ and g normalizes L. It follows that $L = H \cap H^g$. Now $|H : L| = 6$, and as H is a maximal subgroup of T, $\langle H, g \rangle = $

T. Therefore, by Lemma 3.1 and Theorem 2.2, the coset graph $\Gamma = \Gamma(T, H, HgH)$ is a connected 4-transitive graph of valency 6. It also follows from Theorem 2.2 that Aut $\Gamma = T$. Now H has a normal subgroup N such that $5_+^{1+2}.\mathbb{Z}_4 \cong N \lhd L$. Thus $N \lhd L \lhd \mathbf{N}_T(L) \leq \mathbf{N}_T(H_5) = N.\mathrm{S}_4$. Thus $\mathbb{Z}_4 \cong L/N \lhd \mathbf{N}_T(L)/N \leq \mathrm{S}_4$, and so $\mathbf{N}_T(L)/N \cong \mathrm{D}_8$. In particular, $\mathbf{N}_T(L)/L \cong \mathbb{Z}_2$, and so by Lemma 2.3, Γ is a unique $(T, 4)$-transitive graph of valency 6.

Now let T be the Monster simple group M. By the Atlas [8], T has one conjugacy class of maximal subgroups which are isomorphic to $\mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4.\mathrm{PSL}(2, 13).\mathbb{Z}_2)$. Let H be a subgroup of T such that

$$H \cong \mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4.\mathrm{PSL}(2, 13).\mathbb{Z}_2),$$

and let H_{13} be a Sylow 13-subgroup of H. Then

$$L := \mathbf{N}_H(H_{13}) \cong 13_+^{1+2}.(\mathbb{Z}_3 \times (\mathbb{Z}_4.\mathbb{Z}_4)),$$

and however, by the Atlas [8], $\mathbf{N}_T(H_{13}) \cong 13_+^{1+2} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_4.\mathrm{S}_4)$. Let $L_{\{2,3\}}$ and P be Hall $\{2, 3\}$-subgroups of L and $\mathbf{N}_T(H_{13})$, respectively, such that $L_{\{2,3\}} < P$. It follows that either $L_{\{2,3\}} \lhd P$, or $\mathbf{N}_P(L_{\{2,3\}}) \cong \mathbb{Z}_3 \times [2^5]$. In either case, $|P|/|L_{\{2,3\}}|$ is even, and thus there exists a 2-element $g \in \mathbf{N}_T(H_{13}) \setminus L$ such that g normalizes L and $g^2 \in L < H$. Now $|H : L| = 14$, and as H is a maximal subgroup of T, $\langle H, g \rangle = T$. Therefore, by Lemma 3.1 and Theorem 2.2, the coset graph $\Gamma = \Gamma(T, H, HgH)$ is a connected 4-transitive graph of valency 14. Note that H has a normal subgroup N such that $13_+^{1+2}.\mathbb{Z}_{12} \cong N \lhd L \lhd \mathbf{N}_T(L) \leq \mathbf{N}_T(H_{13}) = N.\mathrm{S}_4$. Hence $\mathbb{Z}_4 \cong L/N \lhd \mathbf{N}_T(L)/N \leq \mathrm{S}_4$, and so $\mathbf{N}_T(L)/N \cong \mathrm{D}_8$. In particular, $\mathbf{N}_T(L)/L \cong \mathbb{Z}_2$, and so by Lemma 2.3, Γ is a unique $(T, 4)$-transitive graph of valency 14.

To complete the proof of this lemma, we only need to prove that Aut $\Gamma = T$. Suppose that Aut $\Gamma \neq T$. Since H is a maximal subgroup of T, T is primitive on $V\Gamma$. Thus any group X with $T < X \leq$ Aut Γ is primitive on $V\Gamma$. Choose X such that T is maximal in X. Since $\mathrm{Aut}(T) = T$ and T is primitive on $V\Gamma$, it follows that $\mathbf{N}_{\mathrm{Aut}\ \Gamma}(T) = T$. Since Γ is of valency 14, all prime divisors of $|X_\alpha|$ are less than 14. As 17 exactly divides $|T|$, we have that 17 exactly divides $|X|$ and note that 17 divides $|V\Gamma|$. It then follows from the O'Nan-Scott theorem (see [24]) that X is an almost simple group. Now both T and X_α are maximal subgroups of X, and $X = TX_\alpha$, which is a maximal factorization of X. By Corollary 4 and Theorems B and C of [21], there exists no such triple (T, X_α, X), which is a contradiction. Therefore, Aut $\Gamma = T$, and Lemma 3.10 is proved. $\qquad\square$

4. Proofs of the theorems

This section is devoted to proving Theorems 1.1–1.4. Let Γ be a (G, s)-transitive graph, where $G \leq$ Aut Γ and $s \in \{4, 5, 7\}$, such that G is primitive or biprimitive on $V\Gamma$. If Γ is bipartite with parts Δ_1 and Δ_2, then $G_{\Delta_1} = G_{\Delta_2}$ is primitive on Δ_1 and Δ_2. Let

$$\begin{cases} \Omega = V\Gamma, & G^+ = G, & \text{if } \Gamma \text{ is non-bipartite,} \\ \Omega = \Delta_1, & G^+ = G_{\Delta_1}, & \text{if } \Gamma \text{ is bipartite with parts } \Delta_i. \end{cases}$$

Then G^+ is a primitive permutation group on Ω (see [25, Theorem 2.1]). Thus for $\alpha \in \Omega$, $G_\alpha = G_\alpha^+$ is a maximal subgroup of G^+. The first lemma gives a reduction to almost simple groups.

Lemma 4.1. *Let Γ be a graph such that $G \leq \text{Aut } \Gamma$ is primitive or biprimitive on $V\Gamma$ and Γ is (G, s)-transitive for some $s \in \{4, 5, 7\}$. Then G^+ is an almost simple group, and further, either*

(i) *G is an almost simple group, or*

(ii) *Γ is the double cover of a vertex-primitive (G^+, s)-transitive graph.*

Proof. By Theorem 2.2, for $\alpha \in \Omega$, $G_\alpha^{\Gamma(\alpha)}$ is a 2-transitive permutation group on $\Gamma(\alpha)$ such that $\text{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \text{PSL}(2, q)$, where q is a power of a prime p with $|\Gamma(\alpha)| = q + 1$. Since G^+ is primitive, by Praeger [24, 25] and in terms of [26], G^+ is of type HA, AS, PA or TW.

Suppose first that G^+ is of type HA or TW. Then $\text{soc}(G^+)$ is regular on Ω. Let P be the normal subgroup of G_α of order q^{s-2}, and let $X = \text{soc}(G^+) \rtimes P$. Then $X \lhd G^+$, and so $1 \neq X_\alpha^{\Gamma(\alpha)} \lhd G_\alpha^{\Gamma(\alpha)}$. As $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive, $X_\alpha^{\Gamma(\alpha)}$ is transitive on $\Gamma(\alpha)$, which is a contradiction since $|\Gamma(\alpha)| = q + 1$ and $X_\alpha^{\Gamma(\alpha)}$ is a p-group.

Suppose now that G^+ is of type PA, and let $N = \text{soc}(G^+)$. Then $1 \neq N_\alpha^{\Gamma(\alpha)} \lhd G_\alpha^{\Gamma(\alpha)}$. As $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive, $N_\alpha^{\Gamma(\alpha)}$ is transitive on $\Gamma(\alpha)$, and hence $N_\alpha^{\Gamma(\alpha)} \geq \text{PSL}(2, q)$. It follows that the unique insoluble composition factor of G_α (isomorphic to $\text{PSL}(2, q)$) is a composition factor of N_α. However, as G^+ is of type PA, $N_\alpha = R_1 \times \cdots \times R_k$, where $k \geq 2$ and $R_1 \cong \cdots \cong R_k$. This is not possible. Therefore, G^+ is almost simple, that is, $\text{soc}(G^+) = T$ is a nonabelian simple group.

Assume that G is not an almost simple group. It follows that $\mathbf{C}_G(T) \neq 1$. Since $|G|/|G^+| = 2$ and $\mathbf{C}_{G^+}(T) = 1$, we have that $C := \mathbf{C}_G(T) \cong \mathbb{Z}_2$. Now $C \lhd G$, and it then follows that $G = G^+ \times C$. Let $\Sigma = \Gamma_C$, the quotient of Γ induced by C. Then Γ is the standard double cover of Σ. Further, $G^+ \cong G/C \leq \text{Aut } \Sigma$, G/C is primitive on $V\Sigma$, and by [24, Theorem 4.1], Σ is $(G/C, s)$-transitive. $\qquad\square$

Thus, to complete the proofs of Theorems 1.3 and 1.4, we only need to consider almost simple groups. So we assume that G is an almost simple group with socle $T = \text{soc}(G) = \text{soc}(G^+)$, and that Γ is a (G, s)-transitive graph with $s \in \{4, 5, 7\}$. We analyse the finite nonabelian simple groups class by class.

Lemma 4.2. *Assume that $T = \text{soc}(G) \cong \text{A}_n$ for some $n \geq 5$. Then $n = 6$, and Γ is a bipartite 5-transitive cubic graph, which is isomorphic to Tutte's 8-Cage (see Example 3.6).*

Proof. Assume that $n = 6$. Then $T \cong \text{PSL}(2, 9)$, and $\text{PSL}(2, 9) \leq G \leq \text{P}\Gamma\text{L}(2, 9)$. By information given in the Atlas [8], $\text{P}\Gamma\text{L}(2, 9)$ has a maximal subgroup $\text{S}_4 \times \mathbb{Z}_2$, and further we may conclude that the pair $(\text{P}\Gamma\text{L}(2, 9), \text{S}_4 \times \mathbb{Z}_2)$ gives rise to the smallest graph Γ constructed in Example 3.6, which is bipartite, cubic and 5-transitive.

Suppose that $n \neq 6$. Then either $G = G^+$, or $G^+ = T$ and $G \cong \text{S}_n$. In particular, $G^+ \cong \text{A}_n$ or S_n. Now G^+ naturally acts on $\Omega = \{1, 2, \ldots, n\}$, and G_α^+ is a maximal subgroup of G^+. Consider the G_α^+-action on Ω. Note that G_α^+ has the form of H given in Table 3 of Theorem 2.2, and in particular, the unique insoluble composition factor of G_α^+ is $\text{PSL}(2, q)$. By the O'Nan-Scott theorem (see [21, p. 130]), we may conclude that $n = p^2$ and $G_\alpha^+ = \text{AGL}(2, p) \cap G^+ = (\mathbb{Z}_p^2 \rtimes (\mathbb{Z}_{p-1}.\text{PSL}(2, p).\mathbb{Z}_2)) \cap G^+$. Suppose that $T_\alpha = \mathbb{Z}_p^2 \rtimes (\mathbb{Z}_{p-1}.\text{PSL}(2, p).\mathbb{Z}_2) = \text{AGL}(2, p)$. Then there exists an involution σ of $\text{S}_n \setminus \text{A}_n$ which normalizes T_α. However, by the O'Nan-Scott theorem, $\langle T_\alpha, \sigma \rangle$ is not a subgroup of S_n, which is a contradiction. Thus $T_\alpha = \mathbb{Z}_p^2 \rtimes (\mathbb{Z}_{p-1}.\text{PSL}(2, p))$. If Γ is non-bipartite, then Γ is (T, s)-transitive, which is a

contradiction since T_α does not satisfy Theorem 2.2. If Γ is bipartite, then $G^+ = T$ and so $G_\alpha^+ = T_\alpha = \mathbb{Z}_p^2 \rtimes (\mathbb{Z}_{p-1}.\mathrm{PSL}(2,p))$, which is again a contradiction since G^+ does not satisfy Theorem 2.2. □

Lemma 4.3. *Assume that $T = \mathrm{soc}(G)$ is a sporadic simple group. Then $T = \mathrm{M}_{12}$, J_3, Ru, Th or M, and Theorems 1.3 and 1.4 hold.*

Proof. For a vertex α of Γ, T_α satisfies Theorem 2.2. Thus, by information given in the Atlas [8] and the recent result in [23, 38], the pair (T, T_α) is one of the following:

$(\mathrm{M}_{12}, \mathbb{Z}_3^2 \rtimes \mathrm{GL}(2,3))$; $(\mathrm{M}_{22}, \mathbb{Z}_2^4 \rtimes \mathrm{PSL}(2,4).\mathbb{Z}_2)$; $(\mathrm{M}_{23}, \mathbb{Z}_2^4 \rtimes \mathrm{GL}(2,4).\mathbb{Z}_2)$;

$(\mathrm{J}_3, \mathbb{Z}_2^4 \rtimes \mathrm{GL}(2,4))$; $(\mathrm{McL}, \mathbb{Z}_3^4 \rtimes \mathrm{M}_{10})$; $(\mathrm{He}, \mathbb{Z}_7^2 \rtimes \mathrm{SL}(2,7))$;

$(\mathrm{Ru}, \mathbb{Z}_5^2 \rtimes \mathrm{GL}(2,5))$; $(\mathrm{Th}, \mathbb{Z}_5^2 \rtimes \mathrm{GL}(2,5))$; $(\mathrm{M}, \mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4.\mathrm{PSL}(2,13).\mathbb{Z}_2))$.

Suppose that $T = \mathrm{M}_{22}$ or M_{23}. Then the graph Γ has valency 5, and $|T : T_\alpha|$ is odd, which is a contradiction. Next suppose $(T, T_\alpha) = (\mathrm{McL}, \mathbb{Z}_3^4 \rtimes \mathrm{M}_{10})$. If there exists such a graph Γ, then $T_\alpha \cap T_\alpha^g = T_3 \rtimes \mathrm{Q}_8$ where $T_3 \cong \mathbb{Z}_3^4$ is a Sylow 3-subgroup of T and g normalizes $T_3 \rtimes \mathrm{Q}_8$. Thus we have $\mathbf{N}_T(T_3) = T_3 \rtimes [16]$. However, by the Atlas [8], this is not possible. An argument for $T = \mathrm{He}$ using Sylow 7-subgroups, which is analogous to the argument for McL using Sylow 3-subgroups, proves that $T \neq \mathrm{He}$.

The pair $(T, T_\alpha) = (\mathrm{M}_{12}, \mathbb{Z}_3^2 \rtimes \mathrm{GL}(2,3))$ gives rise to the bipartite graph Γ described in Lemma 3.7 such that $\mathrm{Aut}\,\Gamma = \mathrm{Aut}(\mathrm{M}_{12}) = \mathrm{M}_{12}.\mathbb{Z}_2$. The pair $(T, T_\alpha) = (\mathrm{J}_3, \mathbb{Z}_2^4 \rtimes \mathrm{GL}(2,4))$ gives rise to the Weiss graph (see Lemma 3.8). The pair $(\mathrm{Ru}, \mathbb{Z}_5^2 \rtimes \mathrm{GL}(2,5))$ gives rise to the Stroth-Weiss graph, one of the graphs constructed in Lemma 3.10; while the pairs $(\mathrm{Th}, \mathbb{Z}_5^2 \rtimes \mathrm{GL}(2,5))$ and $(\mathrm{M}, \mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4.\mathrm{PSL}(2,13).\mathbb{Z}_2))$ give rise to the other two graphs constructed in Lemma 3.10. □

We note that, by Lemma 3.10 and Lemma 4.3, we have completed the proof of Theorem 1.1.

Assume next that $\mathrm{soc}(G)$ is a classical simple group of Lie type. Note that H is a maximal local subgroup of G^+. All possibilities for H are described in [16], and we will analyse the classification in [16] to complete this case. For convenience, we first deal with some small groups, using information given in the Atlas [8].

Lemma 4.4. *The socle $T = \mathrm{soc}(G)$ of G is not one of the following groups:*

(1) $\mathrm{PSL}(3,2)$, $\mathrm{PSL}(4,2)$, $\mathrm{PSL}(5,2)$, $\mathrm{PSL}(4,3)$;

(2) $\mathrm{PSU}(4,2)$, $\mathrm{PSU}(5,2)$, $\mathrm{PSU}(6,2)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(4,3)$;

(3) $\mathrm{PSp}(6,2)$, $\mathrm{PSp}(8,2)$, $\mathrm{PSp}(4,3)(\cong \mathrm{PSL}(4,2))$, $\mathrm{PSp}(6,3)$;

(4) $\mathrm{O}_8^+(2)$, $\mathrm{O}_8^-(2)$, $\mathrm{O}_{10}^+(2)$, $\mathrm{O}_{10}^-(2)$, $\mathrm{O}_7(3)$, $\mathrm{O}_8^+(3)$, $\mathrm{O}_8^-(3)$.

Proof. For the groups listed in (1), from information given in the Atlas [8], if $T = \mathrm{PSL}(3,2)$, then $H = \mathrm{S}_4$, but T is 2-transitive on $[T : H]$, a contradiction; while if T is one of the other groups in (1), there is no a subgroup H of T satisfying Theorem 2.2.

Now consider the groups listed in (2). By the Atlas [8], only $\mathrm{PSU}(4,2)$ and $\mathrm{PSU}(4,3)$ have subgroups satisfying Theorem 2.2. If $T = \mathrm{PSU}(4,2)$, then H is isomorphic to $[2^4] \rtimes \mathrm{A}_5$ or $[3^3] \rtimes 2\,\mathrm{A}_4$. Thus the valency $k = 5$ or 4, and $K \cong ([2^6].3).2$ or $[3^4].[4]$, but $\mathrm{PSU}(4,2)$ has no such subgroups. If $T = \mathrm{PSU}(4,3)$, then H is isomorphic to $3^4 \rtimes \mathrm{A}_6$ or $[3^5].2\,\mathrm{S}_4$. Thus K should be isomorphic to $[3^6].[8]$, but $\mathrm{PSU}(4,3)$ has no such subgroups.

Similarly, from the information given in the Atlas [8], we may easily conclude that T is not one of the groups in (3) or (4). □

Next we check subgroups in $\mathcal{C}_i$ for $1 \leq i \leq 8$ described in [16]. Observing the form of the point-stabilizer in G, by Sections 4.2, 4.4, 4.5, 4.7 and 4.8 and by Lemma 4.4, we may conclude that H is not in $\mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_5 \cup \mathcal{C}_7 \cup \mathcal{C}_8$. Thus we only need to consider the remaining three cases.

Lemma 4.5. *Assume that $T = \mathrm{soc}(G)$ is a classical simple group of Lie type. Then $T = \mathrm{A}_6$, $\mathrm{PSL}(3,3)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSp}(4,2^m)$, $\mathrm{PSL}(2,p)$, or $\mathrm{PSp}(4,p)$ with $p \equiv \pm 1$ (mod 8), and Γ is one of the graphs given in Tables 1 and 2.*

Proof. First assume that H is a subgroup of G in $\mathcal{C}_1$, namely a reducible subgroup. Checking Section 4.1 of [16] and using Lemma 4.4, we find that one of the following holds:

$T = \mathrm{PSL}(3,q)$ and $T_\alpha = [q^2].\mathbb{Z}_{\frac{q-1}{(q-1,3)}} \mathrm{PGL}(2,q)$ (see [16, Proposition 4.1.17]);

$T = \mathrm{PSp}(4,q)$ and $T_\alpha = [q^3].\,\mathrm{GL}(2,q)$ where q is a 2-power (see [16, Proposition 4.1.19]).

We note that although for $T = \mathrm{PSU}(4,q)$ where q is a 3-power, T has a subgroup M which has a normal subgroup N of order q^5 and has a unique insoluble composition factor isomorphic to $\mathrm{PSL}(2,q)$, $M/N \not\cong \mathrm{GL}(2,q)$. A similar situation happens for the group $\mathrm{PSL}(4,q)$. Therefore, T is isomorphic to $\mathrm{PSL}(3,q)$ or $\mathrm{PSp}(4,q)$. The former gives rise to the generalized 3-gon graphs, and the latter gives rise to the generalized 4-gon graphs described in Section 2.

Now assume that H is a subgroup of type $\mathcal{C}_3$, that is a field extension subgroup. Then by Section 4.3 of [16], we may conclude that $T = \mathrm{PSL}(2,3^2) \cong \mathrm{A}_6$, and $H \cap T \cong \mathrm{PGL}(2,3) \cong \mathrm{S}_4$. This case gives rise to one of the graphs Γ constructed in Example 3.6 such that $\mathrm{Aut}\,\Gamma = \mathrm{P\Gamma L}(2,3^2)$.

Finally, we assume that H is a subgroup of G in $\mathcal{C}_6$, namely a symplectic-type normalizer. Checking Section 4.6 of [16] and by Lemma 4.4, we may conclude that one of the following cases holds:

(1) $T = \mathrm{PSL}(3,3)$, and $H \cap T \cong \mathrm{PSL}(2,3).\mathbb{Z}_2 \cong \mathrm{S}_4$ (see [16, Proposition 4.6.5]);
(2) $T = \mathrm{PSL}(2,p)$ with $p \equiv \pm 1$ (mod 8) and $H \cong \mathrm{S}_4$ (see [16, Proposition 4.6.7]);
(3) $T = \mathrm{PSp}(4,p)$ with $p \equiv \pm 1$ (mod 8), and $H \cong \mathbb{Z}_2^4.(\Omega_4^-(2).\mathbb{Z}_2) \cong \mathbb{Z}_2^4.\,\mathrm{GL}(2,4)$ (see [16, Proposition 4.6.9]).

Case (1) gives rise to the Wong-graph described in Example 3.3. Case (2) gives rise to the Biggs-Hoare-graphs Γ described in Example 3.5 such that $\mathrm{Aut}\,\Gamma = \mathrm{PGL}(2,p)$; if further $p \equiv \pm 1$ (mod 16), then Case (2) gives rise to the Wong-graphs described in Example 3.4. Case (3) gives rise to the graphs constructed in Lemma 3.9. This completes the proof of the lemma. $\qquad\blacksquare$

We remark that, by Lemma 3.9 and Lemma 4.5, we have completed the proof of Theorem 1.2.

Finally, we deal with exceptional simple groups of Lie type.

Lemma 4.6. *If T is an exceptional simple group of Lie type, then $T = \mathrm{G}_2(q)$ with $q = 3^m$ for some $m \geq 1$, and $T_\alpha \cong [q^5] \rtimes \mathrm{GL}(2,q)$, as in Theorem 1.4.*

Proof. Let $T = \mathrm{soc}(G)$ be an exceptional simple group of Lie type over $GF(t^f)$ where t is a prime. Let H be a maximal subgroup of T which is isomorphic to T_α.

Assume first that $T = \mathrm{F}_4(q)$, $\mathrm{E}_i(q)$ for $i \in \{6, 7, 8\}$, or $^2\mathrm{E}_6(q)$.

Let E be a minimal normal subgroup of H, and let $\mathrm{InnDiag}(T)$ be the group generated by all inner and diagonal automorphisms of T. Then E is elementary abelian and $H = \mathbf{N}_T(E)$. By [5], there are four cases to be considered:

(i) $E < \mathrm{InnDiag}(T)$ and H is of maximal rank;

(ii) $E < \mathrm{InnDiag}(T)$ and H is not of maximal rank;

(iii) $E \not< \mathrm{InnDiag}(T)$;

(iv) $t \,\big|\, |E|$, so that H is a maximal parabolic subgroup of G.

Suppose that case (i) holds. Then all possibilities for H are listed in [5, Tables 5.1 and 5.2]. We may easily conclude that no subgroups H satisfy Theorem 2.2, which is a contradiction.

Suppose that case (ii) holds. Then all possibilities are listed in [5, Table 1]. Checking the list, we find that H has a nonabelian simple composition factor not isomorphic to $\mathrm{PSL}(2, q)$, which is a contradiction.

Suppose that case (iii) holds. Then $E \cap \mathrm{InnDiag}(T) = 1$. It follows from [12, Sections 7 and 9] that $H \cap T = \mathbf{C}_T(\alpha) \neq 1$ for some outer automorphism α of T of prime order. Therefore, by [2, Section 19] and [5, Proposition 2.7], either $\mathbf{C}_T(\alpha)$ is a group of Lie type over a maximal subfield of $GF(t^f)$, or $\mathbf{C}_T(\alpha)$ is $^2\mathrm{F}_4(q)$, $^2\mathrm{E}_6(q^{1/2})$, $\mathrm{C}_4(q)$ with q odd, or $\mathrm{F}_4(q)$. Each of them contains an insoluble composition factor not isomorphic to $\mathrm{PSL}(2, q)$, which is a contradiction.

Suppose that case (iv) holds. Then H is an extension of a t-group by the Chevalley group determined by a maximal subdiagram of the Dynkin diagram of T. It follows that either H contains a nonabelian composition factor not isomorphic to $\mathrm{PSL}(2, q)$, or $T = \mathrm{F}_4(2)$. The former is not the case. By the Atlas [8], $\mathrm{F}_4(2)$ has no maximal subgroup satisfying Theorem 2.2, which is a contradiction.

Now consider the remaining exceptional simple groups T of Lie type. All maximal subgroups of such groups T are completely known: [29] for $T = \mathrm{Sz}(q)$, [22] for $T = {}^2\mathrm{F}_4(q^2)$, [15] for $T = {}^2\mathrm{G}_2(q)$, and [14] for $T = {}^3\mathrm{D}_4(q)$. Checking these lists of maximal subgroups of T, we may conclude that $T = \mathrm{G}_2(q)$ with $q = 3^m$ and $T_\alpha = [q^5] \rtimes \mathrm{GL}(2, q)$. This gives rise to the generalized 6-gon graphs described in Section 3. The proof of the lemma is completed. $\square$

Combining Lemmas 4.1–4.6, we have completed the proofs of Theorem 1.3 and Theorem 1.4.

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