THE UNIVERSITY OF CHICAGO

THE FIRST LAW OF BLACK HOLE MECHANICS FOR FIELDS WITH INTERNAL GAUGE FREEDOM

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To my parents and teachers

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ABSTRACT

We derive the first law of black hole mechanics for physical theories based on a local, covariant and gauge-invariant Lagrangian where the dynamical fields transform non-trivially under the action of some internal gauge transformations. The theories of interest include General Relativity formulated in terms of tetrads, Einstein-Yang-Mills theory and Einstein-Dirac theory. Since the dynamical fields of these theories have some internal gauge freedom, we argue that there is no natural group action of diffeomorphisms of spacetime on such dynamical fields. In general, such fields cannot even be represented as smooth, globally well-defined tensor fields on spacetime. Consequently the derivation of the first law by Iyer and Wald cannot be used directly. Nevertheless, we show how such theories can be formulated on a principal bundle and that there is a natural action of automorphisms of the bundle on the fields. These bundle automorphisms encode both spacetime diffeomorphisms and internal gauge transformations. Using this reformulation we define the Noether charge associated to an infinitesimal automorphism and the corresponding notion of stationarity and axisymmetry of the dynamical fields. We first show that we can define certain potentials and charges at the horizon of a black hole so that the potentials are constant on the bifurcate Killing horizon, giving a generalised zeroth law for bifurcate Killing horizons. We further identify the gravitational potential and perturbed charge as the temperature and perturbed entropy of the black hole which gives an explicit formula for the perturbed entropy analogous to the Wald entropy formula. We then obtain a general first law of black hole mechanics for such theories. The first law relates the perturbed Hamiltonians at spatial infinity and the horizon, and the horizon contributions take the form of a "potential times perturbed charge" term. We also comment on the ambiguities in defining a prescription for the total entropy for black holes.

CHAPTER 1

INTRODUCTION

In [1] Lee and Wald provided a construction of the phase space, Noether current and Noether charge for Lagrangian field theories where the dynamical fields can be viewed as tensor fields on spacetime (or more generally maps from spacetime into another finite-dimensional manifold). Using this construction, Iyer and Wald [2, 3] have given a derivation of the first law of black hole mechanics for arbitrary (in particular non-stationary) perturbations off a stationary axisymmetric black hole in any diffeomorphism covariant theory of gravity where the gravitational dynamical field is a Lorentzian metric and the matter fields are smooth tensor fields on the spacetime manifold. They also identified the corresponding entropy as the integral over a horizon cross-section of the Noether charge associated with the action of diffeomorphisms generated by the horizon Killing field. This formulation of the Noether charges and first law has been useful in analysing solutions to Einstein-matter theories [4, 5, 6, 7, 8, 9], and stability of black holes [10, 11] and perfect fluid stars [12].

Even though the results of [1, 2, 3] encompass a wide variety of theories, there are situations of physical interest where their analysis cannot be directly applied. In particular, we would like to derive a first law of black hole mechanics for gravity formulated in terms of orthonormal coframes (i.e. vielbeins), Einstein-Yang-Mills theory and Einstein-Dirac theory. In all these cases the dynamical fields of the theory have some internal gauge freedom under the action of a group. As we will discuss below this internal gauge freedom is the main reason we cannot directly use the results of [1, 2, 3] to derive a first law. Since all the "fundamental" fields in the Standard Model of particle physics have such gauge freedom, it is of interest to formulate the first law when such dynamical fields are present in the theory. The main obstructions to using the formalism of [1, 2, 3] for charged fields are as follows.

The first obstruction is that fields with internal gauge transformations cannot, in general, be represented as globally smooth tensor fields on spacetime. A typical example from

Maxwell electrodynamics is when the source is a magnetic monopole. In the presence of a magnetic monopole the Maxwell gauge field (or vector potential) A_μ cannot be chosen to be smooth everywhere (in any choice of gauge; see Problem 2. § Vbis [13]). As is well-known, this "singularity" in the Maxwell gauge field is an artefact of trying to make a global choice of gauge. Even in the absence of monopoles, the most "convenient" choice of gauge might make the Maxwell vector potential singular; see [14] for a discussion of the Reissner-Nordström black hole where, in the traditional choice of gauge, the vector potential is singular on the bifurcation surface. Similarly, in Yang-Mills theories the dyanmical gauge fields A^I_μ might not be representable as smooth tensor fields on spacetime. The analysis of [1] assumes from the outset that a global choice of gauge can be made to represent Yang-Mills gauge fields as tensor fields on spacetime. Similarly to derive the first law for Yang-Mills theory, Sudarsky and Wald [4, 5] assume that a choice of gauge has been made so that the "gauge-fixed" fields are smooth everywhere, and moreover, are stationary (i.e. $\pounds_t A^I_\mu = 0$) in that choice of gauge. Similarly, for a coframe formulation of gravity (and also to describe spinor fields), one introduces a Lorentz gauge field (or spin connection) ${\omega_\mu}^a{}_b$ which might not be smooth everywhere in some chosen gauge (see § III. [15]). In fact, for non-parallelisable manifolds, there does not even exist a globally smooth choice of coframes e^a_μ. In all of these cases the obstruction is that a globally smooth choice of gauge can only be made under certain topological restrictions (when the principal bundle is trivial; see § A). Even when we can make a global gauge choice it is far from obvious that a gauge choice can be made such that the gauge-fixed fields are stationary in that gauge (see § 4 [14] for a related discussion). Thus, it is of interest to have a formulation of the first law for theories like Yang-Mills and coframe gravity that does not require a choice of gauge a priori.

The second obstruction arises in defining the action of diffeomorphisms of spacetime on dynamical fields with internal gauge transformations. To formulate a first law of black hole mechanics, we need a notion of a stationary solution to the equations of motion. When the dynamical fields are usual tensor fields on the spacetime M, there is a natural action of the diffeomorphism group of the spacetime M on tensor fields which can be used to define stationarity by the action of the Lie derivative with respect to the corresponding vector fields. However, when the dynamical fields transform under some internal gauge transformation, we cannot distinguish the action of a diffeomorphism of the spacetime manifold from the action of a diffeomorphism along with an arbitrary (spacetime dependent) internal gauge transformation. That is, we only have a notion of "diffeomorphisms up to a gauge transformations". Similarly in general, a stationary (axisymmetric) black hole will only be "stationary (axisymmetric) up to an internal gauge transformation" i.e. $\pounds_t \psi = gauge$ where ψ denotes the charged dynamical fields.[1] Thus, the full group of transformations of the dynamical fields of the theory is not simply a product group of diffeomorphisms and internal gauge transformations. Without such a separation of diffeomorphisms and internal gauge transformations we have to consider the full group of transformations to define the appropriate notion of Noether charge to obtain a first law.

There have been numerous attempts to sidestep this problem. One approach is to use the usual Lie derivative on spacetime, ignoring the internal gauge structure of the fields [20, 21, 22, 23]. A straightforward computation shows that such a Lie derivative (acting on the Yang-Mills gauge fields A^I_μ for instance) depends on the gauge choice made. Thus any definition of Noether charges, stationarity and first law using such Lie derivatives on charged fields will also depend on the choice of gauge used. In [4, 5] stationarity of the Yang-Mills gauge fields was defined by requiring a global choice of gauge so that the gauge-fixed fields A^I_μ on spacetime are annihilated by the Lie derivative along the stationary Killing field. It is far from obvious that such a global choice of gauge exists, even when a global gauge choice can be made, and as we will show, this assumption is actually a restriction on the types of Yang-Mills fields considered in [4, 5].

One could alternatively attempt to define the infinitesimal action of a diffeomorphism of

1. Such a notion of "stationarity up to gauge" was already used in [16, 17, 18, 19] where nevertheless, the dynamical fields are assumed to be tensor fields on spacetime.

spacetime by a "gauge covariant Lie derivative" and use the vanishing of this Lie derivative to define stationarity and axisymmetry. For example in Ch.10 [24] the action of a diffeomorphism along a vector field $\underline{X} \equiv X^\mu \in TM$ is defined through a "gauge covariant Lie derivative" of a Yang-Mills gauge field A^I_μ (here the I index only refers to the Yang-Mills Lie algebra) as

$$\hat{\pounds}_{\underline{X}} A^I_\mu := X^\nu F^I_{\nu\mu} \tag{1.1}$$

where $F^I_{\mu\nu}$ is the curvature 2-form for the gauge field A^I_μ. Similarly, to obtain a first law for a coframe formulation of gravity, Jacobson and Mohd [15] use a *Lorentz-Lie derivative* for the coframes by [2] (see also [25] and references in [15])

$$\begin{aligned} \hat{\pounds}_{\underline{X}} e^a_\mu &:= \pounds_{\underline{X}} e^a_\mu + \lambda^a{}_b e^b_\mu \\ \text{with} \quad \lambda^{ab} &:= E^{\mu[a} \pounds_{\underline{X}} e^{b]}_\mu = X^\mu \omega_\mu{}^{ab} + E^{\mu[a} e^{b]}_\nu \nabla_\mu X^\nu \end{aligned} \tag{1.2}$$

where the $\pounds_{\underline{X}}$ is computed ignoring the internal indices and $\omega_\mu{}^{ab}$ is the Lorentz connection on spacetime. Likewise, there have been many attempts to define a Lie derivative for spinor fields (viewed as fields on spacetime). A definition of Lie derivative for spinors with respect to a Killing field of the metric was put forth by Lichnerowicz [26], and then generalised for arbitrary vector fields by Kosmann [27] by prescribing that one use the same formula as that given by Lichnerowicz but for any vector field (see also [25] and Supplement 2. [28]). The Kosmann prescription can be formalised on a principal bundle through the notion of a *Kosmann lift* (see [29, 30] and the references therein).[3] The *Lichnerowicz-Kosmann-Lie*

2. [15] define the Lorentz-Lie derivative to act on arbitrary tensors, including the Lorentz connection, carrying a representation of the Lorentz group, but we only present the coframes to be brief. They also use the symbol $\mathcal{K}$ to denote the Lorentz-Lie derivative in honour of Kosmann.

3. Also note that the "spinorial Lie derivative" prescription in [31] annihilates the metric for any vector field — making every vector field a "Killing field" and all spacetimes stationary — which is clearly not desirable.

derivative acts on Dirac spinor fields Ψ according to

$$\hat{\pounds}_{\underline{X}}\Psi := X^\mu D_\mu \Psi - \tfrac{1}{8}\nabla_{[\mu}X_{\nu]}[\gamma^\mu, \gamma^\nu]\Psi \tag{1.3}$$

where D_μ is the covariant spin derivative on Dirac spinor fields with respect to a torsionless spin connection (see § A.1). Even though these definitions of Lie derivative are gauge covariant, it is a straightforward computation to verify that

$$[\hat{\pounds}_{\underline{X}}, \hat{\pounds}_{\underline{Y}}] = \hat{\pounds}_{[\underline{X},\underline{Y}]} + gauge \tag{1.4}$$

and the *gauge*-term does not vanish except when (1) $\underline{\boldsymbol{F}}^I = 0$ for "Lie derivative" in Eq. 1.1 (see Remark A.1 for the bundle point of view on this) (2) either X^μ or Y^μ is a conformal Killing field of the metric for the Lorentz-Lie derivative Eq. 1.2 and the Lichnerowicz-Kosmann-Lie derivative Eq. 1.3 (see [25] for a proof). Thus even though, the linear maps $X^\mu \mapsto \hat{\pounds}_{\underline{X}}$ in Eq. 1.1-Eq. 1.3 are gauge-covariant, none of the above prescriptions for the Lie derivative implement the Lie algebra for the diffeomorphism group of spacetime and the Noether currents derived from these notions of a Lie derivative cannot be interpreted as Noether currents associated to diffeomorphisms.

Further, if the dynamical fields ψ of the theory are chosen to be stationary with respect to these modified Lie derivatives, i.e. $\hat{\pounds}_{\underline{X}}\psi = 0$ then they are obviously "stationary up to internal gauge transformations". But it might not be possible to choose a globally smooth gauge representative of the dynamical fields so that they are stationary in this modified sense, particularly when we cannot make a global choice of gauge.

The main aim of this work is to address these issues directly, by formulating physical theories with charged dynamical fields on a *principal bundle* (see § A for details). All the charged fields are legitimate (globally well-defined and smooth) tensor fields on the principal bundle. A smooth global choice of gauge exists only when the principal bundle is trivial, and only in that case can we write the charged fields as smooth tensor fields on spacetime. However, working directly on a principal bundle avoids the issue of making any choice of gauge, and thus, we can handle theories defined on non-trivial principal bundles where there is no way to represent the fields as smooth tensor fields on spacetime. We shall similarly formulate the Lagrangian, Noether currents and charges directly on the principal bundle without making any gauge choices. The principal bundle also provides the necessary structure to consider the full group of transformations of the dynamical fields as the group of automorphisms of the bundle manifold. The automorphisms of the bundle then encode both diffeomorphisms and internal gauge transformations, and we will not need to "artificially" single out the action of just diffeomorphisms. Infinitesimal actions of these automorphisms are then generated by (standard) Lie derivatives with respect to vector fields on the bundle. Using this we define the appropriate notions of "stationarity (and axisymmetry) up to gauge transformations" for charged dynamical fields as automorphisms of the bundle that project down to the stationary (axisymmetric) diffeomorphisms of spacetime (see Def. A.2). We then generalise the constructions of [1] to define the symplectic form on arbitrary (non-stationary) perturbations of the dynamical fields, and the Noether current and Noether charge associated with any bundle automorphism. Next, we describe the main results of this paper, which include a derivation of a generalised zeroth law for bifurcate Killing horizons, and the first law of black hole mechanics for stationary and axisymmetric black holes which arise from solutions to the equations of motion, for theories with charged dynamical fields.

We are interested in static or stationary-axisymmetric, asymptotically flat black hole spacetimes with bifurcate Killing horizons determined by dynamical fields which have non-trivial internal gauge transformations. We expect our results can be generalised to spacetimes with different asymptotics but we stick to the asymptotically flat case. We refer the reader to Ch. 3 for a more detailed description of the spacetimes under consideration.

As our first result we show in Theorem 1 that on any bifurcate Killing horizon in spacetime (not necessarily a solution to any equations of motion) we can define certain *potentials* $\mathscr{V}^{\Lambda}$ at the horizon which are constant along the entire bifurcate Killing horizon. This can be viewed as a generalised zeroth law for bifurcate Killing horizons. These potentials are defined solely in terms of the dynamical gauge fields of the theory (for instance, a Yang-Mills gauge field or the Lorentz gauge field for gravity) and get no direct contributions from any other matter fields. In Cor. 3.1, we show that the perturbed Hamiltonian δH_K associated to the horizon Killing field K^{μ} at the bifurcation surface can be put into a "potential times perturbed charge" form where the *charges* are determined by the dependence of the Lagrangian on the curvatures of the dynamical gauge fields of the theory. Then, in Theorem 2, we provide a new perspective on the temperature $T_{\mathscr{H}}$ and perturbed entropy δS of the black hole by identifying them with the potential and perturbed charge, respectively, corresponding to the Lorentz connection in a first-order formulation of gravity. Thus, the temperature and perturbed entropy can be viewed on the same footing as any other potentials and perturbed charges of any matter gauge fields (like Yang-Mills gauge fields) in the theory. This also gives us an explicit formula for the perturbed entropy in direct parallel with the *Wald entropy* formula [2, 3].

Our main result (Theorem 3) is a general formulation of the first law of black hole mechanics for theories with charged dynamical fields, where the dynamical fields solve the equations of motion obtained from a gauge-invariant Lagrangian. The first law is obtained as an equality between the perturbed boundary Hamiltonian δH_K associated to the horizon Killing field K^{μ} evaluated at the bifurcation surface and at spatial infinity, and takes the form

$$T_{\mathscr{H}}\delta S + \mathscr{V}'^{\Lambda}\delta\mathscr{Q}'_{\Lambda} = \delta E_{can} - \Omega^{(i)}_{\mathscr{H}}\,\delta J_{(i),can} \tag{1.5}$$

for any perturbation which solves the linearised equations of motion off a stationary, axisymmetric (up to internal gauge transformations) black hole background which solves the equations of motion. The left-hand-side terms are the potentials and charges of the black hole on the bifurcation surface defined in Theorem 1 and Cor. 3.1. The first term consists of the temperature and perturbed entropy of the black hole, identified with the gravitational potential and perturbed charge (Theorem 2), while the second term is the contribution of the non-gravitational gauge fields (such as Yang-Mills fields). The quantities on the right-hand-side are the perturbed canonical energy (associated to the stationary Killing field t^μ) and angular momenta (associated to the axial Killing fields $\phi^\mu_{(i)}$; here the index (i) is used to denote the multiple axial Killing fields in greater than 4-spacetime dimensions) defined at spatial infinity (Eq. 4.1). The form of the perturbed canonical energy and angular momenta at infinity depends on the theory under consideration and also the asymptotic fall-off conditions on the fields, and they contain contributions from both the gravitational dynamical fields and other matter dynamical fields in the theory. For instance, in Einstein-Yang-Mills theory, δE_{can} contains both the perturbed ADM mass and a "potential times perturbed charge" term from the Yang-Mills gauge field at infinity (see § 5.2). Similarly, δJ_{can} contains both the perturbed ADM angular momentum and the perturbed angular momentum of the Yang-Mills fields.

Note that for Einstein-Yang-Mills theory, Sudarsky and Wald [4] get a vanishing Yang-Mills potential term at the horizon because of their assumption that there exists a smooth choice of gauge such that the gauge-fixed Yang-Mills fields are stationary $\pounds_K A^I_\mu = 0$. We will argue that in general such a gauge choice cannot be made, and the "potential times perturbed charge" at the horizon can be set to vanish only in special situations (at the cost of changing the contributions to perturbed canonical energy and angular momenta at infinity; see Remark 5.2). The existence of this non-vanishing term at the horizon was also pointed out in [14], though they could not write the term in terms of potentials and perturbed charges for non-abelian Yang-Mills fields.

We also show that the ambiguities in defining the Noether charge for a Lagrangian do not affect the first law and the perturbed entropy. We also discuss the ambiguities in defining a total entropy for a stationary axisymmetric black hole. We argue that a second law of black mechanics could fix at least some of these ambiguities in the total entropy. Since we do not know of a general derivation of the second law for arbitrary theories of gravity (except in the case of General Relativity), we do not make an attempt to define the total entropy or a notion of dynamical black hole entropy in this paper.

Even though we use a first-order coframe formulation for gravity, the general form of the first law described above is applicable to any Lagrangian theory for gravity where some fields with internal gauge freedom are considered as dynamical fields instead of the metric [32, 33, 34] including, higher-derivative theories of gravity [35] with Lagrangians depending on torsion, curvature and finitely many of their derivatives. One can also include "non-metricity", metric-affine theories of gravity [36, 37] by a simple extension of the formalism. On the matter side, we can include all the charged matter fields in the Standard Model of particle physics. We also expect that our results can be generalised to include supersymmetric theories following, for instance, [38].

Despite this generality, there remain some potentially physically interesting theories that are not covered by our formalism. These include higher p-form gauge theories in the presence of magnetic charges (see [39, 40, 17] for work in this direction) and Chern-Simons Lagrangians which are only gauge-invariant up to a total derivative term. The entropy contribution of gravitational Chern-Simons Lagrangians was computed from a spacetime point of view in [41, 42] (using a "modified Lie derivative"), and in [43] (using a modification of the symplectic current). We shall defer the analysis of Chern-Simons theories from a bundle point of view to a forthcoming paper [44].

* * *

The remainder of this work is organised as follows. We describe the principal bundle formulation of dynamical fields and gauge-invariant Lagrangians in Ch. 2 and give a general form for the symplectic potential, symplectic current, and Noether charge for any bundle automorphism (i.e. combined diffeomorphisms and gauge transformations) for such theories. We define the horizon potentials and charges, and derive a generalised zeroth law for bifurcate

Killing horizons in Ch. 3. In Ch. 4 we give a formulation of the first law of black hole mechanics for the theories under consideration. In Ch. 5 we use this formalism to derive a first law for General Relativity in a first-order tetrad formulation, Einstein-Yang-Mills theory and Einstein-Dirac theory. The appendix § A will review some constructions on principal bundles and prove new results on the uniqueness of bundle automorphisms that preserve the connection and coframes which will be used in the main arguments of the paper.

1.1 Notation

We will use an abstract index notation for vector spaces and tensor fields whenever convenient. Tensor fields on spacetime (or some base space for a principal bundle) will be denoted by indices $\mu, \nu, \lambda, \dots$ from the middle of the lower case Greek alphabet, e.g. X^μ is a vector field and σ_μ is a covector field on spacetime. Similarly, lower case Latin indices $m, n, l, \dots$ denote tensors on a principal bundle, e.g. X^m is a vector field and σ_m is a covector field on a principal bundle.

It will be convenient to often use an index-free notation for differential forms and vector fields. We will use the factor and sign conventions of Wald [45] when translating differential forms to and from an index notation and use the symbol $\equiv$ to denote such a translation. When using an index-free notation we denote differential forms by a bold-face symbol, for instance, a differential k-form on a principal bundle is denoted by $\boldsymbol{\sigma} \equiv \sigma_{m_1 \dots m_k} = \sigma_{[m_1 \dots m_k]}$ and for a vector field X^m, we denote the *interior product* by $X \cdot \boldsymbol{\sigma} \equiv X^{m_1} \sigma_{m_1 \dots m_k}$ and the *Lie derivative* by $\pounds_X \boldsymbol{\sigma} = X \cdot d\boldsymbol{\sigma} + d(X \cdot \boldsymbol{\sigma})$. When using an index-free notation (and for scalars which have no indices) we will also use an underline to distinguish between functions, differential forms and vector fields on the base space from those on the principal bundle i.e. $\underline{\varphi}$ is a function, and $\underline{\boldsymbol{\sigma}} \equiv \sigma_{[\mu_1 \dots \mu_k]}$ a k-form, respectively on M, and $\underline{X} \cdot \underline{\boldsymbol{\sigma}}$ is the interior product, on the base space.

Upper case indices $I, J, K, \dots$ from middle of the Latin alphabet will denote elements of a finite-dimensional Lie algebra $\mathfrak{g}$ e.g. X^I is an element of a Lie algebra and $c^I{}_{JK} = c^I{}_{[JK]}$

denotes the *structure constants*. We write the Lie bracket on $\mathfrak{g}$ as $[X,Y]^K = c^K{}_{IJ}X^IY^J$. The *Killing form* on $\mathfrak{g}$ is a bilinear, symmetric form defined by

$$k_{IJ} := c^L{}_{IK}c^K{}_{JL} \tag{1.6}$$

The Killing form is invariant under the adjoint action of the group G on its Lie algebra $\mathfrak{g}$. It is non-degenerate if and only if $\mathfrak{g}$ is *semisimple*, and hence defines a metric on the Lie algebra. Further, when the group G is *compact* the Killing form is *negative* definite. Throughout the paper we stick to the semisimple case and use k_{IJ} and its inverse k^{IJ} to raise and lower the abstract indices on elements of $\mathfrak{g}$.

Upper case letters $A, B, \ldots$ from the beginning of the Latin alphabet denote elements of a vector space $\mathbb{V}$ with some representation R of a finite-dimensional group G. The action of any $g \in G$ on any element $\varphi^A \in \mathbb{V}$ under the representation R is denoted by $R(g)\varphi$; omitting the indices for simplicity. The corresponding action r of a Lie algebra element $X^I \in \mathfrak{g}$ is denoted (using the abstract index notation) by $X^I r_I{}^A{}_B\varphi^B$. We shall also use $\alpha, \beta, \ldots$ as indices to denote some collection of fields, each of which can be tensor fields valued in different vector spaces (for example in Eqs. 2.13 and 2.15).

CHAPTER 2

DYNAMICAL FIELDS AND LAGRANGIAN THEORIES ON A PRINCIPAL BUNDLE

Fields with internal gauge freedom under the action of a group G (which we assume to be semisimple) are usually written as tensor fields on spacetime M valued in some vector space which transform under a representation of G. As noted in the Introduction, in general such fields can only be represented locally as smooth tensor fields. Since we are interested in the first law of black hole mechanics which is a global equality relating quantities defined at the horizon to those defined at spatial infinity, it will be very convenient to have globally well-defined smooth dynamical fields to describe the physical theory. Such fields with internal gauge transformations under some group can be defined globally on a principal bundle as we describe next (see § A for details).

A *principal bundle* with *structure group* G over a *base* manifold M is another manifold P which is *locally* (but not necessarily globally) a product manifold $M \times G$, with a smooth *projection* $\pi : P \to M$ and an action of the structure group G on P. That is, locally any point $u \in P$ can be written as a pair $u = (x, g)$ where $x \in M$ and $g \in G$. The projection map π then acts as $\pi(u) = x$, and for any element $g' \in G$ the structure group maps $u = (x, g) \mapsto u' = (x, gg')$.[1] If the product structure of P extends globally then the bundle is called *trivial*.

Now we describe how to write charged fields with internal gauge transformations as tensor fields on the bundle P instead of the base space M. First let us consider the case of a charged scalar field; we'll describe more general charged tensor fields later. A charged scalar field $\underline{\varphi}^A$ on M is (locally) a function valued in some vector space $\mathbb{V}$ which has a representation R of the internal gauge group G. Here we have used an underline to emphasise that the field is

1. This defines a *right action* of G on P. Some authors define a principal bundle with a *left action* instead; one must be mindful of this difference when comparing formulae directly.

described on the base spacetime. Under internal gauge transformations by an element $g \in G$ (g can vary over the base space) such a charged field transforms as

$$\underline{\varphi}(x) \mapsto R(g^{-1}(x))\underline{\varphi}(x) \tag{2.1}$$

where we have suppressed the internal representation indices. As noted earlier, it might not be possible to write $\underline{\varphi}^A$ globally as a smooth field on the base space M, though we can do so in a small enough neighbourhood of any point. On the intersection of any two such neighbourhoods the corresponding locally defined fields are related by a gauge transformation as in Eq. 2.1. Thus the gauge equivalence class of such fields is well-defined. Even though it is possible to work with such fields on patches of local neighbourhoods on spacetime, it will be far more convenient to define them globally on a principal bundle P with the structure group G. We represent the field $\underline{\varphi}^A$ on the principal bundle P as an *equivariant* function valued in $\mathbb{V}$ i.e. by $\varphi^A \in \Omega^0 P(\mathbb{V}, R)$. Here the term equivariant means that under the group action of G on P, the function φ^A transforms similar to Eq. 2.1 under the representation R. The relation between the field φ^A on P and the $\underline{\varphi}^A$ on M is as follows. Consider a point $u \in P$ such that $\pi(u) = x \in M$. Then the charged field $\underline{\varphi}^A$ on M is related to the function φ^A on P by $\underline{\varphi}^A(x) = \varphi^A(u)$. If we choose a different point u' that also projects to x, then we get a field $\underline{\varphi}'^A(x) = \varphi^A(u')$ which is related to $\underline{\varphi}^A$ by a gauge transformation. Thus, the charged field φ^A on the bundle P corresponds to the entire class of gauge-equivalent fields defined on the base space. Since P is locally a product, we can always locally associate a charged field on the bundle to a charged field on the base M. When the principal bundle is trivial, we can globally choose a smooth gauge i.e. to every point $x \in M$ we can smoothly associate a unique point $u(x) \in P$ (see definition of a *section* in § A) and define the charged scalar field globally on M. If the principal bundle is non-trivial this correspondence does not extend globally in a smooth (or even, continuous) manner which is why $\underline{\varphi}^A$ might not be globally defined on M even though the corresponding field φ^A is globally well-defined on the bundle P. The main example of such a charged scalar field we'll consider is the Dirac spinor field in § 5.3 in which case the group G is the spin group $Spin^0(3,1)$ and the corresponding bundle is the spin bundle $F_{Spin}M$ (see § A.1). Another example of such a field is the Higgs field of the Standard Model where the group G is taken to be $SU(2) \times U(1)$.

Note that if the scalar field $\underline{\varphi}$ is invariant under gauge transformations, then the corresponding field $\varphi \in \Omega^0 P$ is constant along the G-direction of the bundle P. In this case, we have the global correspondence by the pullback through the projection i.e. $\varphi = \pi^* \underline{\varphi}$.

When considering the coframe formulation of gravity, one introduces the local frames $E_a^\mu(x)$ and coframes $e_\mu^a(x)$ which act as a linearly-independent set of basis for the tangent space and cotangent space at a point x of M respectively. Usually the index a is thought as a labelling index for the set of basis, but it is more useful to think of it as an abstract index in $\mathbb{R}^d$ where d is the dimension of M. The frames and coframes are dual in the sense that they satisfy

$$E_a^\mu e_\nu^a = \delta_\nu^\mu \quad ; \quad E_b^\mu e_\mu^a = \delta_b^a \tag{2.2}$$

We can obtain an equivalent set of basis $(E')_a^\mu(x)$ by performing a general linear transformation on the index a in $\mathbb{R}^d$ which corresponds to a different choice of basis — the frames and coframes can be considered as charged tensor fields with an internal gauge freedom under the group $GL(d, \mathbb{R})$. On a non-parallelisable manifold we cannot choose the frames and coframes as globally linearly independent fields; a typical example being a 2-sphere. But, just like scalar field case, we can write the entire $GL(d, \mathbb{R})$-equivalence class of frames and coframes as global fields on a principal bundle (with structure group $G = GL(d, \mathbb{R})$) called the *frame bundle* FM. We defer the details of this frame bundle construction to § A.1 but review the following. Any choice of frames E_a^μ can be described on the frame bundle FM globally as vector fields $E_a^m \in TFM$ valued in $\mathbb{R}^{d*}$, where we consider two frames E_a^m and $(E')_a^m$ to be equivalent if their difference $(E')_a^m - E_a^m$ is a *vertical* vector field on the bundle defined as follows. A vector on the principal bundle is vertical if it projects to the zero vector on the base space i.e. $(\pi_*)^\mu_m X^m = 0$ — thus a vertical vector points purely in the G-direction of the local product $M \times G$.

Similarly, any choice of coframes e^a_μ on spacetime is represented by 1-forms $\boldsymbol{e}^a \equiv e^a_m$ valued in $\mathbb{R}^d$ which are *horizontal* forms in the following sense. A differential form $\boldsymbol{\sigma}$ on the bundle is horizontal if $X \cdot \boldsymbol{\sigma} = 0$ for any vertical vector X^m. Both the frames E^m_a and the coframes e^a_m are equivariant under the $GL(d, \mathbb{R})$ action on the frame bundle FM, where the term equivariant is used in the same sense as for the charged scalar field described above.

If we have a metric $g_{\mu\nu}$ (of Lorentzian signature) on the base space M we can similarly construct the *bundle of orthonormal frames* $F_O M$ where the choice of frames is restricted to satisfy

$$g^{\mu\nu} = \eta^{ab} E^\mu_a E^\nu_b \quad ; \quad g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\mu \tag{2.3}$$

where η_{ab} is a metric (of the same signature as $g_{\mu\nu}$) on the vector space $\mathbb{R}^d$. The bundle $F_O M$ (with the Lorentz group $O(d-1, 1)$ as the structure group) is a subbundle of FM and further, every choice of metric $g_{\mu\nu}$ gives rise to such a subbundle (see § A.1).

Thus, we see that if we work with some choice of orthonormal frame bundle $F_O M$ then we have already picked a fixed metric on spacetime, which is not desirable when formulating theories of gravity. To avoid this, we work with an abstract Lorentz bundle P_O with structure group $O(d-1, 1)$ without a priori identifying it with any orthonormal frame bundle (see details in § A.1). We define the orthonormal frames and coframes on the abstract Lorentz bundle P_O as described above. Only once we have coframes which are solutions to the equations of motion, we identify P_O with the choice of orthonormal frame bundle given by a metric $g_{\mu\nu}$.

With the frames and coframes at our disposal we can write general charged tensor fields (with arbitrary tensorial index structure) on a principal bundle. To do this, we now restrict attention to a principal bundle of the form $P = P_O \oplus P'$, where P_O is the Lorentz bundle described above and P' is another principal bundle with structure group G' accounting for some other internal gauge transformations of the matter fields. The structure group of P is then $G = O(d-1,1) \times G'$. On P we can write the frames, coframes, and scalar fields which transform under G' in the manner described above. To write more general charged tensor fields on P we express them in terms of their frame components as follows. For simplicity, we consider tensor fields that only have internal gauge transformations under the group G' — the construction proceeds in exactly the same way if the tensor fields do transform under the Lorentz group $O(d-1,1)$. Let σ_μ^A be a charged covector field and and η_A^μ a charged vector field on M valued in some vector space $\mathbb{V}$ with some internal gauge transformation by G'. Locally choosing a set of frames and coframes, we write the frame components as

$$\sigma_a^A := \sigma_\mu^A E_a^\mu \quad ; \quad \eta_A^a := \eta_A^\mu e_\mu^a \tag{2.4}$$

Now we can view the frame components σ_a^A and η_A^a as scalar fields valued in $\mathbb{R}^d \otimes \mathbb{V}$ with internal gauge transformations under both $O(d-1,1)$ and G'. Then, we can consider the frame components as globally smooth functions valued in $\mathbb{R}^d \otimes \mathbb{V}$ on the bundle P, in the same manner as the charged scalar field discussed above. Similarly, we can write any charged tensor field — with arbitrary tensor structure and with internal gauge transformations under the full structure group $G = O(d-1,1) \times G'$ — on P in terms of its frame components, the frames and coframes. Henceforth we will always represent charged tensor fields defined on spacetime M by their frame components on the bundle P written as charged scalars φ^A where now the A index includes the frame component indices.

Similar to the scalar field case, if the covector field σ_μ is invariant under internal gauge transformations then we can use the pullback $\sigma_m = (\pi^*)_m^\mu \sigma_\mu$ to globally define the field σ_m as a horizontal 1-form on the bundle P. In particular, this is a global isomorphism between gauge-invariant differential forms on M and horizontal forms on the bundle i.e. $\Omega^k_{hor} P \cong \pi^* \Omega^k M$. We shall use this isomorphism to express gauge-invariant Lagrangians defined on the base space as horizontal differential forms on the bundle.

Note that we could have represented σ^A_μ and η^μ_A on the bundle as differential forms and vector fields in a manner similar to the coframes and frames. As we shall see, the strategy of representing the frame components of dynamical tensor fields as scalar fields on the bundle simplifies many of the intermediate computations, and the form of the various quantities like the Noether charge, needed to derive a first law.

To define covariant derivatives of the charged fields (and also to consider Yang-Mills theory or a first-order coframe formulation of gravity) we need to define the corresponding gauge fields. Locally, on M, the gauge fields $\underline{\boldsymbol{A}}^I \equiv A^I_\mu$ are 1-forms valued in the Lie algebra $\mathfrak{g}$ of the structure group G, and which transform under a (local) gauge transformation as

$$\underline{\boldsymbol{A}}(x) \mapsto g^{-1}(x)\ \underline{\boldsymbol{A}}(x)\ g(x) + g^{-1}(x)\ dg(x) \tag{2.5}$$

where the internal indices have been suppressed and we have used a matrix-type notation for the group action. For the gravitational Lorentz gauge field the structure group is the Lorentz group $O(d-1,1)$ and we can locally write it as $\omega_\mu{}^a{}_b$. For Yang-Mills theory the structure group is usually chosen to be either $SU(N)$ or $U(1)$ but we can pick any semisimple Lie group. Since the transformation law is inhomogenous the gauge fields cannot be represented as horizontal equivariant forms on P as described above. Nevertheless, there is a non-horizontal equivariant called the *connection* which does represent the gauge fields.[2]

A *connection* on a G-principal fibre bundle P is a smooth equivariant 1-form valued in the Lie algebra $\mathfrak{g}$ i.e. $\boldsymbol{A}^I \equiv A^I_m \in \Omega^1 P(\mathfrak{g}, \mathrm{Ad})$ (see § A for a more precise definition). Here Ad is the adjoint representation of the structure group G on its Lie algebra $\mathfrak{g}$. Again, the local gauge fields cannot be defined globally as tensor fields on M unless the principal bundle is trivial. The connection then defines a covariant exterior derivative D on equivariant horizontal differential forms through Eq. A.3. The curvature 2-form $\boldsymbol{F}^I$ of an arbitrary

2. The non-horizontality of the connection on the bundle is responsible for the inhomogenous transformation under an internal gauge transformation for A^I_μ as viewed from the base space.

connection is defined through Eq. A.4 — for a Lorentz connection $\boldsymbol{\omega}^a{}_b$ the curvature $\boldsymbol{R}^a{}_b$ is given by Eq. A.35 and the torsion 2-form $\boldsymbol{T}^a$ by Eq. A.36. Both the curvature and torsion are equivariant horizontal forms on the bundle.

Given a connection $\boldsymbol{A}^I$ we can define, on charged tensor fields on M, a *covariant derivative operator* ∇ which is covariant with respect to internal gauge transformations. As above, we again restrict to a principal bundle of the form $P = P_O \oplus P'$. The connection on such a bundle also splits as $\boldsymbol{A}^I = \left(\boldsymbol{\omega}^a{}_b, \boldsymbol{A}'^{I'}\right)$, where $\boldsymbol{\omega}^a{}_b$ is a Lorentz connection on P_O and $\boldsymbol{A}'^{I'}$ is a conection on P'.

Let us start by considering a charged covector field $\sigma^A_\mu \in T^*M(\mathbb{V})$ where the A is an abstract index on some vector space $\mathbb{V}$ and $r_I{}^A{}_B$ is a representation of the Lie algebra $\mathfrak{g}$ of the structure group G on $\mathbb{V}$. Locally, on spacetime the action of ∇ can be written in terms of the local gauge field A^I_λ as

$$\nabla_\lambda \sigma^A_\mu = \hat{\nabla}_\lambda \sigma^A_\mu + A^I_\lambda r_I{}^A{}_B \sigma^B_\mu \tag{2.6}$$

where $\hat{\nabla}$ is a metric-compatible derivative operator (possibly with torsion) on spacetime tensors that ignores the internal representation indices. Since the local gravitational spin connection can be written in terms of the derivatives of the orthonormal coframes on spacetime as (see Eq. 3.4.13 [45])

$$\omega^{ab}_\mu = E^{a\nu} \hat{\nabla}_\mu e^b_\nu = e^a_\nu \hat{\nabla}_\mu E^{b\nu} \tag{2.7}$$

the derivative ∇ annihilates the frames and coframes. Thus, using the coframes we can rewrite the covariant derivatives of σ^α_μ in terms of their frame components and the covariant exterior derivative D (see Eq. A.2). Let $\underline{D} \equiv D_\mu$ be the locally defined exterior covariant derivative on spacetime M, then on charged scalar fields the action of ∇ in Eq. 2.6 coincides with that of D. Thus, we have

$$\begin{aligned} \nabla_\lambda \sigma^A_\mu &= (D_\lambda \sigma^A_a)\ e^a_\mu = (\underline{E}_b \cdot \underline{D}\sigma^A_a)\ e^b_\lambda e^a_\mu \\ \nabla_\rho \nabla_\lambda \sigma^A_\mu &= D_\rho(\underline{E}_b \cdot \underline{D}\sigma^A_a)\ e^b_\lambda e^a_\mu = \underline{E}_c \cdot \underline{D}(\underline{E}_b \cdot \underline{D}\sigma^A_a)\ e^c_\rho e^b_\lambda e^a_\mu \end{aligned} \tag{2.8}$$

where now $\sigma^A_a \in (\mathbb{R}^d)^* \otimes \mathbb{V}$ is the coframe component of σ^A_μ (see Eq. 2.4). For a charged vector field η^μ_A we have

$$\begin{aligned} \nabla_\lambda \eta^\nu_A &= (D_\lambda \eta^a_A)\ E^\nu_a = (\underline{E}_b \cdot \underline{D}\eta^a_A)\ e^b_\lambda E^\nu_a \\ \nabla_\rho \nabla_\lambda \eta^\nu_A &= D_\rho(\underline{E}_b \cdot \underline{D}\eta^a_A)\ e^b_\lambda E^\nu_a = \underline{E}_c \cdot \underline{D}(\underline{E}_b \cdot \underline{D}\eta^a_A)\ e^c_\rho e^b_\lambda E^\nu_a \end{aligned} \tag{2.9}$$

with $\eta^a_A \in (\mathbb{R}^d) \otimes \mathbb{V}$. We can similarly locally write the covariant derivatives of any charged tensor field in terms of their frame components, the coframes, frames and the covariant exterior derivative on spacetime. As discussed above the frame components can then be represented on the principal bundle as globally well-defined functions φ^A (recall that here the A-index includes the internal incides and the frame component indices) and then, we will write the frame components of the k-derivatives of φ^A on the bundle P using the shorthand $\varphi^A_{a_1 \dots a_k}$ where

$$\varphi^A_{a_1 \dots a_k} := E_{a_k} \cdot D(\dots E_{a_1} \cdot D\varphi^A) = E_{a_k} \cdot D\varphi^A_{a_1 \dots a_{k-1}}$$

Thus we can describe all the dynamical fields with internal gauge transformations and their covariant derivatives as globally smooth tensor fields on the bundle P without making any choice of gauge.

* * *

The other problem that arises if the dynamical fields have some internal gauge freedom is that we can only define a notion of "diffeomorphisms up to a gauge transformation", and consequently there is only notion of "stationarity up to internal gauge transformations". As discussed in the Introduction Ch. 1, if one uses the ordinary Lie derivative on spacetime (ignoring the internal gauge transformations), the result is not gauge-invariant. Also, the various attempts at defining "covariant Lie derivatives" (Eqs. 1.1–1.3) do not implement the Lie algebra of diffeomorphisms of spacetime (see Eq. 1.4).

Since we have defined the dynamical fields on a principal bundle P, the source of this problem becomes more apparent. For theories with dynamical tensor fields defined on spacetime M, the group of transformations is the group of diffeomorphisms of spacetime i.e. Diff(M). Similarly, for theories with dynamical fields defined on the bundle P the group of transformations is the group of automorphisms of the principal bundle $\mathrm{Aut}(P) \cong \mathrm{Diff}(M) \ltimes \mathrm{Aut}_V(P)$. Here $\mathrm{Aut}_V(P)$ is the normal subgroup of vertical automorphisms that do not move the points in the base space M i.e. these correspond to internal gauge transformations (see § A). Since, $\mathrm{Aut}_V(P)$ is a normal subgroup, the action of internal gauge transformations leaving the spacetime points fixed is well-defined. However, $\mathrm{Diff}(M)$ is not a normal subgroup there is only a notion of "diffeomorphisms up to internal gauge transformations" and any attempt to define an action of just diffeomorphisms of M on charged fields is doomed to fail. In fact from Eq. 1.4 we see that even when one defines some "gauge covariant Lie derivative" one has to consider diffeomorphisms and internal gauge transformations simultaneously. Thus, again we are lead to work directly with fields defined on the principal bundle with $\mathrm{Aut}(P)$ acting as the full group of transformations with the action on charged fields given by the usual Lie derivative with respect to vector fields on the bundle. We then define stationary (axisymmetric) charged fields as fields that are preserved under those automorphisms of the bundle P which project to stationary (axisymmetric) diffeomorphisms of spacetime M (see Def. A.2). Viewed from the base spacetime this gives the appropriate notion of dynamical fields being stationary (axisymmetric) up to gauge. This point of view has the further advantage that we can treat both diffeomorphisms and gauge transformations using standard tools of differential calculus on the bundle.

Even though in general there is no unique way to associate a given diffeomorphism of spacetime to an automorphism of the bundle, if we require that the automorphism preserves a given connection on the bundle, we can prove that the non-uniqueness is given by a *global symmetry* (if any exist) of the chosen connection (see Lemma A.1 and Remark A.2). As we will show, in Einstein-Yang-Mills theory, when such a global symmetry of the solution Yang-Mills connection does not exist we *cannot* set the Yang-Mills potential at the horizon to vanish and we get a new non-vanishing "potential times perturbed charge" term at the horizon for the first law, generalising the results of [4, 5, 14] on the first law for Einstein-Yang-Mills theory.

Similarly, if an automorphism of the Lorentz bundle P_O (see § A.1) is required to preserve the coframes, then it is *uniquely* determined by the corresponding isometry of the spacetime metric (see Lemma A.4). This uniqueness essentially implies that for a Killing field of the spacetime metric, the Lie derivative on the bundle coincides with the Lorentz-Lie derivative Eq. 1.2 on coframes and the Lichnerowicz-Kosmann-Lie derivative Eq. 1.3 on spinors (see Eqs. A.50 and A.57). Thus, even though our Noether charges for arbitrary automorphisms differ from those derived by [15] for a coframe formulation of gravity, we get the same first law for stationary spacetimes for the first-order formulation of gravity.

2.1 The form of the gauge-invariant Lagrangian

Following the above discussion, we formulate Lagrangian theories where the dynamical fields have some internal gauge transformation directly on a G-principal bundle $\pi : P \to M$. Since we are interested in theories with gravity described by some orthonormal coframes, we choose P to have the structure[3]

$$P = P_O \oplus P' \tag{2.10}$$

where P_O is a Lorentz bundle M and P' is a principal bundle with structure group G' corresponding to other internal gauge transformations of the matter fields. This also implies

3. Most of our results generalise straightforwardly to the more general case where the Lorentz bundle P_O is simply a subbundle of principal bundle P.

that the connection $\boldsymbol{A}^I$ on P splits as

$$\boldsymbol{A}^I = \left(\boldsymbol{\omega}^a{}_b, \boldsymbol{A}'^{I'}\right) \tag{2.11}$$

where $\boldsymbol{\omega}^a{}_b$ is a $SO(d-1,1)$-connection on P_O and $\boldsymbol{A}'^{I'}$ is a G'-connection on P'.

On spacetime the Lagrangian is a d-form $\underline{\boldsymbol{L}} \in \Omega^d M$ and we further assume that the Lagrangian is invariant under internal gauge transformations.[4] Such gauge-invariant differential forms on spacetime are isomorphic, under pullback through the projection π, to horizontal differential forms on the principal bundle i.e. $\pi^*\Omega^k M \cong \Omega^k_{hor} P$ (see § A). Thus we pullback the Lagrangian $\underline{\boldsymbol{L}}$ from the spacetime M to the bundle P that is, we consider the Lagrangian of the theory as a real horizontal d-form on the P given by $\boldsymbol{L} \in \Omega^d_{hor} P$.

We will take the Lagrangian to depend on the frames E^m_a, the coframes $\boldsymbol{e}^a$, the connection $\boldsymbol{A}^I$ Eq. 2.11, the frame components φ^A of charged tensor fields, and their finitely many covariant derivatives $\varphi^A_{a_1 \ldots a_k}$ (written as functions on P; see the discussion above). We also allow dependence on the curvature $\boldsymbol{F}^I$ and the torsion $\boldsymbol{T}^a$, and finitely many of their covariant derivatives.

Any antisymmetrisation in the derivatives of the tensor fields φ^A can be converted to terms with lower order derivatives and torsion and curvature terms using

$$2\varphi^A_{[ab]} = -T^c{}_{ab}\varphi^A_c + F^I{}_{ab} r_I{}^A{}_B \varphi^B \tag{2.12}$$

where $T^c{}_{ab} = E_b \cdot E_a \cdot \boldsymbol{T}^c$ and $F^I{}_{ab} = E_b \cdot E_a \cdot \boldsymbol{F}^I$ are the frame components of the torsion and curvature. Using this on higher order antisymmetrised derivatives we can write all derivative terms in terms of completely symmetrised derivatives and derivatives of torsion and curvature. Then in a similar manner we can eliminate any antisymmetrised derivatives

4. For instance, in this paper we do not consider theories with a Chern-Simons Lagrangian deferring their analysis to future work [44].

of the torsion and curvature. Finally, using Eq. A.37 we eliminate any dependence of the Lagrangian on $D\boldsymbol{T}^a$ in favour of the Lorentz curvature $\boldsymbol{R}^a{}_b$ and the coframes $\boldsymbol{e}^a$. For later convenience we introduce the shorthand

$$\chi^\alpha := \{\varphi^A, T^c{}_{ab}, F^I{}_{ab}\} \tag{2.13}$$

and the frame components of their completely *symmetrised* derivatives by $\chi^\alpha_{a_1 \ldots a_i}$.

Thus the dependence of the Lagrangian on the dynamical fields can be written as[5]

$$\boldsymbol{L}(E^m_a, \boldsymbol{e}^a, \boldsymbol{A}^I, \{\chi^\alpha_{a_1 \ldots a_i}\}) \in \Omega^d_{hor} P \tag{2.14}$$

where $0 \leq i \leq k$ counts the number of completely symmetrised derivatives of the corresponding fields in Eq. 2.13. As discussed above the frames E^m_a on the bundle are only defined up to vertical vector fields, so we also demand that the Lagrangian depend on the frames so that $\boldsymbol{L}[E^m_a] = \boldsymbol{L}[E'^m_a]$ whenever $E'^m_a - E^m_a$ is a vertical vector field. The full set of *dynamical* fields of the theory then includes the coframes $\boldsymbol{e}^a$, the connection $\boldsymbol{A}^I$ and the frame components of the charged tensor fields φ^A which we collectively denote as a differential form on P valued in a collective vector space $\mathbb{V}$

$$\boldsymbol{\psi}^\alpha := \{\boldsymbol{e}^a, \boldsymbol{A}^I, \varphi^A\} \in \Omega^{\deg(\alpha)} P(\mathbb{V}) \tag{2.15}$$

Here and henceforth, we use the notation $\deg(\alpha)$ to denote the degree of the differential form corresponding to the dynamical field $\boldsymbol{\psi}^\alpha$ with an α index i.e.

$$\deg(\alpha) = \{\deg(a), \deg(I), \deg(A)\} = \{1, 1, 0\} \tag{2.16}$$

5. Note that we simply assume that the Lagrangian is independent of any background fields and do not attempt to prove a "Thomas replacement theorem" as done in Lemma 2.1 [3].

The Lagrangian is further required to be a *local and covariant* functional of the fields in the sense of Def. A.3 i.e. for any automorphism of the bundle $f \in \mathrm{Aut}(P)$ we have

$$f^* \boldsymbol{L}[\boldsymbol{\psi}] = \boldsymbol{L}[f^* \boldsymbol{\psi}] \quad \forall f \in \mathrm{Aut}(P) \tag{2.17}$$

where it is implicit that on the right-hand-side that f also acts on the derivatives of $\boldsymbol{\psi}$. If $X^m \in \mathfrak{aut}(P)$ is the vector field generating the automorphism f then the above equation implies that

$$\pounds_X \boldsymbol{L}[\boldsymbol{\psi}] = \boldsymbol{L}[\pounds_X \boldsymbol{\psi}] \quad \forall X^m \in \mathfrak{aut}(P) \tag{2.18}$$

Note that since we assume that the Lagrangian is gauge-invariant, we have

$$\begin{aligned} f^* \boldsymbol{L}[\boldsymbol{\psi}] &= \boldsymbol{L}[\boldsymbol{\psi}] \quad \forall f \in \mathrm{Aut}_V(P) \\ \pounds_X \boldsymbol{L}[\boldsymbol{\psi}] &= 0 \quad \forall X^m \in \mathfrak{aut}_V(P) \end{aligned} \tag{2.19}$$

2.2 Equations of motion, the symplectic potential and symplectic current

With the above described Lagrangian the equations of motion of the theory are obtained by a variation of the Lagrangian with respect to the dynamical fields Eq. 2.15. To consider such variations, we take any smooth 1-parameter family of dynamical fields $\boldsymbol{\psi}^\alpha(\lambda)$ with $\boldsymbol{\psi}^\alpha(0) = \boldsymbol{\psi}^\alpha$ corresponding to the background dynamical fields of interest. Define the first variation or perturbation about $\boldsymbol{\psi}^\alpha$ by

$$\delta \boldsymbol{\psi}^\alpha := \frac{d}{d\lambda} \boldsymbol{\psi}^\alpha(\lambda) \Big|_{\lambda=0} \tag{2.20}$$

We use the symbol δ to denote variations of any functional $\mathcal{F}$ of the dynamical fields defined in the same way i.e.

$$\delta \mathcal{F}[\psi] := \frac{d}{d\lambda} \mathcal{F}[\psi(\lambda)] \Big|_{\lambda=0} \tag{2.21}$$

Since the difference of two connections is horizontal and all the other dynamical fields are already horizontal the perturbations of the dynamical fields Eq. 2.15 given by $\delta\boldsymbol{\psi}^{\alpha} \in \Omega^{\deg(\alpha)}_{hor}P(\mathbb{V})$ are all horizontal forms on P. Further, since $E_a \cdot \boldsymbol{e}^b = \delta^b_a$ holds at each λ of the 1-parameter family of frames and coframes, we have $\delta E_a \cdot \boldsymbol{e}^b = -E_a \cdot \delta\boldsymbol{e}^b$. Since the frames are considered equivalent if their difference is vertical we have

$$\delta E^m_a = -(E_a \cdot \delta\boldsymbol{e}^b)E^m_b = -E^m_b E^n_a \delta e^b_n \tag{2.22}$$

and we can convert all variations of the Lagrangian of the form Eq. 2.14 obtained from frame variations to variations of the dynamical coframe fields as

$$\frac{\delta\boldsymbol{L}}{\delta E^m_a}\delta E^m_a = \left(-\frac{\delta\boldsymbol{L}}{\delta E^n_b}E^n_a E^m_b\right)\delta e^a_m \tag{2.23}$$

The variation of the Lagrangian can be written in the form (we will prove this in Lemma 2.1):

$$\delta\boldsymbol{L} = \tilde{\boldsymbol{\mathcal{E}}}^{m_1\ldots m_{\deg(\alpha)}}_{\alpha}\delta\psi^{\alpha}_{m_1\ldots m_{\deg(\alpha)}} + d\boldsymbol{\theta}(\psi;\delta\psi) \tag{2.24}$$

where the *equations of motion* $\tilde{\boldsymbol{\mathcal{E}}}_{\alpha} : \Omega^{\deg(\alpha)}_{hor}P(\mathbb{V}) \to \Omega^{d}_{hor}P(\mathbb{V}^*)$ are the following functional derivative

$$\tilde{\boldsymbol{\mathcal{E}}}^{m_1\ldots m_{\deg(\alpha)}}_{\alpha} = \frac{\delta\boldsymbol{L}}{\delta\psi^{\alpha}_{m_1\ldots m_{\deg(\alpha)}}} \tag{2.25}$$

The *symplectic potential* $\boldsymbol{\theta}$ denotes the "boundary term" in the variation and depends locally and covariantly on the background $\boldsymbol{\psi}$ and linearly on the perturbation $\delta\boldsymbol{\psi}$.

For subsequent computations it will be very convenient to express the equations of motion purely as differential forms rather than linear maps valued in differential forms as in Eq. 2.25.

We show that this can be done in Lemma A.3 and then, the variational principle takes the more convenient form

$$\delta \boldsymbol{L} = \boldsymbol{\mathcal{E}}_\alpha(\psi) \wedge \delta \boldsymbol{\psi}^\alpha + d\boldsymbol{\theta}(\psi; \delta\psi) \tag{2.26}$$

where now the equations of motion $\boldsymbol{\mathcal{E}}_\alpha$ corresponding to the variation of each dynamical field $\boldsymbol{\psi}^\alpha$ (Eq. 2.15), with $k \in \deg(\alpha) = \{1, 1, 0\}$ (Eq. 2.16) are given by

$$\boldsymbol{\mathcal{E}}_\alpha \equiv (\mathcal{E}_\alpha)_{m_1 \ldots m_{d-k}} = \frac{(d-k)!k!}{d!} \frac{\delta L_{m_1 \ldots m_{d-k} l_1 \ldots l_k}}{\delta \psi^\alpha_{n_1 \ldots n_k}} \delta^{l_1}_{n_1} \cdots \delta^{l_k}_{n_k} \in \Omega^{d-k}_{hor} P(\mathbb{V}^*) \tag{2.27}$$

This rewriting of the variational principle (as opposed to Eq. 2.24) will simplify a lot of the later computations as compared to similar ones in [1, 3]. That the equations of motion Eq. 2.27 are horizontal forms expresses the well-known fact that gauge-invariant Lagrangians give gauge-covariant equations of motion. The dynamical fields $\boldsymbol{\psi}^\alpha$ which satisfy the equations of motion $\boldsymbol{\mathcal{E}}_\alpha(\psi) = 0$ form the subspace of *solutions*. Given a solution $\boldsymbol{\psi}^\alpha$, any perturbation $\delta\boldsymbol{\psi}^\alpha$ is called a *linearised solution* if it satisfies the *linearised equations of motion* $\delta \boldsymbol{\mathcal{E}}_\alpha = 0$ at $\boldsymbol{\psi}^\alpha$.

The variational principle Eq. 2.26 implies that $d\boldsymbol{\theta}$ is a horizontal form but we can show $\boldsymbol{\theta}$ itself can be chosen to be horizontal (i.e. gauge-invariant) in the following lemma which is an extension of Lemma 3.1 [3] to the bundle.

Lemma 2.1 (horizontal symplectic potential)**.** *For any Lagrangian of the form specified in Eq. 2.14, the symplectic potential* $\boldsymbol{\theta}(\psi; \delta\psi)$ *can be chosen to be a horizontal form on* P *of the form*

$$\boldsymbol{\theta} = (-)^{d-2} \boldsymbol{Z}_I \wedge \delta \boldsymbol{A}^I + (-)^{d-2} \boldsymbol{Z}_a \wedge \delta \boldsymbol{e}^a + \boldsymbol{\theta}' \in \Omega^{d-1}_{hor} P \tag{2.28}$$

with

$$\boldsymbol{\theta}' = - \sum_{i=1}^{k} \left(E_{a_i} \cdot \boldsymbol{Z}^{a_1 \ldots a_i}_\alpha \right) \delta \chi^\alpha_{a_1 \ldots a_{i-1}} \tag{2.29}$$

Here

$$\boldsymbol{Z}_I \equiv (Z_I)_{m_1\dots m_{d-2}} = \frac{(d-2)!2!}{d!}\frac{\delta L_{m_1\dots m_{d-2}l_1l_2}}{\delta F^I_{n_1n_2}}\delta^{l_1}_{n_1}\delta^{l_2}_{n_2} \in \Omega^{d-2}_{hor}P(\mathfrak{g}^*) \tag{2.30a}$$

$$\boldsymbol{Z}_a \equiv (Z_a)_{m_1\dots m_{d-2}} = \frac{(d-2)!2!}{d!}\frac{\delta L_{m_1\dots m_{d-2}l_1l_2}}{\delta T^a_{n_1n_2}}\delta^{l_1}_{n_1}\delta^{l_2}_{n_2} \in \Omega^{d-2}_{hor}P(\mathbb{R}^{d^*}) \tag{2.30b}$$

would be the equations of motion obtained if the curvature $\boldsymbol{F}^I$ and torsion $\boldsymbol{T}^a$, respectively, are viewed as an independent fields. Similarly

$$\boldsymbol{Z}^{a_1\dots a_i}_\alpha = \frac{\delta \boldsymbol{L}}{\delta \chi^\alpha_{a_1\dots a_i}} \tag{2.31}$$

would be the equations of motion if the all the derivatives up to the i-th derivative of χ^α Eq. 2.13 (but not higher derivatives) are viewed as independent fields.

Proof. The proof proceeds by varying the Lagrangian Eq. 2.14 considering all the fields and their derivatives as independent and then "integrating by parts" the variations due to the derivatives. Write the variation of the Lagrangian as

$$\delta \boldsymbol{L} = \sum_{i=0}^{k} \boldsymbol{U}^{a_1\dots a_i}_\alpha \delta\chi^\alpha_{a_1\dots a_i} + [\cdots]$$

where here (and throughout the rest of this proof) $[\cdots]$ denotes terms proportional to $\delta\boldsymbol{e}^a$ and $\delta\boldsymbol{A}^I$ and

$$\boldsymbol{U}^{a_1\dots a_i}_\alpha := \frac{\partial \boldsymbol{L}}{\partial \chi^\alpha_{a_1\dots a_i}} \tag{2.32}$$

is a horizontal d-form valued in the appropriate representation of the structure group and we fix the index permutation symmetries of $\boldsymbol{U}$ to be the same as the corresponding field χ. Note that we have used ∂ in Eq. 2.32 to emphasise that we have not performed any "integration by parts" yet. To get the form of the variational principle we have to rewrite the terms obtained by a variation of the derivatives of χ^α in terms of variations of χ^α by "integrating by parts". Consider the variation due to the i-th derivatives as

$$\begin{aligned}\boldsymbol{U}_\alpha^{a_1\dots a_i}\delta\chi^\alpha_{a_1\dots a_i} &= \boldsymbol{U}_\alpha^{a_1\dots a_i}\delta\left(E_{a_i}\cdot D\chi^\alpha_{a_1\dots a_{i-1}}\right)\\ &= \boldsymbol{U}_\alpha^{a_1\dots a_i}\left[-(E_{a_i}\cdot\delta\boldsymbol{e}^b)E_b\cdot D\chi^\alpha_{a_1\dots a_{i-1}} + E_{a_i}\cdot\delta D\chi^\alpha_{a_1\dots a_{i-1}}\right]\\ &= (-)^{d+1}E_{a_i}\cdot\boldsymbol{U}_\alpha^{a_1\dots a_i}\wedge D\delta\chi^\alpha_{a_1\dots a_{i-1}} + [\cdots]\\ &= \boldsymbol{Y}_\alpha^{a_1\dots a_{i-1}}\delta\chi^\alpha_{a_1\dots a_{i-1}} + d\boldsymbol{\theta}^{(i)} + [\cdots]\end{aligned}$$

where the second line uses Eq. 2.22. The term Y then contributes to the variation with respect to the $(i-1)$-th derivative term. Thus define recursively, for any $0 \le i \le k$, the term obtained by a variation of the Lagrangian considering the derivatives up to the i-th derivative of χ^α, but not higher derivatives, as independent

$$\boldsymbol{Z}_\alpha^{a_1\dots a_i} := \begin{cases}\boldsymbol{U}_\alpha^{a_1\dots a_i} & \text{for} \quad i = k\\ \boldsymbol{U}_\alpha^{a_1\dots a_i} + D\left(E_{a_{i+1}}\cdot\boldsymbol{Z}_\alpha^{a_1\dots a_{i+1}}\right) & \text{for} \quad 0 \le i < k\end{cases}$$

Using this the variation due to the i-th derivative term has the terms

$$\begin{aligned}\boldsymbol{Y}_\alpha^{a_1\dots a_i} &= D\left(E_{a_{i+1}}\cdot\boldsymbol{Z}_\alpha^{a_1\dots a_{i+1}}\right)\\ \boldsymbol{\theta}^{(i)} &= -\left(E_{a_i}\cdot\boldsymbol{Z}_\alpha^{a_1\dots a_i}\right)\delta\chi^\alpha_{a_1\dots a_{i-1}}\end{aligned}$$

Iterating the above computation for each derivative order we can write

$$\delta\boldsymbol{L} = \boldsymbol{Z}_\alpha\delta\chi^\alpha + \sum_{i=1}^{k} d\boldsymbol{\theta}^{(i)} + [\cdots]$$

The second term above gives us the term in $\boldsymbol{\theta}'$ in Eq. 2.29. From the collective notation Eq. 2.13, the first term has variations of the charged tensor fields φ^A (which contribute to the equation of motion) as well as those of the curvature and torsion. We then convert the terms obtained from the variations of the curvature and torsion to variations of the connection and coframes.

$$
\begin{aligned}
\boldsymbol{Z}_I^{ab}\delta F_{ab}^I &= \boldsymbol{Z}_I^{ab}\delta(E_b\cdot E_a\cdot \boldsymbol{F}^I) = (-)^{d-2}d(\boldsymbol{Z}_I\wedge\delta\boldsymbol{A}^I)+[\cdots]\\
\boldsymbol{Z}_c^{ab}\delta T_{ab}^c &= \boldsymbol{Z}_c^{ab}\delta(E_b\cdot E_a\cdot \boldsymbol{T}^c) = (-)^{d-2}d(\boldsymbol{Z}_a\wedge\delta\boldsymbol{e}^a)+[\cdots]
\end{aligned}
\tag{2.33}
$$

where $\boldsymbol{Z}_I = E_b\cdot E_a\cdot \boldsymbol{Z}_I^{ab}$ and $\boldsymbol{Z}_a = E_c\cdot E_b\cdot \boldsymbol{Z}_a^{bc}$, and are explicitly given by Eq. 2.30.

Thus, in the total variation of the Lagrangian we can collect all terms proportional to $\delta\boldsymbol{e}^a$, $\delta\boldsymbol{A}^I$ and $\delta\varphi^A$ into the respective equations of motion and an exact form

$$
\delta\boldsymbol{L} = \boldsymbol{\mathcal{E}}_\alpha\wedge\delta\boldsymbol{\psi}^\alpha + d\boldsymbol{\theta}
$$

with $\boldsymbol{\theta}$ given by the claim of the lemma. □

The above algorithm for choosing a horizontal $\boldsymbol{\theta}$ has some ambiguities which we enumerate next.

(1) For some $\boldsymbol{\mu}(E_a^m, \boldsymbol{e}^a, \boldsymbol{A}^I, \{\chi^\alpha_{a_1\ldots a_i}\}) \in \Omega^{d-1}P$, we can add a exact form $d\boldsymbol{\mu}\in\Omega^d_{hor}P$ to the Lagrangian without changing the equations of motion (i.e. the dynamical content of the theory) as

$$
\boldsymbol{L}\mapsto\boldsymbol{L}+d\boldsymbol{\mu}
\tag{2.34}
$$

Since we restrict to horizontal Lagrangians, we only consider $d\boldsymbol{\mu}$ that are horizontal forms, but we do not demand that $\boldsymbol{\mu}$ itself be horizontal i.e. gauge-invariant in contrast to [3]. For instance, $d\boldsymbol{\mu}$ could be the integrand of a topological invariant of the bundle (for example, the Euler density), in which case $\boldsymbol{\mu}$ itself would not be horizontal. This shifts the symplectic potential as

$$
\boldsymbol{\theta}(\delta\psi)\mapsto\boldsymbol{\theta}(\delta\psi)+\delta\boldsymbol{\mu}
$$

Note that we can apply Lemma 2.1 to the Lagrangian $\boldsymbol{L}+d\boldsymbol{\mu}$ and conclude that $\delta\boldsymbol{\mu}$ is horizontal i.e. invariant under internal gauge transformations, even if $\boldsymbol{\mu}$ is not.

(2) Given a choice of Lagrangian, the variational principle Eq. 2.26 only determines the symplectic potential up to the addition of a local, covariant and horizontal $(d-1)$-form $\boldsymbol{\lambda}'(\psi;\delta\psi)$ which is linear in the perturbation $\delta\boldsymbol{\psi}$ and $d\boldsymbol{\lambda}' = 0$. Using Lemma A.2 (with $\boldsymbol{\psi}^\alpha$ as the "background field" and $\delta\boldsymbol{\psi}^\alpha$ as the "dynamical field") we get, $\boldsymbol{\lambda}' = d\boldsymbol{\lambda}$ for some local and covariant horizontal form $\boldsymbol{\lambda}(\psi;\delta\psi) \in \Omega^{d-2}_{hor}P$. Thus, this additional ambiguity in the symplectic current is

$$\boldsymbol{\theta}(\delta\psi) \mapsto \boldsymbol{\theta}(\delta\psi) + d\boldsymbol{\lambda}(\delta\psi) \tag{2.35}$$

Using the symplectic potential we define the *symplectic current* as an antisymmetric bilinear map on perturbations (see [1, 3])

$$\boldsymbol{\omega}(\psi;\delta_1\psi,\delta_2\psi) := \delta_1\boldsymbol{\theta}(\delta_2\psi) - \delta_2\boldsymbol{\theta}(\delta_1\psi) - \boldsymbol{\theta}([\delta_1,\delta_2]\psi) \in \Omega^{d-1}_{hor}P \tag{2.36}$$

In the above definition we have considered the perturbations $\delta\psi$ as vector fields on the space of field configurations evaluated at the background given by ψ. The commutator $[\delta_1,\delta_2]\psi := \delta_1\delta_2\psi - \delta_2\delta_1\psi$ depends on how one chooses to extend these vector fields away from the background ψ in configuration space, even though the symplectic current $\boldsymbol{\omega}$ at ψ is independent of this choice. If the variations δ_1 and δ_2 are extended to correspond to "independent" one-parameter families of dynamical fields then the commutator vanishes, which suffices for most situations. On the other hand, if the variations are extended to correspond to the same infinitesimal action of bundle automorphisms the commutator is non-vanishing in general (this is useful, for instance, when considering Einstein-fluid systems; see [12]).

Note that the symplectic current Eq. 2.36 is a horizontal form and hence gauge-invariant, in the sense that it is invariant under any vertical automorphism $f \in \mathrm{Aut}_V(P)$ of the bundle. However, it does not vanish if we substitute one of the perturbations, say $\delta_2\boldsymbol{\psi}$, by a perturbation $\pounds_X\boldsymbol{\psi}$ generated by an infinitesimal vertical automorphism $X^m \in \mathfrak{aut}_V(P)$

(see Lemma 2.5).

The symplectic current can be shown to be a closed form on solutions (see also [1, 46]) as follows.

Lemma 2.2 (closed symplectic current)**.** *The symplectic current is closed when restricted to solutions and linearised solutions.*

Proof. Consider a second variation of the Lagrangian Eq. 2.14

$$\delta_1\delta_2\boldsymbol{L} = \delta_1\boldsymbol{\mathcal{E}}_\alpha \wedge \delta_2\boldsymbol{\psi}^\alpha + \boldsymbol{\mathcal{E}}_\alpha \wedge \delta_1\delta_2\boldsymbol{\psi}^\alpha + d\delta_1\boldsymbol{\theta}(\delta_2\psi)$$

Using the identity $(\delta_1\delta_2 - \delta_2\delta_1 - [\delta_1,\delta_2])\boldsymbol{L} = 0$ we have:

$$0 = \delta_1\boldsymbol{\mathcal{E}}_\alpha \wedge \delta_2\boldsymbol{\psi}^\alpha - \delta_2\boldsymbol{\mathcal{E}}_\alpha \wedge \delta_1\boldsymbol{\psi}^\alpha - \boldsymbol{\mathcal{E}}_\alpha \wedge [\delta_1,\delta_2]\boldsymbol{\psi}^\alpha + d\boldsymbol{\omega} \tag{2.37}$$

Restricting the above on solutions $\boldsymbol{\psi}$ and linearised solutions $\delta\boldsymbol{\psi}$ we have $d\boldsymbol{\omega} = 0$. □

Since the sympectic current $\boldsymbol{\omega}$ is a horizontal form on P there is a corresponding gauge-invariant form on spacetime (since $\Omega^k_{hor}P \cong \pi^*\Omega^k M$; see § A) which we denote by $\underline{\boldsymbol{\omega}}$. Given a Cauchy surface Σ, the symplectic current defines a *symplectic form* W_Σ on perturbations as

$$W_\Sigma(\psi;\delta_1\psi,\delta_2\psi) := \int_\Sigma \underline{\boldsymbol{\omega}}(\psi;\delta_1\psi,\delta_2\psi) \tag{2.38}$$

From Lemma 2.2 we can conclude that the symplectic form is conserved on linearised solutions i.e. if Σ_t is a time-evolved Cauchy surface then $W_\Sigma(\psi;\delta_1\psi,\delta_2\psi) = W_{\Sigma_t}(\psi;\delta_1\psi,\delta_2\psi)$, whenever ψ is a solution, $\delta_1\psi$ and $\delta_2\psi$ are linearised solutions with boundary conditions such that there is no symplectic flux at infinity.

The $\boldsymbol{\mu}$-ambiguity in the Lagrangian Eq. 2.34 does not affect the symplectic current and from the $\boldsymbol{\lambda}$-ambiguity Eq. 2.35 we have

$$\boldsymbol{\omega}(\delta_1\psi,\delta_2\psi) \mapsto \boldsymbol{\omega}(\delta_1\psi,\delta_2\psi) + d\left[\delta_1\boldsymbol{\lambda}(\delta_2\psi) - \delta_2\boldsymbol{\lambda}(\delta_1\psi) - \boldsymbol{\lambda}([\delta_1,\delta_2]\psi)\right]$$

This adds a boundary term to the symplectic form W_Σ

$$W_\Sigma \mapsto W_\Sigma + \int_{\partial\Sigma} [\delta_1 \underline{\boldsymbol{\lambda}}(\delta_2\psi) - \delta_2\underline{\boldsymbol{\lambda}}(\delta_1\psi) - \underline{\boldsymbol{\lambda}}([\delta_1, \delta_2]\psi)] \tag{2.39}$$

where $\underline{\boldsymbol{\lambda}}$ is the unique gauge-invariant differential form on M that pullsback to the horizontal form $\boldsymbol{\lambda}$ on the bundle i.e. $\boldsymbol{\lambda} = \pi^*\underline{\boldsymbol{\lambda}}$.

Following [3], we will use the symplectic form to derive the first law of black hole mechanics and show that the above ambiguities do not effect the first law. At this point one can generalise the entire analysis of [1] to construct the phase space and Poisson brackets for such theories which is certainly of independent interest. Since we are primarily interested in the first law of black hole mechanics, we turn next to the definition of the Noether charge for any bundle automorphism. The first law then follows from the relation between the symplectic form defined above and the Noether charge (see Lemma 2.5).

2.3 Noether current, Noether charge and boundary Hamiltonians

As is well known, Noether's theorem associates gauge symmetries of a Lagrangian theory to conserved currents and charges (see [1] for instance). The Lagrangians we are considering are both covariant under diffeomorphisms of the base spacetime M as well as invariant under internal gauge transformations. Though, as discussed earlier, there is no natural group action of the diffeomorphisms of M on the dynamical fields with non-trivial internal gauge transformations and we only have a notion of "diffeomorphism up to internal gauge". Thus, we cannot separately define Noether currents associated to only diffeomorphisms and have to consider the full gauge group of the theory. The full group of gauge transformations is the group of automorphisms $\mathrm{Aut}(P)$ of the principal bundle P (see § A). Thus, we will define Noether currents associated to any automorphism in $\mathrm{Aut}(P)$ by adapting the procedure used in [1] to work directly on the principal bundle instead of the base spacetime.

Automorphisms of the bundle $\mathrm{Aut}(P)$ are generated by the Lie algebra $\mathfrak{aut}(P)$ of vector fields on P. The corresponding action of the automorphism on any differential form $\boldsymbol{\phi}$ on the bundle is given by the Lie derivative along a vector field. We denote the variation obtained by the bundle automorphism generated by a vector field $X^m \in \mathfrak{aut}(P)$ as $\delta_X \boldsymbol{\phi} := \pounds_X \boldsymbol{\phi}$. Since we have assumed that the Lagrangian is covariant under such automorphisms, to each automorphism we can associate a Noether current as follows. For the gauge-invariant Lagrangians under consideration $d\boldsymbol{L} \in \Omega^{d+1}_{hor}P$, and hence $d\boldsymbol{L} = 0$. Then for the variation of the Lagrangian under an automorphism we have

$$\delta_X \boldsymbol{L} = \pounds_X \boldsymbol{L} = X \cdot d\boldsymbol{L} + d(X \cdot \boldsymbol{L}) = d(X \cdot \boldsymbol{L}) \tag{2.40}$$

The *Noether current* corresponding to any $X^m \in \mathfrak{aut}(P)$ is defined by (see [1]):

$$\boldsymbol{J}_X := \boldsymbol{\theta}(\delta_X \psi) - X \cdot \boldsymbol{L} \tag{2.41}$$

Here we note that if $X^m \in \mathfrak{aut}_V(P)$ generates vertical automorphisms of the bundle i.e. internal gauge transformations we have $X \cdot \boldsymbol{L} = 0$.

We can define the Noether charge associated to the Noether current $\boldsymbol{J}_X$ as follows. Consider the following computation:

$$\begin{aligned} d\boldsymbol{J}_X &= d\boldsymbol{\theta}(\delta_X \psi) - d(X \cdot \boldsymbol{L}) = \delta_X \boldsymbol{L} - \boldsymbol{\mathcal{E}}_\alpha \wedge \delta_X \boldsymbol{\psi}^\alpha - \delta_X \boldsymbol{L} \\ &= -\boldsymbol{\mathcal{E}}_\alpha \wedge \pounds_X \boldsymbol{\psi}^\alpha \end{aligned} \tag{2.42}$$

This immediately implies that $\boldsymbol{J}_X$ is a closed horizontal form (i.e. a conserved current) for (1) any $X^m \in \mathfrak{aut}(P)$ if $\boldsymbol{\psi}$ is a solution; or (2) for any $\boldsymbol{\psi}$ if $X^m \in \mathfrak{aut}(P;\psi)$ is an infinitesimal automorphism that preserves $\boldsymbol{\psi}^\alpha$ i.e. $\pounds_X \boldsymbol{\psi}^\alpha = 0$. For case (1), following [3] and using Lemma A.2, we can write $\boldsymbol{J}_X = d\boldsymbol{Q}_X$ where $\boldsymbol{Q}_X$ is the Noether charge.

In fact we can define a Noether charge even without using the equations of motion ("off-shell") by adapting the procedure in [46] to work on the bundle. To do this, we first define the following linear maps from infinitesimal automorphisms of the bundle to horizontal forms, which are generalised versions of the *constraints* and *Bianchi identities* (see [1, 46]).

The *constraints* are linear maps $\boldsymbol{\mathcal{C}} : \mathfrak{aut}(P) \to \Omega^{d-1}_{hor} P$ given by

$$\begin{aligned} \boldsymbol{\mathcal{C}}(X) &:= (-)^{d-\deg(\alpha)+1} \boldsymbol{\mathcal{E}}_\alpha \wedge X \cdot \boldsymbol{\psi}^\alpha \\ &= (-)^d \left[\boldsymbol{\mathcal{E}}_a (X \cdot \boldsymbol{e}^a) + \boldsymbol{\mathcal{E}}_I (X \cdot \boldsymbol{A}^I) \right] \end{aligned} \tag{2.43}$$

and in the second line we have used Eqs. 2.15 and 2.16. Since $\boldsymbol{\mathcal{C}}(X)$ is a horizontal form on the bundle we can consider the corresponding gauge-invariant $(d-1)$-form $\underline{\boldsymbol{\mathcal{C}}}(X)$ on spacetime. The pullback of $\underline{\boldsymbol{\mathcal{C}}}(X)$ to any Cauchy surface then are the constraint equations that hold for any initial data for dynamical fields which correspond to a solution to the equations of motion. Note that none of the charged tensor fields φ^A and their equations of motion contribute to the constraints, since we consider the frame components of tensor fields as the dynamical fields. Further, if $X^m \in \mathfrak{aut}_V(P)$ then from Eq. 2.15 only the connection $\boldsymbol{A}^I$ and its equation of motion $\boldsymbol{\mathcal{E}}_I$ contribute to the constraints corresponding to gauge transformations.

The Bianchi identities $\boldsymbol{\mathcal{B}} : \mathfrak{aut}(P) \to \Omega^d_{hor} P$ are linear maps given explicitly by

$$\begin{aligned} \boldsymbol{\mathcal{B}}(X) &:= -\boldsymbol{\mathcal{E}}_\alpha \wedge X \cdot d\boldsymbol{\psi}^\alpha + (-)^{d-\deg(\alpha)} d\boldsymbol{\mathcal{E}}_\alpha \wedge X \cdot \boldsymbol{\psi}^\alpha \\ &= -\boldsymbol{\mathcal{E}}_a \wedge (X \cdot \boldsymbol{T}^a) + (-)^{d-1} D\boldsymbol{\mathcal{E}}_a (X \cdot \boldsymbol{e}^a) + \boldsymbol{\mathcal{E}}_a \wedge \boldsymbol{e}^b (X \cdot \boldsymbol{\omega}^a{}_b) \\ &\quad - \boldsymbol{\mathcal{E}}_I \wedge (X \cdot \boldsymbol{F}^I) + (-)^{d-1} D\boldsymbol{\mathcal{E}}_I (X \cdot \boldsymbol{A}^I) \\ &\quad - \boldsymbol{\mathcal{E}}_A (X \cdot D\varphi^A) + \boldsymbol{\mathcal{E}}_A \varphi^B (X \cdot \boldsymbol{A}^I) r_I{}^A{}_B \end{aligned} \tag{2.44}$$

where in the second equality we have used Eq. 2.15 and written all terms as manifestly horizontal forms. Note that the second line only depends on the gravitational connection $\boldsymbol{\omega}^a{}_b$ while the rest depend on the full connection $\boldsymbol{A}^I$ on P (see Eq. 2.11). These Bianchi identities above are a generalisation of the ones for diffeomorphism covariant theories given in [1, 46] to include all automorphisms of the bundle i.e. both gauge transformations as well as diffeomorphisms.

Using Eqs. 2.43 and 2.44, we can rewrite Eq. 2.42 in the following form

$$d\left[\boldsymbol{J}_X - \boldsymbol{\mathcal{C}}(X)\right] = \boldsymbol{\mathcal{B}}(X) \tag{2.45}$$

Using the arguments in § IV. [46], we can show that the Bianchi identities $\boldsymbol{\mathcal{B}}(X)$ vanish identically on all dynamical fields even those that do not satisfy the equations of motion.

Proposition 2.1 (Bianchi identities)**.** *The Bianchi identities $\boldsymbol{\mathcal{B}}(X)$ vanish for any dynamical field $\boldsymbol{\psi}^\alpha$ for all $X^m \in \mathfrak{aut}(P)$.*

Proof. Since $\boldsymbol{\mathcal{B}}(X)$ is a horizontal form denote the corresponding gauge-invariant form on spacetime by $\underline{\boldsymbol{\mathcal{B}}}(X) \in \Omega^d M$. Then we can show that $\underline{\boldsymbol{\mathcal{B}}}(X) = 0$ using same argument as in § IV. [46]. Thus $\boldsymbol{\mathcal{B}}(X) = \pi^* \underline{\boldsymbol{\mathcal{B}}}(X) = 0$ for any $\boldsymbol{\psi}$ and all $X \in \mathfrak{aut}(P)$. □

Using the above we can define the Noether charge without using any equations of motion or symmetries as follows

Lemma 2.3 (Noether charge)**.** *For any infinitesimal automorphism $X^m \in \mathfrak{aut}(P)$ there exists a horizontal $(d-2)$-form $\boldsymbol{Q}_X \in \Omega^{d-2}_{hor} P$ called the Noether charge, such that the Noether current $\boldsymbol{J}_X$ can be written in the form*

$$\boldsymbol{J}_X = d\boldsymbol{Q}_X + \boldsymbol{\mathcal{C}}(X) \tag{2.46}$$

Proof. Since $\boldsymbol{\mathcal{B}}(X) = 0$, from Eq. 2.45 we have $d\left[\boldsymbol{J}_X - \boldsymbol{\mathcal{C}}(X)\right] = 0$. Then using Lemma A.2 (with X^m as the "dynamical field" and $\boldsymbol{\psi}^\alpha$ as a "background field") we conclude that there exists a $\boldsymbol{Q}_X \in \Omega^{d-2}_{hor} P$ (which depends linearly on X^m and finitely many of it derivatives) such Eq. 2.46 holds. □

Lemma 2.3 shows that the Noether charge exists but for any theory based on a Lagrangian of the form Eq. 2.14 we can obtain an explicit useful expression for the Noether charge (this generalises Proposition 4.1 [3]) as follows

Lemma 2.4 (Form of the Noether charge)**.** *The Noether charge* $\boldsymbol{Q}_X$ *for* $X^m \in \mathfrak{aut}(P)$ *can be chosen to be of the form*

$$\boldsymbol{Q}_X = \boldsymbol{Z}_I(X \cdot \boldsymbol{A}^I) + \boldsymbol{Z}_a(X \cdot \boldsymbol{e}^a) \in \Omega^{d-2}_{hor}P \tag{2.47}$$

where $\boldsymbol{Z}_I$ *and* $\boldsymbol{Z}_a$ *are as in Lemma 2.1 and can be computed directly from the Lagrangian using Eq. 2.30.*

Proof. To get an explicit form for the Noether charge we start with Eq. 2.41 and use the form of the symplectic current given by Lemma 2.1 to compute

$$\boldsymbol{\theta}(\delta_X\psi) = (-)^{d-2}\boldsymbol{Z}_I \wedge \pounds_X \boldsymbol{A}^I + (-)^{d-2}\boldsymbol{Z}_a \wedge \pounds_X \boldsymbol{e}^a + \boldsymbol{\theta}'(\delta_X\psi)$$

Using Eq. A.7 for the Lie derivatives of the connection and coframes, we can write the first two terms as

$$\begin{aligned}&(-)^{d-2}\boldsymbol{Z}_I \wedge \Big[X \cdot \boldsymbol{F}^I + D(X \cdot \boldsymbol{A}^I)\Big] + (-)^{d-2}\boldsymbol{Z}_a \wedge \Big[X \cdot \boldsymbol{T}^a + D(X \cdot \boldsymbol{e}^a) - (X \cdot \boldsymbol{A}^a{}_b)\boldsymbol{e}^b\Big] \\ &= d\Big[\boldsymbol{Z}_I(X \cdot \boldsymbol{A}^I) + \boldsymbol{Z}_a(X \cdot \boldsymbol{e}^a)\Big] + [\cdots]\end{aligned}$$

where thoughout this proof the $[\cdots]$ represents a local, covariant, and horizontal $d-1$ form which is linear in X^m and independent of its derivatives. A similar computation for the $\boldsymbol{\theta}'$ term only gives $[\cdots]$-type terms since the $\chi^\alpha_{a_1 \ldots a_i}$ are all functions i.e. 0-forms. Thus, using Eq. 2.41 the Noether current can be written as

$$\boldsymbol{J}_X = d\Big[\boldsymbol{Z}_I(X \cdot \boldsymbol{A}^I) + \boldsymbol{Z}_a(X \cdot \boldsymbol{e}^a)\Big] + [\cdots]$$

where we have again absorbed $X \cdot \boldsymbol{L}$ into the $[\cdots]$-term. Adding the constraints $\boldsymbol{\mathcal{C}}(X)$ in Eq. 2.43, which are also of $[\cdots]$-type, to both sides we get

$$\boldsymbol{J}_X - \boldsymbol{\mathcal{C}}(X) - d\left[\boldsymbol{Z}_I(X \cdot \boldsymbol{A}^I) + \boldsymbol{Z}_a(X \cdot \boldsymbol{e}^a)\right] = [\cdots]$$

Since $d\left[\boldsymbol{J}_X - \boldsymbol{\mathcal{C}}(X)\right] = 0$ from Eq. 2.45 and Prop. 2.1, we get that the right-hand-side is a closed horizontal $(d-1)$-form that does not depend on derivatives of X^m. Using Lemma A.2 (with X^m as the "dynamical field" and $\boldsymbol{\psi}^\alpha$ as a "background field") we conclude that right-hand-side vanishes and we get

$$\boldsymbol{J}_X = \boldsymbol{\mathcal{C}}(X) + d\left[\boldsymbol{Z}_I(X \cdot \boldsymbol{A}^I) + \boldsymbol{Z}_a(X \cdot \boldsymbol{e}^a)\right]$$

Thus the Noether charge can be chosen to be of the form in Eq. 2.47. □

Note here that only the coframes and connection contribute explicitly to the form of the Noether charge since we have converted all other tensor fields and their derivatives into functions (using the coframe and frames). Consequently the form of the Noether charge given by Lemma 2.4 is much simpler than the corresponding one in Proposition 4.1 [3]. Further the expression Eq. 2.47 for the Noether charge is completely specified by the dependence of the Lagrangian on the curvature and torsion.

The ambiguities Eqs. 2.34 and 2.35 in the Lagrangian and the symplectic potential lead to the following ambiguities in the Noether current and Noether charge

$$\begin{aligned} \boldsymbol{J}_X &\mapsto \boldsymbol{J}_X + d(X \cdot \boldsymbol{\mu}) + d\boldsymbol{\lambda}(\delta_X \psi) \\ \boldsymbol{Q}_X &\mapsto \boldsymbol{Q}_X + X \cdot \boldsymbol{\mu} + \boldsymbol{\lambda}(\delta_X \psi) + d\boldsymbol{\rho} \end{aligned} \tag{2.48}$$

where $\boldsymbol{\rho}$ is an extra ambiguity in the Noether charge, since the charge is defined by the Noether current only up to a closed and hence exact form (see Lemma A.2). We note that the form of the Noether charge given by Lemma 2.4 is not unambiguous. If the terms $\boldsymbol{\mu}$ and

$\boldsymbol{\lambda}$ have suitable dependence on the curvature and torsion they can contribute non-trivially to the Noether charge.

The utility of the above formalism in deriving a first law stems from the following relation between the symplectic current and the Noether charge. The proof follows by a simple computation on the bundle P, in exact parallel to the ones on spacetime in [1, 3, 46, 10].

Lemma 2.5. *For any perturbation $\delta\boldsymbol{\psi}$ and $X^m \in \mathfrak{aut}(P)$, the symplectic current $\boldsymbol{\omega}(\delta\psi, \pounds_X\psi)$ is related to the Noether charge $\boldsymbol{Q}_X$ by*

$$\boldsymbol{\omega}(\delta\psi, \pounds_X\psi) = d\left[\delta\boldsymbol{Q}_X - X\cdot\boldsymbol{\theta}(\delta\psi)\right] + \delta\boldsymbol{\mathcal{C}}(X) + X\cdot(\boldsymbol{\mathcal{E}}_\alpha \wedge \delta\boldsymbol{\psi}^\alpha) \tag{2.49}$$

Proof. Consider a variation of Eq. 2.41 with a given fixed X^m

$$\begin{aligned}\delta\boldsymbol{J}_X &= \delta\boldsymbol{\theta}(\delta_X\psi) - X\cdot\delta\boldsymbol{L} \\ &= \delta\boldsymbol{\theta}(\delta_X\psi) - X\cdot d\boldsymbol{\theta}(\delta\psi) - X\cdot(\boldsymbol{\mathcal{E}}_\alpha\wedge\delta\boldsymbol{\psi}^\alpha) \\ &= \delta\boldsymbol{\theta}(\pounds_X\psi) - \pounds_X\boldsymbol{\theta}(\delta\psi) + d(X\cdot\boldsymbol{\theta}(\delta\psi)) - X\cdot(\boldsymbol{\mathcal{E}}_\alpha\wedge\delta\boldsymbol{\psi}^\alpha)\end{aligned} \tag{2.50}$$

The first two terms on the right-hand-side can be rewritten in terms of the symplectic current using Eq. 2.36 to get

$$\begin{aligned}\boldsymbol{\omega}(\delta\psi, \pounds_X\psi) &= \delta\boldsymbol{J}_X - d(X\cdot\boldsymbol{\theta}(\delta\psi)) + X\cdot(\boldsymbol{\mathcal{E}}_\alpha\wedge\delta\boldsymbol{\psi}^\alpha) \\ &= d\left[\delta\boldsymbol{Q}_X - X\cdot\boldsymbol{\theta}(\delta\psi)\right] + \delta\boldsymbol{\mathcal{C}}(X) + X\cdot(\boldsymbol{\mathcal{E}}_\alpha\wedge\delta\boldsymbol{\psi}^\alpha)\end{aligned} \tag{2.51}$$

□

From Lemma 2.5 we see that for linearised solutions $\delta\boldsymbol{\psi}^\alpha$ we have

$$\boldsymbol{\omega}(\delta\psi, \pounds_X\psi) = d\left[\delta\boldsymbol{Q}_X - X\cdot\boldsymbol{\theta}(\delta\psi)\right] \tag{2.52}$$

and the corresponding symplectic form Eq. 2.38 on a Cauchy surface Σ is an integral of a boundary term on $\partial\Sigma$.

$$W_\Sigma(\psi;\delta\psi,\pounds_X\psi) = \int_{\partial\Sigma} \delta\underline{\boldsymbol{Q}}_X - \underline{X}\cdot\underline{\boldsymbol{\theta}}(\delta\psi) \tag{2.53}$$

Here we have used the fact that both $\boldsymbol{Q}_X$ and $\boldsymbol{\theta}$ are horizontal (i.e. gauge-invariant) forms on P and thus can be represented as gauge-invariant forms $\underline{\boldsymbol{Q}}_X$ and $\underline{\boldsymbol{\theta}}$ on spacetime M. Further, since $\boldsymbol{\theta}$ is horizontal, only the projection $\underline{X} \equiv X^\mu = (\pi_*)^m_\mu X^\mu$ contributes in the second term, but the Noether charge $\underline{\boldsymbol{Q}}_X$ depends on the full vector field $X^m \in \mathfrak{aut}(P)$.

As discussed in [3], a boundary Hamiltonian for the dynamics generated by X^m exists if and only if there is a function H_X on the space of solutions such that its variation is given by

$$\delta H_X = W_\Sigma(\psi;\delta\psi,\pounds_X\psi) = \int_{\partial\Sigma} \delta\underline{\boldsymbol{Q}}_X - \underline{X}\cdot\underline{\boldsymbol{\theta}}(\delta\psi) \tag{2.54}$$

which is equivalent to the existence of $\underline{\boldsymbol{\Theta}}(\psi) \in \Omega^{d-1}M$ so that

$$\int_{\partial\Sigma} \underline{X}\cdot\underline{\boldsymbol{\theta}}(\delta\psi) = \delta\int_{\partial\Sigma} \underline{X}\cdot\underline{\boldsymbol{\Theta}}(\psi) \tag{2.55}$$

Note here that $\underline{\boldsymbol{\Theta}}$ need not be covariant or gauge-invariant in its dependence on the dynamical fields. Thus the boundary Hamiltonian becomes

$$H_X = \int_{\partial\Sigma} \underline{\boldsymbol{Q}}_X - \underline{X}\cdot\underline{\boldsymbol{\Theta}}(\psi) \tag{2.56}$$

The existence of the Hamiltonian H_X is intimately related to the boundary conditions at $\partial\Sigma$ on the dynamical fields $\boldsymbol{\psi}$, the perturbation $\delta\boldsymbol{\psi}$, and the vector field X^m; for general field configurations and arbitrary perturbations the Hamiltonian might not exist or may not be unique. Imposing boundary conditions so that the symplectic form $W_\Sigma(\psi;\delta\psi,\pounds_X\psi)$ is finite ensures that the perturbed Hamiltonian δH_X is also well-defined. But, even if we choose boundary conditions so that δH_X is well-defined (it is manifestly covariant and gauge-invariant as it is defined in terms of horizontal forms on P) there still might not exist a unique, or covariant, or gauge-invariant Hamiltonian H_X.

CHAPTER 3

HORIZON POTENTIALS AND CHARGES, AND THE ZEROTH LAW FOR BIFURCATE KILLING HORIZONS

We describe next the spacetimes for which we will derive a zeroth law for bifurcate Killing horizons and a first law of blackhole mechanics.

To formulate the zeroth law for bifucate Killing horizons, we will consider dynamical fields $\boldsymbol{\psi}^\alpha$ which determine a d-dimensional spacetime M with a bifurcate Killing horizon $\mathscr{H} := \mathscr{H}^+ \cup \mathscr{H}^-$ and let the bifurcation surface be $B := \mathscr{H}^+ \cap \mathscr{H}^-$. The *horizon Killing field* K^μ is null on $\mathscr{H}$ and vanishes on B. We denote the corresponding infinitesimal automorphism on the bundle by K^m, which preserves the background dynamical fields i.e. $\pounds_K \boldsymbol{\psi}^\alpha = 0$. The possible ambiguity in the choice of such K^m is given in Lemma A.1, Remark A.3 and Lemma A.4. For the zeroth law, we do not demand that the spacetime described above be determined by solutions to the equations of motion $\boldsymbol{\mathcal{E}}_\alpha = 0$ (Eq. 2.27) nor do we require any asymptotic conditions.

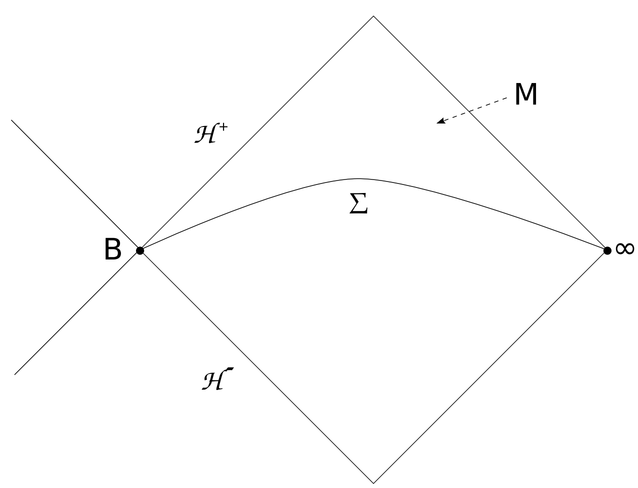

Figure 3.1: Carter-Penrose diagram of the black hole exterior spacetime $(M, g_{\mu\nu})$.

For the first law of black hole mechanics we will consider stationary and axisymmetric dynamical fields $\boldsymbol{\psi}^\alpha$ (Eq. 2.15) which determine a d-dimensional, *asymptotically flat*, stationary and axisymmetric black hole spacetime with Lorentzian metric $g_{\mu\nu}$ with a bifurcate Killing horizon as described above. For the bundle P (Eq. 2.10) on which we have formulated the theory, we choose as the base space M, the exterior (including the horizon) of the black hole (see Fig. 3.1). The metric $g_{\mu\nu}$ on M is determined by the (local) coframes $\boldsymbol{e}^a$ by Eq. 2.3. Thus, we can now identify the abstract Lorentz bundle P_O in Eq. 2.10 with the bundle $F_O M$ of orthonormal frames determined by $g_{\mu\nu}$ (see § A.1).

Since, the dynamical fields $\boldsymbol{\psi}^\alpha$ are defined on the bundle P we define stationarity and axisymmetry of $\boldsymbol{\psi}^\alpha$ using Def. A.2 which we summarise as follows. Let t^μ denote the time translation Killing field, i.e. the Killing field that is timelike near infinity, and $\phi^\mu_{(i)}$ the axial Killing fields (we use the index (i) to account for more than one axial Killing fields at infinity in greater than 4 dimensions). Then, there exist infinitesimal automorphisms, $t^m, \phi^m_{(i)} \in \mathfrak{aut}(P; \boldsymbol{\psi})$ which preserve the dynamical fields i.e. $\pounds_t \boldsymbol{\psi}^\alpha = 0 = \pounds_{\phi_{(i)}} \boldsymbol{\psi}^\alpha$ on the bundle P, which project to the corresponding stationary and axial Killing fields t^μ and $\phi^\mu_{(i)}$ respectively. The ambiguity in the choice of such t^m and $\phi^m_{(i)}$ is given in Lemma A.1 and Lemma A.4. In this case, the horizon Killing field is $K^\mu = t^\mu + \Omega^{(i)}_{\mathscr{H}} \phi^\mu_{(i)}$ (where $\Omega^{(i)}_{\mathscr{H}}$ are constants representing the *horizon angular velocities*) is null on $\mathscr{H}$ and vanishes on B. We denote the corresponding infinitesimal automorphism on the bundle by $K^m = t^m + \Omega^{(i)}_{\mathscr{H}} \phi^m_{(i)}$.

We will define asymptotic flatness as follows. There exist asymptotically Minkowskian coordinates (t, x^i) near spatial infinity, with $r := \sqrt{\sum_i (x^i)^2}$ being the asymptotic radial coordinate and $\eta_{\mu\nu} \equiv \text{diag}(-1, 1, \ldots, 1)$ be the asymptotic flat metric in these coordinates. We require the asymptotic fall-off of the metric to be

$$g_{\mu\nu} = \eta_{\mu\nu} + O(1/r^{d-3}) \tag{3.1}$$

with each derivative of the metric falling-off faster by a factor of $1/r$. To prescribe the fall-off conditions for the dynamical fields defined on the bundle, we lift the asymptotic radial coordinate r (viewed as a function on M near infinity) to the bundle near spatial infinity. Then, the fall-off conditions on dynamical fields on the bundle are prescribed as their behaviour in $1/r$; the precise fall-off conditions are chosen depending on the equations of motion of the theory under consideration so that the solution metric behaves as in Eq. 3.1. We expect our results can be generalised to spacetimes with different asymptotics but we stick to the asymptotically flat case.

In the formulation of the first law we will also consider an asymptotically flat Cauchy surface Σ which smoothly terminates at the bifurcation surface B. We further assume that the embeddings of Σ and B in M are regular i.e. they admit smooth no-where vanishing normals.

* * *

Next, we show that the black hole spacetime described above satisfies a generalisation of *the zeroth law for bifurcate Killing horizons* in the sense that, we can define certain potentials which are constant on the bifurcate Killing horizon $\mathscr{H}$. The term "zeroth law" for this result is justified by Theorem 2 (see Remark 3.4), where we show that the horizon potential contributed by the gravitational Lorentz connection can be identified with the surface gravity of the black hole.

To prove the zeroth law we first show that the Lie-algebra-valued function $K \cdot \boldsymbol{A}^I$ is covariantly constant on the bundle over the bifurcate Killing horizon.

Proposition 3.1. *Let the principal bundle P restricted to the bifurcate Killing horizon $\mathscr{H}$ be $P_{\mathscr{H}}$. Then, the pullback of $D(K \cdot \boldsymbol{A}^I)$ to $P_{\mathscr{H}}$ vanishes i.e.*

$$D(K \cdot \boldsymbol{A}^I)\Big|_{P_{\mathscr{H}}} = 0 \tag{3.2}$$

Proof. First let P_B be the restriction of the principal bundle to the bifurcation surface B. Since the horizon Killing field satisfies $K^\mu|_B = 0$, the corresponding vector field K^m is vertical on P_B. As a result we have $K \cdot \boldsymbol{F}^I\Big|_{P_B} = 0$, since $\boldsymbol{F}^I$ is a horizontal form.

Now, consider any cross-section B' of the bifurcate Killing horizon $\mathscr{H}$. Along the flow generated by the vector field K^μ, B' limits to the bifurcation surface B (see § 2 [47]). Similarly, the principal bundle $P_{B'}$ limits to P_B under the flow along the integral curves of the corresponding vector field K^m on the bundle $P_{\mathscr{H}}$. Clearly, the pullback of $K \cdot \boldsymbol{F}^I$ to the integral curves of K^m vanishes, and so consider then $K \cdot \boldsymbol{F}^I\Big|_{P_{B'}}$. Taking the limit of $K \cdot \boldsymbol{F}^I\Big|_{P_{B'}}$ along the flow of K^m as $P_{B'} \to P_B$, we have $K \cdot \boldsymbol{F}^I\Big|_{P_{B'}} \to 0$. Since, K^m is an infinitesimal automorphism which preserves the connection $\boldsymbol{A}^I$, $\pounds_K(K \cdot \boldsymbol{F}^I)\Big|_{P_{\mathscr{H}}} = 0$ and since the curvature and K^m are smooth on the bundle $P_{\mathscr{H}}$ we must have $K \cdot \boldsymbol{F}^I\Big|_{P_{B'}} = 0$, and thus

$$K \cdot \boldsymbol{F}^I\Big|_{P_{\mathscr{H}}} = 0 \tag{3.3}$$

Thus, using Eq. A.7 for the Lie derivative along K^m of the connection we have

$$\begin{aligned} 0 = \pounds_K \boldsymbol{A}^I\Big|_{P_{\mathscr{H}}} &= K \cdot \boldsymbol{F}^I\Big|_{P_{\mathscr{H}}} + D(K \cdot \boldsymbol{A}^I)\Big|_{P_{\mathscr{H}}} \\ &= D(K \cdot \boldsymbol{A}^I)\Big|_{P_{\mathscr{H}}} \end{aligned} \tag{3.4}$$

□

To define the horizon potentials we will expand the covariantly constant Lie-algebra-valued function $K \cdot \boldsymbol{A}^I$ on $P_{\mathscr{H}}$ in a basis of the Lie algebra $\mathfrak{g}$.[1] To construct a suitable basis we need the notion of a Cartan subalgebra defined as follows.

Definition 3.1 (Cartan subalgebra)**.** A *Cartan subalgebra* $\mathfrak{h}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is a maximal abelian Lie subalgebra such that the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ (given by the Lie bracket) is diagonalisable (see §2.1 [49]).

Properties 3.1 (Properties of a Cartan subalgebra)**.** We list below the key properties of Cartan subalgebras we will need in the following.

1. Our strategy to define the horizon potentials in terms of the Cartan subalgebra parallels the one used in § V. [48] to define global charges in Yang-Mills theory.

(1) Any two Cartan subalgebras of a semisimple complex Lie algebra $\mathfrak{g}$ are isomorphic under the adjoint action of some element of the corresponding group G (Theorem F. §2.10 [49]); the dimension of the Cartan subalgebras is called the *rank* of $\mathfrak{g}$.

(2) The Killing form k_{IJ} is non-degenerate on any Cartan subalgebra of $\mathfrak{g}$ (§ 2.3 [49]).

(3) For any given choice of Cartan subalgebra $\mathfrak{h}$ (of dimension l), there exists a choice of basis for $\mathfrak{g}$ (of dimension n) — the *Weyl-Chevalley basis* — given by $\{h^I_\Lambda, a^I_i\}$. Here, h^I_Λ with $\Lambda = 1, 2, \ldots, l$ are a choice of *simple coroots* (see § 2 [49]) and form a basis of $\mathfrak{h}$. The remaining basis elements a^I_i for $i = 1, 2, \ldots, n - l$ are orthogonal to $\mathfrak{h}$ with respect to the Killing form k_{IJ} (see § 2.8 and 2.9 [49] for details).

(4) Given a choice of the simple coroots h^I_Λ any other choice can be obtained by the action of a *finite* subgroup of G — called the *Weyl group of* $\mathfrak{g}$ (§ 2.11 [49]). The action of the Weyl group elements on the h^I_Λ is generated by certain permutations and sign changes (see § 2.14 [49] for a description of the simple coroots and the Weyl group for simple Lie algebras).

(5) Any given element $X^I \in \mathfrak{g}$ can be mapped into a chosen Cartan subalgebra $\mathfrak{h}$ by the adjoint action of some element of the group G [50]. For a given $X^I \in \mathfrak{g}$, all possible choices of the corresponding element in $\mathfrak{h}$ under the above map, are related by the action of the Weyl group on $\mathfrak{h}$; since the Weyl group is a finite group there are only *finitely many* possible choices.[2]

The above properties of a Cartan subalgebra strictly hold for a complex semisimple Lie algebra. When, the Lie algebra $\mathfrak{g}$ of the structure group G of the theory under consideration is a real semisimple Lie algebra, we first take its complexification $\mathfrak{g}^{\mathbb{C}}$ (which is also semisimple; see § 11.3 [52]) to apply the Cartan subalgebra construction above, and then in the end take the real form of $\mathfrak{g}^{\mathbb{C}}$ corresponding to the original real Lie algebra $\mathfrak{g}$ (see § 11.10 [52]).

2. This last statement can be proved by writing an element of $\mathfrak{h}$ in a basis given by the simple coroots, and then applying the results of the first theorem in § 10.3 [51].

Using the above properties of a Cartan subalgebra of $\mathfrak{g}$ we define the horizon potentials $\mathscr{V}^\Lambda$ at a point of the bifurcate Killing horizon $\mathscr{H}$ in the following.

Proposition 3.2 (Horizon potentials at a point)**.** *Let $\mathfrak{h}$ be some fixed choice of Cartan subalgebra of $\mathfrak{g}$ and let h^I_Λ given by the simple coroots be a choice of basis of $\mathfrak{h}$ (Property (3)). For any point $x \in \mathscr{H}$ on the horizon there exists a point $u \in P_{\mathscr{H}}$ such that $\pi(u) = x$ and $K \cdot \boldsymbol{A}^I(u) \in \mathfrak{h}$. Thus, in the chosen basis of $\mathfrak{h}$ we can write*

$$K \cdot \boldsymbol{A}^I(u) = \mathscr{V}^\Lambda h^I_\Lambda \tag{3.5}$$

The set of coefficients $\mathscr{V}^\Lambda$ is determined up to the action of the Weyl group of $\mathfrak{g}$, irrespective of the chosen point $u \in \pi^{-1}(x)$ and the chosen simple coroots h^I_Λ.

We call the coefficients $\mathscr{V}^\Lambda$ in the above expansion, the horizon potentials.

Proof. For any point $x \in \mathscr{H}$, the fibre of the principal bundle $P_{\mathscr{H}}$ over x is $\pi^{-1}(x) \cong G$. Note that $K \cdot \boldsymbol{A}^I$ is a $\mathfrak{g}$-valued function with the adjoint representation of G. Using Property (5), there exists some point $u \in \pi^{-1}(x)$ in the fibre where $K \cdot \boldsymbol{A}^I(u) \in \mathfrak{h}$. Then, using the chosen basis h^I_Λ of $\mathfrak{h}$ we can define the coefficients $\mathscr{V}^\Lambda$ as in Eq. 3.5 at the point $u \in P_{\mathscr{H}}$.

Suppose there exists another point $\tilde{u} \in \pi^{-1}(x)$ such that $K \cdot \boldsymbol{A}^I(\tilde{u}) \in \mathfrak{h}$ and hence Eq. 3.5 holds with some other set of coefficients $\tilde{\mathscr{V}}^\Lambda$. From Property (5), the new set of potentials $\tilde{\mathscr{V}}^\Lambda$ are related to the original set $\mathscr{V}^\Lambda$ by the action of some element of the Weyl group on $\mathfrak{h}$. Similarly by Property (4), the possible choice of simple coroots are given by the action of the Weyl group on the original choice h^I_Λ.

Thus, the potentials $\mathscr{V}^\Lambda$ are well-defined on the horizon up to the action of the Weyl group of $\mathfrak{g}$. □

Using Prop. 3.2 as the definition of the horizon potentials we next prove a generalised zeroth law for bifurcate Killing horizons.

Theorem 1 (The zeroth law for bifurcate Killing horizons)**.** *The horizon potentials* $\mathscr{V}^\Lambda$ *given by Prop. 3.2 can be chosen to be constant on the horizon*

$$d\mathscr{V}^\Lambda\Big|_{\mathscr{H}} = 0 \tag{3.6}$$

Proof. Let $x, x' \in \mathscr{H}$ be any two points on the horizon connected by a path γ, and let Eq. 3.5 hold at some choice of $u \in \pi^{-1}(x)$ as discussed in Prop. 3.2. Let Γ be the unique path in $P_{\mathscr{H}}$ starting at u which is horizontal with respect to the given connection $\boldsymbol{A}^I$ and projects to the path γ (see Prop. 3.1 § II.3 [53]), and let $u' \in \pi^{-1}(x')$ be the endpoint of Γ. From Prop. 3.1 we have $D(K \cdot \boldsymbol{A}^I)\Big|_{P_{\mathscr{H}}} = 0$. This implies that we can obtain $K \cdot \boldsymbol{A}^I(u')$ by parallel-transporting $K \cdot \boldsymbol{A}^I(u)$ along Γ to always point in the same Lie algebra direction, and further since $K \cdot \boldsymbol{A}^I$ is covariantly constant on $P_{\mathscr{H}}$, the result is independent of the choice of path γ on the horizon. Thus, $K \cdot \boldsymbol{A}^I(u') \in \mathfrak{h}$ and Eq. 3.5 holds at the point $u' \in \pi^{-1}(x')$ with the same set of potentials $\mathscr{V}^\Lambda$. Since the chosen points x, x' are arbitrary and $K \cdot \boldsymbol{A}^I$ is smooth, the potentials $\mathscr{V}^\Lambda$ must be constant on the entire horizon. □

Remark 3.1 (Ambiguity in the horizon potentials). The first set of ambiguities in the horizon potentials arises due to our choice of a fixed Cartan subalgebra $\mathfrak{h}$. From Property (1) we see that different choices of $\mathfrak{h}$ will lead to equivalent sets of horizon potentials.

The other ambiguity in the horizon potentials arises due to our choice of the vector field K^m on the bundle. From Lemma A.1, given the horizon Killing field K^μ on spacetime M, the ambiguity in the corresponding vector field K^m on P is given by $K^m \mapsto \tilde{K}^m = K^m + Y^m$ where Y^m is a vertical vector field so that $Y \cdot \boldsymbol{A}^I \in \mathfrak{g}$ is covariantly constant everywhere (not just on the horizon) on P. Further, if there are other dynamical charged tensor fields (such as the φ^A in Eq. 2.15) in the background, then Y^m is also required to preserve them i.e. Y^m must also satisfy Eq. A.13. If a non-trivial Y^m exists for the given dynamical fields $\boldsymbol{\psi}^\alpha$ (Eq. 2.15), the new choice $\tilde{K}^m$ will define a new set of potentials $\tilde{\mathscr{V}}^\Lambda$ at the horizon. This ambiguity in the potentials does not affect the zeroth law Eq. 3.6 since $\tilde{\mathscr{V}}^\Lambda$ are also constant on the horizon. However, there might exist some Y^m so that we can reduce the number of linearly independent potentials. From Remarks A.2 and A.3 we see that this ambiguity Y^m corresponds to a global symmetry of *all* the dynamical fields $\boldsymbol{\psi}^\alpha$. Thus, the number of linearly independent horizon potentials are ambiguous if the dynamical fields $\boldsymbol{\psi}^\alpha$ have a global symmetry on P. In that case, we can use Y^m to redefine the vector field K^m on P so that some, or all, of the horizon potentials vanish. This redefinition of the horizon potentials also changes the terms at infinity in the first law (see Remark 5.2 for the case of Einstein-Yang-Mills theory). Note however, for some given choice of $\boldsymbol{\psi}^\alpha$ that there might not exist any global symmetries and in general one cannot set the horizon potentials to vanish. Also from Lemma A.4, K^m is uniquely determined on the Lorentz bundle part of P (Eq. 2.10) and so this ambiguity does not affect the potentials due to the gravitational Lorentz connection.

Remark 3.2 (Independent potentials)*.* From Property (1), the maximum number of non-zero horizon potentials $\mathscr{V}^\Lambda$ is the dimension of $\mathfrak{h}$ i.e. the rank of $\mathfrak{g}$. Thus, there are at most l non-zero potentials for each of the simple Lie algebras of rank l in Cartan's classification (see Theorem A § 2.14 [49]). For $SU(2)$-Yang-Mills theory, this reduces to the case considered by Sudarsky and Wald [4], where they find only one Yang-Mills potential. The analysis can be easily extended to include abelian Lie algebras (which are, by definition, neither simple nor semisimple) to find l horizon potentials for an abelian Lie algebra of dimension l.

* * *

Using the horizon potentials defined in Prop. 3.2 we show that the perturbed boundary Hamiltonian δH_K on B associated to the infinitesimal automorphism K^m can be put into a "potential times perturbed charge" form for any theory under consideration, when the dynamical fields $\boldsymbol{\psi}^\alpha$ satisfy the equations of motion and the perturbation $\delta\boldsymbol{\psi}^\alpha$ satisfies the linearised equations of motion.

Corollary 3.1 (Perturbed Hamiltonian and charges on the bifurcation surface)**.** *The perturbed Hamiltonian on the bifurcation surface B associated to K^m can be written as a "potential times perturbed charge" term of the form*

$$\delta H_K|_B = \mathscr{V}^\Lambda \delta \mathscr{Q}_\Lambda \tag{3.7}$$

where the horizon potentials $\mathscr{V}^\Lambda$ are as defined in Prop. 3.2 and the charges $\mathscr{Q}_\Lambda$ are defined by

$$\mathscr{Q}_\Lambda := \int_B \underline{\boldsymbol{Z}_I h^I_\Lambda} \tag{3.8}$$

where $\underline{\boldsymbol{Z}_I h^I_\Lambda}$ is the gauge-invariant $(d-2)$-form on M such that $\boldsymbol{Z}_I h^I_\Lambda = \pi^*\left(\underline{\boldsymbol{Z}_I h^I_\Lambda}\right)$, with $\boldsymbol{Z}_I$ given by Eq. 2.30a.

Proof. We first evaluate the perturbed Hamiltonian on the bifurcation surface using Eq. 2.54. By Lemma 2.1, the symplectic potential $\boldsymbol{\theta}$ is horizontal and the second term in Eq. 2.54 vanishes at the bifurcation surface since $K^m|_{P_B}$ is vertical. Similarly, the second term in the form of the Noether charge Eq. 2.47 for K^m vanishes. Thus, the perturbed horizon Hamiltonian associated to K^m is

$$\delta H_K|_B = \int_B \delta \underline{\boldsymbol{Q}}_K = \int_B \underline{\delta \boldsymbol{Z}_I\ (K \cdot \boldsymbol{A}^I)}$$

where in the last equality we again use the fact that K^m is vertical and $\delta \boldsymbol{A}^I$ is horizontal.

Then, using the definition of the horizon potentials (Eq. 3.5) and the zeroth law (Eq. 3.6) we get the form of the perturbed Hamiltonian in Eq. 3.7 with the charges $\mathscr{Q}_\Lambda$ given by Eq. 3.8. □

Remark 3.3 (Independent charges). We note, from Property (3), that only the projection of $\boldsymbol{Z}_I$ to the chosen Cartan subalgebra contributes to the charges in Eq. 3.8. Further, from Property (2), the maximum number of non-zero charges $\mathscr{Q}_\Lambda$ is the dimension of $\mathfrak{h}$ i.e. the rank of $\mathfrak{g}$. For $SU(2)$-Yang-Mills theory there is only one Yang-Mills charge as found in [4].

We can show that the ambiguities in the symplectic potential Eqs. 2.34 and 2.35 and the Noether charge Eq. 2.48 do not affect the perturbed Hamiltonian $\delta H_K|_B$ (the following argument also apply to the perturbed Hamiltonian at spatial infinity). These ambiguities give rise to the following change

$$\begin{aligned}\delta \boldsymbol{Q}_K - K \cdot \boldsymbol{\theta}(\delta\psi) \mapsto \quad & \delta \boldsymbol{Q}_K - K \cdot \boldsymbol{\theta}(\delta\psi) + \delta\boldsymbol{\lambda}(\delta_K\psi) + d\delta\boldsymbol{\rho} - K \cdot d\boldsymbol{\lambda}(\delta\psi) \\ & = \delta \boldsymbol{Q}_K - K \cdot \boldsymbol{\theta}(\delta\psi) + \delta\boldsymbol{\lambda}(\delta_K\psi) - \pounds_K \boldsymbol{\lambda}(\delta\psi) + d\left[K \cdot \boldsymbol{\lambda}(\delta\psi) + \delta\boldsymbol{\rho}\right]\end{aligned}$$

Since K^m is an infinitesimal automorphism which preserves the background dynamical fields $\boldsymbol{\psi}^\alpha$ we have

$$\delta\boldsymbol{\lambda}(\delta_K\psi) = \delta\boldsymbol{\lambda}[\psi; \pounds_K\psi] = \boldsymbol{\lambda}[\psi; \pounds_K\delta\psi] = \pounds_K\boldsymbol{\lambda}(\delta\psi)$$

and thus

$$\delta \boldsymbol{Q}_K - K \cdot \boldsymbol{\theta}(\delta\psi) \mapsto \delta \boldsymbol{Q}_K - K \cdot \boldsymbol{\theta}(\delta\psi) + d\left[K \cdot \boldsymbol{\lambda}(\delta\psi) + \delta\boldsymbol{\rho}\right]$$

Since $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ are local and covariant horizontal forms, the integral of $\delta\underline{\boldsymbol{Q}}_K - \underline{K} \cdot \underline{\boldsymbol{\theta}}(\delta\psi)$ over a closed surface (the bifurcation surface B), and consequently the perturbed boundary Hamiltonian $\delta H_K|_B$ (from Eq. 2.54), is unambiguous. Since the potentials are defined independently of these ambiguities, from Eq. 3.7 we see that the charges $\mathscr{Q}_\Lambda$ are also unaffected by these ambiguities.

3.1 Temperature and entropy as the horizon potential and charge for gravity

In the following, we show that the gravitational potential and the perturbed gravitational charge corresponding to the Lorentz connection $\boldsymbol{\omega}^a{}_b$ can be identified (up to conventions of numerical factors) with the temperature and perturbed entropy of the black hole respectively. Thus, the first-order formulation of gravity in terms of the coframes $\boldsymbol{e}^a$ and the Lorentz connection $\boldsymbol{\omega}^a{}_b$ on the Lorentz bundle P_O gives a new point of view on the temperature and perturbed entropy of the horizon. In particular, the temperature and perturbed entropy can be viewed on the same footing as the potentials and perturbed charges of any matter gauge fields in the theory. Further, we give an explicit formula for the gravitational charge which is a direct parallel of the Wald entropy formula [2, 3].

Theorem 2 (Temperature and perturbed entropy)**.** *The gravitational potential (corresponding to the Lorentz connection* $\boldsymbol{\omega}^a{}_b$ *on the Lorentz bundle* P_O *in Eq. 2.10) at the bifurcation surface* B *only has non-vanishing values in a* 1*-dimensional vector space spanning the abelian Lie algebra* $\mathfrak{so}(1,1)$ *corresponding to local Lorentz boosts of the frames* $\tilde{E}^m_a$ *normal to* B *in* M*. Let* $\tilde{\epsilon}^{ab} = \tilde{E}^b \cdot \tilde{E}^a \cdot \tilde{\boldsymbol{\varepsilon}}_2$ *be the frame components of the binormal to* B *along normal frames* $\tilde{E}^m_a$*. Then,* $-\tilde{\epsilon}^{ab}$ *forms a basis of* $\mathfrak{so}(1,1)$ *satisfying the normalisation* **??** *so that the gravitational potential and charge can be written as*

$$\mathscr{V}_{grav} = \kappa \quad ; \quad \mathscr{Q}_{grav} = -\int_B \underline{\boldsymbol{Z}_{ab}\tilde{\epsilon}^{ab}} \tag{3.9}$$

where κ *is the surface gravity of* B*. In Eq. 3.9* $\underline{\boldsymbol{Z}_{ab}\tilde{\epsilon}^{ab}}$ *is the unique gauge-invariant form on spacetime that pullsback to* $\boldsymbol{Z}_{ab}\tilde{\epsilon}^{ab}$ *on the bundle.* $\boldsymbol{Z}_{ab}$ *can be computed from Eq. 2.30a for the Lorentz connection* $\boldsymbol{\omega}^a{}_b$*, and thus, the gravitational charge formula is in direct parallel to the Wald entropy formula [2, 3].*

Thus, we can define the temperature and the perturbed entropy of the bifurcate Killing horizon as

$$T_{\mathscr{H}} := \frac{1}{2\pi}\mathscr{V}_{grav} \quad ; \quad \delta S := 2\pi\delta\mathscr{Q}_{grav} \tag{3.10}$$

Proof. Since, for a stationary axisymmetric black hole, $K^m \in \mathfrak{aut}(P; \boldsymbol{e}^a)$ is an infinitesimal automorphism which preserves the coframes, we have from Lemma A.4

$$K \cdot \boldsymbol{\omega}^{ab} = \tfrac{1}{2} E_a \cdot E_b \cdot d\boldsymbol{\xi} + (K \cdot \boldsymbol{e}^c) C_{cab}$$

where $\boldsymbol{\xi} = (K \cdot \boldsymbol{e}^a)\boldsymbol{e}_a$ is the pullback to the bundle of the Killing form $\underline{\boldsymbol{\xi}} \equiv \xi_\mu = g_{\mu\nu}K^\nu$ and the contorsion C_{abc} is defined in Eq. A.42. The vector field K^m is vertical on P_B and using usual definition of the surface gravity κ we can write the Killing form on the bifurcation surface as $d\underline{\boldsymbol{\xi}}|_B = 2\kappa\underline{\tilde{\boldsymbol{\varepsilon}}}_2$, where $\underline{\tilde{\boldsymbol{\varepsilon}}}_2$ is the binormal to B (see § 12.5 [45]). Thus we have

$$K \cdot \boldsymbol{\omega}^{ab}|_{P_B} = -\kappa E^b \cdot E^a \cdot \tilde{\boldsymbol{\varepsilon}}_2 \tag{3.11}$$

where $\tilde{\boldsymbol{\varepsilon}}_2$ is the binormal $\underline{\tilde{\boldsymbol{\varepsilon}}}_2$ lifted to the bundle P_B. Note that the torsion terms vanish due to K^m being a vertical vector field on P_B.

The surface gravity is constant on a bifurcate Killing horizon (see [47] and also Remark 3.5 below) i.e. $d\kappa|_{P_B} = 0$, and using Eq. 3.4 for the Lorentz connection, we get $D(E^b \cdot E^a \cdot \tilde{\boldsymbol{\varepsilon}}_2)|_{P_B} = 0$. Thus the Lorentz bundle $P_O|_B$, identified with the orthonormal frame bundle $F_O M|_B$ of the background solution metric, can be reduced to the *bundle of adapted orthonormal frames* (see § VII.1 [54]) i.e. $P_O|_B \cong F_O B \oplus F_N B$ where $F_O B$ is the orthonormal frame bundle of B and $F_N B$ is the $O(1,1)$-bundle of frames normal to B in M. Denote the adapted normal frames in FB by $\tilde{E}_a^m$.

From Proposition 1.4 in §VII.1 [54] we see that the connection $\boldsymbol{\omega}^a{}_b|_{P_B}$ can be written as a direct sum of a $O(d-2)$-connection on $F_O B$ and an abelian $O(1,1)$-connection $\boldsymbol{\omega}_N$ on $F_N B$. The invariant tensor $\tilde{\epsilon}^{ab} = \tilde{E}^b \cdot \tilde{E}^a \cdot \tilde{\boldsymbol{\varepsilon}}_2$ acts as a projector to this abelian $O(1,1)$-connection as $\boldsymbol{\omega}_N := \frac{1}{2}\boldsymbol{\omega}_{ab}\tilde{\epsilon}^{ab}$. Thus, Eq. 3.11 becomes $K \cdot \boldsymbol{\omega}_N = \kappa$ i.e. the surface gravity is the vertical part of K^m (with respect to the background connection) in the normal frame bundle of B.

A choice of Cartan subalgebra $\mathfrak{h}$ of the Lorentz Lie algebra is spanned by boosts in $\mathfrak{so}(1,1)$ normal to B and some choice of commuting rotations in $\mathfrak{so}(d-2)$. Since, $K \cdot \boldsymbol{\omega}^{ab}$ only points in the $\mathfrak{so}(1,1)$-part we have (choosing $-\tilde{\epsilon}^{ab}$ as a basis of $\mathfrak{so}(1,1)$)

$$\mathscr{V}_{grav} = K \cdot \boldsymbol{\omega}_N = \kappa$$

and the corresponding charge, using Eq. 3.8

$$\mathscr{Q}_{grav} = -\int_B \underline{\boldsymbol{Z}_{ab}\tilde{\epsilon}^{ab}}$$

and $\boldsymbol{Z}_{ab}$ is given by Eq. 2.30a for the Lorentz connection $\boldsymbol{\omega}^a{}_b$. Thus, the gravitational charge is determined in direct parallel to the Wald entropy formula [2, 3] for any theory of gravity formulated in terms of the coframes and a Lorentz connection. □

Note that since the coframes completely fix the form of the infinitesimal automorphism K^m (see Lemma A.4) we cannot eliminate the temperature and the perturbed entropy by a redefinition of K^m in any spacetime. For General Relativity, the gravitational charge can be computed to be $\mathscr{Q}_{grav} = \frac{1}{8\pi}\text{Area}(B)$ (see § 5.1) and we get the usual notion of temperature and perturbed entropy for black holes.

Remark 3.4 (Zeroth law). The gravitational potential i.e. the surface gravity κ, was shown to be constant on bifurcate Killing horizons in [47] without using the Einstein equations. Thus, in view of Theorem 2, we can see that Theorem 1 can be seen as a "generalised zeroth law for bifurcate Killing horizons" (analogous to the result of [47]) showing that the potentials defined in Prop. 3.2 are always constant on a bifurcate Killing horizon without using any equations of motion.

Remark 3.5 (Other versions of the zeroth law). Rácz and Wald [55] showed that if the surface gravity has non-vanishing gradient on any null geodesic generator of a (not necessarily bifurcate) Killing horizon then there necessarily exists a parallel-propagated curvature singularity on the horizon, without using the Einstein equations. If one uses the Einstein equations and the dominant energy condition on matter fields, then the surface gravity was shown to be constant on any (not necessarily bifurcate) Killing horizon in [56]. The results of [55, 56] can also be viewed as different versions of "the zeroth law".

CHAPTER 4

THE FIRST LAW FOR GAUGE-INVARIANT LAGRANGIANS

Next, we formulate the first law of black hole mechanics for a stationary-axisymmetric black hole solution described at the beginning of Ch. 3. The first law is obtained by evaluating the symplectic form on any Cauchy surface Σ for $X^m = K^m$ where K^m is the infinitesimal automorphism that projects to the horizon Killing field K^μ. Since the black hole is stationary and axisymmetric $\pounds_K \boldsymbol{\psi}^\alpha = 0$, using Eq. 2.53 for the symplectic form, the first law is an equality of the perturbed boundary Hamiltonians δH_K evaluated at the bifurcation surface and at spatial infinity.

The perturbed Hamiltonian on the bifurcation surface was already put into a "potential times perturbed charge" form in Cor. 3.1. Near spatial infinity, we can lift the asymptotic Minkowski radial coordinate r (viewed as a gauge-invariant function) to the bundle P. We choose the dynamical fields and their perturbations to fall-off suitably in $1/r$ so that the symplectic form W_Σ is finite. The particular choice of fall-off in general depends on the specific Lagrangian theory under consideration. For the infinitesimal automorphisms t^m and $\phi^m_{(i)}$ we define the *canonical energy* and *canonical angular momentum* as the corresponding Hamiltonians (whenever they exist) at infinity (see [3]).

$$\begin{aligned} E_{can} &:= H_t|_\infty = \int_\infty \underline{\boldsymbol{Q}}_t - \underline{t} \cdot \underline{\boldsymbol{\Theta}} \\ J_{(i),can} &:= -\, H_{\phi_{(i)}}\Big|_\infty = - \int_\infty \underline{\boldsymbol{Q}}_{\phi_{(i)}} - \underline{\phi}_{(i)} \cdot \underline{\boldsymbol{\Theta}} \end{aligned} \tag{4.1}$$

where $\underline{\boldsymbol{\Theta}}$ is as described by Eq. 2.55. Thus, the perturbed Hamiltonian at infinity associated to $K^m = t^m + \Omega^{(i)}_{\mathscr{H}} \phi^m_{(i)}$ becomes $\delta H_K|_\infty = \delta E_{can} - \Omega^{(i)}_{\mathscr{H}} \delta J_{(i),can}$. In all the examples we consider in Ch. 5, only the Einstein-Hilbert Lagrangian contributes a non-zero $\underline{\boldsymbol{\Theta}}$ at infinity and for all other cases $\underline{K} \cdot \underline{\boldsymbol{\theta}}$ falls off fast enough that we can choose $\underline{\boldsymbol{\Theta}} = 0$. Note that in 4-dimensions, we might need to impose faster fall-off conditions or some suitable generalisation of the Regge-Teitelboim parity conditions [57] on the dynamical fields and their perturbations for the $J_{(i),can}$ to be well-defined; we assume that such choices have been made to get a well-defined canonical angular momentum.

This leads to our main result in the following theorem which gives us a general formulation of the first law of black hole mechanics.

Theorem 3 (The first law of black hole mechanics)**.** *Consider any theory with a local, covariant and gauge-invariant Lagrangian of the form Eq. 2.14. Let $\boldsymbol{\psi}^\alpha$ be a solution corresponding to a stationary axisymmetric black hole with a bifurcate Killing horizon (described at the beginning of Ch. 3) and $\delta\boldsymbol{\psi}^\alpha$ be an arbitrary linearised solution. Then the first law of black hole mechanics takes the form*

$$T_{\mathscr{H}}\delta S + \mathscr{V}'^{\Lambda}\delta\mathscr{Q}'_{\Lambda} = \delta E_{can} - \Omega^{(i)}_{\mathscr{H}}\,\delta J_{(i),can} \tag{4.2}$$

where, on left-hand-side the first term consists of the temperature and perturbed entropy of the black hole as described in Theorem 2 and the second term is the potential and perturbed charge of the connection $\boldsymbol{A}'^{I'}$ (see Eq. 2.11) described in Prop. 3.2 and Eq. 3.8, and the terms on the right-hand-side being defined at infinity by Eq. 4.1.

Proof. The first law is obtained by evaluating the expression Eq. 2.53 with $X^m = K^m$ on a hypersurface Σ which goes from the bifurcation surface B to spatial infinity. For a stationary axisymmetric black hole $\pounds_K\boldsymbol{\psi}^\alpha = 0$ and the left-hand-side of Eq. 2.53 vanishes and then the first law equates the perturbed boundary Hamiltonian of K^m at B to the one at infinity.

$$\delta H_K|_B = \delta H_t|_\infty - \Omega^{(i)}_{\mathscr{H}}\delta H_{\phi_{(i)}}|_\infty \tag{4.3}$$

Using Cor. 3.1, Eq. 2.11, Theorem 2 and Eq. 4.1 we get the first law Eq. 4.2. □

At this point, we emphasise that we have not assumed any choice of gauge in the above form of the first law, and in fact this form holds even when the principal bundle is non-trivial and no choice of gauge can be made.

As discussed after Cor. 3.1, the perturbed Hamiltonian at B, and also at infinity is not affected by the ambiguities in the Lagrangian and the symplectic potential. However there is an ambiguity in choosing the vector field K^m on the bundle P (see Lemma A.1 and Remarks A.2 and A.3) corresponding to a global symmetry of the background dynamical fields $\boldsymbol{\psi}^\alpha$ if any exists. Note that the vector field K^m is uniquely determined over the Lorentz bundle P_O part (from Lemma A.4) and thus, the temperature and perturbed entropy, as well as, the gravitational contributions to the canonical energy and canonical angular momentum are unambiguous. Thus, the possible ambiguity in choosing K^m leads to a simultaneous redefinition of the horizon potentials $\mathscr{V}'^\Lambda$ and charges $\mathscr{Q}'_\Lambda$, and the contributions to the canonical energy and angular momenta at infinity of any non-gravitational fields. One can use such an ambiguity to set some, or possibly all, of the horizon potentials at the horizon to vanish (see Remark 3.1) at the cost of changing the contributions to the canonical energy and canonical angular momenta (see also Remark 5.2 for Einstein-Yang-Mills theory). Even though this ambiguity affects the individual terms, the form of the first law Eq. 4.2 holds for any choice of K^m on the bundle.

Even though δH_K is unambiguously defined (given a choice of K^m), the Hamiltonian H_K (if it exists) is not. Consider first the Hamiltonian at the bifurcation surface B (the analysis proceeds similarly for spatial infinity). Since the $\boldsymbol{\theta}$ contribution vanishes at B, the choice of $\underline{\boldsymbol{\Theta}}$ has the ambiguity

$$\int_B \underline{K} \cdot \underline{\boldsymbol{\Theta}} \mapsto \int_B \underline{K} \cdot \underline{\boldsymbol{\Theta}} - \int_B \underline{K} \cdot \underline{\boldsymbol{\Lambda}}$$

where $\int_B \underline{K} \cdot \underline{\boldsymbol{\Lambda}}$ is some topological invariant of B (possibly depending on the embedding of B in M) and does not change under variations. Similarly for the Noether charge at B we have the ambiguities

$$\int_B \underline{\boldsymbol{Q}_K} \mapsto \int_B \underline{\boldsymbol{Q}_K} + \int_B \underline{K \cdot \boldsymbol{\mu}}$$

Thus the ambiguity in the Hamiltonian H_K is of the form

$$H_K \mapsto H_K + \int_B \underline{K} \cdot \underline{\boldsymbol{\Lambda}} + \int_B \underline{K \cdot \boldsymbol{\mu}}$$

But we have already shown that δH_K is unambiguous. Thus, any contribution of the ambiguities $\boldsymbol{\mu}$ and $\underline{\boldsymbol{\Lambda}}$ to the boundary Hamiltonian can be considered as a *topological charge*. Similarly, we can have topological charge contributions to the Hamiltonian at spatial infinity. In the gravitational case, the topological charges at spatial infinity can be fixed by requiring that flat Minkowski spacetime have vanishing ADM mass and ADM angular momentum. Nevertheless, it is possible to have topologically non-trivial solutions to Yang-Mills theory which result in non-trivial topological charges (e.g. magnetic monopole charges) at the horizon. In fact such topological charges do arise in Yang-Mills theory when we add the $\boldsymbol{\mu}$-ambiguity to the Lagrangian (§ 5.2). Similarly, the addition of the Euler density to the Einstein-Hilbert Lagrangian corresponds to the $\boldsymbol{\mu}$-ambiguity Eq. 2.34 where $\boldsymbol{\mu}$ is not horizontal[1] (see § 5.1) and does contribute a topological term to the Noether charge at B [58]. Even though these topological charges do not affect the first law of black hole mechanics for stationary black holes, they do affect any attempt to define a total entropy and charge for stationary black holes purely from the first law, as we shall discuss later.

Since the perturbed entropy is given by the perturbed gravitational charge, Iyer and Wald [2, 3] prescribe that the total entropy (known as the *Wald entropy*) for a stationary

1. Iyer and Wald [3] only considered $\boldsymbol{\mu}$ that were horizontal in which case the $\boldsymbol{\mu}$-ambiguity does not affect the Hamiltonian at B.

axisymmetric black hole be defined as the gravitational charge

$$S_{\text{Wald}} := \mathscr{Q}_{grav} = -\int_B \underline{\boldsymbol{Z}_{ab}\tilde{\varepsilon}^{ab}} \tag{4.4}$$

This prescription has the advantage that the entropy S_{Wald} satisfies the first law and is the same on any cross-section of the horizon of a stationary axisymmetric black hole since, $\boldsymbol{J}_K = 0$ on the horizon [59] . But this prescription is not unambiguous. For instance, an alternative definition of the entropy as $S = S_{\text{Wald}} + C(B)$ where $C(B)$ is a topological invariant of the bifurcation surface B also satisfies the first law, since $C(B)$ does not change under linearised variations [60]. In fact the $\boldsymbol{\mu}$-ambiguity Eq. 2.34 in the Lagrangian contributes a topological charge of precisely this nature. In the case of General Relativity, as pointed out by [58, 60], the gravitational charge does acquire such a topological contribution (the Euler number of B) when the Euler density is added to the Einstein-Hilbert Lagrangian in 4-dimensions even though the equations of motion are unaffected (also see § 5.1). The *area theorem* for General Relativity in 4-dimensions (along with an energy condition) guarantees that the entropy defined as the area of a horizon cross-section always increases and thus satisfies the *second law of black hole mechanics* (see [61], Proposition 9.2.7 [62] or Theorem 12.2.6 [45]). If one includes the topological charges of the horizon in the definition of the entropy then the second law is violated even for General Relativity (see [15, 60, 63]). Thus, it seems that a version of the second law is needed to fix at least some of the ambiguities in defining the total entropy even for a stationary black hole. To consider the second law one has to consider non-stationary black hole configurations where the entropy prescription Eq. 4.4 can have more ambiguities which vanish only in the stationary case [59]. Unfortunately, a general formulation of the second law for an arbitrary theory of gravity including arbitrary matter fields remains out of reach; though it has been investigated in special situations for higher curvature gravity [64, 65]. In the absence of a second law, the first law Eq. 4.2 only determines the perturbed entropy δS. In light of this we will refrain from giving a prescription for the total entropy S for stationary black holes or for a dynamical entropy for non-stationary ones.

CHAPTER 5

EXAMPLES

In this section we use the formalism described above to derive the first law of black hole mechanics for the first-order coframe formulation of General Relativity, Einstein-Yang-Mills theory and Einstein-Dirac theory. The Lagrangians considered in this section are of the form $\boldsymbol{L} = \boldsymbol{L}_{\text{grav}} + \boldsymbol{L}_{\text{matter}}$, where the gravitational Lagrangian $\boldsymbol{L}_{\text{grav}}$ only depends on the coframes $\boldsymbol{e}^a$ and a Lorentz connection $\boldsymbol{\omega}^a{}_b$ on the Lorentz bundle P_O and not on the matter fields.[1] It will be convenient to work in the *first-order formalism* where we consider the coframes $\boldsymbol{e}^a$ and the Lorentz connection $\boldsymbol{\omega}^a{}_b$ as independent fields. We will write the equations of motion obtained by varying the Lagrangian with respect to the coframes and the Lorentz connection as

$$\boldsymbol{\mathcal{E}}_a - \boldsymbol{\mathcal{T}}_a = 0 \quad ; \quad \boldsymbol{\mathcal{E}}_{ab} - \boldsymbol{\mathcal{S}}_{ab} = 0 \tag{5.1}$$

where the gravitational contributions to the equations of motion are

$$\begin{aligned} \boldsymbol{\mathcal{E}}_a \equiv (\mathcal{E}_a)_{m_1 \dots m_{d-1}} &= \frac{1}{d} \frac{\delta (L_{\text{grav}})_{m_1 \dots m_{d-1} l}}{\delta e^a_n} \delta^l_n \\ \boldsymbol{\mathcal{E}}_{ab} \equiv (\mathcal{E}_{ab})_{m_1 \dots m_{d-1}} &= \frac{1}{d} \frac{\delta (L_{\text{grav}})_{m_1 \dots m_{d-1} l}}{\delta \omega^{ab}_n} \delta^l_n \end{aligned} \tag{5.2}$$

and the matter contributions are

$$\begin{aligned} \boldsymbol{\mathcal{T}}_a \equiv (\mathcal{T}_a)_{m_1 \dots m_{d-1}} &= -\frac{1}{d} \frac{\delta (L_{\text{matter}})_{m_1 \dots m_{d-1} l}}{\delta e^a_n} \delta^l_n \\ \boldsymbol{\mathcal{S}}_{ab} \equiv (\mathcal{S}_{ab})_{m_1 \dots m_{d-1}} &= -\frac{1}{d} \frac{\delta (L_{\text{matter}})_{m_1 \dots m_{d-1} l}}{\delta \omega^{ab}_n} \delta^l_n \end{aligned} \tag{5.3}$$

i.e. $\boldsymbol{\mathcal{T}}_a$ is the *energy-momentum* and $\boldsymbol{\mathcal{S}}_{ab}$ is the *spin current* of the matter fields, both written as $(d-1)$-forms. We note here that when the matter Lagrangian depends on the gravitational

1. For the case of dilaton gravity considered in [3] the gravitational Lagrangian does depend on an additional scalar field. We will not consider such examples in this section but they are covered in the more general formulation in Ch. 4.

Lorentz connection used, $\boldsymbol{\mathcal{T}}_a$ does not give the usual symmetric energy-momentum tensor [36, 37].

Remark 5.1 (Second-order formalism). If one insists on having vanishing torsion from the outset (i.e. one works in the *second-order formalism*) then the Lorentz connection is completely determined by the coframes (see Remark A.4) and one can use Eq. A.40 (and Eq. 2.22) to convert all the variations of the torsionless connection to variation of the coframes. In that case, the second equation of motion in Eq. 5.1 is deleted but the "new" energy-momentum tensor $\boldsymbol{\mathcal{T}}_a$ gets contributions from the spin current [36, 37].

We consider the first-order formulation of General Relativity with the gravitational Lagrangian to be given by the Palatini-Holst Lagrangian. We show that the gravitational charge Eq. 3.9 is given by the area of the bifurcation surface and thus we reproduce the usual identification between the perturbed entropy and perturbed area of the bifurcation surface. Similarly, (up to terms involving torsion) Eq. 4.1 reproduces the ADM mass and ADM angular momentum and we get the usual first law of black hole mechanics for General Relativity. For the matter Lagrangian we consider the two cases of Yang-Mills Lagrangian for gauge fields of any semisimple group, and the free Dirac Lagrangian for spinor fields. These examples can be generalised to include chiral spinor fields with Yang-Mills charge, charged scalar fields such as the Higgs field, and thus, the entire Standard Model of particle physics. We shall work out the details in 4-spacetime dimensions but the computations can be easily generalised to other dimensions. We also illustrate the topological charge ambiguities that arise in the definitions of the Hamiltonian and the entropy.

5.1 Palatini-Holst

To start let us consider the *first-order formulation* of General Relativity in 4-spacetime dimensions in vacuum. A derivation of the first law in this case was recently given in [15] by using a generalised notion of Lie derivatives of the coframes called the *Lorentz-Lie deriva-*

tive. As discussed in Ch. 2 the Lorentz-Lie derivative defined in [15] depends non-linearly on the coframes and does not form a Lie algebra for diffeomorphisms of spacetime. Thus, the Lorentz-Lie derivative is not the generator of any group action on the coframes. We will write the first-order Palatini-Holst action on the oriented Lorentz bundle (see § A.1) i.e. for this section $P = P_{SO}$ over spacetime M and use the notion of the Lie derivative on the bundle to obtain a first law. When a vector field X^m is an automorphism which preserves the coframes the bundle notion of Lie derivative coincides with the Lorentz-Lie derivative defined by [15]. Thus, even though the Noether charge we define for arbitrary automorphisms of the frame bundle in not equivalent to that of [15] we get the same results for the first law for stationary axisymmetric black holes.

The dynamical fields for the first-order formulation of General Relativity are the coframes $\boldsymbol{e}^a \in \Omega^1_{hor}P(\mathbb{R}^4)$ and the Lorentz connection $\boldsymbol{\omega}^a{}_b \in \Omega^1 P(\mathfrak{so}(3,1))$ (see § A.1 for details). The *Palatini-Holst Lagrangian* for General Relativity can be written as:

$$\boldsymbol{L}_{PH} = \tfrac{1}{32\pi}\phi_{abcd}\, \boldsymbol{e}^a \wedge \boldsymbol{e}^b \wedge \boldsymbol{R}^{cd} \in \Omega^4_{hor}P \tag{5.4}$$

where $\phi_{abcd} := \epsilon_{abcd} + \frac{2}{\gamma}\eta_{a[c}\eta_{d]b}$ is an $SO(3,1)$-invariant tensor with γ being the *Barbero-Immirzi parameter*. This tensor satisfies the index symmetries $\phi_{abcd} = \phi_{cdab}$ and $\phi_{abcd} = -\phi_{bacd} = -\phi_{abdc}$. The ϵ-term is the usual *Palatini-Einstein-Hilbert Lagrangian* and the γ-term corresponds to the *Holst Lagrangian* [66]. The corresponding Lagrangian form on spacetime is (using Eq. A.39)

$$\underline{\boldsymbol{L}}_{PH} = \frac{1}{16\pi}\boldsymbol{\varepsilon}_4 \left(R - \frac{1}{2\gamma}\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} \right)$$

Note that the last term vanishes (using Eq. A.37) when one restricts, a priori, to a torsionless connection.

A variation of the Lagrangian gives $\delta \boldsymbol{L}_{PH} = \boldsymbol{\mathcal{E}}_a \wedge \delta \boldsymbol{e}^a + \boldsymbol{\mathcal{E}}_{ab} \wedge \delta \boldsymbol{\omega}^{ab} + d\boldsymbol{\theta}$ where:

$$\begin{aligned} \boldsymbol{\mathcal{E}}_a &= -\tfrac{1}{16\pi}\phi_{abcd}\ \boldsymbol{e}^b \wedge \boldsymbol{R}^{cd} \\ \boldsymbol{\mathcal{E}}_{ab} &= \tfrac{1}{16\pi}\phi_{abcd}\ \boldsymbol{e}^c \wedge \boldsymbol{T}^d \\ \boldsymbol{\theta} &= \tfrac{1}{32\pi}\phi_{abcd}\ \boldsymbol{e}^a \wedge \boldsymbol{e}^b \wedge \delta\boldsymbol{\omega}^{cd} \end{aligned} \tag{5.5}$$

while the symplectic current is:

$$\boldsymbol{\omega} = \tfrac{1}{16\pi}\phi_{abcd}\ \boldsymbol{e}^a \wedge (\delta_1 \boldsymbol{e}^b \wedge \delta_2 \boldsymbol{\omega}^{cd} - \delta_2 \boldsymbol{e}^b \wedge \delta_1 \boldsymbol{\omega}^{cd}) \tag{5.6}$$

In vacuum the equation of motion $\boldsymbol{\mathcal{E}}_{ab} = 0$ implies that $\boldsymbol{T}^a = 0$ [66]. Since we define the Noether current and Noether charge "off-shell" we will not make use of this fact. Also, in § 5.3 the torsion can be sourced by the spin current of matter fields and so it will be useful to keep this term to analyse the effects of torsion.

A straightforward computation gives the Noether current Eq. 2.41 for $X^m \in \mathfrak{aut}(P)$ as:

$$\boldsymbol{J}_X = \boldsymbol{\theta}_X - X \cdot \boldsymbol{L} = d\boldsymbol{Q}_X + \boldsymbol{\mathcal{E}}_a(X \cdot \boldsymbol{e}^a) + \boldsymbol{\mathcal{E}}_{ab}(X \cdot \boldsymbol{\omega}^{ab}) \tag{5.7}$$

with the Noether charge $\boldsymbol{Q}_X = \tfrac{1}{32\pi}\phi_{abcd}\ \boldsymbol{e}^a \wedge \boldsymbol{e}^b (X \cdot \boldsymbol{\omega}^{cd}) \in \Omega^2_{hor} P$.

Now, consider $X^m \in \mathfrak{aut}(P; \boldsymbol{e}^a, \boldsymbol{\omega}^a{}_b)$ i.e. an infinitesimal automorphism that preserves the coframes and the connection. From Lemma A.4 such an automorphism is uniquely determined by a Killing field of the spacetime metric determined by the coframes. The Noether charge for such an X^m becomes:

$$\begin{aligned} \boldsymbol{Q}_X &= \frac{1}{16\pi}\frac{1}{4}\phi_{ab}{}^{cd}\boldsymbol{e}^a \wedge \boldsymbol{e}^b\ \ (E_c \cdot E_d \cdot d\boldsymbol{\xi} + 2(X \cdot \boldsymbol{e}^e)C_{ecd}) \\ &= -\frac{1}{16\pi}\left(\star [d\boldsymbol{\xi} - 2(X \cdot \boldsymbol{e}^a)\boldsymbol{C}_a] + \frac{1}{\gamma}[d\boldsymbol{\xi} - 2(X \cdot \boldsymbol{e}^a)\boldsymbol{C}_a] \right) \end{aligned} \tag{5.8}$$

where $\boldsymbol{\xi} = (X \cdot \boldsymbol{e}^a)\boldsymbol{e}_a$, $\star$ is the horizontal Hodge dual operation on differential forms on the

Lorentz bundle Eq. A.33 and we have written the contorsion Eq. A.42 as a horizontal 2-form as $\boldsymbol{C}_a := \frac{1}{2}C_{abc}\boldsymbol{e}^b \wedge \boldsymbol{e}^c$.

To get the first law for a stationary axisymmetric black hole solution (see Ch. 4) we use the infinitesimal automorphism $X^m = K^m = t^m + \Omega_{\mathscr{H}}\phi^m$ which projects to the horizon Killing field K^μ on M. Since K^μ vanishes on the bifurcation surface the $\boldsymbol{\theta}$-term and the contorsion terms do not contribute. Using $*d\underline{\boldsymbol{\xi}}|_B = -2\kappa\underline{\boldsymbol{\varepsilon}}_2$ where $\underline{\boldsymbol{\varepsilon}}_2$ is the volume form on B we have:

$$\int_B \underline{\boldsymbol{Q}}_X = -\frac{1}{16\pi}\int_B \left(*d\underline{\boldsymbol{\xi}} + \frac{1}{\gamma}d\underline{\boldsymbol{\xi}}\right) = \frac{\kappa}{8\pi}\int_B \underline{\boldsymbol{\varepsilon}}_2 = \frac{\kappa}{2\pi}\frac{1}{4}\text{Area}(B) \tag{5.9}$$

Thus, following Theorem 2, the perturbed entropy for General Relativity is

$$\delta S = \frac{1}{4}\delta\text{Area}(B) \tag{5.10}$$

Next we compute the canonical energy and angular momentum using Eq. 4.1 and show that they correspond to the ADM mass and ADM angular momentum up to torsion terms. Since the spacetime is asymptotically flat, near spatial infinity the spacetime is asymptotically Minkowskian, and the Lorentz bundle is asymptotically trivial. Then, there is a section near infinity s_∞ such that the pullbacks $\underline{\boldsymbol{e}}^a \equiv e^a_\mu = (s^*_\infty)^m_\mu e^a_m$ and $\underline{\boldsymbol{\omega}}^a{}_b \equiv \omega^a{}_{b\mu} = (s^*_\infty)^m_\mu \omega^a{}_{bm}$ satisfy the asymptotic conditions

$$\underline{\boldsymbol{e}}^a = \underline{\boldsymbol{e}}^a_{\mathbb{M}} + O(1/r) \quad ; \quad \underline{\boldsymbol{\omega}}^a{}_b = O(1/r^2) \tag{5.11}$$

where the asymptotic coframes $\underline{\boldsymbol{e}}^a_{\mathbb{M}}$ are adapted to the asymptotic Minkowskian coordinates as

$$\underline{\boldsymbol{e}}^a_{\mathbb{M}} = \Big(dt, dx, dy, dz\Big) \tag{5.12}$$

To compute the canonical energy Eq. 4.1, consider the pullback of $t \cdot \boldsymbol{\theta}(\delta\varphi)$ through the section s_∞ at infinity. Using the fall-off conditions on the pullback of the dynamical fields at infinity Eq. 5.11 we can write $s^*_\infty(t\cdot\boldsymbol{\theta}) = \delta(\underline{t}\cdot\underline{\boldsymbol{\Theta}})$ where:

$$\underline{t}\cdot\underline{\boldsymbol{\Theta}} = \underline{t}\cdot\left(\frac{1}{32\pi}\phi_{abcd}\underline{\boldsymbol{e}}^a\wedge\underline{\boldsymbol{e}}^b\wedge\underline{\boldsymbol{\omega}}^{cd}\right) = \frac{1}{16\pi}\phi_{abcd}(\underline{t}\cdot\underline{\boldsymbol{e}}^a)\underline{\boldsymbol{e}}^b\wedge\underline{\boldsymbol{\omega}}^{cd} + \underline{\boldsymbol{Q}}_X \tag{5.13}$$

Note, that $\underline{\boldsymbol{\Theta}}$ is not gauge-invariant under the action of the Lorentz group $O(d-1,1)$. Using Eq. 4.1 The canonical energy at infinity is given by

$$E_{can} = -\tfrac{1}{16\pi}\int_\infty \phi_{abcd}(\underline{t}\cdot\underline{\boldsymbol{e}}^a)\underline{\boldsymbol{e}}^b\wedge\underline{\boldsymbol{\omega}}^{cd}$$

Note here that the Noether charge contribution to the canonical energy actually cancels out. As discussed in [3] this accounts for the "factor of 2" discrepancy in the Komar formula for the ADM mass relative to the one for ADM angular momentum. Also note that this cancellation is much more easily obtained in the computation presented here than the corresponding one in [3]. We next show that the above expression for the canonical energy reproduces the well-known ADM mass formula.

Let r_μ be the outward pointing spatial conormal to a 2-sphere at infinity with the volume form $\underline{\boldsymbol{\varepsilon}}_2$ and $h_{\mu\nu}$ be the asymptotic spatial metric on the Cauchy surface Σ. The Einstein-Hilbert term gives

$$\begin{aligned}
-\tfrac{1}{16\pi}\int_\infty \epsilon_{abcd}(\underline{t}\cdot\underline{\boldsymbol{e}}^a)\underline{\boldsymbol{e}}^b\wedge\underline{\boldsymbol{\omega}}^{cd} &= -\tfrac{1}{16\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\ \epsilon_{abcd}t^\sigma e^a_\sigma e^b_\mu \omega^{cd}_\nu t_\lambda r_\rho \varepsilon^{\lambda\rho\mu\nu} \\
&= -\tfrac{3!}{16\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\ t^\sigma t_\lambda r_\rho \omega^{cd}_\nu \delta^{[\lambda}_\sigma E^\rho_c E^{\nu]}_d \\
&= \tfrac{1}{8\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\ r_\rho E^\rho_c (E^{d\eta}\hat{\nabla}_\nu e^c_\eta)(t^\lambda t_\lambda E^\nu_d - t^\nu t_\lambda E^\lambda_d) \\
&= \tfrac{1}{8\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\ r_\rho E^\rho_c (\hat{\nabla}_\nu e^c_\eta)(-g^{\nu\eta} - t^\nu t^\eta) \\
&= -\tfrac{1}{8\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\ r^{[\lambda}h^{\nu]\mu} e_{\lambda a}\left(\partial_\mu e^a_\nu - \Gamma^\sigma{}_{\mu\nu} e^a_\sigma\right)
\end{aligned} \tag{5.14}$$

where in the third line we have used Eq. 2.7 to write the Lorentz connection in terms of the derivatives of the coframes and in the last line in terms of the Christoffel symbols.

Using the asymptotic conditions Eq. 5.11 the first term can be written as

$$-\tfrac{1}{8\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2 \;\; \partial_\mu\left(r^\lambda h^{\nu\mu}_{\mathbb{M}} e^a_{\mathbb{M}[\lambda}e_{\nu]a}\right) = -\tfrac{1}{16\pi}\int_\infty d *_2 \left[r\cdot\left(\underline{\boldsymbol{e}}^a_{\mathbb{M}}\wedge\underline{\boldsymbol{e}}_a\right)\right] = 0 \tag{5.15}$$

where $h^{\nu\mu}_{\mathbb{M}}$ is the flat spatial Cartesian metric on the asymptotic Cauchy surface and $*_2$ is the 2-dimensional Hodge dual on the asymptotic sphere. The second term, depending on the Christoffel symbols, can be written as

$$\tfrac{1}{8\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\; r^\lambda h^{\nu\mu}\Gamma_{[\lambda\nu]\mu} = \tfrac{1}{16\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\; r^\lambda h^{\nu\mu}\left(\partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\nu\mu} - T_{\mu\nu\lambda}\right) \tag{5.16}$$

using the definition of the Christoffel symbols (with torsion)

$$\Gamma_{\lambda\nu\mu} = \tfrac{1}{2}\left(\partial_\mu g_{\nu\lambda} + 2\partial_{[\nu}g_{\lambda]\mu}\right) + \tfrac{1}{2}\left(-T_{\mu\nu\lambda} + 2T_{(\nu\lambda)\mu}\right)$$

Now computing the Holst-term contribution to the canonical energy we have

$$\begin{aligned}
-\tfrac{1}{8\pi\gamma}\int_\infty (\underline{t}\cdot\underline{\boldsymbol{e}}^a)\underline{\boldsymbol{e}}^b\wedge\underline{\boldsymbol{\omega}}_{ab} &= -\tfrac{1}{8\pi\gamma}\int_\infty (\underline{t}\cdot\underline{\boldsymbol{e}}^a)(d\underline{\boldsymbol{e}}_a - \underline{\boldsymbol{T}}_a)\\
&= -\tfrac{1}{8\pi\gamma}\int_\infty d\left[(\underline{t}\cdot\underline{\boldsymbol{e}}^a_{\mathbb{M}})\underline{\boldsymbol{e}}_a\right] + \tfrac{1}{16\pi\gamma}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\; t^\mu\varepsilon^{\nu\lambda}T_{\mu\nu\lambda}\\
&= \tfrac{1}{16\pi\gamma}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\; t^\mu\varepsilon^{\nu\lambda}T_{\mu\nu\lambda}
\end{aligned} \tag{5.17}$$

Thus we get the total canonical energy as:

$$E_{can} = \tfrac{1}{16\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\; r^\lambda h^{\nu\mu}\left(\partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\nu\mu}\right) + \tfrac{1}{16\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2\;\left(-r^\lambda h^{\nu\mu} + \tfrac{1}{\gamma}t^\mu\varepsilon^{\nu\lambda}\right)\; T_{\mu\nu\lambda} \tag{5.18}$$

The first term can be recognised as the usual formula for the *ADM mass* M_{ADM}, while the second is the canonical energy contributed by the presence of any torsion at infinity.

Now for the canonical angular momentum, the $\phi\cdot\boldsymbol{\theta}$-term does not contribute when pulled back to the sphere at infinity and we get (here $\underline{\boldsymbol{\phi}} \equiv \phi_\mu$)

$$
\begin{aligned}
J_{can} &= -\int_\infty \underline{\boldsymbol{Q}}_\phi = \tfrac{1}{16\pi}\int_\infty * \left[d\underline{\boldsymbol{\phi}} - 2\phi^\mu \underline{\boldsymbol{C}}_\mu\right] + \tfrac{1}{\gamma}\left[d\underline{\boldsymbol{\phi}} - 2\phi^\mu \underline{\boldsymbol{C}}_\mu\right] \\
&= \tfrac{1}{16\pi}\int_\infty * d\underline{\boldsymbol{\phi}} - \tfrac{1}{16\pi}\int_\infty \underline{\boldsymbol{\varepsilon}}_2 \left[-\tilde{\varepsilon}^{\nu\lambda} + \tfrac{1}{\gamma}\varepsilon^{\nu\lambda}\right] \ \phi^\mu C_{\mu\nu\lambda}
\end{aligned}
\tag{5.19}
$$

where the first term is the Komar formula for the *ADM angular momentum* J_{ADM} and second is the angular mometum due to any torsion at infinity ($\tilde{\varepsilon}^{\mu\nu}$ is the binormal to the 2-sphere at infinity).

As noted before in vacuum, the equation of motion $\boldsymbol{\mathcal{E}}_{ab} = 0$ ensures that the torsion vanishes everywhere and the canonical energy and angular momentum are exactly the ones given by the ADM quantities. This is also the case if any matter sources for torsion fall-off suitably at infinity, as happens in the case of the Dirac field § 5.3. We note that, when the torsion due to matter sources does not fall-off fast enough the Barbero-Immirizi parameter γ does contribute to the canonical energy and angular momentum at infinity.

Thus, we get the usual first law for vacuum General Relativity

$$
\frac{\kappa}{2\pi}\frac{1}{4}\delta \text{Area}(B) = \delta M_{ADM} - \Omega_{\mathscr{H}}\delta J_{ADM} \tag{5.20}
$$

To illustrate the $\boldsymbol{\mu}$-ambiguity Eq. 2.34, we consider the following three topological terms, that can be added to the Palatini-Holst Lagrangian in 4 dimensions.

$$
\begin{aligned}
\boldsymbol{L}_{\text{E}} &= \tfrac{1}{2}\epsilon_{abcd}\boldsymbol{R}^{ab} \wedge \boldsymbol{R}^{cd} \\
\boldsymbol{L}_{\text{P}} &= \boldsymbol{R}^a{}_b \wedge \boldsymbol{R}^b{}_a = -\boldsymbol{R}^{ab} \wedge \boldsymbol{R}_{ab} \\
\boldsymbol{L}_{\text{NY}} &= \boldsymbol{T}^a \wedge \boldsymbol{T}_a - \boldsymbol{e}^a \wedge \boldsymbol{e}^b \wedge \boldsymbol{R}_{ab}
\end{aligned}
$$

The first is the *Euler character* of the Lorentz bundle over M (also known as the *Gauss-*

Bonnet invariant), the second is the corresponding *Pontryagin character*, and the third is the *Nieh-Yan character* [67, 68, 69], which exists only for a connection with torsion. Each of these terms are exact forms $\boldsymbol{L} = d\boldsymbol{\mu}$ where the corresponding $\boldsymbol{\mu}$'s can be computed to be[2]

$$\begin{aligned}
\boldsymbol{\mu}_{\rm E} &= \tfrac{1}{2}\epsilon_{abcd}\boldsymbol{\omega}^{ab} \wedge \left(\boldsymbol{R}^{cd} - \tfrac{1}{3}\boldsymbol{\omega}^{c}{}_{e} \wedge \boldsymbol{\omega}^{ed}\right) \\
\boldsymbol{\mu}_{\rm P} &= \boldsymbol{\omega}^{a}{}_{b} \wedge \left(\boldsymbol{R}^{b}{}_{a} - \tfrac{1}{3}\boldsymbol{\omega}^{b}{}_{c} \wedge \boldsymbol{\omega}^{c}{}_{a}\right) \\
\boldsymbol{\mu}_{\rm NY} &= \boldsymbol{e}^{a} \wedge \boldsymbol{T}_{a}
\end{aligned}$$

Note that $\boldsymbol{\mu}_{\rm P}$ is just the Chern-Simons term for the Lorentz bundle, $\boldsymbol{\mu}_{\rm E}$ is similar but with the "trace" taken with a ϵ_{abcd}, and $\boldsymbol{\mu}_{\rm NY}$ can be viewed as the Chern-Simons term for torsion. These terms contribute the following additional terms to the Noether charge at the bifurcation surface

$$\begin{aligned}
\boldsymbol{Q}_K|_{\rm E} &= \epsilon_{cdab}\boldsymbol{R}^{cd}(K \cdot \boldsymbol{\omega}^{ab}) = \tfrac{1}{2}\epsilon_{cdab}\boldsymbol{R}^{cd}(E^a \cdot E^b \cdot d\boldsymbol{\xi}) \\
\boldsymbol{Q}_K|_{\rm P} &= -2\boldsymbol{R}_{ab}(K \cdot \boldsymbol{\omega}^{ab}) = -\boldsymbol{R}_{ab}(E^a \cdot E^b \cdot d\boldsymbol{\xi}) \\
\boldsymbol{Q}_K|_{\rm NY} &= -\boldsymbol{e}_a \wedge \boldsymbol{e}_b(K \cdot \boldsymbol{\omega}^{ab}) = -\tfrac{1}{2}\boldsymbol{e}_a \wedge \boldsymbol{e}_b(E^a \cdot E^b \cdot d\boldsymbol{\xi})
\end{aligned}$$

and hence integrating the corresponding gauge-invariant forms over B gives

$$\begin{aligned}
\int_B \underline{\boldsymbol{Q}}_K\Big|_{\rm E} &= -2\kappa \int_B \star\tilde{\epsilon}_{ab}\underline{\boldsymbol{R}}^{ab} = 2\kappa \int_B \epsilon_{ab}\underline{\boldsymbol{R}}^{ab} \\
\int_B \underline{\boldsymbol{Q}}_K\Big|_{\rm P} &= 2\kappa \int_B \tilde{\epsilon}_{ab}\underline{\boldsymbol{R}}^{ab} \\
\int_B \underline{\boldsymbol{Q}}_K\Big|_{\rm NY} &= \int_B d\underline{\boldsymbol{\xi}} = 0
\end{aligned} \tag{5.21}$$

The Euler contribution is the Euler class of the tangent bundle TB i.e. the Euler characteristic of B. The Pontryagin contribution is, similarly, the Euler class of the normal bundle of B in M. Since, we have smooth, no-where vanishing normals to B, the Euler class of the

2. The explicit expression for $\boldsymbol{\mu}_{\rm E}$ seems to be largely absent from the literature except in [22], and in [70] where it is given in terms of the Dirac matrices.

normal bundle must vanish (see Proposition. 11.17 in [71]).[3] Further, we see that the Holst Lagrangian is $\boldsymbol{L}_{Holst} \sim -\boldsymbol{L}_{NY} + \boldsymbol{T}^a \wedge \boldsymbol{T}_a$, and is thus an exact form up to terms not involving the curvature. This explains why the Holst term does not contribute to the Noether charge at the horizon.[4] The Noether charge contributions Eq. 5.21 are purely topological and do not contribute to the perturbed entropy, and due to the asymptotic fall-off conditions they do not contribute to the canonical energy and angular momentum at infinity. Hence none of the above terms affect the first law. But they do affect a prescription for a total entropy as discussed towards the end of Ch. 4. As shown in [58, 60], the Euler term in the Noether charge leads to violations of the second law in General Relativity if one prescribes that the total entropy be given by the gravitational charge.

5.2 Yang-Mills

Next let us consider Einstein-Yang-Mills theory where the Yang-Mills connection $\boldsymbol{A}'^{I'}$ in Eq. 2.11 is a dynamical field governed by the Yang-Mills Lagrangian Eq. 5.22. Since we have worked out the gravitational contribution in § 5.1 in this section we only deal with the Yang-Mills connection, and so, for simplicity omit the "primes" and write the Yang-Mills connection as $\boldsymbol{A}^I$.

The contribution of Yang-Mills fields to the first law was worked out by Sudarsky and Wald [4, 5] under the assumption that one can pick a global choice of gauge i.e. an everywhere smooth section $s : M \to P$ (see § A).[5] They then consider the pullback $A^I_\mu = (s^*)_\mu^m A^I_m$ of the Yang-Mills connection as a tensor field on spacetime. They further assume that the section s can be chosen so that A^I_μ is stationary with respect to the horizon Killing field K^μ.

As discussed in Ch. 2, these assumptions are too restrictive to cover all Yang-Mills fields

3. This can also be shown by an explicit computation of this term, and the fact that the extrinsic curvature of B in M vanishes (see [15]).

4. See [72] for an Euclidean path integral approach to the Holst and Nieh-Yan terms in the Lagrangian.

5. Sudarsky and Wald also assume that the Yang-Mills group G is compact, and is $SU(2)$, but this restriction can be easily removed.

that are of interest. As noted before, sections on a principal bundle exist if and only if the bundle is trivial. For a non-trivial bundle there exist no global sections and the Yang-Mills connection cannot be considered as a globally well-defined tensor field on spacetime. It is far from clear that even for a trivial bundle a section s can be chosen so that for a given connection $\boldsymbol{A}^I$, the pullback $s^*\boldsymbol{A}^I \equiv A^I_\mu$ is both smooth everywhere and is annihilated by the Lie derivative with the horizon Killing field K^μ. Using this assumption Sudarsky and Wald conclude that the Yang-Mills fields do not contribute to the first law at the bifurcation surface as the Yang-Mills potential on B vanishes since $K^\mu A^I_\mu|_B = 0$. In fact, as argued by Gao [14], the Maxwell gauge field on the Reissner-Norstörm spacetime, in the standard choice of gauge, is singular on the bifurcation surface. Working instead on some other cross-section of the horizon where the Maxwell vector potential is smooth, Gao finds a "potential times perturbed charge" term at the horizon for Maxwell fields. [14] also finds that Yang-Mills fields do contribute at the horizon but their contribution cannot be put into a "potential times perturbed charge" form without additional gauge choices which might be incompatible with the assumed stationarity of A^I_μ (see § 4 [14]).

Our formalism allows us to work on arbitrary non-trivial bundles for Yang-Mills fields without making any choice of gauge, and we only assume that the Yang-Mills connection on the principal bundle is Lie-derived up to a gauge transformation i.e. $\pounds_K \boldsymbol{A}^I = 0$ where K^m is an infinitesimal automorphism of the bundle which projects to the horizon Killing field K^μ. Using this, we find potentials for the Yang-Mills fields that are constant on the horizon (Theorem 1) and that the Yang-Mills fields do contribute a "potential times perturbed charge" term at the bifurcation surface (Cor. 3.1) without assuming any choice of gauge. Thus, the following will be a generalisation of the results of [4, 5, 14].

The Yang-Mills Lagrangian can be written for any structure group G using a non-degenerate, symmetric bilinear form on its Lie algebra $\mathfrak{g}$, which is invariant under the adjoint action of G on its Lie algebra. Since we have assumed that the Lie algebra $\mathfrak{g}$ of the structure group is semisimple we will use the Killing form k_{IJ} (Eq. 1.6) as such a non-degenerate, symmetric, invariant bilinear form.[6] Further, any semisimple Lie algebra can be decomposed uniquely into a direct sum of simple Lie algebras (see § 1.10 [49] or § 11.2 [52]), and thus the Yang-Mills Lagrangian for $\mathfrak{g}$ can be written as a sum of Yang-Mills Lagrangians of the same form for each simple factor (with possibly different coupling constants). We can also include abelian groups (which, by definition, are neither simple nor semi-simple) into the theory by using the natural product on their Lie algebra $\mathbb{R}^n$. For instance, to get Maxwell electromagnetism we can use $k_{IJ} \to -2$ and $g^2 \to \mu_0$ in Eq. 5.22.

With the above discussion, we write the Yang-Mills Lagrangian on the bundle as:

$$\boldsymbol{L}_{YM} = \frac{1}{4g^2}(\star \boldsymbol{F}_I) \wedge \boldsymbol{F}^I = \frac{1}{8g^2}\boldsymbol{\varepsilon}_4\left(F^2\right) \in \Omega^4_{hor}P \tag{5.22}$$

where $F^2 := (E^a \cdot E^b \cdot \boldsymbol{F}^I)(E_a \cdot E_b \cdot \boldsymbol{F}_I)$ and g^2 is the Yang-Mills coupling constant. On spacetime M we get the usual Lagrangian $\underline{\boldsymbol{L}}_{YM} = \frac{1}{8g^2}\underline{\boldsymbol{\varepsilon}}_4 F^I_{\mu\nu}F_I^{\mu\nu}$.

The first variation gives $\delta\boldsymbol{L}_{YM} = -\boldsymbol{\mathcal{T}}_a \wedge \delta\boldsymbol{e}^a + \boldsymbol{\mathcal{E}}_I \wedge \delta\boldsymbol{A}^I + d\boldsymbol{\theta}^{(\mathrm{YM})}$ with:

$$\begin{aligned} \boldsymbol{\mathcal{E}}_I &= -\tfrac{1}{2g^2} D \star \boldsymbol{F}_I \\ \boldsymbol{\theta}^{(\mathrm{YM})} &= \tfrac{1}{2g^2} \star \boldsymbol{F}_I \wedge \delta\boldsymbol{A}^I \end{aligned} \tag{5.23}$$

where we have used the first form of the Lagrangian. The symplectic current contribution takes the form $\boldsymbol{\omega}_{(\mathrm{YM})} = \frac{1}{2g^2}\left[\delta_1(\star\boldsymbol{F}_I) \wedge \delta_2\boldsymbol{A}^I - \delta_2(\star\boldsymbol{F}_I) \wedge \delta_1\boldsymbol{A}^I\right]$.

To compute the energy-momentum 3-form $\boldsymbol{\mathcal{T}}_a$, it is convenient to use the second form of the Lagrangian Eq. 5.22. Varying with the tetrad we have

$$\begin{aligned} \tfrac{1}{8g^2}\delta_e(\boldsymbol{\varepsilon}_4 F^2) &= \tfrac{1}{8g^2}\tfrac{1}{3!}\epsilon_{abcd}\delta\boldsymbol{e}^a \wedge \boldsymbol{e}^b \wedge \boldsymbol{e}^c \wedge \boldsymbol{e}^d\ F^2 + \tfrac{1}{8g^2}\boldsymbol{\varepsilon}_4\delta_e\left[(E^a \cdot E^b \cdot \boldsymbol{F}^I)(E_a \cdot E_b \cdot \boldsymbol{F}_I)\right] \\ &= \tfrac{1}{8g^2}\left[-\tfrac{1}{3!}\epsilon_{abcd}\boldsymbol{e}^b \wedge \boldsymbol{e}^c \wedge \boldsymbol{e}^d\ F^2\right] \wedge \delta\boldsymbol{e}^a + \boldsymbol{\varepsilon}_4\tfrac{1}{2g^2}(E^a \cdot E^b \cdot \boldsymbol{F}^I)(\delta E_a \cdot E_b \cdot \boldsymbol{F}_I) \end{aligned}$$

6. Our convention for the Killing form differs from that of [4] by a sign and a factor of 2.

The first term can be written in the form

$$\frac{1}{8g^2}\left[-\frac{1}{3!}\epsilon_{abcd}\boldsymbol{e}^b\wedge\boldsymbol{e}^c\wedge\boldsymbol{e}^d\ F^2\right]\wedge\delta\boldsymbol{e}^a=\frac{1}{2g^2}\left[\star\boldsymbol{e}_a\left(-\frac{1}{4}F^2\right)\right]\wedge\delta\boldsymbol{e}^a$$

and the second term as (using Eq. 2.22)

$$\begin{aligned}\boldsymbol{\varepsilon}_4\frac{1}{2g^2}(E^a\cdot E^b\cdot\boldsymbol{F}^I)(\delta E_a\cdot E_b\cdot\boldsymbol{F}_I)&=-\boldsymbol{\varepsilon}_4\frac{1}{2g^2}(E^c\cdot E^b\cdot\boldsymbol{F}^I)(E_a\cdot E_b\cdot\boldsymbol{F}_I)E_c\cdot\delta\boldsymbol{e}^a\\&=\frac{1}{2g^2}\left[(E_c\cdot\boldsymbol{\varepsilon}_4)(E^c\cdot E^b\cdot\boldsymbol{F}^I)(E_a\cdot E_b\cdot\boldsymbol{F}_I)\right]\wedge\delta\boldsymbol{e}^a\\&=\frac{1}{2g^2}\left[\star\boldsymbol{e}_c(E^c\cdot E^b\cdot\boldsymbol{F}^I)(E_a\cdot E_b\cdot\boldsymbol{F}_I)\right]\wedge\delta\boldsymbol{e}^a\end{aligned}$$

putting these together we have

$$\boldsymbol{\mathcal{T}}_a=-\frac{1}{2g^2}\star\boldsymbol{e}^c\left[(E_a\cdot E_b\cdot\boldsymbol{F}_I)(E_c\cdot E^b\cdot\boldsymbol{F}^I)-\frac{1}{4}\eta_{ac}F^2\right]$$

To find the Noether current consider the following computation for some $X^m\in\mathfrak{aut}(P)$:

$$\begin{aligned}X\cdot\boldsymbol{L}_{YM}&=\frac{1}{8g^2}X\cdot(\boldsymbol{\varepsilon}_4F^2)=\frac{1}{8g^2}\frac{1}{3!}\epsilon_{abcd}(X\cdot\boldsymbol{e}^a)\ \boldsymbol{e}^b\wedge\boldsymbol{e}^c\wedge\boldsymbol{e}^dF^2\\&=-\frac{1}{2g^2}\left[\star\boldsymbol{e}_a\left(-\frac{1}{4}F^2\right)\right](X\cdot\boldsymbol{e}^a)\end{aligned}\tag{5.24}$$

$$\begin{aligned}\boldsymbol{\theta}_X^{(\mathrm{YM})}&=\frac{1}{2g^2}\star\boldsymbol{F}_I\wedge\pounds_X\boldsymbol{A}^I=\frac{1}{2g^2}\star\boldsymbol{F}_I\wedge\left(X\cdot\boldsymbol{F}^I+D(X\cdot\boldsymbol{A}^I)\right)\\&=\frac{1}{2g^2}\star\boldsymbol{F}_I\wedge X\cdot\boldsymbol{F}^I+\frac{1}{2g^2}D\left(\star\boldsymbol{F}_I(X\cdot\boldsymbol{A}^I)\right)-\frac{1}{2g^2}D\star\boldsymbol{F}_I(X\cdot\boldsymbol{A}^I)\end{aligned}\tag{5.25}$$

The first term in $\boldsymbol{\theta}_X^{(\mathrm{YM})}$ can be written as

$$\begin{aligned}\frac{1}{2g^2}\star\boldsymbol{F}_I\wedge X\cdot\boldsymbol{F}^I&=\frac{1}{2g^2}\star\boldsymbol{F}^I\wedge\frac{1}{2!}X\cdot(\boldsymbol{e}^a\wedge\boldsymbol{e}^b)(E_b\cdot E_a\cdot\boldsymbol{F}_I)\\&=\frac{1}{2g^2}\frac{1}{2!2!}\epsilon_{efcd}\boldsymbol{e}^c\wedge\boldsymbol{e}^d(E^f\cdot E^e\cdot\boldsymbol{F}^I)\wedge\boldsymbol{e}^b(E_b\cdot E_a\cdot\boldsymbol{F}_I)(X\cdot\boldsymbol{e}^a)\\&=\frac{1}{2g^2}\star\boldsymbol{e}^c(E^b\cdot E_c\cdot\boldsymbol{F}^I)(E_b\cdot E_a\cdot\boldsymbol{F}_I)(X\cdot\boldsymbol{e}^a)\end{aligned}$$

This gives the Noether current:

$$
\begin{aligned}
\boldsymbol{J}_X^{(\mathrm{YM})} &= \tfrac{1}{2g^2} \star \boldsymbol{e}^c \left[(E^b \cdot E_c \cdot \boldsymbol{F}^I)(E_b \cdot E_a \cdot \boldsymbol{F}_I) - \tfrac{1}{4}\eta_{ac}F^2 \right] (X \cdot \boldsymbol{e}^a) \\
&\quad + \boldsymbol{\mathcal{E}}_I (X \cdot \boldsymbol{A}^I) + \tfrac{1}{2g^2} D \left(\star \boldsymbol{F}_I (X \cdot \boldsymbol{A}^I) \right) \\
&= d\boldsymbol{Q}_X^{(\mathrm{YM})} + \boldsymbol{\mathcal{E}}_I (X \cdot \boldsymbol{A}^I) - \boldsymbol{\mathcal{T}}_a (X \cdot \boldsymbol{e}^a)
\end{aligned}
\tag{5.26}
$$

The terms with $\boldsymbol{\mathcal{E}}_I$ and $\boldsymbol{\mathcal{T}}_a$ contribute to the constraints of Einstein-Yang-Mills theory (see Eq. 2.46) and the Noether charge contribution is:

$$
\boldsymbol{Q}_X^{(\mathrm{YM})} = \tfrac{1}{2g^2} \star \boldsymbol{F}_I (X \cdot \boldsymbol{A}^I) \tag{5.27}
$$

which is of the general form given in Lemma 2.4.

Now consider a stationary-axisymmetric connection $\boldsymbol{A}^I$ which satisfies the Einstein-Yang-Mills equations on a stationary axisymmetric black hole spacetime. This means that the horizon Killing field K^m is an infinitesimal automorphism that preserves the Yang-Mills conection $\pounds_K \boldsymbol{A}^I = 0$. The extent to which K^m is determined by its projection K^μ on M is given by Lemmas A.1 and A.4. Following the computations in Theorem 1–Cor. 3.1 we can write the Noether charge of Yang-Mills fields at the horizon as

$$
\int_B \underline{\boldsymbol{Q}_K^{(\mathrm{YM})}} = \mathscr{V}^\Lambda \mathscr{Q}_\Lambda
$$

where $\mathscr{V}^\Lambda$ is the Yang-Mills potential and $\mathscr{Q}_\Lambda$ is the Yang-Mills electric charge (also see [48]) given by

$$
\mathscr{Q}_\Lambda = \tfrac{1}{2g^2} \int_B \underline{* \boldsymbol{F}_I h^I_\Lambda} \tag{5.28}
$$

where the h^I_Λ are some fixed basis of a fixed choice of Cartan subalgebra on the horizon as defined in Ch. 3, and the contribution to the first law at B becomes

$$\delta \int_B \underline{\boldsymbol{Q}}_K^{(\mathrm{YM})} = \mathscr{V}^\Lambda \delta \mathscr{Q}_\Lambda \tag{5.29}$$

To compute the contribution at infinity, first consider the vector field t^m which gives the canonical energy E_{can}. We choose the Yang-Mills fields to satisfy the asymptotic conditions

$$\boldsymbol{F}^I = O(1/r^2)$$

which also gives $\delta \boldsymbol{A}^I = O(1/r)$. We immediately see that $\boldsymbol{\theta}^{(\mathrm{YM})} = \frac{1}{2g^2} \star \boldsymbol{F}_I \wedge \delta \boldsymbol{A}^I = O(1/r^3)$ and does not contribute to the first law.

Since t^m is an infinitesimal automorphism which preserves the connection we see that

$$\begin{aligned}
0 &= \left(\pounds_t \boldsymbol{F}^I \right) |_{P_\infty} = D(t \cdot \boldsymbol{F}^I) - c^I{}_{JK}(t \cdot \boldsymbol{A}^J) \boldsymbol{F}^K \\
&= -c^I{}_{JK}(t \cdot \boldsymbol{A}^J) \boldsymbol{F}^K + O(1/r^3) \\
0 &= (\pounds_t \boldsymbol{A}^I)|_{P_\infty} = t \cdot \boldsymbol{F}^I + D(t \cdot \boldsymbol{A}^I) \\
&= D(t \cdot \boldsymbol{A}^I) + O(1/r^2)
\end{aligned}$$

Up to higher order terms in $1/r$, we can repeat the procedure in Theorem 1–Cor. 3.1 to get contribution of Yang-Mills fields to the canonical energy as

$$E_{can}^{(\mathrm{YM})} = \int_\infty \underline{\boldsymbol{Q}}_t^{(\mathrm{YM})} = \mathscr{V}^\Lambda \mathscr{Q}_\Lambda$$

with the Yang-Mills electric charge Eq. 5.28 but evaluated at infinity. Note that aymptotically the Yang-Mills equation of motion at infinity becomes

$$\begin{aligned}
0 = \boldsymbol{\mathcal{E}}_I|_{P_\infty} &= -\frac{1}{2g^2} D \star \boldsymbol{F}_I = O(1/r^3) \\
\implies D(\star \boldsymbol{F}_I h^I_\Lambda) &= d(\star \boldsymbol{F}_I h^I_\Lambda) = O(1/r^3)
\end{aligned}$$

where we used the basis h^I_Λ of the fixed Cartan subalgebra defined in Ch. 3. Thus, the Yang-Mills charge can be computed over any "sufficiently large" surface homologous to a sphere. Further

$$\delta(t \cdot \boldsymbol{A}^I)|_\infty = (t \cdot \delta\boldsymbol{A}^I)|_\infty = O(1/r) \tag{5.30}$$

and the variation of the potential term falls off faster at infinity and we have the first law contribution at infinity as

$$\delta E^{(\text{YM})}_{can} = \delta \int_\infty \underline{\boldsymbol{Q}}^{(\text{YM})}_t = \mathscr{V}^\Lambda \delta\mathscr{Q}_\Lambda \tag{5.31}$$

In a similar manner (assuming faster fall-off for the asymptotic fields if necessary), the Yang-Mills contribution to the canonical angular momentum is

$$J^{(\text{YM})}_{can} = -\int_\infty \underline{\boldsymbol{Q}}^{(\text{YM})}_\phi = -\tfrac{1}{2g^2}\int_\infty \underline{(\phi \cdot \boldsymbol{A}^I)} * \underline{\boldsymbol{F}}_I \tag{5.32}$$

The first law of black hole mechanics in Einstein-Yang-Mills then can be written as

$$T_{\mathscr{H}}\delta S + (\mathscr{V}^\Lambda\delta\mathscr{Q}_\Lambda)\Big|_B = \delta M_{ADM} + (\mathscr{V}^\Lambda\delta\mathscr{Q}_\Lambda)\Big|_\infty - \Omega_{\mathscr{H}}\left(\delta J_{ADM} + \delta J^{(\text{YM})}_{can}\right) \tag{5.33}$$

Remark 5.2 (Yang-Mills potentials at horizon and infinity). When the Yang-Mills structure group is abelian the charge at B and infinity are equal (using the abelian Yang-Mills equation of motion). Thus, the abelian Yang-Mills term in the first law can be written as a "difference in potentials times perturbed charge" $\left(\mathscr{V}^\Lambda\Big|_\infty - \mathscr{V}^\Lambda\Big|_B\right)\delta\mathscr{Q}_\Lambda$. Further, the ambiguity in the choice of the vector field K^m (for a given horizon Killing field K^μ) is given by a $\mathbb{R}^n$-valued constant function λ; here n is the dimension of the abelian structure group (see Remark A.2). By a suitable choice of λ we can always set the potentials at the horizon to vanish, while shifting the potentials at infinity by a constant.

Even for a non-abelian structure group, if there exists a vertical vector field $Y^m \in VP$ corresponding to a global symmetry of $\boldsymbol{A}^I$ (which is a solution to the Einstein-Yang-Mills equations) as described in Lemma A.1, we can use Y^m to redefine our choice of the horizon Killing field K^m on the bundle. Using, this freedom we can set some, or possibly all, of the horizon potentials to zero but at the cost of changing the potentials at infinity. The cases where we can set all potentials at the horizon to vanish correspond to the analysis of Sudarsky and Wald [4, 5]. For arbitrary connections $\boldsymbol{A}^I$, which solve the Einstein-Yang-Mills equations, there may not exist any such global symmetry (which corresponds to the existence of global solutions to $D\lambda^I = 0$, see Remark A.2). Thus, it seems that the first law as derived in [4, 5] applies only to special cases and in general, the first law for Einstein-Yang-Mills takes the form above (also see the related discussion in [14]).

As an illustration of the $\boldsymbol{\mu}$-ambiguity (Eq. 2.34) we add a topological term to the Yang-Mills Lagrangian using the Chern character of the bundle as [7]

$$\boldsymbol{L}_{\text{C}} := \tfrac{1}{2}\boldsymbol{F}_I \wedge \boldsymbol{F}^I = d\boldsymbol{\mu}_{\text{C}} \tag{5.34}$$

where $\boldsymbol{\mu}_{\text{C}}$ is the Chern-Simons form

$$\boldsymbol{\mu}_{\text{C}} = \tfrac{1}{2}\left(\boldsymbol{A}_I \wedge \boldsymbol{F}^I - \tfrac{1}{6}c_{IJK}\boldsymbol{A}^I \wedge \boldsymbol{A}^J \wedge \boldsymbol{A}^K\right) \tag{5.35}$$

The corresponding Noether charge for K^m then gets an additional contribution

$$\int_B \underline{\boldsymbol{Q}}_K = \mathscr{V}^\Lambda \tilde{\mathscr{Q}}_\Lambda \tag{5.36}$$

where the Yang-Mills magnetic charge is

$$\tilde{\mathscr{Q}}_\Lambda = \int_B \underline{\boldsymbol{F}_I h^I_\Lambda} \tag{5.37}$$

The magnetic charge is purely a topological charge that does not vary under perturbations

7. In particle physics literature this is also known as the θ-term; where the θ refers to the conventional coupling constant in front of $\boldsymbol{L}_C$ and is unrelated to the symplectic potential.

and hence does not contribute to the first law.

5.3 Dirac spinor

As the third case of interest we consider spinor matter fields in Einstein-Dirac theory. The construction of Dirac spinor fields on a *spin bundle* $P_{Spin}M$ is reviewed in § A.1. The Dirac Lagrangian can be written as:

$$\boldsymbol{L}_{Dirac} := \boldsymbol{\varepsilon}_4 \left(\frac{1}{2} \overline{\Psi} \not{D} \Psi - \frac{1}{2} \not{D} \overline{\Psi} \Psi - m \overline{\Psi} \Psi \right) \tag{5.38}$$

where $\boldsymbol{\varepsilon}_4$ is the horizontal volume 4-form on P_{Spin} Eq. A.32. Note that we have admitted a connection with torsion and so this Lagrangian is *not* equivalent to usual Dirac Lagrangian which uses the torsionless Levi-Civita connection Eq. A.40 (see § V.B.4 [73]); the dynamics of the Dirac field *does* depend on the choice of spin connection used. If one assumes that the connection is torsionless from the outset, then one has to work in the "second-order formulation". Since the following computations are easier in the first-order formalism, we will continue to use an independent connection with torsion to obtain a first law for Einstein-Dirac theory. The computations in the second-order formalism can be performed in exactly the same manner (see Remark 5.1).

To get the Dirac equation and the symplectic potential compute first the variation with the Dirac spinor fields:

$$\begin{aligned} \boldsymbol{\varepsilon}_4 \left(\overline{\Psi} \not{D} \delta\Psi \right) &= \boldsymbol{\varepsilon}_4 \left(\overline{\Psi} \gamma^a E_a \cdot D\delta\Psi \right) = -(E_a \cdot \boldsymbol{\varepsilon}_4) \overline{\Psi} \wedge \gamma^a D\delta\Psi \\ &= D \left[(E_a \cdot \boldsymbol{\varepsilon}_4) \overline{\Psi} \gamma^a \delta\Psi \right] - D \left[(E_a \cdot \boldsymbol{\varepsilon}_4) \overline{\Psi} \gamma^a \right] \delta\Psi \end{aligned} \tag{5.39}$$

where the second equality in the first line uses the vanishing of a horizontal 5-form. The first term in the last line contributes to the symplectic potential $\boldsymbol{\theta}$ while, using Eq. A.32 the second term can be written as:

$$\begin{aligned} -D\left[(E_a\cdot\boldsymbol{\varepsilon}_4)\overline{\Psi}\gamma^a\right]\delta\Psi &= -\frac{1}{3!}D\left(\overline{\Psi}\gamma^a\epsilon_{abcd}\boldsymbol{e}^b\wedge\boldsymbol{e}^c\wedge\boldsymbol{e}^d\right)\delta\Psi \\ &= -\boldsymbol{\varepsilon}_4\left(\not{D}\overline{\Psi}\delta\Psi\right)-\frac{1}{3!}\overline{\Psi}\gamma^a\epsilon_{abcd}\boldsymbol{T}^b\wedge\boldsymbol{e}^c\wedge\boldsymbol{e}^d\delta\Psi \end{aligned} \tag{5.40}$$

The variation with respect to the Dirac cospinor field $\overline{\Psi}$ is obtained by the condition $\overline{\delta\Psi}=\delta\overline{\Psi}$ and we have $\delta_\Psi\boldsymbol{L}_{Dirac}=\overline{\boldsymbol{\mathcal{E}}}\delta\Psi+\delta\overline{\Psi}\;\boldsymbol{\mathcal{E}}+d\boldsymbol{\theta}^{(\text{Dirac})}$ with

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \left[(\not{D}-m)\Psi-\tfrac{1}{3!}T^b{}_{ba}(\gamma^a\Psi)\right]\boldsymbol{\varepsilon}_4 \\ \overline{\boldsymbol{\mathcal{E}}} &= \left[(-\not{D}-m)\overline{\Psi}+\tfrac{1}{3!}T^b{}_{ba}(\overline{\Psi}\gamma^a)\right]\boldsymbol{\varepsilon}_4 \\ \boldsymbol{\theta}^{(\text{Dirac})} &= \tfrac{1}{2}(E_a\cdot\boldsymbol{\varepsilon}_4)\left(\overline{\Psi}\gamma^a\delta\Psi-\delta\overline{\Psi}\gamma^a\Psi\right) \\ &= \tfrac{1}{2}(\star\boldsymbol{e}_a)\left(\overline{\Psi}\gamma^a\delta\Psi-\delta\overline{\Psi}\gamma^a\Psi\right) \end{aligned} \tag{5.41}$$

where we have used the frame components of the torsion $T^c{}_{ab}:=E^b\cdot E^a\cdot\boldsymbol{T}^c$ and the horizontal Hodge dual Eq. A.33. On spacetime M, the Dirac equation $\underline{\boldsymbol{\mathcal{E}}}$ takes the form

$$\underline{\boldsymbol{\mathcal{E}}}=\left[(\not{D}-m)\Psi-\tfrac{1}{3!}T^\nu{}_{\nu\mu}\;\gamma^\mu\Psi\right]\underline{\boldsymbol{\varepsilon}}_4$$

We again, note that this is not equivalent to the usual Dirac equation since we chose a spin connection with torsion [73] — setting the torsion to vanish however does give us the standard Dirac equation.

For the energy-momentum form we have to compute the variation with the tetrad. For this we rewrite

$$\begin{aligned} \boldsymbol{\varepsilon}_4\overline{\Psi}\not{D}\Psi &= \tfrac{1}{4!}\epsilon_{abcd}\boldsymbol{e}^a\wedge\boldsymbol{e}^b\wedge\boldsymbol{e}^c\wedge\boldsymbol{e}^d\overline{\Psi}\gamma^e E_e\cdot D\Psi \\ &= -\tfrac{1}{3!}\epsilon_{abcd}\boldsymbol{e}^b\wedge\boldsymbol{e}^c\wedge\boldsymbol{e}^d\wedge(\overline{\Psi}\gamma^a D\Psi) \end{aligned}$$

Then

$$\delta_e(\boldsymbol{\varepsilon}_4\overline{\Psi}\not{D}\Psi) = -\tfrac{1}{2}\epsilon_{abcd}\boldsymbol{e}^b \wedge \boldsymbol{e}^c \wedge (\overline{\Psi}\gamma^d D\Psi) \wedge \delta\boldsymbol{e}^a$$

and

$$-\delta_e(\boldsymbol{\varepsilon}_4 m\overline{\Psi}\Psi) = \tfrac{1}{3!}\epsilon_{abcd}\boldsymbol{e}^b \wedge \boldsymbol{e}^c \wedge \boldsymbol{e}^d(m\overline{\Psi}\Psi) \wedge \delta\boldsymbol{e}^a$$

Thus, we have $\delta_e\boldsymbol{L}_{Dirac} = -\boldsymbol{\mathcal{T}}_a \wedge \delta\boldsymbol{e}^a$ with the *energy-momentum*

$$\begin{aligned}\boldsymbol{\mathcal{T}}_a &= \epsilon_{abcd}\boldsymbol{e}^b \wedge \boldsymbol{e}^c \wedge \left[\tfrac{1}{4}\left(\overline{\Psi}\gamma^d D\Psi - D\overline{\Psi}\gamma^d\Psi\right) - \tfrac{1}{3!}\boldsymbol{e}^d m\overline{\Psi}\Psi\right] \\ &= (\star\boldsymbol{e}_a)\left(\tfrac{1}{2}\overline{\Psi}\not{D}\Psi - \tfrac{1}{2}\not{D}\overline{\Psi}\Psi - m\overline{\Psi}\Psi\right) - \tfrac{1}{2}(\star\boldsymbol{e}_b)\left(\overline{\Psi}\gamma^b E_a \cdot D\Psi - E_a \cdot D\overline{\Psi}\gamma^b\Psi\right)\end{aligned}$$

For the spin current compute the variation due to the connection:

$$\delta_\omega\frac{1}{2}\left(\boldsymbol{\varepsilon}_4\overline{\Psi}\not{D}\Psi\right) = -\frac{1}{16}\,\boldsymbol{\varepsilon}_4\left(\overline{\Psi}\gamma^c[\gamma_a,\gamma_b]\Psi E_c \cdot \delta\boldsymbol{\omega}^{ab}\right) = \frac{1}{16}(E_c \cdot \boldsymbol{\varepsilon}_4)\left(\overline{\Psi}\gamma^c[\gamma_a,\gamma_b]\Psi\right) \wedge \delta\boldsymbol{\omega}^{ab} \tag{5.42}$$

where the last equality uses the vanishing of a horizontal 5-form. Thus we have $\delta_\omega\boldsymbol{L} = -\boldsymbol{\mathcal{S}}_{ab} \wedge \delta\boldsymbol{\omega}^{ab}$ where the *spin current* is:

$$\begin{aligned}\boldsymbol{\mathcal{S}}_{ab} &:= -\frac{1}{16}(E_c \cdot \boldsymbol{\varepsilon}_4)\left(\overline{\Psi}\gamma^c[\gamma_a,\gamma_b]\Psi + \overline{\Psi}[\gamma_a,\gamma_b]\gamma^c\Psi\right) \\ &= -\frac{1}{16}(\star\boldsymbol{e}_c)\ \left(\overline{\Psi}\gamma^c[\gamma_a,\gamma_b]\Psi + \overline{\Psi}[\gamma_a,\gamma_b]\gamma^c\Psi\right)\end{aligned} \tag{5.43}$$

To compute the Noether current and Noether charge we need a notion of a "Lie derivative" for spinor fields. As discussed in Ch. 1 the prescriptions for defining a Lie derivative on spinors in the existing literature [29, 30, 31] are not satisfactory. Since we view spinors as fields defined on the spin bundle P_{Spin} we can use the natural notion of Lie derivatives with respect to a vector field $X^m \in \mathfrak{aut}(P)$ on the bundle which we will show (see Eq. A.57) reduces to the definition given by Lichnerowicz [26] in the case X^m projects to a Killing vector field. Using Eq. A.7 the Lie derivative on spinor fields on the bundle can be written as $\pounds_X \Psi := X \cdot d\Psi = X \cdot D\Psi + \frac{1}{8} X \cdot \boldsymbol{\omega}_{ab}[\gamma^a, \gamma^b]\Psi$. The Noether current then is

$$\boldsymbol{J}_X^{(\text{Dirac})} = -\boldsymbol{\mathcal{T}}_a(X \cdot \boldsymbol{e}^a) - \boldsymbol{\mathcal{S}}_{ab}(X \cdot \boldsymbol{\omega}^{ab}) \tag{5.44}$$

The energy-momentum and spin current terms on the right-hand-side contribute to the constraints. Thus the Noether charge contribution of the Dirac fields can be chosen to vanish i.e. $\boldsymbol{Q}_X^{(\text{Dirac})} = 0$, as we expect from the general formula in Lemma 2.4.

For the first law, stationary axisymmetric Dirac fields i.e. $\pounds_K \Psi = 0$ do not explicitly contribute to the black hole entropy since the Dirac field contribution to the Noether charge vanishes identically. Near infinity, the Dirac field Ψ falls off faster than $1/r^{3/2}$, in which case $t \cdot \boldsymbol{\theta}^{(\text{Dirac})}$ falls-off faster than $1/r^3$. Since the Noether charge contribution of the Dirac field vanishes, the Dirac field does not explicitly contribute to boundary integral defining the canonical energy (Eq. 4.1). These fall-offs also ensure that the torsion terms in the gravitational canonical energy in Eq. 5.18 vanish. Similarly, there is no explicit Dirac contribution to the boundary integral for the canonical angular momentum and the torsion terms in the gravitational canonical angular momentum Eq. 5.19 also vanish. Thus, the first law of black hole mechanics with spinor fields governed by the Einstein-Dirac Lagrangian Eq. 5.38 retains the form Eq. 5.20.

APPENDIX A

PRINCIPAL FIBRE BUNDLES

In this section we review some differential geometry on principal bundles following the classic treatment of [74, 53, 54, 13, 28].[1] For any smooth manifold N, we denote the tangent bundle by TN, the cotangent bundle by T^*N and the bundle of differential forms of degree k by $\Omega^k N$. Diffeomorphisms of N form an infinite-dimensional Lie group $\mathrm{Diff}(N)$ and vector fields on N are the corresponding Lie algebra $TN \cong \mathfrak{diff}(N)$. $\mathrm{Diff}(N)$ has a natural action by pullbacks/pushforwords on tensor fields on N, and the corresponding Lie algebra action of $\mathfrak{diff}(N)$ is implemented by the Lie derivative.

* * *

A G-principal bundle over M is a manifold P with an action of the group G (which we take to be semisimple) and a projection $\pi : P \to M$ such that in some neighbourhood U_x of every point $x \in M$, $\pi^{-1}U_x$ is diffeomorphic to the product $U_x \times G$ (see § I.5. [53] and § II.B.2. [13] for a precise definition). The manifold P is called the *total space*; M is the *base space* and G, the *structure group*. When formulating gauge theories, M will be the spacetime manifold and the structure group G will be the group of internal gauge transformations that acts on the dynamical fields at a point of M.

The projection $\pi : P \to M$ induces the push-forward $\pi_* : TP \to TM$ which, using abstract indices, we denote by $(\pi_*)^\mu_m$. The kernel of π_* defines the *vertical vector fields* denoted by $VP := \ker \pi_* \subset TP$. There is a canonical isomorphism between vertical vector fields and $\mathfrak{g}$-valued functions $VP \cong C^\infty P(\mathfrak{g})$ (see § I.5 [53] and § Vbis.A [13]). This also implies that $TP/VP \cong TM$ i.e. if $X^m, X'^m \in TP$ and $X^m - X'^m \in VP$ then $(\pi_*)^\mu_m X^m = (\pi_*)^\mu_m X'^m = X^\mu$. The space of *horizontal* k-forms, $\Omega^k_{hor} P \subset \Omega^k P$ is defined as the subspace

1. Note that these references may use different conventions for numerical factors and signs for differential forms when converting to and from an index notation. Throughout this paper we use the conventions of [45].

of k-forms that vanish on vertical vector fields i.e. $\boldsymbol{\sigma} \in \Omega^k_{hor} P \iff X \cdot \boldsymbol{\sigma} = 0 \; \forall \; X^m \in VP$. Without specifying any further structure on P, there is no natural notion of "horizontal vectors" or "vertical forms" as there is no natural embedding of TP/VP into TP. Thus, there is no natural unique way to lift vector fields on M to those on P.

A general principal bundle is only locally a product; a bundle is called *trivial* if the product structure extends globally. That is, there exists a diffeomorphism $f : P \to M \times G$ compatible with the action of G and the projection π. The diffeomorphism f is called a *trivialisation* (§ I.5. of [53]). The physics idea of globally choosing a gauge corresponds to choosing a section of the principal bundle defined as follows. A *section* (or *cross-section*) of a principal bundle is a smooth mapping $s : M \to P$ such that $\pi \circ s = \mathbb{1}_M$. A section s determines a trivialisation f and vice versa. A section of a principal bundle exists if and only if it is a trivial bundle (§ I.5. [53] and § III.B.3. [13]). Thus, we can pick a global choice of gauge only when the principal bundle is trivial. Since principal bundles are locally trivial (by definition) in some neighbourhood U_x of a point $x \in M$ we can always pick a *local section* $s_x : U_x \mapsto P$ and we can always locally pick a choice of gauge in physics.

Consider the space $\Omega^k_{hor} P(\mathbb{V}, R)$ consisting of horizontal equivariant differential forms on P which transform under the R-representation under the group action of G on P (see § II.5 of [53]). Locally, such differential forms represent differential forms on the base space M that transform homogenously under the R-representation of G (see Example 5.2 in § II.5 [53]). In particular differential forms on the base space that are *invariant* under internal gauge transformations are isomorphic to horizontal differential forms on the bundle i.e. $\pi^* \Omega^k M \cong \Omega^k_{hor} P$ (see Example 5.1 in § II.5 [53]). As described in Ch. 2, when we consider fields with arbitrary tensor structure on the base space, we convert the tensor indices into internal frame component indices and represent the frame components on the bundle as scalar functions.

To consider Yang-Mills theory or a first-order coframe formulation of gravity we need to consider the corresponding gauge fields as dynamical fields. On the principal bundle the gauge fields are described by a connection. A *connection* on a G-principal fibre bundle P is a smooth equivariant 1-form $\boldsymbol{A}^I \in \Omega^1 P(\mathfrak{g}, \mathrm{Ad})$ that implements the canonical isomorphism $VP \cong C^\infty(\mathfrak{g})$ through $X \cdot \boldsymbol{A}^I \in C^\infty(\mathfrak{g})$ for all $X^m \in VP$ (see Definition (c) in § Vbis.A. [13]). Here Ad is the adjoint representation of the structure group on its Lie algebra. The relation between local gauge fields and the connection is described in § Vbis.A.2 [13]. If we pick a local section s_x around some point $x \in M$ then the pullback of the connection through the section defines the local gauge fields $A^I_\mu = (s^*_x)^m_\mu A^I_m$ in the neighbourhood of x. Unless we can globally pick a choice of gauge (the principal bundle is trivial) the gauge fields cannot be defined globally as tensor fields on M. Thus in general, it is more useful to work directly with the connection as the dynamical field instead of the corresponding gauge fields on spacetime.

Note, that any two connections $\boldsymbol{A}^I$ and $\boldsymbol{A}'^I$, implement the canonical isomorphism $VP \cong C^\infty(\mathfrak{g})$ on vertical vector fields, and thus the difference of any two connections is a horizontal form valued in the Lie algebra $\mathfrak{g}$ (this can be seen as the analog of the fact that the action of two derivative operators on spacetime differs by a tensor field; see § 3.1 [45])

$$\boldsymbol{A}^I - \boldsymbol{A}'^I \in \Omega^1_{hor} P \tag{A.1}$$

The connection induces a split of the tangent bundle TP into vertical and horizontal subspaces. For any $X^m \in TP$ we have $X \cdot \boldsymbol{A}^I \in C^\infty P(\mathfrak{g}) \cong VP$. We can define the space $H_A P$ of *horizontal vector fields relative to the connection* by the kernel i.e. $X^m \in H_A P \iff X \cdot \boldsymbol{A}^I = 0$. Thus, the choice of a connection $\boldsymbol{A}^I$ can be seen as a G-equivariant choice of decomposition $TP = V_A P \oplus H_A P$ (see § II.1 of [53] and Definition (b) in [13]). Given a connection, we denote the vertical and horizontal projections of a vector field $X^m \in TP$ by $(ver_A X)^m \equiv X \cdot \boldsymbol{A}^I$ and $(hor_A X)^m$. Similarly, for a differential form $\boldsymbol{\sigma} \in \Omega^k P$ we can define the horizontal projection $hor_A \boldsymbol{\sigma} \in \Omega^k_{hor} P$ using:

$$X_{(1)} \cdot \ldots \cdot X_{(k)} \cdot (hor_A \boldsymbol{\sigma}) = (hor_A X_{(1)}) \cdot \ldots \cdot (hor_A X_{(k)}) \cdot \boldsymbol{\sigma} \quad \forall X^m_{(i)} \in TP$$

A connection $\boldsymbol{A}^I$ defines the *covariant exterior derivative* $D : \Omega^k P(\mathbb{V}; r) \to \Omega^{k+1}_{hor} P(\mathbb{V}; r)$ on equivariant differential forms valued in a vector space $\mathbb{V}$ on P, by projecting the usual exterior derivative to its horizontal part

$$D\boldsymbol{\sigma}^A := hor_A \; d\boldsymbol{\sigma}^A \tag{A.2}$$

If $\boldsymbol{\sigma}^A \in \Omega^k_{hor} P(\mathbb{V}; R)$ is a horizontal equivariant form (corresponding to gauge covariant fields on spacetime) then

$$D\boldsymbol{\sigma}^A = d\boldsymbol{\sigma}^A + (\boldsymbol{A}^I {r_I}^A{}_B) \wedge \boldsymbol{\sigma}^B \tag{A.3}$$

where r is the representation of the Lie algebra $\mathfrak{g}$ of the structure group $\mathbb{V}$. Note that on the space $\Omega^k P$ representing gauge-invariant differential forms, the covariant exterior derivative D coincides with the exterior derivative d.

Acting on the connection itself, the covariant derivative D, defines the *curvature* 2-form $\boldsymbol{F}^I \in \Omega^2_{hor} P(\mathfrak{g}, \mathrm{Ad})$ (see § II.5. [53] and § Vbis.A.4 [13]) as

$$\boldsymbol{F}^I := D\boldsymbol{A}^I = d\boldsymbol{A}^I + \tfrac{1}{2} c^I{}_{JK} \boldsymbol{A}^J \wedge \boldsymbol{A}^K \tag{A.4}$$

where $c^I{}_{JK}$ are the *structure constants* of the Lie algebra. Combining Eq. A.3 and Eq. A.4 we see that the curvature measures the failure of D^2 to vanish on equivariant horizontal forms i.e.

$$D^2\boldsymbol{\sigma}^A = (\boldsymbol{F}^I {r_I}^A{}_B) \wedge \boldsymbol{\sigma}^B \tag{A.5}$$

The curvature further satisfies the *Bianchi identity*

$$D\boldsymbol{F}^I = d\boldsymbol{F}^I + c^I{}_{JK} \boldsymbol{A}^J \wedge \boldsymbol{F}^K = 0 \tag{A.6}$$

which can be directly checked using Eq. A.4.

* * *

As discussed in Ch. 2, diffeomorphisms of the base space do not have a well-defined action on charged fields. Since, the charged fields are more naturally defined as equivariant differential forms on the bundle P, the automorphisms of a principal bundle do have a well-defined action, which will take the place of diffeomorphisms on spacetime in theories with charged fields. An *automorphism of the principal bundle* P is a diffeomorphism $f : P \to P$ that is equivariant with respect to the action of G on P and is fibre-preserving i.e. there exists a diffeomorphism $\underline{f} : M \to M$ so that $\pi \circ f = \underline{f} \circ \pi$ (see § I.5 [53]). Since the bundle automorphisms are diffeomorphisms of the bundle they act on all equivariant differential forms by the usual pullback action and map horizontal forms to horizontal forms and vertical vectors into vertical vectors.

Automorphisms of P form a group $\mathrm{Aut}(P)$ under composition which is generated by the corresponding Lie algebra of vector fields $\mathfrak{aut}(P) \subset TP$. A *vertical automorphism* is an $f \in \mathrm{Aut}(P)$ which projects to the identity on the base space M i.e. $\underline{f} = \mathbb{1}_M$; vertical automorphisms form a *normal* subgroup $\mathrm{Aut}_V(P) \cong C^\infty(M, G)$ and $\mathfrak{aut}_V(P) \subset VP$ are a *ideal* subalgebra of $\mathfrak{aut}(P)$. The vertical automorphisms $\mathrm{Aut}_V(P)$ acting on differential forms are precisely the internal gauge transformations. The projection π gives us the relations $\pi : \mathrm{Aut}(P) \to \mathrm{Aut}(P)/\mathrm{Aut}_V(P) \cong \mathrm{Diff}(M)$. The group of automorphisms of the bundle is a semi-direct product of diffeomorphisms of the base and the internal gauge transformations $\mathrm{Aut}(P) = \mathrm{Diff}(M) \ltimes \mathrm{Aut}_V(P)$. Thus we see that there is only a notion of "diffeomorphism up to internal gauge transformations" on charged fields.

For a trivial bundle P and a choice of trivialisation (or section) every automorphism of P factorises into a diffeomorphism of the base space M and an internal gauge transformation i.e. $\mathrm{Aut}(P) \cong \mathrm{Diff}(M) \times \mathrm{Aut}_V(P)$. Thus, only in the trivial bundle case and once we have picked a choice of gauge (i.e. trivialisation) there is a notion of separate diffeomorphisms and internal gauge transformations. We emphasise that this identification depends on the gauge choice. Thus for charged dynamical fields we must consider the entire group of bundle automorphisms as the gauge group instead of just the diffeomorphism group of the base

The vector fields in $\mathfrak{aut}(P)$ act on equivariant differential forms by the usual Lie derivative. Using Eqs. A.3 and A.4 we can write the Lie derivative of equivariant forms in terms of the covariant exterior derivative as

$$\begin{aligned} \pounds_X \boldsymbol{\sigma}^A &= X \cdot D\boldsymbol{\sigma}^A + D(X \cdot \boldsymbol{\sigma}^A) - (X \cdot \boldsymbol{A}^I)\, r_I{}^A{}_B \boldsymbol{\sigma}^B \\ \pounds_X \boldsymbol{A}^I &= X \cdot \boldsymbol{F}^I + D(X \cdot \boldsymbol{A}^I) \end{aligned} \tag{A.7}$$

for $\boldsymbol{\sigma}^A \in \Omega^k_{hor} P(\mathbb{V}; R)$ and the connection $\boldsymbol{A}^I$. Note both $\pounds_X \boldsymbol{\sigma}^A$ and $\pounds_X \boldsymbol{A}^I$ are horizontal forms (and hence gauge covariant). When $X^m \in \mathfrak{aut}_V(P)$ this corresponds to the infinitesimal version of the internal gauge transformations Eqs. 2.1 and 2.5.

Remark A.1. The subspace VP of vertical vector fields is a Lie subalgebra of TP, i.e. $X, Y \in VP \implies [X, Y] \in VP$. Since $H_A P \cong TM$, one can attempt to identify horizontal vectors (with respect to a given connection) as representing infinitesimal action of diffeomorphisms. The gauge-covariant Lie derivative Eq. 1.1 defined in [24] is precisely this identification. But, for a given choice of connection, the subspace $H_A P$ is not a Lie algebra. In fact the curvature measures the failure of $H_A P$ to be integrable i.e. for $X^m, Y^m \in H_A P$ we have

$$ver_A[X, Y] \equiv [X, Y] \cdot \boldsymbol{A}^I = X \cdot Y \cdot \boldsymbol{F}^I \tag{A.8}$$

or in terms of Lie derivative of $\boldsymbol{\varphi}^A \in \Omega^k P(\mathbb{V})$

$$\Big([\pounds_{hor_A X}, \pounds_{hor_A Y}] - \pounds_{hor_A [X,Y]} \Big) \boldsymbol{\sigma}^A = (X \cdot Y \cdot \boldsymbol{F}^I)\, r_I{}^A{}_B \boldsymbol{\sigma}^B \tag{A.9}$$

Thus even when we have a trivial bundle and a connection, we cannot identify TM as a Lie algebra with $H_A P$ except in the case where the curvature vanishes.

Given some equivariant differential form $\boldsymbol{\sigma}^A$ there might exist some bundle automorphisms which keep $\boldsymbol{\sigma}^A$ invariant.

Definition A.1 (Automorphism which preserves $\boldsymbol{\sigma}^A$)**.** An automorphism of the bundle $f \in \text{Aut}(P)$ is an automorphism which preserves some given equivariant differential form $\boldsymbol{\sigma}^A \in \Omega^k P(\mathbb{V}; R)$ if $f^*\boldsymbol{\sigma}^A = \boldsymbol{\sigma}^A$. Similarly $X^m \in \mathfrak{aut}(P)$ is an *infinitesimal automorphism* which preserves $\boldsymbol{\sigma}^A$ if $\pounds_X \boldsymbol{\sigma}^A = 0$. For a given $\boldsymbol{\sigma}^A$, denote the subgroup of the automorphisms preserving $\boldsymbol{\sigma}^A$ by $\text{Aut}(P; \boldsymbol{\sigma}^A) \subseteq \text{Aut}(P)$, and the corresponding Lie subalgebra of infinitesimal automorphisms $\mathfrak{aut}(P; \boldsymbol{\sigma}^A) \subseteq \mathfrak{aut}(P)$.

Since for the first law we are interested in stationary and axisymmetric field configurations, we define a notion of stationarity (axisymmetry) for a charged field $\boldsymbol{\sigma}^A$ defined on a bundle P over a base space M with a stationary (axisymmetric) metric as bundle automorphisms which preserve $\boldsymbol{\sigma}^A$ and project to the stationary (axisymmetric) isometries of some given metric $g_{\mu\nu}$ on M.

Definition A.2 (Stationary and/or axisymmetric field)**.** For a stationary and/or axisymmetric spacetime M with a metric $g_{\mu\nu}$, a charged field $\boldsymbol{\sigma}^A$ defined on a bundle $P \to M$ is called *stationary* if there exists a one-parameter family $f_t \in \text{Aut}(P; \boldsymbol{\sigma}^A)$ which projects to a one-parameter family $\underline{f}_t$ of stationary isometries of $g_{\mu\nu}$. Similarly, $\boldsymbol{\sigma}^A$ is *axisymmetric* if there exists a one-parameter family $f_\phi \in \text{Aut}(P; \boldsymbol{\sigma}^A)$ which projects to a one-parameter family $\underline{f}_\phi$ of axisymmetric isometries of $g_{\mu\nu}$.

The corresponding stationary (and axial) infinitesimal automorphism vector field t^m (and ϕ^m) projects to the stationary (and axial) Killing vector t^μ (and ϕ^μ) of $g_{\mu\nu}$, respectively.

As discussed above a diffeomorphism of the base space does not uniquely determine an automorphism of the bundle. But if we require that the automorphism preserve the connection (for example, if the Yang-Mills connection is stationary) then we can classify this ambiguity as follows.

Lemma A.1 (Uniqueness of an infinitesimal automorphism that preserves a connection)**.** *For a given connection $\boldsymbol{A}^I$ on P, any $X^m \in \mathfrak{aut}(P; \boldsymbol{A}^I)$ is uniquely determined by its* *projection* $(\pi_*)^\mu_m X^m \in TM$ *up to a vertical vector field* $Y^m \in \mathfrak{aut}_V(P)$ *such that*

$$D(Y \cdot \boldsymbol{A}^I) = 0 \tag{A.10}$$

if any such non-trivial Y^m *exists on* P.

Proof. Any $X^m \in \mathfrak{aut}(P; \boldsymbol{A}^I)$ satisfies (using Eq. A.7)

$$0 = \pounds_X \boldsymbol{A}^I = X \cdot \boldsymbol{F}^I + D(X \cdot \boldsymbol{A}^I) \tag{A.11}$$

If $X^m \in VP$ i.e. $(\pi_*)^\mu_m X^m = 0$ this means $X \cdot \boldsymbol{A}^I$ is a covariantly constant $\mathfrak{g}$-valued function (since $\boldsymbol{F}^I$ is a horizontal form). So if X^m and X'^m are such that $X^\mu = (\pi_*)^\mu_m X^m = (\pi_*)^\mu_m X'^m$, then $Y^m = X^m - X'^m \in VP$ and $\lambda^I = Y \cdot \boldsymbol{A}^I$ satisfies Eq. A.10. Since the connection is an isomorphism between VP and $\Omega^0 P(\mathfrak{g})$, any such choice of λ^I uniquely fixes the ambiguity Y^m. □

Remark A.2. The ambiguity Y^m in the above lemma corresponds to a *global symmetry* of the chosen connection in the following sense. Taking Lie derivative of the connection with respect to Y^m we have

$$\pounds_Y \boldsymbol{A}^I = D(Y \cdot \boldsymbol{A}^I) = 0$$

i.e. the vertical automorphism (i.e. gauge transformation) $f_Y \in \mathrm{Aut}_V(P)$ generated by Y^m keeps the connection invariant and hence $\boldsymbol{A}^I$ and $f_Y^* \boldsymbol{A}^I$ correspond to the same physical field configuration at every point. Connections with such a non-trivial automorphism f_Y are called *reducible* while connections for which no such non-trivial automorphism exists are called *irreducible* (see § 4.2.2 [75]). For a compact structure group, the space of irreducible connections is known to be an open and dense subspace of the space of all connections [76, 77, 78].

When the structure group is abelian of dimension n, the non-uniqueness above reduces to the addition of a $\mathbb{R}^n$-valued constant function.

Remark A.3 (Infinitesimal automorphism that preserves some charged fields). Consider the case where we have both a connection $\boldsymbol{A}^I$ and some charged field σ^A (we consider a scalar field for simplicity) which transforms under a representation r of the Lie algebra $\mathfrak{g}$ on the bundle P. If X^m is an infinitesimal automorphism that preserves both $\boldsymbol{A}^I$ and σ^A then, in addition to Eq. A.11, we have

$$0 = \pounds_X \sigma^A = X \cdot D\sigma^A - (X \cdot \boldsymbol{A}^I)\, r_I{}^A{}_B \sigma^B \tag{A.12}$$

This imposes a further restriction on the ambiguity $Y^m \in \mathfrak{aut}_V(P)$ in the choice of X^m given by Lemma A.1 i.e. Y^m has to satisfy the additional condition

$$0 = (Y \cdot \boldsymbol{A}^I) r_I{}^A{}_B \sigma^B \tag{A.13}$$

i.e. the gauge transformation generated by Y^m keeps both the connection $\boldsymbol{A}^I$ and the field σ^A invariant at every point. The question of whether any non-trivial Y^m exists depends on the chosen connection $\boldsymbol{A}^I$ and field σ^A. As we will see in Lemma A.4, for the Lorentz connection $\boldsymbol{\omega}^a{}_b$ and the coframes $\boldsymbol{e}^a$, there is no non-trivial ambiguity.

* * *

In constructing physical theories on the bundle we will require that the Lagrangian be a locally and covariantly constructed functional of the dynamical fields on the bundle which we define as follows.

Definition A.3 (Local and covariant functional)**.** A functional $\mathcal{F}[\Phi]$ on a G-principal bundle P depending on a set of fields Φ and finitely many of its derivatives (with respect to an arbitrary derivative operator which is taken to be part of Φ) is a local and covariant functional if for any $f \in \mathrm{Aut}(P)$ we have

$$(f^*\mathcal{F})[\Phi] = \mathcal{F}[f^*\Phi] \tag{A.14}$$

where it is implicit that on the right-hand-side f also acts on the derivatives of Φ. If X^m is the vector field generating the automorphism f then the above equation implies that

$$\pounds_X \mathcal{F}[\Phi] = \mathcal{F}[\pounds_X \Phi] \tag{A.15}$$

Each of $\mathcal{F}$ and Φ can have arbitrary tensorial structure on P and be valued in some representation of the structure group.

For many of the crucial results in the main body we will need to ensure that a closed differential form on P is in fact, (globally) exact. For instance, such a result is used in the classification of the ambiguities in the symplectic potential (Eq. 2.35), and is needed to ensure that a horizontal (i.e. gauge-invariant) Noether charge exists for any Noether current (Lemma 2.3). Under certain assumptions on a differential form $\boldsymbol{\sigma}$ and its dependence on certain fields Φ, which we detail next, we show that if $\boldsymbol{\sigma}$ is closed then it is exact. We shall show this in direct analogy to Lemma 1 [79] (this result can also be derived using jet bundle methods and the *variational bicomplex*; see Theorem 3.1 [80]) and the assumptions on $\boldsymbol{\sigma}$ given below are geared towards generalising the algorithm of Lemma 1 [79] to work with differential forms on the bundle P.

Assumptions A.1 (Assumptions for Lemma A.2)**.** Let $\Phi = \{\phi, \psi\}$ be a collection of two types of fields — ϕ are the "dynamical fields" and ψ are the "background fields" (distinguished by the assumptions listed below), and let $\boldsymbol{\sigma}[\Phi] \in \Omega^p P$ with $p < d$ (where d is the dimension of the base space M) be a p-form on a principal bundle P so that

(1) $\boldsymbol{\sigma}$ is a horizontal form on P which is invariant under the action of the structure group G on P i.e. $\boldsymbol{\sigma}[\Phi] \in \Omega^p_{hor} P$.

(2) $\boldsymbol{\sigma}[\Phi]$ is a local and covariant functional of the fields $\Phi = \{\phi, \psi\}$ as in Def. A.3.

(3) The "dynamical fields" ϕ are sections of a vector bundle over P which is equivariant under the group action G on P, and the action of any automorphism $f \in \mathrm{Aut}(P)$ on

ϕ is linear (i.e. preserves the vector bundle structure).

(4) $\boldsymbol{\sigma}[\Phi]$ depends *linearly* on up to k-derivatives of the "dynamical fields" ϕ.

With the above assumptions on $\boldsymbol{\sigma}$ we can prove the following (generalising Lemma 1 [79])

Lemma A.2 (Generalised Lemma 1 [79])**.** *Let $\boldsymbol{\sigma}[\Phi] \in \Omega^p_{hor}P$ and $\Phi = \{\phi, \psi\}$ be as assumed in Assumptions A.1. If $d\boldsymbol{\sigma}[\Phi] = 0$ for all "dynamical fields" ϕ and any given "background fields" ψ, then there exists (globally) a differential form $\boldsymbol{\eta}[\Phi] \in \Omega^{p-1}_{hor}P$ which, similarly satisfies Assumptions A.1 but depends linearly on at most $(k-1)$-derivatives of ϕ such that $\boldsymbol{\sigma} = d\boldsymbol{\eta}$.*

Further, if $k = 0$ i.e. $\boldsymbol{\sigma}$ does not depend on derivatives of the "dynamical fields" ϕ, then $\boldsymbol{\sigma} = 0$.

Proof. To begin, we note that connections on a principal bundle P are not sections of a vector bundle, since the difference of two connections is a horizontal form (Eq. A.1) and thus, not a connection (see also Remark 1, Ch. IV [74]). Thus, any choice of connection on P would be part of the "background fields" ψ.[2] We assume that a choice of such a connection has been made and denote the corresponding (horizontal) covariant derivative operator on P by D_m and the covariant exterior derivative by D.

By Assump. (3), the derivative operator D_m can be used to take horizontal derivatives of ϕ; the derivatives of ϕ along the vertical directions are fixed by the equivariance requirement. Furthermore, any antisymmetric derivatives of ϕ can be written in terms of lower order derivatives and the curvature (and possibly torsion on a Lorentz bundle; see § A.1) of the chosen connection; thus, we only need to consider totally symmetrised derivatives of ϕ. Thus, using Assump. (2) and Assump. (4) we can write the p-form $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} \equiv \sigma_{m_1 \dots m_p} = \sum_{i=0}^{k} S^{(i)}{}_{m_1 \dots m_p}{}^{n_1 \dots n_i}{}_A D_{(n_1} \cdots D_{n_i)} \phi^A \tag{A.16}$$

2. Note, that even though [79] assumes in the beginning that the "background fields" ψ also are sections of a vector bundle, it is not required for the proof of Lemma 1 [79] as discussed towards the end of § II [79].

where, each of the tensors $S^{(i)}$ are local and covariant functionals of the "background fields" ψ and we have used an abstract index notation on the "dynamical fields" $\phi \equiv \phi^A$. Eq. A.16 is the direct analogue of Eq. 2 [79] on the principal bundle P. Note here that, since all of the m-indices are horizontal and the n, A-indices are contracted away, $\boldsymbol{\sigma}$ tranforms as a horizontal form under the action of any automorphism $f \in \mathrm{Aut}(P)$ as required in Assump. (1).

Again using Assump. (1) we have

$$d\boldsymbol{\sigma} = D\boldsymbol{\sigma} \equiv (p+1)\sum_{i=0}^{k} D_{[l}\left\{ S^{(i)}{}_{m_1 \ldots m_p]}{}^{n_1 \ldots n_i}{}_A D_{(n_1} \cdots D_{n_i)}\phi^A \right\} \tag{A.17}$$

Since, $d\boldsymbol{\sigma} = 0$ for all "dynamical fields" ϕ and the horizontal symmetrised derivatives of ϕ can be specified independently at any point of P, we can directly apply the arguments of Lemma 1 [79] to Eq. A.17. Note, that in each step of the algorithm of Lemma 1 [79], the m-indices are always horizontal and the n, A-indices are contracted away. Thus the algorithm of [79] gives us a horizontal $(p-1)$-form $\boldsymbol{\eta}$ on P so that $\boldsymbol{\sigma} = d\boldsymbol{\eta}$ where $\boldsymbol{\eta}$ has an expansion similar to Eq. A.16 except that it depends linearly on at most, $(k-1)$-derivatives of ϕ. Thus, the form $\boldsymbol{\eta}$ manifestly satisfies Assumptions A.1 and the claim of this lemma.

The algorithm of [79] further shows that when $k = 0$ we have $\boldsymbol{\sigma} = 0$. □

We point out that all of Assumptions A.1 are crucial to prove Lemma A.2. Assump. (1) was used in the first equality in Eq. A.17 to convert the exterior derivative d to the covariant exterior derivative D (which does not hold if $\boldsymbol{\sigma}$ is, either not invariant under the G-action, or not horizontal) to get an expansion in terms of derivatives of ϕ which is the first step in using the algorithm of [79]. As already noted, Assumps. (2) and (4) are needed to write down the expansion Eq. A.16. Finally, Assump. (3) is already used to formulate Assump. (4), since a vector bundle structure is necessary for the notion of $\boldsymbol{\sigma}$ depending linearly on ϕ and its derivatives as in Eq. A.16. Similarly, the assumptions of equivariance of ϕ and linear action of $f \in \mathrm{Aut}(P)$ on ϕ in Assump. (3), are necessary to ensure that the expansion Eq. A.16 — and the corresponding expansion for $\boldsymbol{\eta}$ obtained by applying the algorithm of [79] — give us horizontal forms on P. Assump. (3) also ensures that we can use any connection, as part of the background fields ψ, to define the horizontal covariant derivatives of ϕ. This is crucial since only the horizontal derivatives of ϕ can be freely specified at any point of P and one needs a connection to define horizontal derivatives.

In our applications of Lemma A.2 in the main body of the paper, the "dynamical fields" ϕ will either be (1) the perturbations $\delta\boldsymbol{\psi}^\alpha$ of the dynamical fields $\boldsymbol{\psi}^\alpha$ (Eq. 2.15) of the theory which are equivariant horizontal forms valued in some vector space carrying a representation of G,[3] or (2) infinitesimal automorphisms $X^m \in \mathfrak{aut}(P)$ of the principal bundle P. In both cases the "dynamical fields" ϕ can be considered as sections of vector bundles over P in accordance with Assump. (3).

A.1 Frame and spin bundles

To derive a first law for a coframe formulation of gravity we write the orthonormal coframes and the Lorentz gauge field on a principal bundle over spacetime M. We review the relevant constructions in this section.

First, we describe the linear frame bundle FM over the base space M; the details can be found in § I.5 [53], § III.B.2 and Vbis.A.5. [13]. At a point x on a manifold M, the *frames* E_a^μ are an ordered choice of d linearly-independent basis vectors (labelled by the index a) for the tangent space T_xM. We can equivalently consider the frames as a linear isomorphism $E_a^\mu : \mathbb{R}^d \to T_xM$ so that $X^a \mapsto X^a E_a^\mu \in T_xM$ where we now consider the index a as an abstract index in $\mathbb{R}^d$. Similarly, *vielbeins* or *coframes* are an ordered choice of linearly-independent basis of the cotangent bundle T_x^*M which can be written as e_μ^a and viewed as linear isomorphisms $e_\mu^a : \mathbb{R}^{d^*} \to T_x^*M$. The duality between the frames and coframes is expressed by the relations

$$e_\mu^a E_b^\mu = \delta_b^a \quad ; \quad e_\mu^a E_a^\nu = \delta_\mu^\nu \tag{A.18}$$

3. Even though the connection $\boldsymbol{A}^I$ in the dynamical field $\boldsymbol{\psi}^\alpha$ is not a section of a vector bundle, the perturbation $\delta\boldsymbol{A}^I$ is (see Eq. A.1).

When the manifold M is not *parallelisable* there is no global choice of frames which are linear-independent everywhere. Thus, we cannot treat the frames and the coframes as globally well-defined tensor fields on M. To avoid this issue we can on the *linear frame bundle* $\pi : FM \to M$. The fibre of FM over any point $x \in M$ is the set of all possible choices of linearly-independent frames E_a^μ at x. Since any two choices of frames are related by a general linear transformation, FM is a principal bundle over M with the structure group $GL(d, \mathbb{R})$ (see Example 5.2 Ch. I [53]). Thus any point $u \in FM$ consists of a point $x \in M$ together with a choice of frame i.e. locally $u = (x, E_a^\mu)$ such that the projection acts as $\pi(u) = x$. When writing Lagrangians that depend explicitly on the frames E_a^μ it would be inconvenient to have an explicit dependence on the point u in FM. To avoid this, we define the frames as vector fields (instead of points) on the frame bundle as follows. Locally, at any point $u = (x, E_a^\mu) \in FM$ and for any $X^a \in \mathbb{R}^d$ define a $\mathbb{R}^{d^*}$-valued vector field $E_a^m \in T_u FM(\mathbb{R}^{d^*})$ by

$$(\pi_*)^\mu_m (X^a E_a^m) = X^a E_a^\mu \tag{A.19}$$

This construction can be extended globally to define frames E_a^m as smooth vector fields on FM. Note that only the equivalence class of $E_a^m \in TFM/VFM$ contributes to frames E_a^μ on the base that is, we consider two frames E_a^m and $E_a'^m$ as equivalent (defined as vector fields on FM) iff $E_a'^m - E_a^m \in VP(\mathbb{R}^{d^*})$. Thus the frames E_a^m are non-degenerate on horizontal forms in the sense

$$E_a \cdot \boldsymbol{\sigma} = 0 \iff \boldsymbol{\sigma} = 0 \quad \forall\, \boldsymbol{\sigma} \in \Omega^k_{hor} P \tag{A.20}$$

We can define the coframes on FM as the *canonical form* or *soldering form* as follows (also see § 3.2 [53]). Locally, at any point $u = (x, E_a^\mu) \in FM$ define the 1-form $\boldsymbol{e}^a$ by

$$(X \cdot \boldsymbol{e}^a) E_a^\mu := (\pi_*)^\mu_m X^m \quad \forall\, X^m \in T_u FM \tag{A.21}$$

It can be shown that the this can be extended to define the coframes as a smooth equivariant horizontal 1-form valued in $\mathbb{R}^d$ i.e. $\boldsymbol{e}^a \in \Omega^1_{hor} FM(\mathbb{R}^d)$ (see Proposition 2.1 § 3.2 [53]) and that it is non-degenerate in the sense

$$X \cdot \boldsymbol{e}^a = 0 \iff X^m \in VP \tag{A.22}$$

and the coframes are dual to the frames so that $E_a \cdot \boldsymbol{e}^b = \delta^b_a$.

The frames E^m_a and coframes $\boldsymbol{e}^a$ defined on FM are related to the ones on M through the projection π as follows. Given a point $u \in FM$ so that $\pi(u) = x$

$$e^a_m(u) = (\pi^*)^\mu_m e^a_\mu(x) \quad ; \quad E^\mu_a(x) = (\pi_*)^\mu_m E^m_a(u) \tag{A.23}$$

To extend this correspondence globally we would have to smoothly pick a point $u \in FM$ for every point $x \in M$ i.e. a smooth section $s : M \to FM$. If M is not parallelisable, FM is not trivial and there does not exist any such global section. Thus, even though the frames and coframes are globally well-defined and linearly-independent on the frame bundle there may not exist any corresponding global choice of frames and coframes on M.

We can reduce the frame bundle by demanding that certain $\mathbb{R}^d$-valued tensors be preserved by the reduced structure group as follows.

(1) Choosing a preferred Lorentzian metric[4] $\eta_{ab} = \text{diag}(-1, 1, \ldots, 1)$ on $\mathbb{R}^d$ the bundle FM can be reduced to an *orthonormal frame bundle* $F_O M$ with structure group $O(d-1,1)$ i.e the *Lorentz group*.

(2) Further choosing an orientation $\epsilon_{a_1 \ldots a_d}$ the bundle reduces to an *oriented frame bundle* $F_{SO}M$ with structure group $SO(d-1,1)$.

(3) Choosing a time-orientation t^a reduces $F_{SO}M$ to the *oriented and time-oriented i.e. proper frame bundle* $F^0_{SO}M$ with structure group $SO^0(d-1,1)$ which is the identity component of the Lorentz group.

4. The frame bundle reduction proceeds similarly for other signatures of the metric.

Note that the Lie algebras of $O(d-1,1)$, $SO(d-1,1)$ and $SO^0(d-1,1)$ are all the Lorentz Lie algebra $\mathfrak{so}(d-1,1)$.

Any choice of a reduction of the frame bundle FM to an orthonormal frame bundle $F_O M$ gives rise to a metric on M and conversely a choice of metric $g_{\mu\nu}$on M gives a reduction of the frame bundle to some subbundle of orthonormal frames (see Example 5.7 Ch. I [53]). Since we have picked η_{ab} to be Lorentzian, the metric $g_{\mu\nu}$ on M will also be of Lorentzian signature. The orthonormal frame bundle $F_O^{(g)}M$ determined by $g_{\mu\nu}$ then consists precisely of those frames that are orthonormal in the sense

$$g_{\mu\nu} E_a^\mu E_b^\nu = \eta_{ab} \tag{A.24}$$

Further, if $X^\mu = (\pi_*)^\mu_m X^m$ is the projection of X^m, then the lift to the bundle of the 1-form $\underline{\boldsymbol{\xi}} \equiv \xi_\mu = g_{\mu\nu} X^\nu$ is given by $\boldsymbol{\xi} = (X \cdot \boldsymbol{e}^a)\boldsymbol{e}_a = (\pi^*)^\mu_m \xi_\mu$.

Thus, to formulate a theory of gravity in terms of orthonormal coframes, it seems one should work on some choice of orthonormal frame bundle $F_O^{(g)}M$, but such a choice will necessarily give us a fixed metric $g_{\mu\nu}$ on spacetime. Since the orthonormal coframes e^a_μ are dynamical fields of the theory, we do not have an a priori fixed metric on spacetime. Moreover, consider an automorphism f of the frame bundle FM. In general, the corresponding projection $\underline{f} \in \mathrm{Diff}(M)$ need not preserve a given metric $g_{\mu\nu}$ on M. Thus, an arbitrary automorphism $f \in \mathrm{Aut}(FM)$ will map the subbundle $F_O^{(g)}M$ determined by $g_{\mu\nu}$ to $F_O^{(g')}M$ determined by $g'_{\mu\nu} = (\underline{f}^*)^{-1} g_{\mu\nu}$. Thus, it will be problematic to pick a particular orthonormal frame bundle if one wants to consider the coframes as dynamical fields and the action of diffeomorphisms on the dynamical fields of the theory.

We circumvent this issue as follows. We will consider an abstract principal *Lorentz bundle* P_O with structure group $O(d-1,1)$ which has globally well-defined coframes $\boldsymbol{e}^a = e^a_m$ and frames E_a^m, which are non-degenerate in the sense of Eqs. A.20 and A.22, similar to the ones defined on the frame bundle above. Once, we have some specific choice of coframes

$\boldsymbol{e}^a$ obtained by solving the equations of motion of the theory, we can identify the abstract bundle P_O with the orthonormal frame bundle $F_O^{(g)}M$ determined by the solution metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. We can similarly construct the *oriented Lorentz bundle* P_{SO} and the *proper Lorentz bundle* P^0_{SO} as abstract principal bundles with structure groups $SO(d-1,1)$ and $SO^0(d-1,1)$. Henceforth, we will always work on the above defined Lorentz bundle instead of the frame bundle. We will use the Lorentz bundle P_O to formulate general theories of gravity in terms of the coframes. For the Lagrangian of General Relativity (§ 5.1) we will need to introduce an orientation and hence we work on the oriented Lorentz bundle P_{SO}. Similarly to define spinor fields we will need the proper Lorentz bundle P^0_{SO} (§ 5.3).

Since the frames and coframes are globally linearly-independent on P_O, we can use them to express the frame components of tensor fields on M as globally well-defined equivariant fields on P_O.[5] This procedure was detailed in Ch. 2 but we recall some special cases. First, consider a horizontal k-form $\boldsymbol{\sigma} \equiv \sigma_{m_1 \dots m_k}$; we can write the frame components of $\boldsymbol{\sigma}$ as

$$\begin{aligned} \sigma_{a_1 \dots a_k} &:= (E_{a_k} \cdot \ldots \cdot E_{a_1} \cdot \boldsymbol{\sigma}) \quad \text{so that} \\ \boldsymbol{\sigma} &= \tfrac{1}{k!} \boldsymbol{e}^{a_1} \wedge \ldots \wedge \boldsymbol{e}^{a_k} \sigma_{a_1 \dots a_k} \end{aligned} \tag{A.25}$$

Similarly, let $\mathcal{L}$ be any linear map acting on horizontal forms as

$$\mathcal{L}(\boldsymbol{\sigma}) = \mathcal{L}^{m_1 \dots m_k} \sigma_{m_1 \dots m_k} \tag{A.26}$$

Using Eq. A.25 we have the frame components of L as

$$\begin{aligned} \mathcal{L}^{a_1 \dots a_k} &:= \tfrac{1}{k!} \mathcal{L}(\boldsymbol{e}^{a_1} \wedge \ldots \wedge \boldsymbol{e}^{a_k}) \quad \text{so that} \\ \mathcal{L}(\boldsymbol{\sigma}) &= \mathcal{L}^{a_1 \dots a_k} \sigma_{a_1 \dots a_k} \end{aligned} \tag{A.27}$$

Next we show that on the Lorentz bundle P_O linear maps acting on horizontal forms

5. This construction can also be carried out on the frame bundle FM.

Eq. A.26 can be expressed uniquely as horizontal forms. We will introduce an orientation in the intermediate steps of the proof but the final result is independent of this orientation and holds even for the Lorentz bundle P_O. We make use of this isomorphism in the main body of the paper to simplify a lot of the computations (for instance, compare Eq. 2.24 to Eq. 2.26).

Lemma A.3. *Let $\tilde{\mathcal{L}} : \Omega^k_{hor} P_O \to \Omega^d_{hor} P_O$ be a linear map on horizontal forms i.e. for any horizontal k-form $\boldsymbol{\sigma} \equiv \sigma_{m_1 \dots m_k}$ the map $\tilde{\mathcal{L}}(\boldsymbol{\alpha})$ acts as*

$$\tilde{\boldsymbol{\mathcal{L}}}^{n_1 \dots n_k} \sigma_{n_1 \dots n_k} \equiv \tilde{\mathcal{L}}_{m_1 \dots m_d}{}^{n_1 \dots n_k} \sigma_{n_1 \dots n_k} \tag{A.28}$$

Then there exists $\boldsymbol{\mathcal{L}} \in \Omega^{d-k}_{hor} P_O$ such that

$$\tilde{\boldsymbol{\mathcal{L}}}^{n_1 \dots n_k} \sigma_{n_1 \dots n_k} = \boldsymbol{\mathcal{L}} \wedge \boldsymbol{\sigma} \quad \forall\ \boldsymbol{\sigma} \in \Omega^k_{hor} P_O \tag{A.29}$$

Further, $\boldsymbol{\mathcal{L}}$ is unique given by the formula

$$\boldsymbol{\mathcal{L}} \equiv \mathcal{L}_{m_1 \dots m_{d-k}} = \frac{(d-k)!k!}{d!} \tilde{\mathcal{L}}_{m_1 \dots m_{d-k} n_1 \dots n_k}{}^{n_1 \dots n_k} \tag{A.30}$$

Proof. Since the linear map acts only on horizontal forms, using Eqs. A.25 and A.27 we can write the action of the linear map Eq. A.28 in terms of components with respect to the coframes and frames on P_O as

$$\tilde{\mathcal{L}}_{a_1 \dots a_d}{}^{b_1 \dots b_k} \sigma_{b_1 \dots b_k}$$

This can be rewritten in the following form by introducing an arbitrary (local) orientation

$\epsilon_{a_1\dots a_d}$ and using the identity Eq. A.31

$$
\begin{aligned}
&\tilde{\mathcal{L}}_{a_1\dots a_d}{}^{b_1\dots b_k}\sigma_{b_1\dots b_k} \\
&= \frac{1}{d!(d-k)!k!}\left(\epsilon_{a_1\dots a_d}\epsilon^{c_1\dots c_d}\right)\tilde{\mathcal{L}}_{c_1\dots c_d}{}^{b_1\dots b_k}\left(\epsilon^{d_1\dots d_d}\epsilon_{d_1\dots d_{d-k}b_1\dots b_k}\right)\sigma_{d_{d-k+1}\dots d_d} \\
&= \frac{1}{d!(d-k)!k!}\left(\epsilon_{d_1\dots d_{d-k}b_1\dots b_k}\epsilon^{c_1\dots c_d}\right)\tilde{\mathcal{L}}_{c_1\dots c_d}{}^{b_1\dots b_k}\left(\epsilon_{a_1\dots a_d}\epsilon^{d_1\dots d_d}\right)\sigma_{d_{d-k+1}\dots d_d} \\
&= \tilde{\mathcal{L}}_{[a_1\dots a_{d-k}|b_1\dots b_k}{}^{b_1\dots b_k}\sigma_{|a_{d-k+1}\dots a_d]}
\end{aligned}
$$

The last line is precisely the frame component of $\boldsymbol{\mathcal{L}} \wedge \boldsymbol{\sigma}$ with $\boldsymbol{\mathcal{L}}$ given by Eq. A.30, and implies that, $\boldsymbol{\mathcal{L}}$ is unique, and in fact independent of the chosen orientation. □

On the oriented Lorentz bundle P_{SO} we have an orientation $\epsilon_{a_1\dots a_d}$ as the completely anti-symmetric symbol with $\epsilon_{01\dots d-1} := 1$ satisfying the standard identity

$$\epsilon^{a_1\dots a_k a_{k+1}\dots a_d}\epsilon_{a_1\dots a_k b_{k+1}\dots b_d} = s\ (d-k)!k!\ \delta^{[a_{k+1}}_{b_{k+1}}\dots\delta^{a_d]}_{b_d} \tag{A.31}$$

where s is the signature of the metric η_{ab}. Using this orientation we can define the *horizontal volume form* on P_{SO} as

$$\boldsymbol{\varepsilon}_d := \tfrac{1}{d!}\epsilon_{a_1\dots a_d}\boldsymbol{e}^{a_1}\wedge\dots\wedge\boldsymbol{e}^{a_d} \in \Omega^d_{hor}P_{SO} \tag{A.32}$$

which is the lift through π of the volume form $\underline{\boldsymbol{\varepsilon}}_d \equiv \varepsilon_{\mu_1\dots\mu_d}$ on M. For a horizontal form $\boldsymbol{\sigma}^A \in \Omega^k_{hor}P_{SO}(\mathbb{V},R)$ define the *horizontal Hodge dual* $\star : \Omega^k_{hor}P_{SO}(\mathbb{V},R) \to \Omega^{d-k}_{hor}P_{SO}(\mathbb{V},R)$ as:

$$\star\boldsymbol{\sigma}^A := \frac{1}{(d-k)!k!}\epsilon^{a_1\dots a_k}{}_{b_1\dots b_{d-k}}\ \left(E_{a_k}\cdot\ldots\cdot E_{a_1}\cdot\boldsymbol{\sigma}^A\right)\ \boldsymbol{e}^{b_1}\wedge\dots\wedge\boldsymbol{e}^{b_{d-k}} \tag{A.33}$$

or in terms of the frame components as

$$(\star\sigma)^A_{b_1\dots b_{d-k}} = \frac{1}{k!}\epsilon^{a_1\dots a_k}{}_{b_1\dots b_{d-k}}\ \sigma^A_{a_1\dots a_k} \tag{A.34}$$

Note, that the horizontal Hodge dual maps equivariant horizontal forms to equivariant horizontal forms, and is a duality in the sense that $\star\star\boldsymbol{\sigma}^A = (-)^{k(d-k)}s\, \boldsymbol{\sigma}^A$. It is straightforward to verify that for an invariant form $\boldsymbol{\sigma}$, if $\boldsymbol{\sigma} = \pi^*\underline{\boldsymbol{\sigma}}$ then $\star\boldsymbol{\sigma} = \pi^*(*\underline{\boldsymbol{\sigma}})$ where $*$ is the Hodge dual acting on differential forms on M.

The local Lorentz gauge field $\omega_\mu{}^a{}_b$ can be represented by a $O(d-1,1)$-connection on the Lorentz bundle P_O. As is well-known the Lie algebra $\mathfrak{g} = \mathfrak{so}(d-1,1)$ is isomorphic to antisymmetric matrices with unit determinant[6] on $\mathbb{R}^d$ and hence the connection can be written as $\boldsymbol{\omega}^a{}_b \in \Omega^1 P_O(\mathfrak{g})$ with $\boldsymbol{\omega}^{ab} = -\boldsymbol{\omega}^{ba}$ where the indices are raised with the metric η^{ab}. Following Eq. A.4 the curvature of $\boldsymbol{\omega}^a{}_b$ is

$$\boldsymbol{R}^a{}_b = D\boldsymbol{\omega}^a{}_b = d\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b \tag{A.35}$$

and we define the *torsion* as

$$\boldsymbol{T}^a := D\boldsymbol{e}^a = d\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b \tag{A.36}$$

The torsion then satisfies the *torsional Bianchi identity*

$$D\boldsymbol{T}^a = D^2\boldsymbol{e}^a = \boldsymbol{R}^a{}_b \wedge \boldsymbol{e}^b \tag{A.37}$$

If we choose a local section we can pullback the connection to a local gauge field of the form ω_μ^{ab}. The local gauge field can be written in terms of the derivatives of the coframes on spacetime as

$$\omega_\mu^{ab} = E^{a\nu}\hat{\nabla}_\mu e_\nu^b = -E^{b\nu}\hat{\nabla}_\mu e_\nu^a \tag{A.38}$$

where $\hat{\nabla}$ is the (metric-compatible, torsionful) derivative operator on spacetime tensors that

6. The antisymmetry of the connection also follows from requiring $D\eta_{ab} = 0$ and the unit determinant from $D\epsilon_{a_1\ldots a_d} = 0$.

ignores the internal representation indices (see Eq. 3.4.13 [45]). Similarly, the pullbacks through a local section of the torsion $\boldsymbol{T}^a$ and curvature $\boldsymbol{R}^a{}_b$ are related to the spacetime torsion $T^\lambda{}_{\mu\nu}$ and curvature $R_{\mu\nu\rho}{}^\lambda$ (using the local frames E_a^μ) by

$$\begin{aligned} T^\lambda{}_{\mu\nu} &= E_a^\lambda T^a{}_{\mu\nu} \\ R_{\mu\nu\rho}{}^\lambda &= -E_a^\lambda e_\rho^b R^a{}_{b\mu\nu} \end{aligned} \tag{A.39}$$

where we have used the conventions of [45] for the Riemann tensor (also see § 3.4 [45]).

Remark A.4 (Levi-Civita connection). In the torsionless case $\boldsymbol{T}^a = 0$, using Eq. A.39 we see that, Eq. A.37 reduces to $R_{[\mu\nu\rho]}{}^\lambda = 0$. Further, the torsionless condition determines a unique Lorentz connection, the *Levi-Civita connection* $\tilde{\boldsymbol{\omega}}^a{}_b$. Using, Eq. A.36 we can get an explicit expression for the Levi-Civita connection in terms of the frames and coframes as

$$\tilde{\boldsymbol{\omega}}^a{}_b = -E^{[a} \cdot d\boldsymbol{e}^{b]} + \frac{1}{2}\left(E^a \cdot E^b \cdot d\boldsymbol{e}^c\right)\boldsymbol{e}_c \tag{A.40}$$

Any Lorentz connection with torsion can be written in terms of the Levi-Civita connection and a horizontal form (see Eq. A.1) as

$$\boldsymbol{\omega}^a{}_b = \tilde{\boldsymbol{\omega}}^a{}_b + C_c{}^a{}_b \boldsymbol{e}^c \tag{A.41}$$

where *contorsion* $C_{cab} = C_{c[ab]}$ is defined by

$$C_{cab} := \tfrac{1}{2}\left(T_{cab} - T_{abc} - T_{bca}\right) \tag{A.42}$$

with $T_{cab} = E_b \cdot E_a \cdot \boldsymbol{T}_c$ being the frame components of the torsion form Eq. A.36.

One can work with coframes and the contorsion (or the torsion) as independent dynamical fields instead of the coframes and the connection $\boldsymbol{\omega}^a{}_b$. However, the computations are much simpler in the latter case. Thus, in the main body of the paper we will always consider the Lorentz connection $\boldsymbol{\omega}^a{}_b$ as independent of the coframes $\boldsymbol{e}^a$ i.e. we work in the first-order formalism, but we provide Eqs. A.40–A.42 for readers interested in the second-order formalism for gravity.

Next we consider the possible automorphisms of the Lorentz bundle P_O that preserve the orthonormal coframes. From Lemma A.1 we know that an infinitesimal automorphism preserving the connection is determined only up to a covariantly constant function but we can show that if X^m is an infinitesimal automorphism that preserves the coframes, it is completely determined by a Killing vector field on M as follows (see [18] for a spacetime version of this result).

Lemma A.4 (Uniqueness of the infinitesimal automorphism which preserves orthonormal coframes)**.** *Given an automorphism which preserves some chosen orthonormal coframes $X^m \in \mathfrak{aut}(P_O; \boldsymbol{e}^a)$ so that $\pounds_X \boldsymbol{e}^a = 0$, the projection $X^\mu = (\pi_*)^\mu_m X^m$ is a Killing vector field for the metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ on M determined by the chosen coframes.*

Further, given any connection $\boldsymbol{\omega}^a{}_b$ on P_O, X^m is uniquely determined by X^μ as follows. Denote the Killing form corresponding to the Killing vector field X^μ as $\underline{\boldsymbol{\xi}} \equiv g_{\mu\nu} X^\nu$ and its lift to P_O as $\boldsymbol{\xi} := \pi^* \underline{\boldsymbol{\xi}} = (X \cdot \boldsymbol{e}^a)\boldsymbol{e}_a$ then the vertical part of X^m with respect to the chosen connection $\boldsymbol{\omega}^a{}_b$ is given by

$$ver_\omega X \equiv X \cdot \boldsymbol{\omega}_{ab} = \tfrac{1}{2} E_a \cdot E_b \cdot d\boldsymbol{\xi} + (X \cdot \boldsymbol{e}^c) C_{cab} \tag{A.43}$$

where the contorsion C_{cab} is defined by Eq. A.42.

Proof. For some chosen coframes $\boldsymbol{e}^a$ let, $X^m \in \mathfrak{aut}(P_O; \boldsymbol{e}^a)$ be an infinitesimal automorphism preserving $\boldsymbol{e}^a$ i.e. $\pounds_X \boldsymbol{e}^a = 0$. It immediately follows that the projection $\underline{X} \equiv X^\mu = (\pi_*)^\mu_m X^m$ satisfies

$$\pounds_{\underline{X}}(\eta_{ab} e^a_\mu e^b_\nu) = \pounds_{\underline{X}}(g_{\mu\nu}) = 0 \tag{A.44}$$

Thus, X^m projects to a Killing vector for $g_{\mu\nu}$.

Using Eqs. A.7 and A.36 we have

$$0 = \pounds_X \boldsymbol{e}^a = X \cdot \boldsymbol{T}^a + D(X \cdot \boldsymbol{e}^a) - (X \cdot \boldsymbol{\omega}^a{}_b)\boldsymbol{e}^b \tag{A.45}$$

and taking the interior product of Eq. A.45 with E_c^m we immediately get

$$X \cdot \boldsymbol{\omega}^{ab} = E^{[b} \cdot D(X \cdot \boldsymbol{e}^{a]}) + E^{[b} \cdot X \cdot \boldsymbol{T}^{a]} \tag{A.46}$$

For the Killing form $\boldsymbol{\xi} = (X \cdot \boldsymbol{e}^a)\boldsymbol{e}_a$ we have

$$\begin{aligned} d\boldsymbol{\xi} = D\boldsymbol{\xi} &= D(X \cdot \boldsymbol{e}^a) \wedge \boldsymbol{e}_a + (X \cdot \boldsymbol{e}^a)\ \boldsymbol{T}_a \\ \implies E^a \cdot E^b \cdot d\boldsymbol{\xi} &= 2E^{[b} \cdot D(X \cdot \boldsymbol{e}^{a]}) + (X \cdot \boldsymbol{e}^c)E^a \cdot E^b \cdot \boldsymbol{T}_c \end{aligned} \tag{A.47}$$

Substituting this into Eq. A.46 we can write

$$X \cdot \boldsymbol{\omega}^{ab} = \tfrac{1}{2}E^a \cdot E^b \cdot d\boldsymbol{\xi} - \tfrac{1}{2}(X \cdot \boldsymbol{e}^c)E^a \cdot E^b \cdot \boldsymbol{T}_c - E^{[a} \cdot X \cdot \boldsymbol{T}^{b]} \tag{A.48}$$

Using the frame components of the torsion 2-form and Eq. A.42, we get Eq. A.43. The right-hand-side depends only on X^μ (and its first derivative) and the torsion of the chosen connection. So we see that any $X^m \in \mathfrak{aut}(P_O; \boldsymbol{e}^a)$ is uniquely determined by its projection.

□

Using Lemma A.4 we show that for an automorphism preserving some orthonormal coframes, the Lie derivative on the bundle coincides with the Lorentz-Lie derivative of [25, 15]. We consider a scalar field σ^a that transforms under the local Lorentz transformations for simplicity. The Lie derivative on the bundle with respect to $X^m \in \mathfrak{aut}(P_O; \boldsymbol{e}^a)$ is then (using Eqs. A.7 and A.43)

$$\pounds_X \sigma^a = X \cdot D\sigma^a - \left(\tfrac{1}{2}E_a \cdot E_b \cdot d\boldsymbol{\xi} + (X \cdot \boldsymbol{e}^c)C_{cab}\right)\sigma^b \tag{A.49}$$

Picking a local section and denoting the projection $\underline{X} \equiv X^\mu = (\pi_*)^\mu_m X^m$ we get (in the torsionless case)

$$\begin{aligned}\hat{\pounds}_{\underline{X}}\sigma^a &= X^\mu D_\mu \sigma^a + \left(E^{a\mu}E^{b\nu}\nabla_{[\mu}X_{\nu]}\right)\sigma_b \\ &= \pounds_{\underline{X}}\sigma^a + \left(X^\mu {\omega_\mu}^{ab} + E^{\mu[a}e_\nu^{b]}\nabla_\mu X^\nu\right)\sigma_b\end{aligned} \tag{A.50}$$

The first line is (up to differences in sign and factor conventions) the Lie derivative on Lorentz tensors defined in [25]. In the second line, $\pounds_{\underline{X}}$ is the Lie derivative computed by ignoring the internal index a and the second term coincides with ${\lambda^a}_b$ (Eq. 1.2) used by [15].

* * *

Next we briefly review the construction of spinor fields referring to [81, 82, 83], Problem 4 of §Vbis. [13], and Ch.1 [28] for details. We note that these references use the "mostly minus" signature for the Lorentzian metric but to conform to the earlier sections we stick to the "mostly plus" signature making appropriate changes in signs according to Remark 3.8 [81].

Spinors on a d-dimensional Lorentzian spacetime are tensors valued in a representation of the $Spin^0(d-1,1)$ group, which is the double cover of $SO^0(d-1,1)$ the identity component of the Lorentz group i.e. there is a 2-to-1 group homomorphism $Spin^0(d-1,1) \to SO^0(d-1,1)$. The representation theory of $Spin^0(d-1,1)$ can be described using the *Clifford algebra* $Cl(d-1,1)$ generated by an identity element $\mathbb{1}$ and the *Dirac matrices*[7] γ^a which satisfy the Clifford relation (see [81]):

$$\{\gamma^a, \gamma^b\} := \gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab}\mathbb{1} \tag{A.51}$$

The group $Spin^0(d-1,1)$ is embedded in the Clifford algebra according to Definition 2.4 of

7. The term Dirac matrices is historical and at this point we only view them as elements of an abstract algebra without choosing any matrix representation.

[81]. The Dirac matrices also implement the double cover homomorphism $Spin^0(d-1,1) \to SO^0(d-1,1)$ as detailed in Proposition 2.6 of [81].

At this time to be concrete, let us stick to 3 + 1-dimensions (the representation theory for general dimensions and signature can be found in [28]). The Clifford algebra $Cl(3,1)$, also called the *Dirac algebra*,[8] has a unique irreducible complex representation as a matrix group on $\mathbb{C}^4$ (up to equivalence; see Theorem 2.2 of [81]). This induces a representation of $Spin^0(3,1)$ on $\mathbb{D} \cong \mathbb{C}^4$ which is the vector space of *Dirac spinors*.[9] Similarly, we denote the dual vector space of *Dirac cospinors* by $\mathbb{D}^* \cong \mathbb{C}^4$. In this representation, the Dirac matrices γ^a can be viewed as linear maps (or matrices) on $\mathbb{D}$ or $\mathbb{D}^*$. To avoid a proliferation of indices, we will use the standard "matrix-type notation" where elements of $\mathbb{D}$ are column-vectors, elements of $\mathbb{D}^*$ are row-vectors, and the γ^a are matrices.

The *Dirac adjoint* map or *Dirac conjugation* $^- : \mathbb{D} \to \mathbb{D}^* : v \mapsto \overline{v}$ is an anti-isomorphism of complex vector spaces (see Theorem 3.5 [81] where it is denoted by A). Denote the inverse by the same symbol, so that for $u \in \mathbb{D}^*$ and $v \in \mathbb{D}$

$$\overline{uv} = \overline{v}\ \overline{u} \quad ; \quad \overline{(\gamma^a)} = -\gamma^a \quad ; \quad \overline{(\overline{v}v)} = \overline{v}v \in \mathbb{R} \tag{A.52}$$

To consider spinor fields on spacetime we need the notion of a *spin structure*. A spin structure on M is a $Spin^0(3,1)$-principal fibre bundle $F_{Spin}M$ of *spin frames* together with a 2-to-1 bundle homomorphism to the proper orthonormal frame bundle $F^0_{SO}M$ which is equivariant with respect to the double cover map on the respective structure groups.

Remark A.5 (Existence and uniqueness of spin structures). The existence and uniqueness of a spin structure over M depends on the absence of certain topological obstructions (See § IV.2-

8. Note that $Cl(3,1) \ncong Cl(1,3)$ but, $Spin^0(3,1) \cong Spin^0(1,3)$ (see § I.4-I.7 [28]), and so our choice of $Cl(3,1)$ as the Dirac algebra, instead of $Cl(1,3)$, does not affect the discussion of spinors.

9. In 3 + 1-dimensions even though the representation of the Dirac algebra on $\mathbb{C}^4$ is irreducible, the induced representation of $Spin^0(3,1) \cong SL(2,\mathbb{C})$ is reducible to $\mathbb{C}^2 \oplus \bar{\mathbb{C}}^2$ which are the spaces of *chiral Weyl 2-spinors*.

IV.3 of [28]). A spin structure over M always exists if the orthonormal frame bundle admits a global section, though the converse is not true in general. The necessary and sufficient condition for the existence of a spin structure is given by the vanishing of the *second Stiefel-Whitney class* of M, $w_2(M) \in H^2(M, \mathbb{Z}_2)$. Even then, the possible non-isomorphic spin structures are labelled by the *first Stiefel-Whitney class* of M, $w_1(M) \in H^1(M, \mathbb{Z}_2)$. A theorem due to Geroch [82] shows that for a non-compact $3+1$-dimensional spacetime a spin structure exists if and only if the bundle of oriented orthonormal frames over M admits a global section. As noted in [82], this theorem is trivial in lower dimensions and false in higher.

To formulate Dirac fields on a spacetime M with a fixed metric $g_{\mu\nu}$ and orientation, we can choose some spin structure $F_{Spin}M$ corresponding to the proper frame bundle $F^0_{SO}M$ determined by the given metric and orientation on M. However, as discussed before, this is problematic when considering theories where the metric (or the coframes) themselves are dynamical fields. As before we circumvent this, by considering a choice of spin bundle P_{Spin} corresponding to a proper Lorentz bundle P^0_{SO} in a manner similar to the spin structure $F_{Spin}M$ corresponding to $F^0_{SO}M$. Once we have solved the equations of motion to get a metric, we can identify P_{Spin} with a spin structure $F_{Spin}M$ given by that metric.

On the spin bundle P_{Spin}, we can define the coframes e^a_m by lifting the coframes from P^0_{SO} and also define frame fields E^m_a which project to the frame fields in P^0_{SO}. We can lift the Lorentz connection $\boldsymbol{\omega}^a{}_b$ on P^0_{SO}, through the double cover map, to a connection on P_{Spin} as $\boldsymbol{\omega}_{spin} := -\frac{1}{8}\boldsymbol{\omega}_{ab}[\gamma^a, \gamma^b]$. We can similarly lift other structures such as the horizontal volume form Eq. A.32 and the horizontal Hodge dual Eq. A.33.

A *Dirac spinor field* $\Psi \in \Omega^0 P_{Spin}(\mathbb{D})$ is a function on the spin bundle P_{Spin} valued in the Dirac spinor representation $\mathbb{D}$ and similarly, a *Dirac cospinor field* is $\Phi \in \Omega^0 F_{Spin}M(\mathbb{D}^*)$. The spin covariant exterior derivative for $\Psi \in \Omega^0 P_{Spin}(\mathbb{D})$ and $\Phi \in \Omega^0 P_{Spin}(\mathbb{D}^*)$ is given by:

$$D\Psi = d\Psi - \frac{1}{8}(\boldsymbol{\omega}_{ab}[\gamma^a, \gamma^b])\ \Psi \quad ; \quad D\Phi = d\Phi + \frac{1}{8}\Phi\ (\boldsymbol{\omega}_{ab}[\gamma^a, \gamma^b]) \tag{A.53}$$

and the *Dirac operator* $\not{D}$ on Dirac spinor fields and cospinor fields is given by

$$\not{D}\Psi = \gamma^a E_a \cdot D\Psi \quad ; \quad \not{D}\Phi = (E_a \cdot D\Phi)\gamma^a \tag{A.54}$$

Note that in our conventions $\overline{\not{D}\Psi} = -\not{D}\overline{\Psi}$.

Any $X^m \in \mathfrak{aut}(P_{Spin})$ then acts on the Dirac spinor field through the Lie derivative

$$\pounds_X \Psi := X \cdot d\Psi = X \cdot D\Psi + \tfrac{1}{8}(X \cdot \boldsymbol{\omega}_{ab})[\gamma^a, \gamma^b]\Psi \tag{A.55}$$

Consider now an X^m which projects to an automorphism $X^{m'} \in \mathfrak{aut}(P^0_{SO}; \boldsymbol{e}^a)$ preserving some orthonormal frames through the 2-to-1 bundle homomorphism $P_{Spin} \to P^0_{SO}$. By Lemma A.4, such an $X^{m'}$ always projects to a Killing field X^μ of the metric on M determined by the chosen coframes, and is uniquely determined using Eq. A.43. For such vector fields the Lie derivative Eq. A.55 on the bundle of the Dirac spinor field reads (in the torsionless case)

$$\pounds_X \Psi = X \cdot D\Psi + \tfrac{1}{8}\left(\tfrac{1}{2}E^a \cdot E^b \cdot d\boldsymbol{\xi}\right)[\gamma_a, \gamma_b]\Psi \tag{A.56}$$

Viewed on spacetime with $\underline{X} \equiv X^\mu$ being a Killing field of the given metric, this becomes

$$\pounds_{\underline{X}} \Psi = X^\mu D_\mu \Psi - \tfrac{1}{8}\left(\nabla_{[\mu} X_{\nu]}\right)[\gamma^\mu, \gamma^\nu]\Psi \tag{A.57}$$

which is Eq. 1.3, the Lie derivative of spinors with respect to the Killing field X^μ as defined by Lichnerowicz [26] (see also [27, 25]).

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