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THE FIRST EIGENVALUE OF THE LAPLACIAN ON TORI

KAZUMI TSUKADA

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1. Introduction. Let M be an n-dimensional compact connected differentiable manifold. For every Riemannian metric g on M, let Δ_g be the Laplacian acting on differentiable functions on M. We denote the first eigenvalue of Δ_g by $\lambda_1(g)$ and the volume of (M, g) by $\mathrm{Vol}(M, g)$. Berger [1] posed the following problem: *Does there exist a positive constant $k(M)$ such that*

$$(*) \qquad \lambda_1(g)\,\mathrm{Vol}(M, g)^{2/n} \leqq k(M)\ ,$$

for every Riemannian metric g on M? Hersch [4] showed that if M is diffeomorphic to the 2-dimensional sphere S^2, then for every Riemannian metric g on S^2,

$$\lambda_1(g)\,\mathrm{Vol}(S^2, g) \leqq 8\pi\ .$$

The equality holds if and only if g is a metric with the constant curvature.

On the other hand, recently the following people constructed examples which admit a family of Riemannian metrics $g(t)$ $(0 < t < \infty)$ such that

$$\begin{cases} \lambda_1(g(t))\,\mathrm{Vol}(M, g(t))^{2/n} \to \infty & \text{as} \quad t \to \infty \\ \lambda_1(g(t))\,\mathrm{Vol}(M, g(t))^{2/n} \to 0 & \text{as} \quad t \to 0\ . \end{cases}$$

(i) Urakawa [8] constructed such a family of metrics on a compact connected Lie group with a non-trivial commutator subgroup.

(ii) Tanno [7] constructed such on any odd dimensional sphere S^{2n+1} $(n \geqq 1)$.

(iii) Urakawa and Muto [10] constructed such on compact homogeneous spaces which satisfy some conditions.

(iv) Muto [5] constructed such on any even dimensional sphere S^{2n} $(n \geqq 2)$.

For an n-dimensional torus T^n, it is known that there exists a constant $k(T^n)$ such that $(*)$ holds for every "flat" metric (cf. [9]). In this paper we prove that there exists no constant $k(T^n)$ such that $(*)$ holds for any metric on T^n $(n \geqq 3)$. Namely we show the following.

THEOREM. *On any n-dimensional torus T^n $(n \geqq 3)$, there exists a family of metrics $g(t)$ $(0 < t < \infty)$ such that*

$$\begin{cases}\lambda_1(g(t)) \to \infty & as \quad t \to \infty \\ \lambda_1(g(t)) \to 0 & as \quad t \to 0\end{cases}$$

and $\mathrm{Vol}(T^n, g(t)) = constant$.

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2. Some formulas for a Riemannian submersion. In [6] O'Neill studied fundamental equations of a Riemannian submersion. We review some formulas in it which are useful in the sequel. Given a Riemannian submersion $\pi: M \to B$, we denote by $\mathscr{V}E$ (resp. $\mathscr{H}E$) a vertical part (resp. a horizontal part) of a vector field E on M. Following O'Neill, we define two tensor fields T and A for arbitrary vector fields E and F by

$$T_E F = \mathscr{H}\tilde{\nabla}_{\mathscr{V}E}\mathscr{V}F + \mathscr{V}\tilde{\nabla}_{\mathscr{V}E}\mathscr{H}F$$

and

$$A_E F = \mathscr{H}\tilde{\nabla}_{\mathscr{H}E}\mathscr{V}F + \mathscr{V}\tilde{\nabla}_{\mathscr{H}E}\mathscr{H}F$$

respectively, where we denote by $\tilde{\nabla}$ the Riemannian connection on M.

We review some formulas for the tensor field A which will be used in the sequel. The tensor field A is called an integrability tensor associated with the submersion.

DEFINITION. A basic vector field is a horizontal vector field X^* which is π-related to a vector field X on B, i.e., $\pi X^*_u = X_{\pi(u)}$ for all $u \in M$.

LEMMA 2.1. *Suppose X^* and Y^* are basic vector fields on M which are related to X and Y on B. Then*

(1) $\mathscr{H}([X^*, Y^*])$ *is basic and is π-related to* $[X, Y]$.

(2) $\mathscr{H}\tilde{\nabla}_{X^*}Y^*$ *is basic and π-related to $\nabla_X Y$ where ∇ is the Riemannian connection on B.*

LEMMA 2.2. *Let $\tilde{X}$ and $\tilde{Y}$ be horizontal vector fields on M. Then we have*

$$A_{\tilde{X}}\tilde{Y} = \mathscr{V}([\tilde{X}, \tilde{Y}])/2 .$$

The proof of these results is found in [6].

3. The Laplacian of a metric g on $M \times S^1$. In this section, in the same way as Vilms [11], we introduce a Riemannian metric g on a product manifold $M \times S^1$ and calculate its Laplacian Δ_g.

Let (M, h) be an n-dimensional ($n \geqq 2$) compact connected Riemannian manifold and ω be a 1-form on M. We denote $\boldsymbol{R}/2\pi\boldsymbol{Z}$ by S^1 and its coordinate system by $\{s\}$. We consider a product manifold $M \times S^1$ with natural projections $\pi: M \times S^1 \to M$ and $\eta: M \times S^1 \to S^1$. We define a Riemannian metric g on $M \times S^1$ by

$$g = \pi^* h + (\omega + ds) \otimes (\omega + ds) ,$$

where we simply denote $\pi^*\omega$ and $\eta^* ds$ by ω and ds, respectively. We remark that $(M \times S^1, g)$ may be regarded as a trivial S^1-bundle with a connection $\omega + ds$.

We denote by ζ the vector field d/ds which is naturally regarded as a vector field on $M \times S^1$. We denote by ξ a contravariant form of ω on M. We may naturally regard ξ as a vector field on $M \times S^1$. We denote by L_X the Lie derivation with respect to X. We consider the Laplacian Δ_M on (M, h) as a differential operator acting on differentiable functions on $M \times S^1$ in the following sense: For $\varphi \in C^\infty(M \times S^1)$, $\Delta_M \varphi(x, s) = \Delta_M \iota_s^* \varphi(x)$ at (x, s), where ι_s denotes the natural imbedding $\iota_s: M \to M \times S^1$ given by $\iota_s(x) = (x, s)$.

We easily get:

LEMMA 3.1. *The metric g on $M \times S^1$ has the following properties:*

(1) *The vector field ζ is a unit Killing vector field on $(M \times S^1, g)$.*

(2) *The projection π is a Riemannian submersion from $(M \times S^1, g)$ to (M, h) with totally geodesic fibres.*

PROPOSITION 3.2. *For $\varphi \in C^\infty(M \times S^1)$, we have*

$$\Delta_g \varphi = \Delta_M \varphi - (1 + |\omega|^2) L_\zeta L_\zeta \varphi + 2 L_\xi L_\zeta \varphi - (\delta\omega) L_\zeta \varphi ,$$

where we calculate the norm of ω and the co-differential operator δ with respect to the metric h.

PROOF. For an arbitrary point $x \in M$, let U be a neighborhood of x in M and $\{X_1, X_2, \cdots, X_n\}$ be a local field of orthonormal frames on U. We naturally regard X_j as a vector field on $U \times S^1$ and define a vector field X_j^* on $U \times S^1$ by $X_j^* = X_j - \omega(X_j)\zeta$. Then X_j^* is a basic vector field which is related to X_j. We easily see that $\{X_1^*, X_2^*, \cdots, X_n^*, \zeta\}$ is a local field of orthonormal frames on $U \times S^1$. By the definition of the Laplacian, for $\varphi \in C^\infty(M \times S^1)$ we have

$$-\Delta_g \varphi = \sum_{j=1}^n (X_j^* X_j^* \varphi - \tilde{\nabla}_{X_j^*} X_j^* \varphi) + \zeta\zeta\varphi - \tilde{\nabla}_\zeta \zeta \varphi \quad \text{on} \quad U \times S^1 .$$

We see that $\tilde{\nabla}_\zeta \zeta = 0$ since ζ is a unit Killing vector field. By Lemma 2.1 and Lemma 2.2 we have

$$\mathscr{V}(\tilde{\nabla}_{X_j^*}X_j^*) = A_{X_j^*}X_j^* = \mathscr{V}([X_j^*, X_j^*])/2 = 0$$
$$\mathscr{H}(\tilde{\nabla}_{X_j^*}X_j^*) = (\nabla_{X_j}X_j)^* = \nabla_{X_j}X_j - \omega(\nabla_{X_j}X_j)\zeta ,$$

where $\nabla_{X_j}X_j$ is regarded as a vector field on $U \times S^1$. Hence we get $\tilde{\nabla}_{X_j^*}X_j^* = \nabla_{X_j}X_j - \omega(\nabla_{X_j}X_j)\zeta$. Noticing that $[X_j, \zeta] = 0$ and $\zeta\omega(X_j) = 0$, we have

$$\begin{aligned} X_j^*X_j^*\varphi - \tilde{\nabla}_{X_j^*}X_j^*\varphi &= (X_j - \omega(X_j)\zeta)(X_j - \omega(X_j)\zeta)\varphi - (\nabla_{X_j}X_j - \omega(\nabla_{X_j}X_j)\zeta)\varphi \\ &= X_jX_j\varphi - X_j\omega(X_j)\cdot\zeta\varphi - \omega(X_j)X_j\zeta\varphi \\ &\quad - \omega(X_j)\zeta X_j\varphi + \omega(X_j)\zeta\cdot\omega(X_j)\cdot\zeta\varphi \\ &\quad + \omega(X_j)^2\zeta\zeta\varphi - (\nabla_{X_j}X_j)\varphi + \omega(\nabla_{X_j}X_j)\zeta\varphi \\ &= X_jX_j\varphi - (\nabla_{X_j}X_j)\varphi + \omega(X_j)^2\zeta\zeta\varphi \\ &\quad - \{X_j\omega(X_j) - \omega(\nabla_{X_j}X_j)\}\zeta\varphi - 2\omega(X_j)X_j\zeta\varphi . \end{aligned}$$

Therefore we have

$$\begin{aligned} -\Delta_g\varphi &= \sum_{j=1}^n (X_jX_j\varphi - \nabla_{X_j}X_j\varphi) + \Big(1 + \sum_{j=1}^n \omega(X_j)^2\Big)\zeta\zeta\varphi \\ &\quad - 2\sum_{j=1}^n \omega(X_j)X_j\zeta\varphi - \sum_{j=1}^n (X_j\omega(X_j) - \omega(\nabla_{X_j}X_j))\zeta\varphi \\ &= -\Delta_M\varphi + (1 + |\omega|^2)L_\zeta L_\zeta\varphi - 2L_\xi L_\zeta\varphi + \delta\omega L_\zeta\varphi . \end{aligned}$$

Following Tanno [7], we define a family of Riemannian metrics $g(t)$ $(0 < t < \infty)$ by

$$g(t) = t^{-1}g + (t^n - t^{-1})(\omega + ds) \otimes (\omega + ds) \qquad 0 < t < \infty .$$

By ${}^{(t)}\tilde{\nabla}$ and $\Delta_{g(t)}$, we denote the Riemannian connection and the Laplacian with respect to $g(t)$.

LEMMA 3.3. *$(M \times S^1, g(t))$ has the following properties.*

(1) *Volume elements with respect to $g(t)$ and $g(1) = g$ are identical; $dV_{g(t)} = dV_g$, and* $\mathrm{Vol}(M \times S^1, g(t)) = \mathrm{Vol}(M \times S^1, g)$.

(2) *The vector field ζ is a Killing vector field with constant length $t^{n/2}$.*

(3) *The projection π is a Riemannian submersion from $(M \times S^1, g(t))$ to $(M, t^{-1}h)$ with totally geodesic fibres.*

(4) *Horizontal distributions associated with the submersion π: $(M \times S^1, g(t)) \to (M, t^{-1}h)$ and the submersion π: $(M \times S^1, g) \to (M, h)$ are identical.*

(5) If *$\tilde{X}$ and $\tilde{Y}$ are horizontal vector fields, then we have ${}^{(t)}A_{\tilde{X}}\tilde{Y} = A_{\tilde{X}}\tilde{Y}$, where ${}^{(t)}A$ denotes the integrability tensor associated with the submersion π: $(M \times S^1, g(t)) \to (M, t^{-1}h)$.*

(6) *Suppose X^* and Y^* are basic vector fields which are related*

to X and Y. Then we get ${}^{(t)}\tilde{\nabla}_{X^*}Y^* = \tilde{\nabla}_{X^*}Y^*$.

PROOF. (1), (2), (3), and (4) are easily checked. (5) By Lemma 2.2, we get ${}^{(t)}A_{\tilde{X}}\tilde{Y} = \mathscr{V}([\tilde{X}, \tilde{Y}])/2 = A_{\tilde{X}}\tilde{Y}$. (6) By Lemma 2.1 and Lemma 2.2, we have $\mathscr{V}({}^{(t)}\tilde{\nabla}_{X^*}Y^*) = {}^{(t)}A_{X^*}Y^* = A_{X^*}Y^* = \mathscr{V}(\tilde{\nabla}_{X^*}Y^*)$, $\mathscr{H}({}^{(t)}\tilde{\nabla}_{X^*}Y^*) = ({}^{(t)}\nabla_X Y)^*$, where ${}^{(t)}\nabla$ denotes the Riemannian connection with respect to $(M, t^{-1}h)$. Since $(M, t^{-1}h)$ is a homothetic deformation of (M, h), ${}^{(t)}\nabla$ coincides with ∇. Therefore we have $({}^{(t)}\nabla_X Y)^* = (\nabla_X Y)^* = \mathscr{H}(\tilde{\nabla}_{X^*}Y^*)$. Hence we get (6).

As for the relation between Δ_g and $\Delta_{g(t)}$, we show the following.

PROPOSITION 3.4. *For* $\varphi \in C^\infty(M \times S^1)$, *we have* $\Delta_{g(t)}\varphi = t\Delta_g\varphi + (t - t^{-n})L_\zeta L_\zeta\varphi$.

PROOF. We use again a local frame field $\{X_1^*, \cdots, X_n^*, \zeta\}$ given in the proof of Proposition 3.2. By Lemma 3.3 (4), X_j^* is a basic vector field associated with the submersion $\pi: (M \times S^1, g(t)) \to (M, t^{-1}h)$. We easily see that $\{t^{1/2}X_1^*, \cdots, t^{1/2}X_n^*, t^{-n/2}\zeta\}$ is an orthonormal frame field on $U \times S^1$ with respect to the metric $g(t)$. Noticing that ${}^{(t)}\tilde{\nabla}_{X_j^*}X_j^* = \tilde{\nabla}_{X_j^*}X_j^*$, we have

$$\begin{aligned} -\Delta_{g(t)}\varphi &= \sum_{j=1}^n (t^{1/2}X_j^* t^{1/2}X_j^*\varphi - {}^{(t)}\tilde{\nabla}_{t^{1/2}X_j^*}t^{1/2}X_j^*\varphi) + t^{-n/2}\zeta t^{-n/2}\zeta\varphi \\ &= t\left\{\sum_{j=1}^n (X_j^*X_j^*\varphi - {}^{(t)}\tilde{\nabla}_{X_j^*}X_j^*\varphi) + \zeta\zeta\varphi\right\} - (t - t^{-n})\zeta\zeta\varphi \\ &= t\left\{\sum_{j=1}^n (X_j^*X_j^*\varphi - \tilde{\nabla}_{X_j^*}X_j^*\varphi) + \zeta\zeta\varphi\right\} - (t - t^{-n})\zeta\zeta\varphi \\ &= -t\Delta_g\varphi - (t - t^{-n})L_\zeta L_\zeta\varphi\,. \end{aligned}$$

4. **Proof of Theorem in the 3-dimensional case.** 4.1. *The Laplacian of warped product.* Ejiri [3] studied the Laplacian of a warped product. Here we review his results. Let (B, g) and (F, h) be Riemannian manifolds and f be a positive differentiable function on B. Consider the product manifold $B \times F$ with projections $\pi: B \times F \to B$ and $\eta: B \times F \to F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ furnished with the Riemannian structure $\bar{g}$ defined by

$$\bar{g}(X, Y) = g(\pi_* X, \pi_* Y) + f^2(\pi u)h(\eta_* X, \eta_* Y)$$

for tangent vectors $X, Y \in T_u M$. We denote by Δ_M, Δ_B, and Δ_F the Laplacians of $(M, \bar{g})$, (B, g) and (F, h), respectively. By grad f we denote the gradient of f defined by the metric tensor g and we regard grad f as a vector field on M. Ejiri found the following relation among Δ_B, Δ_F and Δ_M.

LEMMA 4.1. [3]

$$\Delta_M = \Delta_B - (n/f)\ \mathrm{grad}\, f + (1/f)^2 \Delta_F\ ,$$

where n is the dimension of F.

In this note we deal with a warped product $S^1 \times_f S^1$, where S^1 denotes $\boldsymbol{R}/2\pi\boldsymbol{Z}$.

COROLLARY 4.2.

$$\Delta_{S^1 \times_f S^1} = -\partial^2/\partial t^2 - (f'/f)(\partial/\partial t) - (1/f)^2 \partial^2/\partial u^2\ ,$$

where t (resp. u) is the coordinate for the first (resp. second) S^1 *and* $f' = df/dt$.

4.2. *A construction of a Riemannian metric on* T^3. We introduce a Riemannian metric g on T^3 as follows. We consider T^3 as $T^2 \times S^1$ and we apply the method in §3. We define (T^2, h) as the warped product $T^2 = S^1 \times_f S^1$, where f is a positive function on S^1. By S^1 we mean $\boldsymbol{R}/2\pi\boldsymbol{Z}$ and we use $\{t, u\}$ as the coordinate system on $T^2 = S^1 \times_f S^1$. Put $\xi = \partial/\partial u$. Then its dual 1-form on $S^1 \times_f S^1$ is $f^2 du$, which is denoted by ω. Following §3, we define a Riemannian structure g on $T^3 = T^2 \times S^1$ by $g = \pi^* h + (\omega + ds) \otimes (\omega + ds)$. Then the Riemannian metric is represented as

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f^2 + f^4 & f^2 \\ 0 & f^2 & 1 \end{pmatrix}$$

in terms of the coordinate system $\{t, u, s\}$.

Therefore we get:

LEMMA 4.3. *The volume element* dV_g *of* (T^3, g) *is given by* $dV_g = f dt \wedge du \wedge ds$.

Now we calculate the Laplacian of (T^3, g).

PROPOSITION 4.4.

$$\Delta_g = -\partial^2/\partial t^2 - (f'/f)(\partial/\partial t) - (1/f)^2 L_\xi L_\xi - (1 + f^2) L_\zeta L_\zeta + 2 L_\xi L_\zeta\ .$$

PROOF. It is easily checked that ξ is a Killing vector field. So we have $\delta\omega = \mathrm{div}\, \xi = 0$. Applying Proposition 3.2 and Corollary 4.2 we obtain Proposition 4.4 immediately.

4.3. *Eigenvalues and eigenfunctions of* (T^3, g). By $C^\infty(T^3)$ we denote the space of complex-valued differentiable functions on T^3. We define a scalar product on $C^\infty(T^3)$ by

$$\langle \varphi, \psi \rangle_1 = \int_{T^3} \varphi \bar{\psi} dV_g = \int_{T^3} \varphi \bar{\psi} f dt \wedge du \wedge ds \qquad \text{for} \quad \varphi, \psi \in C^\infty(T^3) .$$

On the other hand, we introduce on T^3 another Riemannian metric g_0 which is the natural Riemannian product on $S^1 \times S^1 \times S^1$. We define a scalar product with respect to g_0 by

$$\langle \varphi, \psi \rangle_0 = \int_{T^3} \varphi \bar{\psi} dV_{g_0} = \int_{T^3} \varphi \bar{\psi} dt \wedge du \wedge ds .$$

We denote the minimum of f and the maximum of f by m and M, respectively. Then we have $m \| \varphi \|_0^2 \leqq \| \varphi \|_1^2 \leqq M \| \varphi \|_0^2$ for $\varphi \in C^\infty(T^3)$, where as usual $\| \quad \|_0$ and $\| \quad \|_1$ denote the norms on $C^\infty(T^3)$ defined by $\langle \, , \rangle_0$ and $\langle \, , \rangle_1$, respectively. Therefore we get:

LEMMA 4.5. *If $\{\varphi_j\}_{j=1}^\infty$ is a complete basis for $(C^\infty(T^3), \langle \, , \rangle_1)$, then it is also a complete basis for $(C^\infty(T^3), \langle \, , \rangle_0)$, and vice versa.*

By $C^\infty(S^1)$, we denote the space of complex-valued differentiable functoins on S^1 with a scalar product $\langle \varphi, \psi \rangle = \int_{S^1} \varphi \bar{\psi} f dt$. For integers k and l, we define a differential operator acting on $C^\infty(S^1)$ by

$$L(k; l)\varphi = -d^2\varphi/dt^2 - (f'/f)(d\varphi/dt) + (l/f - kf)^2\varphi + k^2\varphi .$$

LEMMA 4.6. *$L(k; l)$ is a strongly elliptic self-adjoint operator acting on $C^\infty(S^1)$.*

PROOF. We will show that it is a self-adjoint operator. For $\varphi, \psi \in C^\infty(S^1)$, we have

$$\begin{aligned}
&\langle L(k; l)\varphi, \psi \rangle \\
&= \int_{S^1} f\{-d^2\varphi/dt^2 - (f'/f)(d\varphi/dt) + (l/f - kf)^2\varphi + k^2\varphi\}\bar{\psi} dt \\
&= \int_{S^1} \Big\{ -\frac{d}{dt}\Big(f \frac{d\varphi}{dt} \bar{\psi}\Big) + \frac{df}{dt}\frac{d\varphi}{dt}\bar{\psi} + f \frac{d\varphi}{dt}\frac{d\bar{\psi}}{dt} - \frac{df}{dt}\frac{d\varphi}{dt}\bar{\psi} \\
&\qquad + f(l/f - kf)^2 \varphi\bar{\psi} + fk^2\varphi\bar{\psi} \Big\} dt \\
&= \int_{S^1} \Big\{ f \frac{d\varphi}{dt}\frac{d\bar{\psi}}{dt} + f(l/f - kf)^2\varphi\bar{\psi} + fk^2\varphi\bar{\psi} \Big\} dt .
\end{aligned}$$

Similarly we have

$$\langle \varphi, L(k; l)\psi \rangle = \int_{S^1} \Big\{ f \frac{d\varphi}{dt}\frac{d\bar{\psi}}{dt} + f(l/f - kf)^2\varphi\bar{\psi} + fk^2\varphi\bar{\psi} \Big\} dt .$$

Let $\{\mu_1(k; l) \leqq \mu_2(k; l) \leqq \cdots\}$ be the eigenvalues of $L(k; l)$, and $\varphi_j(k; l)$ be the eigenfunction such that $L(k; l)\varphi_j(k; l) = \mu_j(k; l)\varphi_j(k; l)$. By Lemma

4.6, for each pair (k, l), $\{\varphi_j(k; l)\}_{j=1}^{\infty}$ is a complete basis of $C^\infty(S^1)$. As is well known, e^{iks} $(k \in \boldsymbol{Z})$ is an eigenfunction of $-d^2/ds^2$ on S^1. We write $\theta_k(s) = e^{iks}$ and $\psi_l(u) = e^{ilu}$ for $k, l \in \boldsymbol{Z}$.

LEMMA 4.7. *$\varphi_j(k; l)\psi_l\theta_k$ is an eigenfunction of Δ_g and its eigenvalue is $\mu_j(k; l)$:*

$$\Delta_g \varphi_j(k; l)\psi_l\theta_k = \mu_j(k; l)\varphi_j(k; l)\psi_l\theta_k .$$

PROOF. We see that $L_\xi\psi_l = il\psi_l$ and $L_\zeta\theta_k = ik\theta_k$. Applying Proposition 4.4, we obtain the result.

Next we have:

PROPOSITION 4.8. *$\{\varphi_j(k; l)\psi_l\theta_k,\ k, l \in \boldsymbol{Z},\ j = 1, 2, \cdots\}$ is a complete basis for $(C^\infty(T^3), \langle\ ,\ \rangle_1)$ and hence $\{\mu_j(k; l);\ k, l \in \boldsymbol{Z},\ j = 1, 2, \cdots\}$ is the spectrum of (T^3, g).*

PROOF. Let $u_h(t) = e^{iht}$, $h \in \boldsymbol{Z}$, be an eigenfunction of $-d^2/dt^2$ on S^1. Since for each (k, l), $\{\varphi_j(k; l)\}_{j=1}^{\infty}$ is a complete basis for $C^\infty(S^1)$, for u_h there exist $a_j \in \boldsymbol{C}$, $j = 1, 2, \cdots$, such that $\lim_{p\to\infty} \| u_h - \sum_{j=1}^{p} a_j\varphi_j(k; l)\| = 0$, where $\|\ \ \|$ denotes the norm on $C^\infty(S^1)$ defined by the scalar product $\langle\ ,\ \rangle$ with respect to the measure $f dt$. Therefore we have

$$\begin{aligned}
&\left\| u_h\psi_l\theta_k - \sum_{j=1}^{p} a_j\varphi_j(k; l)\psi_l\theta_k \right\|_1 \\
&= \left\| \left(u_h - \sum_{j=1}^{p} a_j\varphi_j(k; l)\right)\psi_l\theta_k \right\|_1 \\
&= \left\| u_h - \sum_{j=1}^{p} a_j\varphi_j(k; l) \right\| \left\{\int_{S^1} \psi_l\bar{\psi}_l du\right\}^{1/2} \left\{\int_{S^1} \theta_k\bar{\theta}_k ds\right\}^{1/2} ,
\end{aligned}$$

from which it follows that $\lim_{p\to\infty} \| u_h\psi_l\theta_k - \sum_{j=1}^{p} a_j\varphi_j(k; l)\psi_l\theta_k \|_1 = 0$, where $\|\ \ \|_1$ denotes the norm on $C^\infty(T^3)$ defined by $\langle\ ,\ \rangle_1$. On the other hand, it is well known that $\{u_h\psi_l\theta_k;\ h, l, k \in \boldsymbol{Z}\}$ is a complete basis for $(C^\infty(T^3), \langle\ ,\ \rangle_0)$ (cf. [2]). By Lemma 4.5 $\{u_h\psi_l\theta_k;\ h, l, k \in \boldsymbol{Z}\}$ is also a complete basis for $(C^\infty(T^3), \langle\ ,\ \rangle_1)$. The above arguments imply that $\{\varphi_j(k; l)\psi_l\theta_k;\ k, l \in \boldsymbol{Z}$ $j = 1, 2, \cdots\}$ is a complete basis for $(C^\infty(T^3), \langle\ ,\ \rangle_1)$.

4.4. *Estimates of eigenvalues of the operator $L(k; l)$.* In this part, making use of the minimum principle we estimate eigenvalues of $L(k; l)$ from below. First of all, we apply the minimum principle to the self-adjoint operator $L(k; l)$. Then we have

$$\begin{aligned}
\mu_1(k; l) &= \inf_{\varphi} \langle L(k; l)\varphi, \varphi\rangle / \langle\varphi, \varphi\rangle \\
&= \inf \int_{S^1} \{f\varphi'\bar{\varphi}' + f(l/f - kf)^2\varphi\bar{\varphi} + fk^2\varphi\bar{\varphi}\} dt \Big/ \int_{S^1} f\varphi\bar{\varphi} dt
\end{aligned}$$

$$= \inf \int_{S^1} \{f\varphi'\bar{\varphi}' + f(l/f - kf)^2\varphi\bar{\varphi}\}dt \Big/ \int_{S^1} f\varphi\bar{\varphi}dt + k^2 ,$$

where $\varphi' = d\varphi/dt$ and $\bar{\varphi}' = d\bar{\varphi}/dt$ and the infimum is taken over all non-zero φ in $C^\infty(S^1)$.

LEMMA 4.9. *If f is not constant on S^1 and at least one of k and l is not zero, then there exists a positive constant $\varepsilon > 0$ which does not depend on k and l such that*

$$\inf_\varphi \left\{\int_{S^1} f\varphi'\bar{\varphi}'dt + \int_{S^1} f(l/f - kf)^2\varphi\bar{\varphi}dt\right\} \Big/ \int_{S^1} f\varphi\bar{\varphi}dt \geqq \varepsilon ,$$

where the infimum is taken over all φ as above.

PROOF. Let m and M be the minimum and the maximum of f, respectively. In the proof of this lemma, for simplicity we omit S^1 in the integral sign. We have

$$(**) \qquad \left\{\int f\varphi'\bar{\varphi}'dt + \int f(l/f - kf)^2\varphi\bar{\varphi}dt\right\} \Big/ \int f\varphi\bar{\varphi}dt$$
$$\geqq \frac{m}{M}\left\{\int \varphi'\bar{\varphi}'dt + \frac{1}{Mm}\int (l - kf^2)^2\varphi\bar{\varphi}dt\right\} \Big/ \int \varphi\bar{\varphi}dt .$$

When $k = 0$, since l is not zero, we have

$$\frac{m}{M}\left\{\int \varphi'\bar{\varphi}'dt + \frac{1}{Mm}l^2\int \varphi\bar{\varphi}dt\right\} \Big/ \int \varphi\bar{\varphi}dt$$
$$\geqq \frac{m}{M}\left\{\int \varphi'\bar{\varphi}'dt + \frac{1}{Mm}\int \varphi\bar{\varphi}dt\right\} \Big/ \int \varphi\bar{\varphi}dt .$$

Let $\varepsilon_1 = \inf_\varphi\left\{\int \varphi'\bar{\varphi}'dt + (1/Mm)\int \varphi\bar{\varphi}dt\right\} \Big/ \int \varphi\bar{\varphi}dt$. ε_1 is positive. Then, in the case $k = 0$, we have

$$(**) \geqq m\varepsilon_1/M \qquad \text{for any} \quad \varphi \in C^\infty(S^1) , \quad \varphi \not\equiv 0 .$$

When $k \neq 0$, we have

$$(**) \geqq \frac{m}{M}\left\{\int \varphi'\bar{\varphi}'dt + \frac{k^2}{Mm}\int (f^2 - l/k)^2\varphi\bar{\varphi}dt\right\} \Big/ \int \varphi\bar{\varphi}dt$$
$$\geqq \frac{m}{M}\left\{\int \varphi'\bar{\varphi}'dt + \frac{1}{Mm}\int (f^2 - l/k)^2\varphi\bar{\varphi}dt\right\} \Big/ \int \varphi\bar{\varphi}dt .$$

We put $\alpha = (M^2 - m^2)/2$. Since f is not constant, α is positive. Let t_1 be a point which attains the maximum of f. Then there exists a positive number $\delta > 0$ such that $f^2(t) - \alpha > 0$ for $t \in (t_1 - \delta, t_1 + \delta)$. There exists a non-negative differentiable function g_1 such that $\mathrm{supp}(g_1) \subset (t_1 - \delta, t_1 + \delta)$,

$(f^2-\alpha)^2 \geqq g_1^2$ on S^1, and g_1 is not identically zero. Let t_2 be a point which attains the minimum of f. Then there exists a positive number $\delta' > 0$ such that $\alpha - f^2(t) > 0$ for $t \in (t_2 - \delta', t_2 + \delta')$. Similarly there exists a non-negative function g_2 on S^1 such that $\mathrm{supp}(g_2) \subset (t_2 - \delta', t_2 + \delta')$, $(\alpha - f^2)^2 \geqq g_2^2$ on S^1, and g_2 is not identically zero. If l/k is not greater than α, then we have $(f^2 - l/k)^2 \geqq g_1^2$. When l/k is not less than α, then we have $(f^2 - l/k)^2 \geqq g_2^2$. We put

$$\varepsilon_2 = \inf_{\varphi} \left\{\int \varphi'\bar{\varphi}'dt + \frac{1}{Mm}\int g_1^2\varphi\bar{\varphi}dt\right\} \Big/ \int \varphi\bar{\varphi}dt ,$$

and

$$\varepsilon_3 = \inf_{\varphi} \left\{\int \varphi'\bar{\varphi}'dt + \frac{1}{Mm}\int g_2^2\varphi\bar{\varphi}dt\right\} \Big/ \int \varphi\bar{\varphi}dt ,$$

where the infimum is taken over all non-zero φ in $C^\infty(S^1)$. Since g_1 and g_2 are not identically zero, we have $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$. Therefore, when $l/k \leqq \alpha$, we have

$$(**) \geqq m\varepsilon_2/M \qquad \text{for any} \quad \varphi \in C^\infty(S^1)\,, \quad \varphi \not\equiv 0\,.$$

Similarly, when $l/k \geqq \alpha$, we have

$$(**) \geqq m\varepsilon_3/M \qquad \text{for any} \quad \varphi \in C^\infty(S^1)\,, \quad \varphi \not\equiv 0\,.$$

By putting $\varepsilon =$ the minimum of $\{m\varepsilon_1/M, m\varepsilon_2/M, m\varepsilon_3/M\}$, we get Lemma 4.9.

When $k = l = 0$, we see that $\mu_1(0; 0) = 0$ and its eigenfunction is constant. Moreover, we have $\mu_2(0; 0) > 0$.

PROPOSITION 4.10. *Let $\tilde{\varepsilon}$ be the minimum of $\mu_2(0; 0)$ and ε in Lemma 4.9. We have $\mu_j(k; l) - k^2 \geqq \tilde{\varepsilon} > 0$ for any j when at least one of k and l is not zero, and for $j \geqq 2$ when $k = l = 0$.*

4.5. *Proof of Theorem in the 3-dimensional case.* Following §3, we define a family of Riemannian metrics on T^3 by

$$g(t) = t^{-1}g + (t^2 - t^{-1})(\omega + ds) \otimes (\omega + ds)\,, \qquad 0 < t < \infty\,.$$

By Proposition 3.4, we have $\Delta_{g(t)}\varphi = t\Delta_g\varphi + (t - t^{-2})L_\zeta L_\zeta\varphi$. We put $\Psi_{jkl} = \varphi_j(k; l)\psi_l\theta_k$, $k, l \in \boldsymbol{Z}$, $j = 1, 2, \cdots$. Then we have

$$\Delta_{g(t)}\Psi_{jkl} = \{t(\mu_j(k; l) - k^2) + t^{-2}k^2\}\Psi_{jkl}\,.$$

By Lemma 3.3 (1), $\{\Psi_{jkl}; k, l \in \boldsymbol{Z}, j = 1, 2, \cdots\}$ is a complete basis with respect to $g(t)$ for the space of differentiable functions on T^3. So $\{t(\mu_j(k; l) - k^2) + t^{-2}k^2; k, l \in \boldsymbol{Z}, j = 1, 2, \cdots\}$ is the spectrum of $(T^3, g(t))$.

When $k = l = 0$ and $j = 1$, $\Psi_{1,0,0}$ is a constant function with 0 as its eigenvalue. By Proposition 4.10, for a non-zero eigenvalue of $\Delta_{g(t)}$ we have $t(\mu_j(k; l) - k^2) + t^{-2}k^2 \geqq t\tilde{\varepsilon}$. Therefore we obtain $\lambda_1(g(t)) \to \infty$ as $t \to \infty$. We easily get $\lambda_1(g(t)) \to 0$ as $t \to 0$. On the other hand, by Lemma 3.3 (1) we see that $\mathrm{Vol}(T^3, g(t)) = \mathrm{Vol}(T^3, g)$. Thus Theorem in the 3-dimensional case is proved.

5. Proof of Theorem for T^n ($n \geqq 4$). When $n \geqq 5$, we can prove Theorem by following the same process as in the 4-dimensional case. So we will prove Theorem only in the 4-dimensional case in this section.

5.1. *A construction of a Riemannian metric g on T^4.* Following §3 again, we define a Riemannian metric g on T^4. We consider T^4 as a product manifold $T^3 \times S^1$ with the natural projection $\tilde{\pi}: T^3 \times S^1 \to T^3$. T^3 is furnished with the Riemannian metric g given in §4. By $\tilde{\omega}$ we denote the 1-form dual to the vector field ζ in (T^3, g). Then we have $\tilde{\omega} = \omega + ds$. We define $\tilde{g}$ on T^4 by

$$\tilde{g} = \tilde{\pi}^* g + (\tilde{\omega} + d\tilde{s}) \otimes (\tilde{\omega} + d\tilde{s}) \ ,$$

where $\{\tilde{s}\}$ is a normal coordinate system in S^1. By $\tilde{\zeta}$ we denote the vector field $\partial/\partial\tilde{s}$ in T^4.

Noticing that ζ is a unit Killing vector field in (T^3, g), by Proposition 3.2 we easily get:

PROPOSITION 5.1. *For $\varphi \in C^\infty(T^4)$ we have*

$$\Delta_{\tilde{g}}\varphi = \Delta_g\varphi - 2L_{\tilde{\zeta}}L_{\tilde{\zeta}}\varphi + 2L_{\zeta}L_{\tilde{\zeta}}\varphi \ .$$

Contrary to the arguments in §4, we denote by λ_j $(0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots)$ the j-th eigenvalue of (T^3, g) with its eigenspace $V(\lambda_j)$. Since ζ is a unit Killing vector field on (T^3, g), L_ζ and Δ_g commute, which implies that L_ζ is a linear transformation of $V(\lambda_j)$. By the results in §4, the following is clear.

LEMMA 5.2. *For each eigenvalue λ_j of Δ_g, $V(\lambda_j)$ has the orthogonal decomposition:*

$$V(\lambda_j) = \sum_{k \in \boldsymbol{Z}} V_k(\lambda_j) \ ,$$

where $L_\zeta\varphi = ik\varphi$ for $\varphi \in V_k(\lambda_j)$ $k \in \boldsymbol{Z}$. (Here we do not care if some $V_k(\lambda_j)$ is trivial or not.) Moreover, the above decomposition has the following property. If λ_j is not zero and $V_k(\lambda_j)$ is not trivial, then there exists a positive number $\tilde{\varepsilon} > 0$ such that $\lambda_j - k^2 \geqq \tilde{\varepsilon}$ and $\tilde{\varepsilon}$ does not depend on j and k.

By ϕ_h, $h \in \boldsymbol{Z}$, we denote an eigenfunction $e^{ih\tilde{s}}$ on S^1. By Proposition 5.1, we have:

PROPOSITION 5.3. *If $V_k(\lambda_j)$ is not trivial, $\varphi\phi_h$ is an eigenfunction of $(T^4, \tilde{g})$ with its eigenvalue $\lambda_j + 2h^2 - 2kh$ for $\varphi \in V_k(\lambda_j)$ and ϕ_h. The set of eigenfunctions of this form is a complete basis for $C^\infty(T^4)$ with respect to $\tilde{g}$.*

5.2. *Proof of Theorem.* Following §3, we define a family of Riemannian metrics $\tilde{g}(t)$ by

$$\tilde{g}(t) = t^{-1}\tilde{g} + (t^3 - t^{-1})(\tilde{\omega} + d\tilde{s}) \otimes (\tilde{\omega} + d\tilde{s}) \qquad 0 < t < \infty .$$

By Proposition 3.4, we have $\Delta_{\tilde{g}(t)}\Phi = t\Delta_{\tilde{g}}\Phi + (t - t^{-3})L_{\tilde{\xi}}L_{\tilde{\xi}}\Phi$. Then we have $\Delta_{\tilde{g}(t)}\varphi\phi_h = \{t(\lambda_j + h^2 - 2kh) + t^{-3}h^2\}\varphi\phi_h$. By the same arguments as in §4, each eigenvalue of $(T^4, \tilde{g}(t))$ has the above form, i.e., $t(\lambda_j + h^2 - 2kh) + t^{-3}h^2$. If $j \neq 0$, by Lemma 5.2 we have

$$t(\lambda_j + h^2 - 2kh) + t^{-3}h^2 \geqq t(k^2 + \tilde{\varepsilon} + h^2 - 2kh) \geqq t(\tilde{\varepsilon} + (k - h)^2) \geqq t\tilde{\varepsilon} .$$

If $j = 0$, then $k = 0$ and we have $t(\lambda_0 + h^2 - 2kh) + t^{-3}h^2 = th^2 + t^{-3}h^2$. Therefore for every positive eigenvalue λ of $(T^4, \tilde{g}(t))$ we have $\lambda \geqq t\varepsilon'$, where ε' is the minimum of 1 and $\tilde{\varepsilon}$. So we have $\lambda_1(\tilde{g}(t)) \to \infty$ as $t \to \infty$. We easily get $\lambda_1(\tilde{g}(t)) \to 0$ as $t \to 0$. On the other hand, by Lemma 3.3 (1) we see that $\mathrm{Vol}(T^4, \tilde{g}(t)) = \mathrm{Vol}(T^4, \tilde{g})$.

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DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
SETAGAYA, TOKYO 158
JAPAN