(ii) $$d(A, B) \geqslant \varepsilon(A) - \varepsilon(B) \quad \text{if} \quad A, B \in \Gamma.$$

It then follows from Theorem 2 that X/Π is pseudo-metrizable.

Define $\varepsilon(A) = \eta(A)/K$, where

$$\eta(A) = \sup\{\varepsilon \mid \varepsilon > 0 \text{ and } N_\varepsilon[A] \subset \bigcup \Gamma\} \quad \text{if} \quad A \in \Gamma.$$

We first prove that $\eta(A) > 0$ for all $A \in \Gamma$. Let $a \in A$ and let $\varepsilon = (1/K)d(a, X - \bigcup \Gamma)$. Clearly $\varepsilon > 0$. To show $\eta(A) > 0$, it is sufficient to show that $N_\varepsilon[A] \subset \bigcup \Gamma$, or, equivalently, that $d(A, X - \bigcup \Gamma) \geqslant \varepsilon$. So let $x \in A$, $u \in X - \bigcup \Gamma$, and let $\varphi \in \Pi$ be such that $\varphi(x) = a$. Then

$$Kd(x, u) \geqslant d(\varphi(x), \varphi(u)) = d(a, \varphi(u)) \geqslant d(a, X - \bigcup \Gamma) = K\varepsilon,$$

so $d(x, u) \geqslant \varepsilon$, and consequently $d(A, X - \bigcup \Gamma) \geqslant \varepsilon$.

It is trivial that $N_{\eta(A)}[A] \subset \bigcup \Gamma$ for all $A \in \Gamma$; thus (i) is true since $\varepsilon(A) \leqslant \eta(A)$ for all $A \in \Gamma$. It remains to prove (ii). Let $A, B \in \Gamma$. By the definition of $\varepsilon(A)$ and $\varepsilon(B)$, it is sufficient to prove that $Kd(A, B) \geqslant \eta(A) - \eta(B)$, i.e., that $\eta(B) \geqslant \eta(A) - Kd(A, B)$; and to do this it is sufficient to show that the following implication is true:

$$d(x, B) < \eta(A) - Kd(A, B) \Rightarrow x \in \bigcup \Gamma.$$

Suppose $d(x, B) < \eta(A) - Kd(A, B)$. Then there exists $\varepsilon > 0$ such that

$$d(x, B) < \eta(A) - Kd(A, B) - \varepsilon.$$

Now let $\delta > 0$ be arbitrary. Then there exists $b \in B$ such that

$$d(x, b) \leqslant d(x, B) + \delta,$$

and by the lemma there exists $a \in A$ such that

$$d(a, b) \leqslant Kd(A, B) + \delta.$$

It follows that

$$d(x, A) \leqslant d(x, a) \leqslant d(x, B) + Kd(A, B) + 2\delta < \eta(A) - \varepsilon + 2\delta.$$

Hence $d(x, A) \leqslant \eta(A) - \varepsilon < \eta(A)$ and $x \in \bigcup \Gamma$.

This concludes the proof of Theorem 5.

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Reçu par la Rédaction le 5. 1. 1967

The first order properties of Dedekind finite integers

by

Erik Ellentuck* (New Brunswick, N. J.)

1. Introduction. It is well known that mathematics is often simplified by the introduction of ideal elements. In the past it has been said that even when their existence is entirely fictitious (points at infinity in geometry, for example), theorems about the original structure which are proved with their aid may be interpreted as relative consistency results. More recently, our firm belief in set theory has led us to take ideal elements which are constructed in set theory as bonafide mathematical objects. In this paper such notions are applied to the Dedekind finite cardinals Δ (cf. [4]). In theorem 1 we show that just as the finite cardinals $\mathcal{E}$ can be extended to the ring of rational integers $\mathcal{E}^*$, Δ can be extended to the ring of Dedekind finite integers Δ^*. Of course all of this is going on in a set theory $\mathfrak{S}^0$ which does not include the axiom of choice. Next, a series of lemmas shows that every function defined on $\mathcal{E}^*$ can be extended to a function defined on Δ^*. Since this extension procedure depends in an essential way on the methods of [4], we must require that $\mathfrak{S}^0$ contains the axiom of choice for sets of finite sets. This does not force $\Delta^* = \mathcal{E}^*$ as is shown in [4]. In order to study the first order properties of Δ^* we define a language L which contains equations between terms, which are built up by composition of function symbols, as atomic formula. L is interpreted in $\mathcal{E}^*$ by letting the function symbols denote functions on $\mathcal{E}^*$, and interpreted in Δ^* by letting the function symbols denote extensions to Δ^* of functions defined on $\mathcal{E}^*$. The bulk of our work is concerned with giving necessary and sufficient conditions that a sentence $\mathfrak{A}$ which holds in $\mathcal{E}^*$ will also hold in Δ^*. Our main sufficiency result is given by corollary 2, which says in essence that if $\mathfrak{A}$ is equivalent in $\mathcal{E}^*$ to a Horn sentence, then $\mathfrak{A}$ will also hold in Δ^*. This theorem easily follows by a routine transcription of [4], theorem 8. The more interesting part of our paper is concerned with necessity. We use metamathematical tools. In lemma 5 we show that in the Fraenkel–Mostowski model $\mathfrak{M}^+$ (cf. [11]), Δ^* is isomorphic to a direct limit of reduced powers of $\mathcal{E}^*$. In lemmas 6

* Research for this paper was supported in part by National Science Foundation contract number GP 5786.

and 7 we analyze the first order theory of this direct limit and in lemma 8 we show that it has the same theory as the system $\mathfrak{F}$ which is defined to be a countable direct power of $\mathcal{E}^*$ reduced modulo the cofinite sets. Hence in $\mathfrak{W}^+$ the first order theory of Δ^* is the same as that of $\mathfrak{F}$. This result is exploited in corollary 3 to show that there is an extension of $\mathfrak{S}^0$, and a class of sentences for which a necessary condition that $\mathfrak{A}$ hold in Δ^* is that $\mathfrak{A}$ be equivalent in $\mathcal{E}^*$ to a Horn sentence. In theorems 5 and 6 certain results about $\mathfrak{W}^+$ are actually modeled in $\mathfrak{S}^0$ by using a rather pretty lifting technique of Kreisel. A section on applications follows.

2. The extension. Let $\mathfrak{S}$ be a version of class-set theory whose axioms are (i) A-D of [7] modified so as to allow for the existence of proper individuals (urelemente), and (ii) an additional axiom.

(1) *There exists an infinite set* $\boldsymbol{K}$ *of all urelemente.*

$\mathfrak{S}^0$ is obtained from $\mathfrak{S}$ by adding the axiom of choice for sets of finite sets, and $\mathfrak{S}^*$ is obtained from $\mathfrak{S}$ by adding the full axiom of choice. It is known that $\mathfrak{S}^*$, $\mathfrak{S}^0$, and $\mathfrak{S}$, in that order, are of strictly decreasing strengths (cf. [11]). Each of the following lemmas, theorems, and corollaries, will be labeled as to the theory in which it occurs.

Define ordinal numbers by modified [7] in such a way that each *ordinal number* is the set of all its predecessors, and denote them by lower case Greek letters. An ordinal number is *finite* if both it and each of its predecessors contains a largest element. Finite ordinals are denoted by lower case Latin letters, in particular by 'i', 'j', and 'k'. ω is the smallest ordinal which is not finite, and as a set, is the set of all finite ordinals.

Define cardinal numbers by the abstractive method of [14] basing our rank theory on $\boldsymbol{K}$ (cf. [4], p. 228), and denote them by lower case German letters. Use '$\sim$' for set theoretic equivalence and $|A|$ for the cardinal number of the set A. Let Γ be the class of all cardinals. A cardinal number is *finite* if it is the cardinal of a finite ordinal. Finite cardinals are denoted by lower case Latin letters, in particular by 'i', 'j', and 'k', and identified with finite ordinals in the usual way. Let $\mathcal{E}$ be the set of all finite cardinals. A cardinal $\mathfrak{x}$ is *Dedekind finite* if $\mathfrak{x} \neq \mathfrak{x}+1$. Let Δ be the class of all Dedekind finite cardinals. Clearly $\mathcal{E} \subseteq \Delta$, however, the converse inclusion is not a theorem of $\mathfrak{S}^0$. The algebraic theory of Δ has received an extensive treatment in [4], which will serve as the principal reference throughout this section.

For any class A and $0 < k < \omega$ let X^kA be the class of all functions whose domain is k and whose range is contained in A. Let V be the class of all sets. Elements $x \in X^kV$ are called *k-tuples*. Write x_i for $x(i)$ and exhibit them as $x = \langle x_0, \ldots, x_{k-1}\rangle$. Denote members of $X^k\Gamma$ by lower case German letters. We extend certain notions componentwise from V to X^kV and use the same symbol for the extension as for the original notion. For $x \in X^kV$ let $|x| = \langle |x_0|, \ldots, |x_{k-1}|\rangle$, and for $\mathfrak{x}, \mathfrak{y} \in X^k\Gamma$ let $\mathfrak{x} \leqslant \mathfrak{y}$ if $\mathfrak{x}_i \leqslant \mathfrak{y}_i$ for $i < k$. Context will always make it clear when a symbol is to be understood in its extended sense.

Let $\mathcal{E}^*$ be the rational integers: positive, negative, and zero. There is a standard construction by means of which the system $\langle \mathcal{E}, +, \cdot\rangle$ can be extended to the ring $\langle \mathcal{E}^*, +, \cdot\rangle$ in such a way that each $x \in \mathcal{E}^*$ can be expressed as a difference $x = y-z$ where $y, z \in \mathcal{E}$. This construction can be carried out using only the following properties of $\langle \mathcal{E}, +, \cdot\rangle$:

(2) *it is a commutative cancellation semigroup with zero element under addition, it is a commutative semigroup with identity element under multiplication, multiplication is distributive over addition.*

We now list the principal steps of this extension, leaving the details to the reader (cf. [3], p. 147).

(a) Define a relation $\approx$ on $X^2\mathcal{E}$ by $\langle x_0, x_1\rangle \approx \langle y_0, y_1\rangle$ if $x_0+y_1 = x_1+y_0$, and prove that it is an equivalence relation. Let $[x_0, x_1]_{\mathcal{E}}$ be the equivalence class determined by $\langle x_0, x_1\rangle$, and let $\mathcal{E}^*$ be the set of all such equivalence classes.

(b) Define addition and multiplication on $\mathcal{E}^*$ by

$$[x_0, x_1]_{\mathcal{E}} + [y_0, y_1]_{\mathcal{E}} = [x_0+y_0, x_1+y_1]_{\mathcal{E}},$$

and

$$[x_0, x_1]_{\mathcal{E}} \cdot [y_0, y_1]_{\mathcal{E}} = [x_0y_0+x_1y_1, x_0y_1+x_1y_0]_{\mathcal{E}}$$

and prove that these definitions are unique.

(c) Show that $\langle \mathcal{E}^*, +, \cdot\rangle$ is a ring with $[0, 0]_{\mathcal{E}}$ playing the role of the zero element and $[1, 0]_{\mathcal{E}}$ playing the role of the identity element.

(d) Show that $f\colon \mathcal{E} \to \mathcal{E}^*$ given by $f(x) = [x, 0]_{\mathcal{E}}$ is an algebraic isomorphism into and use a transfer theorem to embed $\mathcal{E}$ in $\mathcal{E}^*$. Notice that $[x, y]_{\mathcal{E}} = [x, 0]_{\mathcal{E}} - [y, 0]_{\mathcal{E}} = x-y$.

$\mathfrak{S}$-THEOREM 1. *The system* $\langle \Delta, +, \cdot\rangle$ *can be extended to a ring in essentially the same way that the system* $\langle \mathcal{E}, +, \cdot\rangle$ *can be extended to the ring* $\langle \mathcal{E}^*, +, \cdot\rangle$.

Proof. By [4], p. 226 (iii), the system $\langle \Delta, +, \cdot\rangle$ also satisfies (2). Hence the same construction (a)-(d) should extend $\langle \Delta, +, \cdot\rangle$ to a ring $\langle \Delta^*, +, \cdot\rangle$. The only difficulty is that we cannot prove in $\mathfrak{S}$ that the equivalence classes in (a) are actually sets. [1] We remedy this situation by

[1] From a personal communication with J. D. Halpern. See a forthcoming paper of J. D. Halpern and A. Lévy where it is shown by Cohen's method that even in set theories without urelemente, a proper class of Dedekind finite cardinals can be introduced. Note, however, that (10) and (11) of this paper imply the existence of models (with urelemente) in which $\Delta \neq \mathcal{E}$ but Δ is a set.

using the abstractive method of [14] and defining $[\mathfrak{x}_0, \mathfrak{x}_1]_\Delta$ for $\langle \mathfrak{x}_0, \mathfrak{x}_1\rangle \in X^2\Delta$ as the set of all those $\langle \mathfrak{y}_0, \mathfrak{y}_1\rangle \in X^2\Delta$ of least rank such that $\langle \mathfrak{x}_0, \mathfrak{x}_1\rangle \approx \langle \mathfrak{y}_0, \mathfrak{y}_1\rangle$ (cf. [4], p. 228). The rest of our construction closely parallels (a)-(d) above. q.e.d.

DEFINITION 1. Let Δ^* be obtained from Δ by the construction outlined in the proof of theorem 1. Elements of Δ^* are called *Dedekind finite integers*.

$\langle \Delta^*, +, \cdot\rangle$ is a commutative ring with 0 as zero element relative to addition and 1 as identity element relative to multiplication. Subtraction is defined in the usual way. Since the function $f\colon \mathcal{E}^* \to \Delta^*$ given by $f([\mathfrak{x}_0, \mathfrak{x}_1]_\mathcal{E}) = [\mathfrak{x}_0, \mathfrak{x}_1]_\Delta$ is an algebraic isomorphism into, we may use a transfer theorem to embed $\mathcal{E}^*$ in Δ^*. Consequently our structures stand in the following relation: $\mathcal{E} \subseteq \Delta$, $\mathcal{E} \subseteq \mathcal{E}^*$, $\Delta \subseteq \Delta^*$, and $\mathcal{E}^* \subseteq \Delta^*$. The usual clasification of rational integers into three types (positive, negative, and zero) does not carry over to Δ^*, since it is based on the comparability of any two finite cardinals, which does not carry over to Δ (even in $\mathfrak{S}^0$). The elements of Δ^* can be classified into four types: for $\mathfrak{x}_0, \mathfrak{x}_1 \in \Delta$, $\mathfrak{x}_0 - \mathfrak{x}_1$ is *positive* if $\mathfrak{x}_0 > \mathfrak{x}_1$, *zero* if $\mathfrak{x}_0 = \mathfrak{x}_1$, *negative* if $\mathfrak{x}_0 < \mathfrak{x}_1$, and *neutral* if $\mathfrak{x}_0$ is incomparable with $\mathfrak{x}_1$. Denote members of $\mathcal{E}^*$ $(X^k\mathcal{E}^*)$ by lower case Latin letters, and members of Δ^* $(X^k\Delta^*)$ by lower case German letters.

The following discussion is based on the theory $\mathfrak{S}^0$. Let $0 < k < \omega$ and $f\colon X^k\mathcal{E} \to \mathcal{E}^*$. If $c(i) \in \mathcal{E}^*$ are the Stirling coefficients of f (cf. [4], p. 231), let $c^+(i) = \max\big(c(i), 0\big)$, $c^-(i) = \max\big(-c(i), 0\big)$ and define

$$f^+, f^-\colon X^k\mathcal{E} \to \mathcal{E} \tag{3}$$

to be the functions whose Stirling coefficients are $c^+(i)$, $c^-(i)$, respectively. Clearly f^+ and f^- are a pair of k-ary combinatorial functions whose difference $f^+(x) - f^-(x) = f(x)$.

$\mathfrak{S}^0$-LEMMA 1. *If $f\colon X^k\mathcal{E} \to \mathcal{E}^*$ and g^0, g^1 are any pair of k-ary combinatorial functions whose difference is f, then*

$$g^0_\Delta(\mathfrak{x}) - g^1_\Delta(\mathfrak{x}) = f^+_\Delta(\mathfrak{x}) - f^-_\Delta(\mathfrak{x}) \quad \textit{for every } \mathfrak{x} \in X^k\Delta\,.$$

Proof. Since $g^0(x) + f^-(x) = g^1(x) + f^+(x)$ holds for $x \in X^k\mathcal{E}$, a corresponding equation (with functions replaced by their extensions to Δ) holds for $\mathfrak{x} \in X^k\Delta$ (cf. [4], theorem 8). q.e.d.

DEFINITION 2. We extend every function $f\colon X^k\mathcal{E} \to \mathcal{E}^*$ to a function $f_\Delta\colon X^k\Delta \to \Delta^*$ by requiring that $f_\Delta(\mathfrak{x}) = f^+_\Delta(\mathfrak{x}) - f^-_\Delta(\mathfrak{x})$ for every $\mathfrak{x} \in X^k\Delta$, where f^+_Δ, f^-_Δ extend f^+, f^- as in [4].

For $f\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ define $\widetilde{f}\colon X^{2k}\,\mathcal{E} \to \mathcal{E}^*$ by

$$\widetilde{f}(x) = f(x_0 - x_1, \ldots, x_{2k-2} - x_{2k-1}) \quad \text{for} \quad x \in X^{2k}\mathcal{E}\,. \tag{4}$$

$\mathfrak{S}^0$-LEMMA 2. *If $f\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ and $\mathfrak{x}, \mathfrak{y} \in X^{2k}\Delta$ are any elements such that $\mathfrak{x}_0 - \mathfrak{x}_1 = \mathfrak{y}_0 - \mathfrak{y}_1, \ldots, \mathfrak{x}_{2k-2} - \mathfrak{x}_{2k-1} = \mathfrak{y}_{2k-2} - \mathfrak{y}_{2k-1}$, then $\widetilde{f}_\Delta(\mathfrak{x}) = \widetilde{f}_\Delta(\mathfrak{y})$ where $\widetilde{f}$ is as in definition 2.*

Proof. Since $(x_0 + y_1 = x_1 + y_0 \wedge \ldots \wedge x_{2k-2} + y_{2k-1} = x_{2k-1} + y_{2k-2}) \to \to \widetilde{f}^+(x) + \widetilde{f}^-(y) = \widetilde{f}^-(x) + \widetilde{f}^+(y)$ holds for $x, y \in X^{2k}\mathcal{E}$ a corresponding implication holds for $\mathfrak{x}, \mathfrak{y} \in X^{2k}\Delta$ (cf. [4], theorem 8). q.e.d.

DEFINITION 3. We extend every function $f\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ to a function $f_{\Delta^*}\colon X^k\Delta^* \to \Delta^*$ by requiring that for every $\mathfrak{x} \in X^k\Delta^*$, $f_{\Delta^*}(\mathfrak{x}) = \widetilde{f}_\Delta(\mathfrak{y})$ where $\mathfrak{y}$ is any member of $X^{2k}\Delta$ such that $\mathfrak{x}_0 = \mathfrak{y}_0 - \mathfrak{y}_1, \ldots, \mathfrak{x}_{k-1} = \mathfrak{y}_{2k-2} - \mathfrak{y}_{2k-1}$.

On the basis of definitions 1-3 ([4], theorem 8), and a great deal of computation, we have the following lemmas which we state without proof.

$\mathfrak{S}^0$-LEMMA 3. (i) *If $f\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ is a constant function, then f_{Δ^*} is also a constant function with the same value.* (ii) *If $f\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ is a projection onto the $i < k$ component, then f_{Δ^*} is also a projection onto the same component.* (iii) *If $f\colon X^2\mathcal{E}^* \to \mathcal{E}^*$ is the arithmetical plus or times function then f_{Δ^*} is also the plus or times function (as in theorem 1).*

$\mathfrak{S}^0$-LEMMA 4. (i) *Let $f\colon X^n\mathcal{E}^* \to \mathcal{E}^*$, and for each $i < n$, $f^i\colon X^k\mathcal{E} \to \mathcal{E}^*$. If $h = f \circ (f^0, \ldots, f^{n-1})$ is the composition, then $h_\Delta = f_{\Delta^*} \circ (f^0_\Delta, \ldots, f^{n-1}_\Delta)$.* (ii) *Let $f\colon X^n\mathcal{E}^* \to \mathcal{E}^*$, and for each $i < n$, $f^i\colon X^k\mathcal{E}^* \to \mathcal{E}^*$. If $h = f \circ (f^0, \ldots, f^{n-1})$, then $h_{\Delta^*} = f_{\Delta^*} \circ (f^0_{\Delta^*}, \ldots, f^{n-1}_{\Delta^*})$.*

The purpose of this paper is to discover general transfer principles which will tell us when a first order property of $\mathcal{E}^*$ also holds in Δ^*. We will apply the same techniques to this problem as we did to the preceding lemmas. Namely, we convert a property φ of $\mathcal{E}^*$ to an equivalent property $\widetilde{\varphi}$ of $\mathcal{E}$, use [4], theorem 8, to show that $\widetilde{\varphi}$ holds in Δ, and then reconvert $\widetilde{\varphi}$ in Δ to an equivalent φ in Δ^*. In order to carry out this program it is necessary to introduce an auxiliary language. This language is described in the same metalanguage which is used for the description of the syntax of $\mathfrak{S}$. Let us suppose that $\mathfrak{S}$ contains: (i) an infinite list of variables $\mathfrak{x}_i$, (ii) an infinite list of function constants, $\mathfrak{f}^k_i$, $k > 0$, where under interpretation each $\mathfrak{f}^k_i\colon X^k\mathcal{E}^* \to \mathcal{E}^*$. Then we define language L as follows. The *terms* of L is the smallest set which: (i) contains each $\mathfrak{x}_i$, (ii) contains for each $0 < k$ and terms $\tau_0, \ldots, \tau_{k-1}$ the expression $\mathfrak{f}^k_i(\tau_0, \ldots, \tau_{k-1})$. An *atomic formula* of L is an expression $\tau_0 = \tau_1$ where τ_0 and τ_1 are terms. The *formulas* of L is the closure of its set of atomic formulas under truth table connectives and quantification with respect to the $\mathfrak{x}_i$. Let us take $\wedge$ (and), $\vee$ (or), and $\sim$ (not) as our complete set of connectives, and define $\to$ (implies) and $\equiv$ (if and only if) in terms of them. Our quantifiers are $\forall$ (for all) and $\exists$ (there exists). A *sentence* of L is a formula in which no variable $\mathfrak{x}_i$ appears freely. We always suppose that our sentences appear in prenex conjunctive normal form, that is, as a string of quantifiers followed by a conjunction of disjunctions of atomic formula and their negations. For such a sentence $\mathfrak{A}$, a sentence $\mathfrak{A}'$ is called a *Horn reduction* of $\mathfrak{A}$ if $\mathfrak{A}'$ can be obtained from $\mathfrak{A}$ by striking out in each conjunct, with at least two occurrences of unnegated atomic formulas, all but one unnegated atomic formula. $\mathfrak{A}$ is a *Horn sentence* if it coincides with one of its Horn reductions, and a *universal sentence* if its prefix consists entirely of universal quantifiers. Let $\mathfrak{Sft}(\mathfrak{A})$ be the disjunction of all the (finitely many) Horn reductions of $\mathfrak{A}$. Denote a restricted quantifier by appending the symbol for the restriction as a subscript to the quantifier bracket. For any sentence $\mathfrak{A}$ let $\mathfrak{A}_{\mathcal{E}^*}$ be obtained from $\mathfrak{A}$ by restricting each quantifier to $\mathcal{E}^*$, and let $\mathfrak{A}_{\Delta^*}$ be obtained from $\mathfrak{A}$ by restricting each quantifier to Δ^* and replacing each function constant $\mathfrak{f}$ which appears in $\mathfrak{A}$ by $\mathfrak{f}_{\Delta^*}$. Thus $\mathfrak{A}_{\mathcal{E}^*}$ and $\mathfrak{A}_{\Delta^*}$ are both meaningful sentences of $\mathfrak{S}$.

If $\mathfrak{A}$ is a universal Horn sentence then

$\mathfrak{S}^0$-THEOREM 2. $\mathfrak{A}_{\mathcal{E}^*}$ *implies* $\mathfrak{A}_{\Delta^*}$.

Proof. We introduce still another language. Let us suppose that $\mathfrak{S}$ contains: (i) an infinite list of variables $\mathfrak{x}_i^+$ and $\mathfrak{x}_i^-$, (ii) an infinite list of composite constants $\tilde{\mathfrak{f}}_i^{k+}, \tilde{\mathfrak{f}}_i^{k-}$ which under interpretation are related to the previous constants $\mathfrak{f}_i^k$ as in (3) and (4). Then we define language $\tilde{L}$ as follows. The *terms* of $\tilde{L}$ is the smallest set which: (i) contains each $\mathfrak{x}_i^+$ and $\mathfrak{x}_i^-$, (ii) contains for each $0 < k$ and terms $\tau_0, \ldots, \tau_{2k-1}$ the expressions $\tilde{\mathfrak{f}}_i^{k\bullet}(\tau_0, \ldots, \tau_{2k-1})$ where $\bullet$ is either $+$ or $-$, (iii) contains for terms τ_0, τ_1 the expression $\tau_0 + \tau_1$. Atomic formulas, formulas, and sentences of $\tilde{L}$ are defined in exactly the same way that they are defined for L. For any sentence $\mathfrak{A}$ of $\tilde{L}$ let $\mathfrak{A}_{\mathcal{E}}$ be obtained from $\mathfrak{A}$ by restricting each quantifier to $\mathcal{E}$, and let $\mathfrak{A}_{\Delta}$ be obtained from $\mathfrak{A}$ by restricting each quantifier to Δ and replacing each function constant $\tilde{\mathfrak{f}}^\bullet$ which appears in $\mathfrak{A}$ by $\tilde{\mathfrak{f}}^\bullet_\Delta$. Thus $\mathfrak{A}_{\mathcal{E}}$ and $\mathfrak{A}_{\Delta}$ are both meaningful sentences of $\mathfrak{S}$. By induction we define a map from terms τ of L into ordered pairs of terms $\langle \tau^+, \tau^- \rangle$ of $\tilde{L}$ as follows. If τ is $\mathfrak{x}_i$, then τ^+ is $\mathfrak{x}_i^+$ and τ^- is $\mathfrak{x}_i^-$. If τ is $\mathfrak{f}_i^k(\tau_0, \ldots, \tau_{k-1})$, then $\tau^\bullet$ is $\tilde{\mathfrak{f}}_i^{k\bullet}(\tau_0^+, \tau_0^-, \ldots, \tau_{k-1}^+, \tau_{k-1}^-)$ where $\bullet$ is either $+$ or $-$. By induction we define a map from formulas $\mathfrak{A}$ of L into formulas $\tilde{\mathfrak{A}}$ of $\tilde{L}$ as follows. If $\mathfrak{A}$ is $\tau_0 = \tau_1$, then $\tilde{\mathfrak{A}}$ is $\tau_0^+ + \tau_1^- = \tau_0^- + \tau_1^+$. If $\mathfrak{A}$ is $\mathfrak{A}_0 \wedge \mathfrak{A}_1$, $\mathfrak{A}_0 \vee \mathfrak{A}_1$, $\sim\mathfrak{A}_0$, $(\exists \mathfrak{x}_i)\mathfrak{A}_0$, $(\forall \mathfrak{x}_i)\mathfrak{A}_0$ then $\tilde{\mathfrak{A}}$ is $\tilde{\mathfrak{A}}_0 \wedge \tilde{\mathfrak{A}}_1$, $\tilde{\mathfrak{A}}_0 \vee \tilde{\mathfrak{A}}_1$, $\sim\tilde{\mathfrak{A}}_0$, $(\exists \mathfrak{x}_i^+)(\exists \mathfrak{x}_i^-)\tilde{\mathfrak{A}}_0$, $(\forall \mathfrak{x}_i^+)(\forall \mathfrak{x}_i^-)\tilde{\mathfrak{A}}_0$, respectively. This construction should make it clear to the reader that for any sentence $\mathfrak{A}$ of L we can prove in $\mathfrak{S}^0$ that

$$\text{(5)} \qquad \text{(i)}\ \mathfrak{A}_{\mathcal{E}^*} \equiv \tilde{\mathfrak{A}}_{\mathcal{E}}, \qquad \text{(ii)}\ \mathfrak{A}_{\Delta^*} \equiv \tilde{\mathfrak{A}}_{\Delta}.$$

We complete our proof as follows. By hypothesis, $\mathfrak{A}$ is a universal Horn sentence such that $\mathfrak{A}_{\mathcal{E}^*}$. By (5i) $\mathfrak{A}_{\mathcal{E}^*} \to \tilde{\mathfrak{A}}_{\mathcal{E}}$. But $\tilde{\mathfrak{A}}$ is also a universal Horn sentence. Hence by [4], theorem 8, $\tilde{\mathfrak{A}}_{\mathcal{E}} \to \tilde{\mathfrak{A}}_{\Delta}$. Finally, by (5ii), $\tilde{\mathfrak{A}}_{\Delta} \to \mathfrak{A}_{\Delta^*}$. q.e.d.

If $\mathfrak{A}$ is an arbitrary Horn sentence then

$\mathfrak{S}^0$-COROLLARY 1. $\mathfrak{A}_{\mathcal{E}^*}$ *implies* $\mathfrak{A}_{\Delta^*}$.

Proof. Suppose $\mathfrak{A}_{\mathcal{E}^*}$. Since $\mathcal{E}^*$ is well orderable, we can find Skolem functions for the existential quantifiers which appear in $\mathfrak{A}_{\mathcal{E}^*}$. Choose constants $\mathfrak{f}$ which do not already occur in $\mathfrak{A}$ and identify them with these Skolem functions. Let $\mathfrak{A}'$ be the sentence obtained from $\mathfrak{A}$ by deleting existential quantifiers and replacing existentially quantified variables by appropriate $\mathfrak{f}$'s. Then $\mathfrak{A}'$ is a universal Horn sentence and $\mathfrak{A}'_{\mathcal{E}^*}$. By theorem 2, $\mathfrak{A}'_{\Delta^*}$, from which $\mathfrak{A}_{\Delta^*}$ follows by restoration of quantifiers. q.e.d.

If $\mathfrak{A}$ is an arbitrary sentence, then

$\mathfrak{S}^0$-COROLLARY 2. $\mathfrak{Sft}(\mathfrak{A})_{\mathcal{E}^*}$ *implies* $\mathfrak{A}_{\Delta^*}$.

Proof. By hypothesis there is a Horn reduction $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}'_{\mathcal{E}^*}$. By corollary 1, $\mathfrak{A}'_{\Delta^*}$, from which $\mathfrak{A}_{\Delta^*}$ follows by predicate calculus. q.e.d.

We delay specific applications of theorem 2 and its corollaries to section 4, and instead continue with our theoretical development. Having discovered a sufficient condition for sentences to extend from $\mathcal{E}^*$ to Δ^*, it is natural to ask whether it is necessary as well. Obviously we cannot expect a converse to corollary 2. For otherwise, by adding the full axiom of choice to $\mathfrak{S}^0$, we force $\Delta^* = \mathcal{E}^*$ but can certainly find an example of a sentence $\mathfrak{A}$ such that $\mathfrak{A}_{\mathcal{E}^*}$ but not $\mathfrak{Sft}(\mathfrak{A})_{\mathcal{E}^*}$ (for example a sentence asserting the non-existence of zero divisors). Rather, the necessity conditions we have in mind are of a metamathematical nature. Consequently, their investigation involves entirely different techniques than those used in theorem 2. Even so, we do not obtain a complete solution to the problem. What we do get is a set of sentences and an extension of $\mathfrak{S}^0$ for which a converse to corollary 2 holds. This extension is found, and shown to be consistent relative to $\mathfrak{S}$ by an investigation of Fraenkel–Mostowski models.

3. The model.

It is well known that if $\mathfrak{S}$ is consistent, then so is the theory $\mathfrak{S}^*$ which is obtained from $\mathfrak{S}$ by adding the axiom of choice. In [11] a model $\mathfrak{W}^+$ of $\mathfrak{S}^0$ is constructed which does not satisfy the axiom of choice. The construction takes place in the theory $\mathfrak{S}^*$, which we will take as the theory underlying the following informal discussion.

Let $\lessdot$ be a dense linear ordering of $\boldsymbol{K}$, without first, but with last element t, such that for any two finite subsets A and B of $\boldsymbol{K}$ of the same cardinality, with $t \in A \cap B$ there is a $\lessdot$-monotone permutation of $\boldsymbol{K}$ which maps A onto B. Such orderings readily follow from the axiom of choice. Let $\mathfrak{G}^+$ be the set of all $\lessdot$-monotone permutations of $\boldsymbol{K}$ and let M^+ be the set of all finite subsets of $\boldsymbol{K}$ which contain t. Then exactly as in [11] we build a model $\mathfrak{W}^+$ based on the group $\mathfrak{G}^+$ and the $\mathfrak{G}^+$-ring M^+. A brief résumé of this construction will be found in [4] or [5]. Let us indicate notions relativized to $\mathfrak{W}^+$ by appending a superscript '+' to the symbol for that notion. For any two elements $a, b \in \boldsymbol{K}$ with $a \precsim b$ let $(a, b) = \{x: a \precsim x \precsim b\}$ and $(-\infty, a) = \{x: x \precsim a\}$. For any $A = \{a_0, \ldots, a_{k-1}\} \in M^+$ with $a_0 \precsim \ldots \precsim a_{k-1}$ let $\boldsymbol{K}^A$ be the k-tuple with components $\boldsymbol{K}_0^A = (-\infty, a_0)$ and $\boldsymbol{K}_{i+1}^A = (a_i, a_{i+1})$ for $i < k-1$. Finally let $\mathfrak{k}^A = |\boldsymbol{K}^A| = \langle |\boldsymbol{K}_0^A|, \ldots, |\boldsymbol{K}_{k-1}^A| \rangle$. In general $\mathfrak{k}^A$ will be interesting only when it is understood as relativized to $\mathfrak{W}^+$, and in the following discussion we assume that it is.

In [5] we developed a theory of combinatorial series. A combinatorial series is a formal expression

(6) $$f(u_0, \ldots, u_{k-1}) = \sum c(i_0, \ldots, i_{k-1}) \binom{u_0}{i_0} \cdots \binom{u_{k-1}}{i_{k-1}}$$

where the u_i are indeterminates and c is a function with domain $X^k\mathcal{E}$ and assuming values which are well ordered cardinals (finite and alephs). The series is determined by the function c, (6) being simply a suggestive representation of how combinatorial series are to be manipulated. With (6) we associate a function f_Γ: $X^k\Gamma \to \Gamma$ in much the same way as we extend a combinatorial function. The principal results of [5] are that the following assertions hold in a relativized sense for $\mathfrak{W}^+$.

(7) *$\precsim$ is a dense ordering of $\boldsymbol{K}$, without first but with last element t, and $M^+ =$ the finite subsets of $\boldsymbol{K}$ which contain t.*

(8) *For any cardinal $\mathfrak{m}$ there is an $A \in M^+$ and an $|A|$-ary combinatorial series f such that $\mathfrak{m} = f_\Gamma(\mathfrak{k}^A)$.*

(9) *If $A \in M^+$ and f, g are a pair of $|A|$-ary combinatorial series, then $f_\Gamma(\mathfrak{k}^A) = g_\Gamma(\mathfrak{k}^A)$ if and only if there is an $s \in X^{|A|}\mathcal{E}$ such that $f(u+s) = g(u+s)$.*

Here $f(u+s)$ means the formal composition $f(u_0+s_0, \ldots, u_{|A|-1}+s_{|A|-1})$ as defined in [5]. In order to apply (8) and (9) in the present context we make a number of observations about combinatorial series. These are merely stated, but could be easily supplied by any reader familiar with [4] and [5]. First, if f is a combinatorial series whose coefficients $c(i) \in \mathcal{E}$, then f can be uniquely associated with a combinatorial function. In this case the extension methods of [4] and [5] yield the same function f_Γ: $X^k\Gamma \to \Gamma$, and the formal composition of series agrees with the ordinary composition of functions. The last phrase of (9) could then be replaced by 'there is an $s \in X^{|A|}\mathcal{E}$ such that $f(x) = g(x)$ for all $x \in X^{|A|}\mathcal{E}$ with $x \geqslant s$'. Second, $f_\Gamma(\mathfrak{k}^A) \in \Delta$ implies that no $c(i)$ is an aleph. Hence if in (8) we are only interested in representing $\mathfrak{m} \in \Delta$, then the combinatorial series mentioned in (8) could be replaced by a combinatorial function. Third, $\mathfrak{k}^A \in X^{|A|}\Delta$ in the sense of the model (cf. [4], p. 243) and consequently the extension f_Γ in (8) may be restricted to f_Δ. Thus, we have in the sense of $\mathfrak{W}^+$

(10) *For any cardinal $\mathfrak{m} \in \Delta$ there is an $A \in M^+$ and an $|A|$-ary combinatorial function f such that $\mathfrak{m} = f_\Delta(\mathfrak{k}^A)$.*

(11) *If $A \in M^+$ and f, g are a pair of $|A|$-ary combinatorial functions, then $f_\Delta(\mathfrak{k}^A) = g_\Delta(\mathfrak{k}^A)$ if and only if there is an $s \in X^{|A|}\mathcal{E}$ such that $f(x) = g(x)$ for all $x \in X^{|A|}\mathcal{E}$ with $x \geqslant s$.*

Next let us see how the set A is chosen in (10). If we examine the proof of [5], theorem 4, we see that A can be any support of some representative set in the cardinal $\mathfrak{m}$. Hence if we wish to represent two cardinals simultaneously, a single A will suffice. Now suppose that $\mathfrak{m} \in \Delta^*$. Choose $\mathfrak{m}_0, \mathfrak{m}_1 \in \Delta$ such that $\mathfrak{m} = \mathfrak{m}_0 - \mathfrak{m}_1$ and represent them by $\mathfrak{m}_0 = f_\Delta^0(\mathfrak{k}^A)$, $\mathfrak{m}_1 = f_\Delta^1(\mathfrak{k}^A)$ (all in the sense of $\mathfrak{W}^+$) in order to get $\mathfrak{m} = f_\Delta^0(\mathfrak{k}^A) - f_\Delta^1(\mathfrak{k}^A)$. If $f = f^0 - f^1$ is the difference function, then by lemma 1 we have $\mathfrak{m} = f_\Delta(\mathfrak{k}^A)$. Hence in $\mathfrak{W}^+$

(12) *If $\mathfrak{m} \in \Delta^*$, then there is an $A \in M^+$ and f: $X^{|A|}\mathcal{E} \to \mathcal{E}^*$ such that $\mathfrak{m} = f_\Delta(\mathfrak{k}^A)$.*

Now (11) may be brought in line with (12) by observing that if f, g: $X^{|A|}\mathcal{E} \to \mathcal{E}^*$, then $f_\Delta(\mathfrak{k}^A) = g_\Delta(\mathfrak{k}^A)$ if and only if $f_\Delta^+(\mathfrak{k}^A) + g_\Delta^-(\mathfrak{k}^A) = f_\Delta^-(\mathfrak{k}^A) + g_\Delta^+(\mathfrak{k}^A)$, where the $+$ and $-$ is that of (3), and then applying (11) to show that $\mathfrak{W}^+$ satisfies

(13) *If $A \in M^+$ and f, g: $X^{|A|}\mathcal{E} \to \mathcal{E}^*$, then $f_\Delta(\mathfrak{k}^A) = g_\Delta(\mathfrak{k}^A)$ if and only if there is an $s \in X^{|A|}\mathcal{E}$ such that $f(x) = g(x)$ for all $x \in X^{|A|}\mathcal{E}$ with $x \geqslant s$.*

Finally in order to guarantee the non-triviality of (12) and (13), we explicitly state the fact that $\mathfrak{W}^+$ satisfies

(14) $$\textit{If } A \in M^+ \textit{ then } \mathfrak{k}^A \in X^{|A|}\Delta .$$

Let $\mathfrak{S}_1^0(\Delta^*)$ be the theory which is obtained from $\mathfrak{S}^0$ by adding a binary predicate constant $\precsim$, and taking (7), (12), (13), and (14) as additional axioms. Since these axioms all hold in $\mathfrak{W}^+$, our informal discussion demonstrates.

METATHEOREM 3. *If $\mathfrak{S}$ is consistent, then so is $\mathfrak{S}_1^0(\Delta^*)$.*

Our next job is to try and simplify the tangle of representations given by (12) and (13). It turns out that the structure of Δ^* in $\mathfrak{S}_1^0(\Delta^*)$ can best be described in terms of a direct limit of reduced powers. Although at first glance the construction appears to be unmotivated, it quickly becomes apparent that we are defining an algebraic system which is isomorphic to Δ^* as given by (12) and (13). Take $\mathfrak{S}$ and (7) as the theory underlying the following discussion.

For $A \in M^+$ let $P^{|A|} = \{f: f: X^{|A|}\mathcal{E} \to \mathcal{E}^*\}$ and let $\mathcal{F}^{|A|}$ be the set of all $R \subseteq X^{|A|}\mathcal{E}$ such that for some element $s \in X^{|A|}\mathcal{E}$, $x \in R$ for every $x \in X^{|A|}\mathcal{E}$ with $x \geqslant s$. Members of $\mathcal{F}^{|A|}$ are called *cofinite sets* (generalizing the usual one-dimensional nomenclature), and a property which holds for all elements of some $R \in \mathfrak{F}^{|A|}$ is said to *eventually hold*. $\mathfrak{F}^{|A|}$ is a filter of subsets of $X^{|A|}\mathcal{E}$ and therefore we may form the reduced power $P^{|A|}/\mathfrak{F}^{|A|}$ whose members are the equivalence classes $f/\mathfrak{F}^{|A|}$ for $f \in P^{|A|}$. In order to keep these powers distinct for different A with the same $|A|$ put $f^A = \langle f/\mathfrak{F}^{|A|}, A\rangle$ and let $\mathfrak{I}^A = \{f^A\colon f \in P^{|A|}\}$. In general we will use the letters 'x', 'y', and 'z' to denote elements in $P^{|A|}$. If $x = \langle x_0, \ldots, x_{k-1}\rangle \in X^k P^{|A|}$, let $x^A = \langle x_0^A, \ldots, x_{k-1}^A\rangle \in X^k\mathfrak{I}^A$. Functions $f\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ will be extended to functions $f_{\mathfrak{I}^A}\colon X^k\mathfrak{I}^A \to \mathfrak{I}^A$ in the following way. If $x \in X^k P^{|A|}$ put $y^A = f_{\mathfrak{I}^A}(x^A)$ where y is

(15) $$y = f \circ (x_0, \ldots, x_{k-1})\,.$$

It is not difficult to show that y^A is independent of the representative x of x^A which justifies the definition of $f_{\mathfrak{I}^A}$.

Let $A, B \in M^+$ with $A \subseteq B$. If $A = \{a(0), \ldots, a(|A|-1)\}$ with $a(0) \lneqq \ldots \lneqq a(|A|-1)$, $B = \{b(0), \ldots, b(|B|-1)\}$ with $b(0) \lneqq \ldots \lneqq b(|B|-1)$, define a function u_A^B by $a(i) = b(u_A^B(i))$ for each $i < |A|$. $u_A^B\colon |A| \to |B|$ and gives us the index, in the ordering for B, of any element in A. With u_A^B we associate a function $H_B^A\colon X^{|B|}\Gamma \to X^{|A|}\Gamma$ in the following way. If $\mathfrak{x} = \langle \mathfrak{x}_0, \ldots, \mathfrak{x}_{|B|-1}\rangle \in X^{|B|}\Gamma$, let $H_B^A(\mathfrak{x}) = \langle \mathfrak{y}_0, \ldots, \mathfrak{y}_{|A|-1}\rangle$ where for each $j < |A|-1$, and $\sum$ standing for summation

(16) $$\mathfrak{y}_0 = u_A^B(0) + \sum\{\mathfrak{x}_i\colon\ i \leqslant u_A^B(0)\}\,,$$
$$\mathfrak{y}_{j+1} = u_A^B(j+1) - u_A^B(j) - 1 + \sum\{\mathfrak{x}_i\colon\ u_A^B(j) < i \leqslant u_A^B(j+1)\}.$$

H has been defined in such a way that we have the important

(17) $$H_B^A(\mathfrak{k}^B) = \mathfrak{k}^A.$$

With H we define a function $\pi_A^B\colon P^{|A|} \to P^{|B|}$ by requiring that for $f \in P^{|A|}$, $\pi_A^B(f)$ is that function $g \in P^{|B|}$ such that

(18) $$g(x) = f(H_B^A(x)) \quad \text{for} \quad x \in X^{|B|}\mathcal{E}\,.$$

It is not difficult to see that if $f, g \in P^{|A|}$ determine the same equivalence class $f^A = g^A$, then $\pi_A^B(f), \pi_A^B(g)$ determine the same equivalence class in $P^{|B|}$. Consequently we may think of $\pi_A^B\colon \mathfrak{I}^A \to \mathfrak{I}^B$ as given by

(19) $$\pi_A^B(x^A) = \big(\pi_A^B(x)\big)^B \quad \text{for} \quad x \in P^{|A|}\,.$$

Examination of (19) leads us to conclude that π_A^B is a one-one mapping of $\mathfrak{I}^A$ into $\mathfrak{I}^B$. Now suppose that $f\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ and $x \in X^k P^{|A|}$. Then the chain of equalities

$$\begin{aligned}\pi_A^B\big(f_{\mathfrak{I}^A}(x_0^A, \ldots, x_{k-1}^A)\big) &= \pi_A^B\big((f \circ (x_0, \ldots, x_{k-1}))^A\big)\\ &= \big(\pi_A^B(f \circ (x_0, \ldots, x_{k-1}))\big)^B = \big(f \circ (\pi_A^B(x_0), \ldots, \pi_A^B(x_{k-1}))\big)^B\\ &= f_{\mathfrak{I}^B}\big((\pi_A^B(x_0))^B, \ldots, (\pi_A^B(x_{k-1}))^B\big) = f_{\mathfrak{I}^B}\big(\pi_A^B(x_0^A), \ldots, \pi_A^B(x_{k-1}^A)\big)\end{aligned}$$

demonstrates that π_A^B is actually an algebraic embedding of $\mathfrak{I}^A$ into $\mathfrak{I}^B$ with respect to extensions of functions f.

Now let $P^{|K|} = \bigcup\{\mathfrak{I}^A\colon A \in M^+\}$. Define an equivalence relation $\approx$ on $P^{|K|}$ by putting $x^A \approx y^B$ if and only if there is a $C \in M^+$ with $A \cup B \subseteq C$ such that $\pi_A^C(x^A) = \pi_B^C(y^B)$, and let $[x^A]$ be the equivalence class determined by x^A. Finally let $\mathfrak{I}^K = \{[x]\colon x \in P^{|K|}\}$. The reader will recognize $\mathfrak{I}^K$ as a direct limit of the systems $\mathfrak{I}^A$ with respect to M^+ directed by inclusion. Functions $f\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ will be extended to functions $f_{\mathfrak{I}^K}\colon X^k\mathfrak{I}^K \to \mathfrak{I}^K$ in the following way. If $y \in X^k\mathfrak{I}^K$, then for some $A \in M^+$ and $x \in X^k\mathfrak{I}^A$ we have $y = \langle [x_0], \ldots, [x_{k-1}]\rangle = [x]$. If this is the case define

(20) $$f_{\mathfrak{I}^K}(y) = [f_{\mathfrak{I}^A}(x)]\,.$$

Since π_A^B is an embedding, the expression on the right in (20) is independent of the particular A used to represent y. Hence $f_{\mathfrak{I}^K}$ is well defined by (20). For $x \in \mathfrak{I}^A$ let $\pi_A(x) = [x]$. Again by the properties of π_A^B the function $\pi_A\colon \mathfrak{I}^A \to \mathfrak{I}^K$ is an algebraic embedding. This completes the construction of our fundamental algebraic object $\mathfrak{I}^K$.

Let f $(= f_i^k, k > 0)$ be a double sequence of functions such that

(21) $$f_i^k\colon\ X^k\mathcal{E}^* \to \mathcal{E}^*$$

and let $f_{\Delta^*}, f_{\mathfrak{I}^K}$ denote the corresponding sequences of $f_{i\Delta^*}^k, f_{i\mathfrak{I}^K}^k$, respectively. If we think of $\langle \Delta^*, f_{\Delta^*}\rangle$ and $\langle \mathfrak{I}^K, f_{\mathfrak{I}^K}\rangle$ as algebraic systems, then we have

$\mathfrak{S}_1^0(\Delta^*)$-LEMMA 5. *$\langle \Delta^*, f_{\Delta^*}\rangle$ is isomorphic to $\langle \mathfrak{I}^K, f_{\mathfrak{I}^K}\rangle$.*

Proof. Define a function $\theta\colon \mathfrak{I}^K \to \Delta^*$ by

(22) $$\theta([x^A]) = x_A(\mathfrak{k}^A) \quad \text{for} \quad x \in P^{|A|}\,.$$

We justify (22) as follows. If $y \in P^{|A|}$ is another element such that $x^A = y^A$, then as functions x and y are eventually equal, and consequently by (13), $x_A(\mathfrak{k}^A) = y_A(\mathfrak{k}^A)$. Now suppose that $y \in P^{|B|}$ is another element such that $[x^A] = [y^B]$. Without loss of generality we may suppose that $A \subseteq B$ and $y = \pi_A^B(x)$. This means that $y(i) = x(H_B^A(i))$ for $i \in X^{|B|}\mathcal{E}$. Since composition generally commutes with extension, (17) implies that $y_A(\mathfrak{k}^B) = x_A(H_B^A(\mathfrak{k}^B)) = x_A(\mathfrak{k}^A)$. Thus θ is well defined. Conversely, in order to show that θ is one-one, suppose that $x_A(\mathfrak{k}^A) = y_A(\mathfrak{k}^B)$. As before we may assume that $A \subseteq B$. Then $x_A(\mathfrak{k}^A) = x_A(H_B^A(\mathfrak{k}^B)) = (\pi_B^A(x))_A(\mathfrak{k}^B)$. But by (13) this implies that $\pi_A^B(x)$ and y are eventually equal. Hence $[x^A] = [y^B]$. It follows from (12) that θ is onto Δ^*. If $g\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ and $g_{\mathfrak{I}^K}([x^A]) = [y^A]$, then $g \circ (x^0, \ldots, x^{k-1})$ is eventually equal to y. Hence by (13) and lemma 4, $g_{\Delta^*}(x_A^0(\mathfrak{k}^A), \ldots, x_A^{k-1}(\mathfrak{k}^A)) = y_A(\mathfrak{k}^A)$, and by (22), $g_{\Delta^*}(\theta([x^A])) = \theta([y^A])$. q.e.d.

By lemma 5, first order properties of Δ^* in $\mathfrak{S}_1^0(\Delta^*)$ are equivalent to first order properties of $\mathfrak{I}^K$. Consequently we extend our interpretations of sentences of language L to include the relativizations $\mathfrak{A}_{\mathfrak{J}A}$, $\mathfrak{A}_{\mathfrak{J}K}$ in the obvious way. We would like to do some model theory in $\mathfrak{S}_1^0(\Delta^*)$. Let $\boldsymbol{L}$ be a formalization of L in $\mathfrak{S}$ such that every formula $\mathfrak{A}$ of L has a name $\ulcorner\mathfrak{A}\urcorner$ in $\boldsymbol{L}$. Since all of our structures are sets in $\mathfrak{S}_1^0(\Delta^*)$, it is possible to give a definition of satisfaction for them. For any set A let $X^{(\omega)}A$ be the set of ω-tuples $\langle x_0, x_1, \ldots\rangle$ each $x_i \in A$, such that for some n, all entries x_k, $k > n$ are equal to one another. If f is as in (21) (and fixed in the following discussion), $x \in X^{(\omega)}\Delta^*$, and $\mathfrak{A}$ is a formula in $\boldsymbol{L}$, let us define a notion $\Delta^* \models \mathfrak{A}(x)$ which will intuitively mean that x satisfies $\mathfrak{A}$ in the system $\langle\Delta^*, f_{\Delta^*}\rangle$. Formally we require that if $\mathfrak{A}$ is a formula of L, and Φ asserts that $\mathfrak{f}_i^k = f_i^k$ and $\mathfrak{x}_i = x_i$ for every $\mathfrak{f}_i^k$ and $\mathfrak{x}_i$ which occurs in $\mathfrak{A}$, then we can prove in $\mathfrak{S}_1^0(\Delta^*)$ that

(23) $$\Phi \textit{ implies } \big(\mathfrak{A}_{\Delta^*} \textit{ if and only if } \Delta^* \models \ulcorner\mathfrak{A}\urcorner(x)\big)\,.$$

Define notions $\mathfrak{J}^A \models$ and $\mathfrak{J}^K \models$ which satisfy (23) in an analogous fashion. Then we have

$\mathfrak{S}_1^0(\Delta^*)$-LEMMA 6. *If $A, B \in M^+$ with $A \subseteq B$, then π_A^B is an elementary embedding of $\langle\mathfrak{J}^A, f_{\mathfrak{J}A}\rangle$ into $\langle\mathfrak{J}^B, f_{\mathfrak{J}B}\rangle$.* (Note that (7) is the only property beyond $\mathfrak{S}$ that we use.)

Proof. We divide our proof into three parts.

Part (i). $\mathcal{S}^A = \langle S^{|A|}, \emptyset, \cup, \cap, \neg\rangle$ (where $S^{|A|}$ is the power set of $X^{|A|}\mathcal{E}$ and $\neg$ is the relative complement) is a Boolean algebra which contains $\mathcal{F}^{|A|}$ as a proper filter. Hence the quotient $\mathcal{T}^A = \mathcal{S}^A/\mathcal{F}^{|A|} = \langle T^{|A|}, \emptyset, \cup, \cap, \neg\rangle$ is also a Boolean algebra which we readily see is atomless. For $x \in S^{|A|}$ let $[x]$ be the element of $T^{|A|}$ to which x belongs. Algebras $\mathcal{S}^B$ and $\mathcal{T}^B$ are defined in a similar fashion. Now $H_B^A\colon X^{|B|}\mathcal{E} \to X^{|A|}\mathcal{E}$. Hence we may define a map $I_A^B\colon S^{|A|} \to S^{|B|}$ given by $I_A^B(x) = \breve{H}_B^A(x)$ for $x \in S^{|A|}$ (where '$\smile$' means inverse). I_A^B is a Boolean homomorphism for which it is easy to see that if $x \in S^{|A|}$, $x \in \mathcal{F}^{|A|}$ if and only if $I_A^B(x) \in \mathcal{F}^{|B|}$. Consequently I_A^B induces a Boolean isomorphism $\bar{I}_A^B\colon T^{|A|} \to T^{|B|}$ given by $\bar{I}_A^B([x]) = [I_A^B(x)]$ for $x \in S^{|A|}$. Note that the isomorphism is merely into. Let $\boldsymbol{N}$ be a language suitable for the elementary theory of Boolean algebras which contains symbols for the Boolean operations as well as symbols $\boldsymbol{v}_i$ to serve as variables. For $x \in X^{(\omega)}T^{|A|}$ and Φ in $\boldsymbol{N}$ define a notion $\mathcal{T}^A \models \Phi(x)$ which intuitively means that x satisfies Φ in $\mathcal{T}^A$. By a straight elimination of quantifiers it can be shown (without choice, cf. [18]) that every atomless Boolean algebra is an elementary extension of each of its atomless Boolean subalgebras. Consequently $\bar{I}_A^B$ is an elementary embedding of $\mathcal{T}^A$ into $\mathcal{T}^B$.

Part (ii). If $x \in X^{(\omega)}P^{|A|}$, put $x^A = \langle x_0^A, x_1^A, \ldots\rangle$ and $x(i) = \langle x_0(i), x_1(i), \ldots\rangle$ for $i \in X^{|A|}\mathcal{E}$. Also, if $\mathfrak{B}$ is any formula of $\boldsymbol{L}$, define $J(\mathfrak{B}, x) = \{i \in X^{|A|}\mathcal{E}\colon \mathcal{E}^* \models \mathfrak{B}(x(i))\}$. By a slight modification of [6], theorem 3.1 we can associate with every formula $\mathfrak{A}$ of $\boldsymbol{L}$ a sequence $\langle\Phi, \mathfrak{B}_0, \ldots, \mathfrak{B}_{n-1}\rangle$ where Φ is a formula of $\boldsymbol{N}$ with at most the free variables $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{n-1}$, where $\mathfrak{B}_0, \ldots, \mathfrak{B}_{n-1}$ are formulas of $\boldsymbol{L}$ containing no free variables which do not appear in $\mathfrak{A}$, and such that for every $x \in X^{(\omega)}P^{|A|}$

(24) $$\mathfrak{J}^A \models \mathfrak{A}(x^A) \textit{ if and only if } \mathcal{T}^A \models \Phi([J(\mathfrak{B}_0, x)], \ldots, [J(\mathfrak{B}_{n-1}, x)])\,.$$

Part (iii). If $x \in X^{(\omega)}P^{|A|}$, put $\pi_A^B(x) = \langle\pi_A^B(x_0), \pi_A^B(x_1), \ldots\rangle$ and $\pi_A^B(x^A) = \langle\pi_A^B(x_0^A), \pi_A^B(x_1^A), \ldots\rangle$. Then part (ii) gives

(25) $$\mathfrak{J}^B \models \mathfrak{A}\big(\pi_A^B(x^A)\big) \textit{ if and only if } \mathcal{T}^B \models \Phi\big(\big[J\big(\mathfrak{B}_0, \pi_A^B(x)\big)\big], \ldots, \big[J\big(\mathfrak{B}_{n-1}, \pi_A^B(x)\big)\big]\big).$$

Next we claim that for any formula $\mathfrak{B}$ of $\boldsymbol{L}$ and $x \in X^{(\omega)}P^{|A|}$

(26) $$\bar{I}_A^B\big([J(\mathfrak{B}, x)]\big) = \big[J\big(\mathfrak{B}, \pi_A^B(x)\big)\big]\,.$$

For $i \in \breve{H}_B^A\big(J(\mathfrak{B}, x)\big)$ if and only if $H_B^A(i) \in J(\mathfrak{B}, x)$ if and only if $\mathcal{E}^* \models \mathfrak{B}\big(x(H_B^A(i))\big)$ if and only if $\mathcal{E}^* \models \mathfrak{B}\big(\big(\pi_A^B(x)\big)(i)\big)$ if and only if $i \in J(\mathfrak{B}, \pi_A^B(x))$. Then (26) follows by taking quotients. Our lemma follows from (24), (25) and (26) and the fact that $\bar{I}_A^B$ is an elementary embedding. q.e.d.

$\mathfrak{S}_1^0(\Delta^*)$-LEMMA 7. *If $A \in M^+$, then π_A is an elementary embedding of $\langle\mathfrak{J}^A, f_{\mathfrak{J}A}\rangle$ into $\langle\mathfrak{J}^K, f_{\mathfrak{J}K}\rangle$.* (Note that (7) is the only property beyond $\mathfrak{S}$ that we use.)

Proof. Since each π_A^B is an elementary embedding, the result follows by [10], theorem 4.1. q.e.d.

If t is the terminal element in the ordering $\precsim$, then $\{t\} \in M^+$ and $\mathfrak{J}^{\{t\}}$ is simply the reduced power of unary functions modulo the cofinite sets, indexed by t. Let $\mathfrak{J}$ be the isomorphic system obtained from $\mathfrak{J}^{\{t\}}$ by deleting the index t.

$\mathfrak{S}_1^0(\Delta^*)$-LEMMA 8. *The systems $\langle\Delta^*, f_{\Delta^*}\rangle$ and $\langle\mathfrak{J}, f_{\mathfrak{J}}\rangle$ are elementarily equivalent.*

Proof. The result follows by lemmas 5, 6, and 7. q.e.d.

For any sentence $\mathfrak{A}$ of language L

$\mathfrak{S}_1^0(\Delta^*)$-THEOREM 4. *$\mathfrak{A}_{\Delta^*}$ if and only if $\mathfrak{A}_{\mathfrak{J}}$.*

Proof. The result follows by (23) and lemma 8. q.e.d.

Let $\mathfrak{A}$ be a prenex conjunctive normal form sentence of L. Remember that $\mathfrak{A}$ was called *universal* if its prefix consisted only of universal quantifiers. Call $\mathfrak{A}$ a *positive sentence* if it only contains unnegated atomic formula and call $\mathfrak{A}$ a *disjunctive sentence* if its matrix consists of a single conjunct. Finally call $\mathfrak{A}$ a *Bing sentence* if it is either universal, positive, or disjunctive.

If $\mathfrak{A}$ is a Bing sentence, then

$\mathfrak{S}_1^0(\Delta^*)$-COROLLARY 3. *$\mathfrak{A}_{\Delta^*}$ implies $\mathfrak{Sft}(\mathfrak{A})_{\mathcal{E}^*}$.*

Proof. In [1] it is shown that every arithmetical class (in the sense of Tarski) which is determined by a Bing sentence but by no Horn sentence is not closed under direct products. An easy transcription of that proof shows that if $\mathfrak{A}$ is a Bing sentence and $\mathfrak{A}_{\mathcal{F}}$, then $\mathfrak{A}$ must be equivalent to one of its Horn reductions, in $\mathcal{E}^*$, i.e., $\mathfrak{S}\mathfrak{k}\mathfrak{t}(\mathfrak{A})_{\mathcal{E}^*}$. Hence our corollary follows by theorem 4. q.e.d.

This is the converse to corollary 2 that was mentioned at the end of the last section. Its hypothesis is sharp, for in the next section we will give an example of a non-Bing sentence $\mathfrak{A}$ such that in $\mathfrak{S}_1^0(\Delta^*)$ we have $\mathfrak{A}_{\Delta^*}$ and $\sim\mathfrak{S}\mathfrak{k}\mathfrak{t}(\mathfrak{A})_{\mathcal{E}^*}$. Consequently theorem 4 is by far the better result.

We conclude this section by applying an ingenious lifting method (of Kreisel, and appearing in [10], p. 235 to our problem. A function $f\colon X^k\mathcal{E}^*\to\mathcal{E}^*$ is called *absolutely definable* if it is definable, and its definition is absolute (cf. [7], p. 42) with respect to the model $\mathfrak{L}$ of [7] and the model $\mathfrak{W}^+$ of [11]. Clearly such functions are constructible. Let '$\vdash$' denote proveability. Then we have

METATHEOREM 5. *If $\mathfrak{A}$ is a sentence of language L which is provided with an interpretation by specifying definite absolutely definable functions for the function constants which appear in $\mathfrak{A}$, then $\mathfrak{S}^0\vdash\mathfrak{A}_{\Delta^*}$ implies $\mathfrak{S}^0\vdash\mathfrak{A}_{\mathcal{F}}$.*

Proof. We divide our proof into two parts.

Part (i). Let $f=\langle f_0,\dots,f_{n-1}\rangle$ be a sequence of absolutely definable functions such that $f_i\colon X^{k_i}\mathcal{E}^*\to\mathcal{E}^*$ for some k_i. Let Z be an arithmetical language which contains constants for plus, times, and for the functions which appear in f. $g\colon\mathcal{E}\to\mathcal{E}^*$ is arithmetical in f if there is a formula Φ of $\boldsymbol{Z}$ with exactly two free variables such that $g(x)=y$ if and only if $\mathcal{E}^*\models\Phi(\langle x,y\rangle)$ for every $x\in\mathcal{E}$ and $y\in\mathcal{E}^*$. It is well known from the literature that there is a hyperarithmetical predicate H such that $\mathcal{E}^*\models\Phi(\langle x,y\rangle)$ if and only if $H(f,\ulcorner\Phi\urcorner,x,y)$. Since f is constructible, the latter formula (for fixed f) is absolute with respect to $\mathfrak{L}$ (cf. [16]), and trivially with respect to $\mathfrak{W}^+$. Consequently it is clear that the set of all functions $g\colon\mathcal{E}\to\mathcal{E}^*$ which are arithmetic in the functions of f is an absolutely definable (set). Let $\mathcal{F}_a$ be the quasi-reduced power consisting of all such functions g, reduced modulo the cofinite subsets of $\mathcal{E}$. Let $f_{\mathcal{F}_a}$ denote the sequence of extensions to $\mathcal{F}_a$ of functions in f. By the preceding remarks the system $\langle\mathcal{F}_a,f_{\mathcal{F}_a}\rangle$ is absolute with respect to the models $\mathfrak{L}$ and $\mathfrak{W}^+$, and by [12], p. 114, or [6], $\langle\mathcal{F}_a,f_{\mathcal{F}_a}\rangle$ is elementarily equivalent to $\langle\mathcal{F},f_{\mathcal{F}}\rangle$.

Part (ii). Suppose $\mathfrak{S}^0\vdash\mathfrak{A}_{\Delta^*}$ where f of part (i) is the list of absolutely definable functions denoted by the function constants of $\mathfrak{A}$. In $\mathfrak{S}^0$ build a model $\mathfrak{L}$ by the method of [7] which contains no urelemente. $\mathfrak{L}$ satisfies the axiom of choice. Then build a model $\mathfrak{W}_{\mathfrak{L}}^+$ in $\mathfrak{L}$ by the method of [11] where urelemente are introduced as the sets $\Delta_k=\omega-\{k\}$, $k>0$, and $\emptyset$ is introduced as the set $\Delta_0=\omega-\{0\}$. The integers are not absolute with respect to this construction, however those of $\mathfrak{W}_{\mathfrak{L}}^+$ and $\mathfrak{L}$ are isomorphic and consequently may be identified. $\mathfrak{W}_{\mathfrak{L}}^+$ satisfies the axioms of $\mathfrak{S}_1^0(\Delta^*)$, moreover, it satisfies these axioms in the strong sense that their relativizations to $\mathfrak{W}_{\mathfrak{L}}^+$ are actually theorems of $\mathfrak{S}^0$. For any model $\mathcal{M}$ and set theoretic sentence φ, let $\mathfrak{Rel}(\varphi,\mathcal{M})$ be the relativization of φ to $\mathcal{M}$. Then by theorem 4 and by part (i) we have the following sequence of implications:

$$\mathfrak{S}^0\vdash\mathfrak{A}_{\Delta^*}\ \text{implies}\ \mathfrak{S}^0\vdash\mathfrak{Rel}(\mathfrak{A}_{\Delta^*},\mathfrak{W}_{\mathfrak{L}}^+)\ \text{implies}\ \mathfrak{S}^0\vdash\mathfrak{Rel}(\mathfrak{A}_{\mathcal{F}},\mathfrak{W}_{\mathfrak{L}}^+)\ \text{implies}\ \mathfrak{S}^0\vdash\mathfrak{Rel}(\mathfrak{A}_{\mathcal{F}_a},\mathfrak{W}_{\mathfrak{L}}^+)\ \text{implies}\ \mathfrak{S}^0\vdash\mathfrak{Rel}(\mathfrak{A}_{\mathcal{F}_a},\mathfrak{L})\ \text{implies}\ \mathfrak{S}^0\vdash\mathfrak{A}_{\mathcal{F}_a}\ \text{implies}\ \mathfrak{S}^0\vdash\mathfrak{A}_{\mathcal{F}}.\ \text{q.e.d.}$$

METACOROLLARY 4. *If $\mathfrak{A}$ is a Bing sentence, interpreted as in theorem 5, then $\mathfrak{S}^0\vdash\mathfrak{A}_{\Delta^*}$ implies $\mathfrak{S}^0\vdash\mathfrak{S}\mathfrak{k}\mathfrak{t}(\mathfrak{A})_{\mathcal{E}^*}$.*

METATHEOREM 6. *If $\mathfrak{A}$ is a sentence of language L which is provided with an interpretation by specifying definite absolutely definable functions for the function constants which appear in $\mathfrak{A}$, then $\mathfrak{S}^0\vdash\mathfrak{A}_{\Delta^*}$ implies $\mathfrak{S}^0\vdash\mathfrak{A}_{\mathcal{E}^*}$.*

Proof. We use a simpler version of the lifting method that was used in the proof of theorem 5. In $\mathfrak{S}^0$ build a model $\mathfrak{L}$ by the method of [7]. In $\mathfrak{L}$ we have $\Delta^*=\mathcal{E}^*$, consequently:

$$\mathfrak{S}^0\vdash\mathfrak{A}_{\Delta^*}\ \text{implies}\ \mathfrak{S}^0\vdash\mathfrak{Rel}(\mathfrak{A}_{\Delta^*},\mathfrak{L})\ \text{implies}\ \mathfrak{S}^0\vdash\mathfrak{Rel}(\mathfrak{A}_{\mathcal{E}^*},\mathfrak{L})\ \text{implies}\ \mathfrak{S}^0\vdash\mathfrak{A}_{\mathcal{E}^*}.$$

The last implication follows from the absoluteness of the functions in $\mathfrak{A}$. q.e.d.

For theorems 5 and 6 to be non-trivial we must show that some interesting class of functions is absolutely definable. In [16] it is shown that every $\Sigma_2^1\cup\Pi_2^1$ function (functions whose diagram can be expressed in either two function quantifier form) is absolute with respect to $\mathfrak{L}$. It is immediate that such functions are also absolute with respect to $\mathfrak{W}^+$. Hence we may use $\Sigma_2^1\cup\Pi_2^1$ functions in theorems 5 and 6 instead of absolutely definable ones. Incidentally, it has recently been shown in [17] that under the hypothesis of a Ramsey cardinal, there exist non-constructible Δ_3^1 sets. It would be interesting to know whether versions of theorems 5 and 6 hold in the Δ_3^1 case. Certainly, theorem 6 looks as though it ought to hold under much less restrictive hypotheses.

Actually our applications do not require the full force of the last two theorems. Since we are interested in non-proveability, it is not necessary to get sentences to be theorems of $\mathfrak{S}^0$. Various consistent extensions of $\mathfrak{S}^0$ will do. We state a single instance of such a theorem. Note that the word 'definable' appears below (cf. 'absolutely definable' in the hypothesis of theorem 5).

METATHEOREM 7. *If $\mathfrak{A}$ is a sentence of language L which is provided with an interpretation by specifying definite definable functions for the function constants which appear in $\mathfrak{A}$, then, $\mathfrak{S}$ consistent and $\mathfrak{S}^0\vdash\sim\mathfrak{A}_{\mathcal{F}}$, imply non $\mathfrak{S}^0\vdash\mathfrak{A}_{\Delta^*}$.*

Proof. For otherwise $\mathfrak{S}^0$ has a consistent extension $\mathfrak{S}^0_1(\Delta^*)$ in which we can prove a statement contradicting the corresponding statement given by theorem 4. q.e.d.

METACOROLLARY 5. *If $\mathfrak{A}$ is a Bing sentence, interpreted as in theorem* 7, *then,* $\mathfrak{S}$ *consistent and* $\mathfrak{S}^0 \vdash \sim \mathfrak{S}\mathfrak{f}\mathfrak{t}(\mathfrak{A})_{\mathcal{E}^*}$, *imply non* $\mathfrak{S}^0 \vdash \mathfrak{A}_{\Delta^*}$.

METATHEOREM 8. *If $\mathfrak{A}$ is a sentence, interpreted as in theorem* 7, *then* $\mathfrak{S}$ *consistent and* $\mathfrak{S}^0 \vdash \sim \mathfrak{A}_{\mathcal{E}^*}$, *imply non* $\mathfrak{S}^0 \vdash \mathfrak{A}_{\Delta^*}$.

4. Applications. In this section it will be convenient to have a method by which relations $R \subseteq X^k\mathcal{E}^*$ can be extended to relations $R_{\Delta^*} \subseteq X^k\Delta^*$. A function $r\colon X^k\mathcal{E}^* \to \{0, 1\}$ is called the *characteristic function* of R if $x \in R$ if and only if $x(r) = 0$ for every $x \in X^k\mathcal{E}^*$. Then we have

DEFINITION 4: We extend every relation $R \subseteq X^k\mathcal{E}^*$ to a relation $R_{\Delta^*} \subseteq X^k\Delta^*$ by putting $R_{\Delta^*} = \{\mathfrak{x} \in X^k\Delta^*\colon r_{\Delta^*}(\mathfrak{x}) = 0\}$ where r is the characteristic function of R.

According to this definition, statements $\mathfrak{x} \in R_{\Delta^*}$ can be replaced by equalities $r_{\Delta^*}(\mathfrak{x}) = 0$, and the 0 can be replaced by $f_{\Delta^*}(\mathfrak{x})$ where f is a function which is identically zero. Consequently statements involving relations can be transcribed into language L. Let us enlarge L to a language LR which contains an additional list of constants $\mathfrak{R}^k_i$, $0 < k$, where under interpretation each $\mathfrak{R}^k_i \subseteq X^k\mathcal{E}^*$. Define the various notions of LR exactly as for L except that we include expressions $\mathfrak{R}^k_i(\tau_0, \ldots, \tau_{k-1})$, where $\tau_0, \ldots, \tau_{k-1}$ are terms of L, as additional atomic formula. Interpret $\mathfrak{R}^k_i(\tau_0, \ldots, \tau_{k-1})$ to mean $\langle\tau_0, \ldots, \tau_{k-1}\rangle \in \mathfrak{R}^k_i$ so as to avoid an $\in$ in LR. Define $\mathfrak{A}_{\mathcal{E}^*}$ and $\mathfrak{A}_{\Delta^*}$ for LR exactly as for L except in the latter case add the clause 'and replacing each relation constant $\mathfrak{R}$ which appears in $\mathfrak{A}$ by $\mathfrak{R}_{\Delta^*}$'. According to the remarks following definition 4 all results of the preceding two sections apply to LR as well as to L.

We are going to discuss Δ^* in an extremely informal way, using $\mathfrak{S}^0$ as our underlying theory. LR will be used in a heuristic rather than a formal sense. Most of this material has already appeared in [3], section 3 for the analogous Λ^*, hence our presentation will take a brief form.

The fact that a function $f\colon X^k\mathcal{E}^* \to \mathcal{E}^*$ is one-one can be expressed by a Horn sentence. Hence f_{Δ^*} is one-one as well. The fact that f maps onto $\mathcal{E}^*$ can also be expressed by a Horn sentence. Hence, in this case, f_{Δ^*} maps onto Δ^*. Let $F = \{\langle x, y\rangle \in X^{k+1}\mathcal{E}^*\colon f(x) = y\}$ be the $k+1$-ary relation which diagrams f. Statements saying that F diagrams some function can be expressed as Horn sentences in LR. Hence F_{Δ^*} is the diagram of a function. Another Horn sentence says that F diagrams f. Hence F_{Δ^*} is the diagram of f_{Δ^*}. This method of extending functions by their diagrams suggests a way of extending functions whose domain is a proper subset of $X^k\mathcal{E}^*$. Let $A \subseteq X^k\mathcal{E}^*$, $B \subseteq \mathcal{E}^*$, and f a function mapping A onto B. If F diagrams f, then by Horn sentences, F_{Δ^*} diagrams a function mapping A_{Δ^*} onto B_{Δ^*}. Call this function f_{Δ^*}. By our previous remarks, if $A = X^k\mathcal{E}^*$ then this new f_{Δ^*} is identical with the one given in definition 3.

We would like to characterize certain algebraic objects in Δ^*. An idempotent is an element $\mathfrak{x} \in \Delta^*$ such that $\mathfrak{x}^2 = \mathfrak{x}$. Since $x^2 = x$ if and only if $x \in \{0, 1\}$ for $x \in \mathcal{E}^*$, a corresponding result holds in Δ^*, i.e., $\{0, 1\}_{\Delta^*}$ is the set of idempotents of Δ^*. $\{0, 1\}$ is a Boolean algebra in $\mathcal{E}^*$ under the operations $x \wedge y = xy$, $x \vee y = x + y - xy$, and $\neg x = 1 - x$. Since the axioms of a Boolean algebra with respect to these operations can be expressed by identities, $\{0, 1\}_{\Delta^*}$ is also a Boolean algebra with respect to the extensions of $\wedge$, $\vee$, and $\neg$ to Δ^*. By lemmas 3 and 4 these extensions read in Δ^* exactly as they do in $\mathcal{E}^*$ except that the ring operations in $\mathcal{E}^*$ must be replaced by their extensions to Δ^*. Let $\mathcal{B}_{\mathcal{E}^*}$, $\mathcal{B}_{\Delta^*}$ be the Boolean algebra of idempotents in $\mathcal{E}^*$, Δ^* respectively.

A nilpotent is an element $\mathfrak{x} \in \Delta^*$ such that for some integer n, $\mathfrak{x}^n = 0$. Now $x^n = 0$ implies $x = 0$ for $x \in \mathcal{E}^*$ and therefore a corresponding result holds in Δ^*, i.e., 0 is the only nilpotent in Δ^*. If the axiom of choice (in the form of Zorn's lemma) were available we could then show that $\{0\}$ is the intersection of the minimal prime ideals in Δ^*. Since it is not, we must do some extra work to obtain the result. A ring R is said to have enough idempotents if there is a function $e\colon R \to R$ such that for all $x, y \in R$ we have

$$(27) \qquad e(0) = 0\,, \quad e(xy) = e(x)e(y)\,, \quad e\big(e(x)\big) = e(x)\,, \quad \text{and} \quad e(x)x = x.$$

It is easy to show from these conditions that $e(x)^2 = e(x)$, that if $x^2 = x$, then $e(x) = x$, and that the function e is unique. By [15], corollary 2.3 the function e gives a one-one correspondence between the minimal prime ideals of R and the prime ideals in the Boolean algebra of idempotents of R. This follows without the axiom of choice. We apply these results to Δ^* by defining a function $e\colon \mathcal{E}^* \to \mathcal{E}^*$ with $e(0) = 0$ and $e(x) = 1$ for $x \neq 0$. e clearly satisfies (27) and since identities extend from $\mathcal{E}^*$ to Δ^*, the function e_{Δ^*} will also satisfy (27). Thus Δ^* has enough idempotents. It is shown in [8] that $\mathfrak{W}^+$ satisfies the prime ideal theorem. If we are willing to enlarge $\mathfrak{S}^0_1(\Delta^*)$ so as to include this fact then $\mathcal{B}_{\Delta^*}$ contains at least one prime ideal. Simple algebra then shows that for every $\mathfrak{x} \in \mathcal{B}_{\Delta^*}$, $\mathfrak{x} \neq 0$, there is a prime ideal in $\mathcal{B}_{\Delta^*}$ which excludes $\mathfrak{x}$. Thus the intersection of all prime ideals in $\mathcal{B}_{\Delta^*}$ is $\{0\}$. Application of the function e shows that Δ^* contains a minimal prime ideal and that the intersection of the minimal prime ideals in Δ^* is $\{0\}$. Consequently Δ^* is a subdirect product of integral domains. In [12] it is shown that $\Lambda^*(A)$ modulo a minimal prime ideal is a model of all the true (in language L) statements of the arithmetic of $\mathcal{E}^*$. Since the only fact used to obtain this result is an isolic version of our corollary 2, a similar result holds for Δ^* in the theory $\mathfrak{S}^0$. Combining this with our previous result we see that Δ^* is a subdirect product of (non-standard) models of the arithmetic of $\mathcal{E}^*$ in the theory consisting of $\mathfrak{S}_1^0(\Delta^*)+\{\text{prime ideal theorem}\}$.

A unit is an element $\mathfrak{x}\in\Delta^*$ such that for some $\mathfrak{y}\in\Delta^*$, $\mathfrak{x}\mathfrak{y}=1$. Since $xy=1$ implies $x^2=1$ for all $x,y\in\mathcal{E}^*$, a similar result holds in Δ^*. Thus the units of Δ^* are simply the elements whose square is 1. Now $x^2=1$ if and only if $x\in\{1,-1\}$ for $x\in\mathcal{E}^*$. Hence a corresponding result holds in Δ^*, i.e., $\{1,-1\}_{\Delta^*}$ is the set of all units of Δ^*.

A prime is an element $\mathfrak{x}\in\Delta^*$, $\mathfrak{x}\neq 0$, and $\mathfrak{x}$ not a unit such that for every $\mathfrak{y},\mathfrak{z}\in\Delta^*$. $\mathfrak{x}=\mathfrak{y}\mathfrak{z}$ implies that either $\mathfrak{y}$ is a unit or $\mathfrak{z}$ is a unit. A divisor of zero is an element $\mathfrak{x}\in\Delta^*$ such that for some $\mathfrak{y}\in\Delta^*$, $\mathfrak{y}\neq 0$ we have $\mathfrak{x}\mathfrak{y}=0$.

$\mathfrak{S}_1^0(\Delta^*)$-LEMMA 9. (i) *$\Delta^*$ contains non-zero divisors of zero.* (ii) *Δ^* contains no primes.* (iii) *$\mathfrak{B}_{\Delta^*}$ is an atomless Boolean algebra,* (iv) *Δ^* contains non-trivial square roots of unity.*

Proof. Use theorem 4 and the fact that each of these statements holds in $\mathfrak{T}$. q.e.d.

As a consequence of this lemma we can show that there exists a sentence $\mathfrak{A}$ of L provided with an interpretation by specifying definite definable functions for the function constants which appear in $\mathfrak{A}$ such that

$\mathfrak{S}_1^0(\Delta^*)$-THEOREM 9. *$\mathfrak{A}_{\Delta^*}$ and $\sim\mathfrak{Sft}(\mathfrak{A})_{\mathcal{E}^*}$.*

Proof. Let $\mathfrak{A}$ be a sentence of L which says that the Boolean algebra of idempotents is either atomless or contains exactly two elements. The Boolean operations are certainly definable. By the previous lemma $\mathfrak{A}_{\Delta^*}$. In order to show $\sim\mathfrak{Sft}(\mathfrak{A})_{\mathcal{E}^*}$ we use an idea of [2]. If $\mathfrak{A}'_{\mathcal{E}^*}$ for some Horn reduction $\mathfrak{A}'$ of $\mathfrak{A}$ then by [9] $\mathfrak{A}'$ would also hold in the direct product $X^2\mathcal{E}^*$. Consequently $\mathfrak{A}$ would hold in $X^2\mathcal{E}^*$ as well. But the Boolean algebra of idempotents in $X^2\mathcal{E}^*$ contains exactly four elements. Thus every Horn reduction fails in $\mathcal{E}^*$. q.e.d.

This shows that corollaries 3, 4, and 5 will fail if we leave out the hypothesis that our sentences are Bing.

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THE STATE UNIVERSITY, RUTGERS
New Brunswick, New Jersey

Reçu par la Rédaction le 10. 4. 1967