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THE FIRST EIGENVALUE OF THE LAPLACIAN ON TWO DIMENSIONAL RIEMANNIAN MANIFOLDS

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1. **Introduction.** Let M be a two dimensional compact Riemannian manifold without boundary. Let w be a fixed point on M. For any sufficiently small $\varepsilon > 0$, let B_ε be the geodesic disk of radius ε with the center w. We put $M_\varepsilon = M \setminus \bar{B}_\varepsilon$. Let $\lambda_1(\varepsilon)$ be the first positive eigenvalue of the Laplacian $\Delta = -\operatorname{div} \operatorname{grad}$ in M_ε under the Dirichlet condition on ∂B_ε.

The main result of this paper is the following:

THEOREM 1. *Assume* $n = 2$. *Then*

$$\lambda_1(\varepsilon) = -2\pi |M|^{-1} (\log \varepsilon)^{-1} + O((\log \varepsilon)^{-2}) \tag{1.1}$$

holds as ε *tends to zero. Here* $|M|$ *denotes the area of* M.

Chavel-Feldman [3] showed that $\lambda_1(\varepsilon) \to 0$ as ε tends to zero. Theorem 1 improves their result for the case $n = 2$. The readers may also refer to Matsuzawa-Tanno [5] where the case $M = (S^2$, the standard metric) was studied.

In §2, we give the Schiffer-Spencer variational formula for the resolvent kernels of the Laplacian with the Dirichlet condition on the boundary. For the Schiffer-Spencer formula, the reader may refer to Schiffer-Spencer [6] and Ozawa [7]. In [7], the author gave an asymptotic formula for the j-th eigenvalue of the Laplacian when we cut off a small ball of radius ε from a given bounded domain in $\boldsymbol{R}^n$ $(n = 2, 3)$. In §3, we prove Theorem 1. In §4, we make a remark on the inequality of Cheeger.

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2. **A variant of the Schiffer-Spencer formula.** Let $L^2(M)$ (resp. $L^2(M_\varepsilon)$) denote the Hilbert space of square integrable functions on M (resp. M_ε). By A we denote the self-adjoint operator in $L^2(M)$ associated with the Laplacian on M. Let $A(\varepsilon)$ denote the self-adjoint operator in $L^2(M_\varepsilon)$ associated with the Laplacian in M_ε under the Dirichlet condition

on ∂M_ε.

Let $K_\varepsilon(x, y)$ be the integral kernel function of the operator $(A(\varepsilon) + 1)^{-1}$ satisfying

$$K_\varepsilon(x, y) = 0 \qquad x \in M_\varepsilon,\ y \in \partial M_\varepsilon ,$$

and

$$\int_{M_\varepsilon} K_\varepsilon(x, y) \cdot (\Delta_y + 1)\varphi(y) *_y 1 = \varphi(x)$$

for any fixed $x \in M_\varepsilon$ and for $\varphi \in \mathscr{C}_0^\infty(M_\varepsilon)$. Here $*_y 1$ denotes the volume element. Let $K(x, y)$ be the integral kernel of the operator $(A + 1)^{-1}$ satisfying

$$\int_M K(x, y) \cdot (\Delta_y + 1)\psi(y) *_y 1 = \psi(x)$$

for any fixed $x \in M$ and for $\psi \in \mathscr{C}^\infty(M)$.

In this section we give the following proposition which is a variant of the formula in [6, p. 290].

PROPOSITION 1. *Let M and w be as above. Then, for any fixed $x, y \in M \setminus \{w\}$*

(2.1) $\quad K_\varepsilon(x, y) - K(x, y) = (2\pi)(\log \varepsilon)^{-1} K(x, w)K(y, w) + O((\log \varepsilon)^{-2})$

holds as ε tends to zero.

REMARK. It should be remarked that the remainder term $O((\log \varepsilon)^{-2})$ in (2.1) is not uniform with respect to x, y even if w is fixed. As for further generalizations of the formula (2.1), we refer the reader to [7], [8]. See also [9].

PROOF OF PROPOSITION 1. Let $d(x, w)$ denote the distance between x and w. Then it is easy to see that $K(x, w) + (2\pi)^{-1} \log d(x, w)$ is continuously differentiable with respect to x all over M. Put

$$\lim_{x \to w} (K(x, w) + (2\pi)^{-1} \log d(x, w)) = C_w ,$$

and

$$q(x, w) = K(x, w) + (2\pi)^{-1} \log d(x, w) - C_w .$$

Then there exists $C' > 0$ independent of x such that

$$|q(x, w)| \leqq C' d(x, w) \tag{2.2}$$

holds. Let

$$L_\varepsilon(x, y) = K_\varepsilon(x, y) - K(x, y) - 2\pi(-2\pi C_w + \log \varepsilon)^{-1} K(x, w)K(y, w) .$$

Then $L_\varepsilon(x, y) \in \mathscr{C}^\infty(M_\varepsilon \times M_\varepsilon)$,

$$(\Delta_y + 1)L_\varepsilon(x, y) = 0 \qquad x, y \in M_\varepsilon , \tag{2.3}$$

and

$$L_\varepsilon(x, y)|_{y\in\partial M_\varepsilon} = -K(x, y)|_{y\in\partial M_\varepsilon} + K(x, w)(1 + p(y, w))|_{y\in\partial M_\varepsilon} , \tag{2.4}$$

where

$$p(y, w) = -2\pi q(y, w)(-2\pi C_w + \log\varepsilon)^{-1} .$$

From (2.2), (2.4), it follows that

$$\max_{y\in\partial M_\varepsilon} |L_\varepsilon(x, y)| \leqq C(x)\varepsilon$$

as ε tends to zero, where $C(x)$ denotes a continuous function of $x \in \Omega \setminus w$. Applying now the Hopf maximum principle to the solution $L_\varepsilon(x, y)$ of the elliptic equation (2.3), we get

$$\max_{y\in\partial M_\varepsilon} |L_\varepsilon(x, y)| \leqq C(x)\varepsilon ,$$

which implies the desired result. q.e.d.

3. Proof of Theorem 1. We put

$$h_\varepsilon(x, y) = K(x, y) + (2\pi)(-2\pi C_w + \log\varepsilon)^{-1}K(x, w)K(y, w) .$$

Let F_ε be the bounded linear operator in $L^2(M_\varepsilon)$ defined by

$$(F_\varepsilon f)(x) = \int_{M_\varepsilon} h_\varepsilon(x, y)f(y) *_y 1$$

for any $f \in L^2(M_\varepsilon)$.

Let $\|T\|_{2,\varepsilon}$ denote the operator norm of a bounded operator T in $L^2(M_\varepsilon)$. We have the following:

LEMMA 1. *There exists a positive constant C independent of ε such that*

$$\|F_\varepsilon - (A(\varepsilon) + 1)^{-1}\|_{2,\varepsilon} \leqq C\varepsilon|\log\varepsilon|^{1/2} \tag{3.1}$$

holds for any sufficiently small $\varepsilon > 0$.

PROOF. We put $Q_\varepsilon = F_\varepsilon - (A(\varepsilon) + 1)^{-1}$. Q_ε has the integral kernel $-L_\varepsilon(x, y)$. Thus (2.3) implies that $Q_\varepsilon f$ satisfies the following:

$$(\Delta_x + 1)(Q_\varepsilon f)(x) = 0 \qquad x \in M_\varepsilon . \tag{3.2}$$

In view of (2.4) and $K_\varepsilon(x, y) = 0$ for $x \in \partial M_\varepsilon$, there exists a constant E independent of ε such that

$$\max_{x\in\partial M_\varepsilon} |Q_\varepsilon f(x)|$$
$$\leqq \max_{x\in\partial M_\varepsilon} \int_{M_\varepsilon} |K(x, y) - K(y, w)||f(y)| *_y 1 + E\varepsilon \int_{M_\varepsilon} |K(y, w)f(y)| *_y 1 .$$

By Schwarz's inequality we get

$$(3.3)\qquad \max_{x\in\partial M_\varepsilon} |Q_\varepsilon f(x)| \leqq (|I(\varepsilon)| + C'E\varepsilon)\|f\|_{2,\varepsilon}$$

for some constant C' independent of ε, where $\|f\|_{2,\varepsilon}$ denotes the $L^2(M_\varepsilon)$ norm of f and

$$I(\varepsilon)^2 = \max_{x\in\partial M_\varepsilon}\int_{M_\varepsilon} |K(x, y) - K(y, w)|^2 *_y 1 .$$

We now claim

$$(3.4)\qquad |I(\varepsilon)| \leqq C''\varepsilon|\log\varepsilon|^{1/2} ,$$

with a constant C'' independent of ε. Once this is proved, then the Hopf maximum principle gives us

$$\max_{x\in M_\varepsilon} |Q_\varepsilon f(x)| \leqq 2C''\varepsilon|\log\varepsilon|^{1/2} ,$$

which implies (3.1).

We now show (3.4). Let r be a small positive number so that there exists a diffeomorphism $\Psi\colon \bar{B}_r \cong \bar{D}_1$, where D_s is the disk in $\boldsymbol{R}^2$ defined by $D_s = \{x\in\boldsymbol{R}^2; |x| < s\}$. We may assume that

$$(3.5)\qquad \varepsilon < |\Psi(x)| < 2\varepsilon$$

for any $x\in\partial M_\varepsilon$ provided ε $(< r)$ is sufficiently small. We have $|I(\varepsilon)| \leqq |I_1(\varepsilon)| + |I_2(\varepsilon)| + |I_3(\varepsilon)|$, where

$$(3.6)\qquad I_1(\varepsilon)^2 = \max_{x\in\partial M_\varepsilon}\int_{M\setminus B_r} |K(x, y) - K(y, w)|^2 *_y 1 ,$$

$$(3.7)\qquad I_2(\varepsilon)^2 = \max_{x\in\partial M_\varepsilon}\int_{B_r\setminus\bar{B}_\varepsilon} (K(x, y) + 2\pi\log d(x, y) - (K(x, w) + 2\pi\log d(x, w)))^2 *_y 1$$

and

$$(3.8)\qquad I_3(\varepsilon)^2 = (2\pi)^2\max_{x\in\partial M_\varepsilon}\int_{B_r\setminus\bar{B}_\varepsilon} |\log d(x, y) - \log d(x, w)|^2 *_y 1 .$$

It is easy to see that $I_1(\varepsilon) = O(\varepsilon)$ as ε tends to zero. Since we have $K(x, y) + 2\pi\log d(x, y)\in\mathscr{C}^\infty(\partial M_\varepsilon\times B_r)$, we also have $I_2(\varepsilon) = O(\varepsilon)$. (3.4) then follows from

$$(3.9)\qquad I_3(\varepsilon) = O(\varepsilon|\log\varepsilon|^{1/2}) ,$$

which we shall prove below.

By a change of coordinates using the diffeomorphism Ψ, (3.8) is majorized by

$$(3.10)\qquad C\max_{x\in D_{2\varepsilon}\setminus\bar{D}_\varepsilon}\int_{D_1\setminus D_\varepsilon} (\log|x - y| - \log|y|)^2 dy ,$$

with a constant C independent of ε. Here we used (3.5). It is easy to see that

$$\int_{D_1 \setminus D_\varepsilon} (\log|x-y| - \log|y|)^2 dy = \frac{1}{4}\int_0^{2\pi} d\theta \int_\varepsilon^1 (\log((|x|^2 + r^2 - 2|x| r\cos\theta)/r^2)^2) r dr . \tag{3.11}$$

By changing further the variable $r = r^{-1}|x| = \eta$, the term (3.11) is transformed into the following:

$$\frac{1}{4}|x|^2 \int_0^{2\pi} d\theta \int_{|x|}^{|x|/\varepsilon} (\log(1+\eta^2 - 2\eta\cos\theta))^2 \eta^{-3} d\eta .$$

We here have

$$(\log(1+\eta^2 - 2\eta\cos\theta))^2 \leqq \max((\log|1-\eta|)^2, (\log|1+\eta|)^2)$$

for any $0 \leqq \theta \leqq 2\pi$, $\eta \in [|x|, |x|/\varepsilon]$. Hence the term (3.11) is $O(\varepsilon^2|\log \varepsilon|)$. We thus get (3.9), and thus (3.4). q.e.d.

We consider the following equations:

$$((A+1)^{-1} - 1)\xi(x) = |M|^{-1/2}(K(x, w) - |M|^{-1}) \tag{3.12}$$

$$\int_M \xi(x) *_x 1 = 0 . \tag{3.13}$$

Since

$$\int_M K(x, w) *_x 1 = 1 ,$$

the right hand side of (3.12) is orthogonal to 1 in $L^2(M)$, while it is easy to see that the kernel of $(A+1)^{-1} - 1$ is spanned by 1. Therefore, the unique solution ξ of (3.12), (3.13) exists in $L^2(M)$.

Let $\tilde{F}_\varepsilon$ be the linear operator defined by

$$(\tilde{F}_\varepsilon g)(x) = \int_M h_\varepsilon(x, y) g(y) *_y 1 . \tag{3.14}$$

Then $\tilde{F}_\varepsilon$ is a compact self-adjoint operator in $L^2(M)$, since $(A+1)^{-1}$ is a compact linear mapping from $L^2(M)$ to $\mathscr{C}^0(M)$. We have the following:

LEMMA 2. *If we put* $\tilde{\mu}(\varepsilon) = 1 + 2\pi(-2\pi C_w + \log \varepsilon)^{-1}|M|^{-1}$ *and* $\tilde{\varphi}_\varepsilon(x) = |M|^{-1/2} - 2\pi(-2\pi C_w + \log\varepsilon)^{-1}\xi(x)$, *then*

$$\|(\tilde{F}_\varepsilon - \tilde{\mu}(\varepsilon))\tilde{\varphi}_\varepsilon\|_{L^2(M)} \leqq C(\log\varepsilon)^{-2} \tag{3.15}$$

holds as ε *tends to zero. Here* C *is a constant independent of* ε.

PROOF. We have

$$(3.16)\qquad ((\tilde{F}_\varepsilon - \tilde{\mu}(\varepsilon))\tilde{\varphi}_\varepsilon)(x) = 4\pi^2(-2\pi C_w + \log\varepsilon)^{-2}(|M|^{-1}\xi(x) - K(x, w)((A + 1)^{-1}\xi)(w)) .$$

Since $\xi \in L^2(M)$ and $(A + 1)^{-1}\xi \in \mathscr{C}^0(M)$, we see that the $L^2(M)$ norm of (3.16) is $O((\log\varepsilon)^{-2})$. q.e.d.

Let $\chi_\varepsilon(x)$ be the characteristic function of M_ε. Now we want to prove the following:

$$(3.17)\qquad \|\tilde{F}_\varepsilon\tilde{\varphi}_\varepsilon - F_\varepsilon(\chi_\varepsilon\tilde{\varphi}_\varepsilon)\|_{2,\varepsilon} \leqq C\varepsilon|\log\varepsilon| ,$$

where C is a constant independent of ε. We put $v_\varepsilon(x) = (\tilde{F}_\varepsilon\tilde{\varphi}_\varepsilon)(x) - (F_\varepsilon(\chi_\varepsilon\tilde{\varphi}_\varepsilon))(x)$ for $x \in M_\varepsilon$. Then,

$$v_\varepsilon(x) = \int_{B_\varepsilon} h_\varepsilon(x, y)\tilde{\varphi}_\varepsilon(y) *_y 1 .$$

Also

$$(3.18)\qquad (-\Delta + 1)v_\varepsilon(x) = 0 \qquad x \in M_\varepsilon$$

and

$$(3.19)\qquad |v_\varepsilon(x)|_{x\in\partial M_\varepsilon} \leqq \left(\int_{B_\varepsilon} h_\varepsilon(x, y)^2 *_y 1\right)^{1/2}\Big|_{x\in\partial M_\varepsilon} \|\tilde{\varphi}_\varepsilon\|_{L^2(B_\varepsilon)} .$$

Since $|h_\varepsilon(x, y)| \leqq C|\log|x - y||$ for some constant C independent of ε, we get

$$\max_{x\in\partial M_\varepsilon} |v_\varepsilon(x)| \leqq C'\varepsilon|\log\varepsilon| \|\tilde{\varphi}_\varepsilon\|_{L^2(B_\varepsilon)} .$$

Here C' is a constant independent of ε. By the Hopf maximum principle we obtain (3.17).

By (3.1), (3.15) and (3.17), we get the following:

LEMMA 3. *There exists a constant C independent of ε such that*

$$(3.20)\qquad \|((A(\varepsilon) + 1)^{-1} - \tilde{\mu}(\varepsilon))(\chi_\varepsilon\tilde{\varphi}_\varepsilon)\|_{2,\varepsilon} \leqq C(\log\varepsilon)^{-2}$$

holds as ε tends to zero. Also $\|\chi_\varepsilon\tilde{\varphi}_\varepsilon\|_{2,\varepsilon} > 1/2$ holds for any sufficiently small ε.

We use the following:

LEMMA 4. *Let Y be a complex Hilbert space. Let T be a compact self-adjoint operator in Y. We fix $\tau \in \boldsymbol{R}\setminus\{0\}$ and $\delta > 0$. Assume that there exists $\psi \in Y$ satisfying $\|\psi\| > 1/2$ and $\|T\psi - \tau\psi\| < \delta$. Then there exists at least one eigenvalue τ^* of T which satisfies $|\tau^* - \tau| \leqq 2\delta$.*

PROOF. If the set $\{\tilde{\tau}; |\tilde{\tau} - \tau| \leqq 2\delta\}$ does not contain any eigenvalue, then $\|(T - \tau)^{-1}\| < 1/2\delta$. However, this leads to a contradiction

$$1/2 < \|(T-\tau)^{-1}(T\psi - \tau\psi)\| \leqq 1/2 .$$

q.e.d.

By Lemmas 3 and 4, we have the following: There exists at least one eigenvalue $\hat{\mu}(\varepsilon)$ of $(A(\varepsilon)+1)^{-1}$ satisfying

$$|\hat{\mu}(\varepsilon) - \tilde{\mu}(\varepsilon)| \leqq C'(\log \varepsilon)^{-2} , \tag{3.21}$$

where C' is a constant independent of ε.

Let $\lambda_2(\varepsilon)$ be the second positive eigenvalue of the Laplacian in M with the Dirichlet condition on ∂B_ε. By the Courant-Fischer mini-max principle for eigenvalues, we have

$$\lambda_2(\varepsilon) \geqq \lambda_1 > 0 , \tag{3.22}$$

where λ_1 denotes the first positive eigenvalue of the Laplacian on M. Therefore, by (3.20) and (3.21) we see that

$$(\lambda_1(\varepsilon)+1)^{-1} = \hat{\mu}(\varepsilon) .$$

Now the proof of Theorem 1 is complete.

4. **A remark on Cheeger's inequality.** Let N be an n-dimensional compact Riemannian manifold with smooth boundary $\partial N \neq \emptyset$. Let $\lambda_1(N)$ be the first positive eigenvalue of the Laplacian under the Dirichlet condition on ∂N. Then the inequality of Cheeger asserts that

$$\lambda_1(N) \geqq h_D(N)^2/4 , \tag{4.1}$$

where

$$h_D(N) = \inf_Z V_{n-1}(\partial Z)/V_n(Z) . \tag{4.2}$$

Here Z runs through all compact n-dimensional bordered submanifolds of N satisfying $Z \cap \partial N = \emptyset$, and $V_{n-1}(\partial Z)$ and $V_n(Z)$ denote the $(n-1)$-dimensional volume of ∂Z and the n-dimensional volume of Z, respectively. Cheeger gave (4.1) in [4] and also treated the case $\partial N = \emptyset$. In that case $h_D(N)$ should be replaced with another geometric quantity. See also Berger-Gauduchon-Mazet [1] and Buser [2]. It is well known that Cheeger' inequality is sharp, that is, we cannot replace the constant 1/4 with any larger number for general N. See, for example, Buser [2].

If we apply Cheeger's inequality to the manifold M_ε, we get a lower bound for $\lambda_1(\varepsilon)$. Since $n=2$, it is easy to see that there exists a constant $C_o > 1$ such that

$$C_o^{-1}\varepsilon < h_D(M_\varepsilon) < C_o\varepsilon \tag{4.3}$$

holds for any sufficiently small $\varepsilon > 0$. Then by Cheeger's inequality we get

$$\lambda_1(\varepsilon) > C_o^{-2}\varepsilon^2/4 \ . \tag{4.4}$$

Since we have (1.1), (4.4) does not give a good lower bound for $\lambda_1(\varepsilon)$ when ε is sufficiently small. Hence the following question arises: Can we replace the right hand side of (4.1) with another geometric quantity which will give a good bound for $\lambda_1(N)$ from below when the boundary ∂N is sufficiently small?

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