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The first eigenvalue of Laplacians on minimal surfaces in S^3

Dedicated to Professor Naomi Mitsutsuka on his 60th birthday

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1. Introduction.

There are many complete surfaces with constant mean curvature in the Euclidean 3-space $\boldsymbol{R}^3$ and in the hyperbolic 3-space $\boldsymbol{H}^3$ (see [2], [4]). But in the Euclidean 3-sphere $\boldsymbol{S}^3$ there have been few results on such surfaces except umbilic ones and flat tori (cf. [5]).

In this paper, we shall construct a one-parameter family of complete, rotational surfaces in $\boldsymbol{S}^3$ with constant mean curvature, including a flat torus as an initial one. In particular, there is a one-parameter family of complete, rotational, minimal surfaces in $\boldsymbol{S}^3$, including the Clifford torus. And we shall show that none of closed, rotational, minimal surfaces in $\boldsymbol{S}^3$ is embedded and the first eigenvalues of some ones relative to the Laplacian are smaller than *two* except for the Clifford torus.

2. Preliminaries.

In this section, we shall review rotational surfaces in $\boldsymbol{S}^3$. At first, we note that $\boldsymbol{S}^3$ is realized as a hypersurface of the Euclidean 4-space $\boldsymbol{R}^4$:

$$\boldsymbol{S}^3=\{(x_1, \cdots, x_4)\in\boldsymbol{R}^4\,;\ \sum_j x_j^2=1\}.$$

In what follows, we denote by $\boldsymbol{S}^2(c)$ the Euclidean 2-sphere of constant Gaussian curvature c (or equivalently, the 2-sphere in $\boldsymbol{R}^3$ of radius $1/\sqrt{c}$), and by $\boldsymbol{S}^1(r)$ the circle in $\boldsymbol{R}^2$ of radius r. And we put $\boldsymbol{S}^1=\boldsymbol{S}^1(1)$ and $\boldsymbol{R}=\boldsymbol{S}^1(\infty)$ for convenience's sake. We note that $\boldsymbol{S}^1(r)\equiv\boldsymbol{R}/2\pi r\boldsymbol{Z}$ for a positive number r, where $\boldsymbol{Z}$ is the set of all integers.

Up to an isometry of $\boldsymbol{S}^3$, an umbilic surface and a flat torus in $\boldsymbol{S}^3$ are represented as follows. For each real number H, the isometric embedding $f:\boldsymbol{S}^2(H^2+1)\to\boldsymbol{S}^3$, $f(x, y, z)=(x, y, z, H/\sqrt{(H^2+1)})$ of $\boldsymbol{S}^2(H^2+1)$ into $\boldsymbol{S}^3$ defines an umbilic surface $\boldsymbol{M}^2(H)$ in $\boldsymbol{S}^3$ with constant mean curvature H, and for $a=\sqrt{[\{1-H/\sqrt{(H^2+1)}\}/2]}$ and $b=\sqrt{(1-a^2)}$, the isometric embedding $f:\boldsymbol{S}^1(a)\times\boldsymbol{S}^1(b)\to\boldsymbol{S}^3$, $f((x, y), (u, v))=(x, y, u, v)$ of $\boldsymbol{S}^1(a)\times\boldsymbol{S}^1(b)$ into $\boldsymbol{S}^3$ defines a flat torus

$\boldsymbol{T}^2(H)$ in $\boldsymbol{S}^3$ with constant mean curvature H.

We shall construct rotational surfaces in $\boldsymbol{S}^3$. Let $\gamma:\boldsymbol{J}\to\boldsymbol{S}^3$, $\gamma(s)=(x(s), y(s), z(s), 0)$, be any C^2-curve in $\boldsymbol{S}^3$ which is parametrized by arc length, whose domain of definition $\boldsymbol{J}$ is an open interval including zero, and for which the following relations hold on $\boldsymbol{J}$.

(i) $$x(s)^2+y(s)^2+z(s)^2=1\,,$$

(ii) $$x'(s)^2+y'(s)^2+z'(s)^2=1\,.$$

We now consider the C^2-mapping $f:\boldsymbol{J}\times\boldsymbol{S}^1\to\boldsymbol{S}^3$,

$$f(s,\,\theta)=(x(s),\; y(s),\; z(s)\cos\theta,\; z(s)\sin\theta)\,.$$

It can be easily shown that the first and the second fundamental forms of f are given by

$$\begin{aligned}\boldsymbol{I}&=ds^2+z^2d\theta^2,\\ \boldsymbol{II}&=\{x''(yz'-y'z)+y''(zx'-z'x)+z''(xy'-x'y)\}\,ds^2\\ &\quad -z(xy'-x'y)d\theta^2.\end{aligned}$$

3. Rotational surfaces in S^3 with constant mean curvature.

From the previous section we see that the C^2-mapping f is an immersion and is of constant mean curvature H if and only if on the interval $\boldsymbol{J}$, the following relations hold.

(1) $$x^2+y^2+z^2=1\,,$$

(2) $$x'^2+y'^2+z'^2=1\,,$$

(3) $$z^2(x'y''-x''y')-zz'(xy''-x''y)+(zz''-1)(xy'-x'y)=2Hz\,,$$

(4) $$0<z\,.$$

We now try to solve the above system explicitly. From (1) we may put x and y by

(5) $$x=\sqrt{(1-z^2)}\cdot\cos\phi(s)\,,$$

(6) $$y=\sqrt{(1-z^2)}\cdot\sin\phi(s)\,,$$

and then determine the function $\phi=\phi(s)$ satisfying (2).

A short computation shows that

(7) $$\phi'^2=(1-z^2-z'^2)(1-z^2)^{-2}.$$

We assume that

(8) $$1-z^2-z'^2>0 \quad \text{on } \boldsymbol{J}.$$

From (7) and (8) we may put $\phi(s)$ as

(9) $$\phi(s)=\int_0^s[1-z(t)^2-z'(t)^2]^{1/2}[1-z(t)^2]^{-1}dt.$$

Putting (5), (6) and (9) into (3) we can show (cf. [3]) that

(10) $$zz''+z'^2+2z^2-1=2Hz(1-z^2-z'^2)^{1/2}.$$

Defining $u(s)$ by

(11) $$u(s)=z(s)^2-1/2,$$

we can show (cf. [4]) that the equation (10) with the conditions (4) and (8) is equivalent to the equation

(12) $$u'^2=-4(H^2+1)u^2+8aHu+1-4a^2$$

with the conditions

(13) $$|u|<1/2, \quad \text{and}$$

(14) $$a-Hu>0, \quad a: \text{constant}.$$

From (12) we may define $u(s)$ by

(15) $$u(s)=(1+H^2)^{-1}\left[aH+\sqrt{\left(\frac{1+H^2}{4}-a^2\right)}\cdot\cos 2\sqrt{(1+H^2)}\,s\right],$$

provided

(16) $$a^2\leqq(1+H^2)/4.$$

It follows from (15) that $\boldsymbol{J}$, the domain of definition of $u(s)$, may be extended to $\boldsymbol{S}^1(r)$, $r=1/2\sqrt{(1+H^2)}$. Denote the extended function by the same symbol. Then, for the extended function $u(s)$ we see that the conditions (13), (14) and (16) are equivalent to the following inequality

(17) $$|H|<2a\leqq\sqrt{(1+H^2)}.$$

Putting (15) into (11), (9), (5) and (6) we have the triple of solutions of the system (1), (2), (3) and (4).

Reversing the above argument, replacing the constant a by $\sqrt{[(1+H^2)/4-a^2]}$, and taking the completeness into consideration we have the following result.

THEOREM 1. *Let H be a constant, and for each constant a, $0\leqq a<1/2$, we define the function $z(s)$ by*

$$z(s)=\sqrt{\left[\frac{1}{2}+\{H\sqrt{((1+H^2)/4-a^2)}+a\cos 2\sqrt{(1+H^2)}s\}/(1+H^2)\right]}, \qquad s\in\boldsymbol{R},$$

and the function $\phi(s)$ *by* (9). *We define* r *by* $r=\sqrt{[\{1-H/\sqrt{(1+H^2)}\}/2]}$ *for* $a=0$, *or,* $r=\inf\{k/2\sqrt{(1+H^2)}$; k *and* $\phi(k\pi/\sqrt{(1+H^2)})/2\pi$ *are positive integers*$\}$ *for* $a>0$. *Then the analytic mapping* $f: S^1(r)\times S^1\to S^3$,

$$f(s, \theta)=(\sqrt{(1-z(s)^2)}\cdot\cos\phi(s),\ \sqrt{(1-z(s)^2)}\cdot\sin\phi(s),\ z(s)\cos\theta,\ z(s)\sin\theta)\,,$$

defines a complete, rotational surface $M(a, H)$ *in* S^3 *with constant mean curvature* H.

Putting $H=0$ in the theorem we have the following result.

COROLLARY. *For each constant* a, $0\leqq a<1/2$, *we define the function* $\phi(s, a)$ *by*

$$\phi(s, a)=\sqrt{\left(\frac{1}{4}-a^2\right)}\int_0^s\left(\frac{1}{2}+a\cos 2t\right)^{-1/2}\left(\frac{1}{2}-a\cos 2t\right)^{-1}dt\,,\qquad s\in R\,.$$

We define r_a *by* $r_a=1/\sqrt{2}$ *for* $a=0$, *or,* $r_a=\inf\{k/2$; k *and* $\phi(k\pi, a)/2\pi$ *are positive integers*$\}$ *for* $a>0$. *Then the analytic mapping* $f: S^1(r_a)\times S^1\to S^3$,

$$f(s, \theta)=\left(\sqrt{\left(\frac{1}{2}-a\cos 2s\right)}\cdot\cos\phi(s, a),\quad \sqrt{\left(\frac{1}{2}-a\cos 2s\right)}\cdot\sin\phi(s, a),\right.$$
$$\left.\sqrt{\left(\frac{1}{2}+a\cos 2s\right)}\cdot\cos\theta,\quad \sqrt{\left(\frac{1}{2}+a\cos 2s\right)}\cdot\sin\theta\right),$$

defines a complete, rotational, minimal surface M_a *in* S^3.

REMARK 1. For $a=0$, the surface $M(a, H)$ (resp. M_a) is nothing but the flat torus $T^2(H)$ (resp. the Clifford torus). In case where $\phi(\pi/\sqrt{(1+H^2)})/\pi$ (resp. $\phi(\pi, a)/\pi$) is irrational for $a>0$, r (resp. r_a) is defined to be infinity and $S^1(r)=R$ (resp. $S^1(r_a)=R$). From the proof of Theorem 2 below we can show that for different a, b in $[0, 1/2)$, M_a is not isometric to M_b. It follows from Lemma 1 below that there exists a countable set of numbers a such that M_a is a closed minimal surface in S^3.

4. Geometric properties of M_a.

In this section we shall prove the following results.

THEOREM 2. *Let* M_a *be a closed, rotational, minimal surface in* S^3 *as in Corollary. If* $0<a<1/2$, *then* M_a *is not embedded in* S^3 *and whose Gaussian curvature varies in a neighborhood of zero in* R.

THEOREM 3. *Let* M_a *be as in Theorem* 2. *There exists a constant* δ *in* $(0, 1/2)$ *such that if* $0<a<\delta$, *then the first eigenvalue of the closed surface* M_a *relative to the Laplacian is smaller than two.*

We shall prepare the following lemmas for proving the above theorems.

LEMMA 1. *Let* $\phi(s, a)$ *be as in Corollary and put* $g(a)=\phi(\pi, a)$, $0\leqq a<1/2$. *Then it follows that* $g(a)$ *is strictly decreasing and continuous in* a, *and that*

$\pi < g(a) < g(0) = \sqrt{2}\pi$ *for* a, $0 < a < 1/2$.

REMARK 2. We can show that $g(a) \to c \leqq \pi^2/3$, $(a \to 1/2)$.

PROOF. Putting $b = 2a$ and changing variables by $t = 2s$ we have that for each b, $0 \leqq b = 2a < 1$,

$$h(b) \equiv g(a)$$

$$:= \sqrt{[2(1-b^2)]} \int_0^{\pi/2} \{(1-b\cos t)^{-1}(1+b\cos t)^{-1/2} + (1+b\cos t)^{-1}(1-b\cos t)^{-1/2}\}\, dt . \tag{18}$$

Since $0 \leqq b < 1$ we get the following expansion of absolutely convergent series

$$(1-b\cos t)^{-1}(1+b\cos t)^{-1/2}$$

$$= \sum_{k=0}^{\infty} (b\cos t)^k \sum_{m=0}^{\infty} \frac{(-1)^m (2m-1)!!}{(2m)!!} (b\cos t)^m$$

$$= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \frac{(-1)^k (2k-1)!!}{(2k)!!} \right) (b\cos t)^m .$$

From this and the same expansion for the second term of the integrand in (18) we obtain that

$$h(b) = \sqrt{[8(1-b^2)]} \int_0^{\pi/2} \sum_{m=0}^{\infty} S_m (b\cos t)^{2m} dt \tag{19}$$

$$= \sqrt{[8(1-b^2)]} \sum_{m=0}^{\infty} b^{2m} S_m \frac{(2m-1)!!}{(2m)!!} \frac{\pi}{2} ,$$

where $S_m = \sum_{k=0}^{2m} (-1)^k (2k-1)!!/(2k)!!$. It can be easily seen that

$$S_0 = 1 , \qquad S_m < 1 \quad (m \geqq 1). \tag{20}$$

And, from the fact that for each constant c, $0 < c \leqq 1$, the sequence $S_m(c) = \sum_{k=0}^{2m} ((2k-1)!!)/(2k)!!)(-c)^k$ is strictly decreasing and converges to $1/\sqrt{(1+c)}$ it follows that

$$1/\sqrt{2} < S_m \qquad (m \geqq 0). \tag{21}$$

From the fact that for each b, $0 \leqq b < 1$, $\sum_{m=0}^{\infty} ((2m-1)!!/(2m)!!) b^{2m} = 1/\sqrt{(1-b^2)}$ together with (19), (20) and (21) we see that

$$\pi < h(b) \leqq \sqrt{2}\pi \qquad \text{for} \quad 0 \leqq b < 1. \tag{22}$$

We shall now prove that $h(b)$ is strictly decreasing and continuous in b, $0 \leqq b < 1$. For each non-negative integer m we denote $(2m-1)!!S_m/(2m)!!$ by T_m and consider the function

$$g(x) = \sqrt{(1-x)} \cdot \sum_{m=0}^{\infty} T_m x^m , \qquad |x| < 1. \tag{23}$$

We notice that the series $\sum_{m=0}^{\infty} T_m x^m$ is absolutely convergent in x, $|x| < 1$, from which $g(x)$ is a C^∞ function of x and its derivative $g'(x)$ is given by

$$g'(x)=-1/2\sqrt{(1-x)}\cdot\sum_{m=0}^{\infty}T_m x^m+\sqrt{(1-x)}\cdot\sum_{m=0}^{\infty}mT_m x^{m-1}$$

$$=1/2\sqrt{(1-x)}\cdot\sum_{m=0}^{\infty}[2(m+1)T_{m+1}-(2m+1)T_m]x^m. \tag{24}$$

From the fact that $2(m+1)T_{m+1}-(2m+1)T_m=(2m+1)!!(S_{m+1}-S_m)/(2m)!!<0$ together with (24) we see that the function $g(x)$ is strictly decreasing in x, $0\leqq x<1$. From this together with (18), (19), (22) and (23) we see that our assertion is valid.

We shall review a distance on the set $\mathfrak{M}$ of all C^∞ Riemannian metrics on a closed n-manifold $\boldsymbol{M}$ (see [**6**] for detail) for proving Lemma 2 below. For each point x in $\boldsymbol{M}$, let $\boldsymbol{P}_x$ (resp. $\boldsymbol{S}_x$) be the set of all symmetric positive definite (resp. merely symmetric) bilinear forms on $\boldsymbol{T}_x\boldsymbol{M}\times\boldsymbol{T}_x\boldsymbol{M}$, where $\boldsymbol{T}_x\boldsymbol{M}$ is the tangent space of $\boldsymbol{M}$ at x. We can define a distance ρ_x on $\boldsymbol{P}_x$, $x\in\boldsymbol{M}$, by

$$\rho_x(\phi,\psi)=\inf\{\delta>0\ ;\ \exp(-\delta)\cdot\phi<\psi<\exp\delta\cdot\phi\},$$

where, for ϕ, ψ in $\boldsymbol{S}_x$, $\phi<\psi$ means that $\psi-\phi\in\boldsymbol{S}_x$ is positive definite on $\boldsymbol{T}_x\boldsymbol{M}\times\boldsymbol{T}_x\boldsymbol{M}$. And we can define a distance ρ on $\mathfrak{M}$ by

$$\rho(g,h)=\sup\{\rho_x(g_x,h_x)\ ;\ x\in\boldsymbol{M}\},\qquad g,h\in\mathfrak{M}.$$

For each g in $\mathfrak{M}$ we denote by $\lambda_m(g)$ the m-th eigenvalue of $(\boldsymbol{M},g)$ relative to the Laplacian Δ_g. Here the eigenvalues are counted repeatedly as many times as their multiplicities:

$$0=\lambda_0(g)<\lambda_1(g)\leqq\lambda_2(g)\leqq\cdots\leqq\lambda_m(g)\leqq\cdots\uparrow\infty.$$

S. Bando and H. Urakawa have proved the following result.

PROPOSITION 1. *Let $\boldsymbol{M}$ and $\mathfrak{M}$ be as above. Let g be in $\mathfrak{M}$ and δ a positive number. Then $h\in\mathfrak{M}$, $\rho(h,g)<\delta$ implies $|\lambda_m(h)-\lambda_m(g)|\leqq\{\exp((n+1)\delta)-1\}\lambda_m(g)$, for $m\geqq 0$.*

We shall use this proposition in the following situation. For each natural number k we may regard the closed 2-manifold $\boldsymbol{T}^2(k):=\boldsymbol{S}^1(k/2)\times\boldsymbol{S}^1$ with the Riemannian metric $\boldsymbol{I}_a=ds^2+(1/2+a\cos 2s)d\theta^2$, $|a|<1/2$, as the k-fold Riemannian covering manifold of the torus $\boldsymbol{S}^1(1/2)\times\boldsymbol{S}^1$ with the metric $\boldsymbol{I}_a$.

LEMMA 2. *Let $\boldsymbol{T}^2(k)$ and $\boldsymbol{I}_a$ be as above. There exists a constant δ, $0<\delta<1/2$, which is independent of k, such that if $|a|<\delta$ and $k\geqq 2$, then the first eigenvalue $\lambda_{1,k}(a)$ of $(\boldsymbol{T}^2(k),\boldsymbol{I}_a)$ relative to the Laplacian is smaller than two.*

PROOF. At first, it is known (see [**1**]) that the first eigenvalue of the Laplacian for the Riemannian product metric of $\boldsymbol{T}^2(k)$ is $4/k^2$, namely,

$$\lambda_{1,k}(0)=4/k^2\qquad\text{for}\quad k\geqq 2. \tag{25}$$

Next, we shall compute the distance $\rho(\boldsymbol{I}_a, \boldsymbol{I}_b)$, $a, b\in(-1/2, 1/2)$, explicitly. Let a, b be in $(-1/2, 1/2)$ and (s, θ) a point in $\boldsymbol{T}^2(k)$. Then it can be easily shown that at the point (s, θ) the condition that $\exp(-\delta)\boldsymbol{I}_a<\boldsymbol{I}_b<\exp\delta\cdot\boldsymbol{I}_a$ is equivalent to the condition that $|\log[(1+2b\cos 2s)/(1+2a\cos 2s)]|<\delta$. It follows from this fact that

$$\rho_{(s,\theta)}(\boldsymbol{I}_a, \boldsymbol{I}_b)=|\log[(1+2b\cos 2s)/(1+2a\cos 2s)]|. \tag{26}$$

From $\boldsymbol{S}^1(k/2)\equiv\boldsymbol{R}/k\pi\boldsymbol{Z}$ and (26) we see that

$$\begin{aligned}\rho(\boldsymbol{I}_a, \boldsymbol{I}_b)&:=\sup\{\rho_{(s,\theta)}(\boldsymbol{I}_a, \boldsymbol{I}_b)\ ;\ (s, \theta)\in\boldsymbol{T}^2(k)\}\\ &=\sup\{|\log[(1+2b)/(1+2a)]|,\ |\log[(1-2b)/(1-2a)]|\}.\end{aligned} \tag{27}$$

It follows from Proposition 1, (25) and (27) that there exists a constant δ, $0<\delta<1/2$, which is independent of k, such that

$$\lambda_{1,k}(a)<2 \qquad \text{for } a,\ |a|<\delta, \quad \text{and} \quad k\geqq 2.$$

This completes the proof.

PROOF OF THEOREM 2. From the minimality of $\boldsymbol{M}_a$ in $\boldsymbol{S}^3$ and the equation of Gauss it follows that at each point (s, θ) in $\boldsymbol{S}^1(r_a)\times\boldsymbol{S}^1$, the domain of definition of the immersion f, the Gaussian curvature $\boldsymbol{K}_a$ of $\boldsymbol{M}_a$ is

$$\boldsymbol{K}_a=4a(a\cos^2 2s+\cos 2s+a)(1+2a\cos 2s)^{-2}. \tag{28}$$

Using (28) we can easily show that the range of $\boldsymbol{K}_a$ is the closed interval $[-4a/(1-2a), 4a/(1+2a)]$ which implies that the second assertion of this theorem is true.

Next, we notice that

$$\phi(k\pi, a)=k\phi(\pi, a) \qquad \text{for } a,\ 0\leqq a<1/2, \quad k: \text{integer}. \tag{29}$$

From (29) and Lemma 1 we can easily show that $r_a=k/2$ for some integer $k\geqq 3$, or $r_a=\infty$, where r_a is defined to be as in Corollary. And it is easily seen that for such r_a, the mapping $\phi(\cdot, a): \boldsymbol{S}^1(r_a)\to\boldsymbol{R}$, $s\to\phi(s, a)$, is not one-to-one. This implies that the first assertion of this theorem is true.

PROOF OF THEOREM 3. From the proof of Theorem 2 we see that the closed, rotational, minimal surface $\boldsymbol{M}_a$ in $\boldsymbol{S}^3$ is isometric to $\boldsymbol{T}^2(k)=\boldsymbol{S}^1(k/2)\times\boldsymbol{S}^1$ with the Riemannian metric $\boldsymbol{I}_a=ds^2+(1/2+a\cos 2s)d\theta^2$ for some integer $k\geqq 3$. From this observation together with Lemmas 1 and 2 it follows that our assertion is true.

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