The first order properties of products of algebraic systems

by

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Introduction

In modern algebra and set theory, a variety of ways of forming sums or products of finitely or infinitely many algebraic systems have been considered. Examples are cardinal and ordinal sums, ordinary and weak direct products, and ordinal products. In this paper we shall introduce a notion of *generalized product*, which comprehends (in a sense) all of these examples, and a number of other products. By means of its use, we shall investigate the relation between the elementary (i. e., first order) properties possessed by the product algebraic system and those possessed by its factors.

The method applied here has its origin in the work of Mostowski in [13] on finite or infinite, ordinary or weak, direct powers of algebraic systems. The first work on "products" other than direct was the study made by Beth [1] of the ordinal sum of finitely many ordered systems. Our own work has occurred in a series of steps over several years. Some of these were reported in abstracts [4], [5], and [28]. Recently, a summary of our results in nearly their present state was given in [6] and [7] (1).

In many cases the definition of a product operation, which is to be applied in the general situation to an indexed family of systems $\langle \mathfrak{A}^{(i)} \mid i \in I \rangle$, takes into account some sort of "structure" on the index set I in the form of another system $\langle I, R_0, R_1, \ldots \rangle$. (For example, when dealing with ordinal sums or products, it is necessary to consider systems $\langle I, R \rangle$ where R is a binary relation ordering the set I.) However, it has turned

(1) Many of the results described in the present paper were obtained while both authors were students of Professor Alfred Tarski. We are indebted to him for many stimulating remarks and suggestions. The work summarized in [28] formed a chapter of the second author's doctoral disseration at the University of California, Berkeley, 1954.

Part of the work of the first author on this paper was supported by a grant from the National Science Foundation.

out to be somewhat more natural in the metamathematical investigations of products to provide a definition of generalized product which is referred to the (more comprehensive) systems of the form

$$\langle S(I), \subseteq, M_0, M_1, \ldots\rangle \tag{1}$$

where $S(I)$ is the family of all subsets of I, and $M_0, M_1, \ldots$ are certain relations among these subsets — whose nature depends on the particular type of product being considered.

In § 3 and § 9 we obtain two basic theorems on the elimination of quantifiers in the theory of the generalized product or power. The result of these theorems is to provide a complete analysis of the elementary properties of such a product or power in terms of the elementary properties of the factors, on the one hand, and of an algebra of subsets of the form (1), on the other. These theorems and their proofs generalize the results of Mostowski ([13], theorems 5.12 and 5.22). Even if considered only for the cases of ordinary or weak direct powers, which were dealt with by Mostowski, the analysis described above provides a vantage point from which his method can be viewed in somewhat clearer form. In these cases, the algebras of subsets involved are just the simplest, namely $\langle S(I), \subseteq\rangle$, and $\langle S(I), \subseteq, Fin\rangle$, where Fin is the set of all finite subsets of I.

The basic theorem of § 3 has a number of interesting consequences, which are given in § 5. One is that the decision problem for the theory of a system which is a finite product (of any of the types comprehended in the generalized product) reduces to the decision problems for the factor systems. Another consequence is that the first order theory of any generalized product system is completely determined by the theories of the factors involved, in other words, that the generalized product preserves elementary equivalence of systems.

A third consequence of the basic theorem is that the decision problem for the theory of a generalized power (finite or infinite) reduces to the decision problem for the theory of the factor and to that for the theory of the system (1). At the conclusion of § 5 we obtain, in addition, a reduction procedure for the theory of the class of all generalized products of systems of a given kind.

The second basic theorem (of § 9) shows that for weak generalized powers, (1) may be replaced by an algebra of finite subsets of I.

To obtain further results one must examine the theories of algebras of subsets such as (1). In §§ 6, 7, and 8, we examine the cases where (1) is of the forms $\langle S(I), \subseteq\rangle$, $\langle S(I), \subseteq, Fin\rangle$ or $\langle S(I), \subseteq, \sim\rangle$. (Here $\sim$ is the relation of set-theoretical equivalence.) A decision method for the theory of a system of the first type is known (cf. [17]), and we shall obtain decision methods for theories of the second and third types.

Applying these results in conjunction with the basic theorem of § 3, we obtain the results of Mostowski on the decision problem for the theory of ordinary or weak powers; we also obtain a number of new results or new proofs concerning decision problems and the theory of models. In particular, we answer affirmatively a question raised by Łoś: If a (first order) sentence is true in each of the (ordinary) direct products $\mathfrak{A}^{(0)}, \mathfrak{A}^{(0)}\times\mathfrak{A}^{(1)}, \ldots, \mathfrak{A}^{(0)}\times\ldots\times\mathfrak{A}^{(n)}, \ldots$, then is it true in the product $\mathfrak{A}^{(0)}\times\ldots\times\mathfrak{A}^{(n)}\times\ldots$?

The further study of the nature of the first order theories of various systems of the form (1) appears to present an interesting class of problems. Many of these theories correspond, in a natural way, to certain special kinds of second order theories.

§ 1. Preliminaries

We use the ordinary notations ϵ, $\{i \mid \ldots\}$, $\{a_0, \ldots, a_{n-1}\}$, $\sim$, $\subseteq$, Λ, $\cup$, $\cap$, $^{-}$, $\bigcup$, and $\bigcap$, respectively, for membership, the set of i such that ..., the set consisting of $a_0, \ldots, a_{n-1}$, set-theoretical equivalence, inclusion, the empty set, union, intersection, difference, arbitrary union and arbitrary intersection. The set of all subsets of a set I will be denoted by $S(I)$.

The notions of ordinal and cardinal are taken to be defined in such a way that each ordinal is the set of all smaller ordinals, and a cardinal is an ordinal not set-theoretically equivalent to any smaller ordinal. We denote by $\mathfrak{c}(X)$ the cardinal number of a set X. The finite ordinals are identified with the natural numbers, so that $0 = \Lambda$, $1 = \{\Lambda\}$, etc., and the set of natural numbers is denoted by ω.

We denote by $\mathcal{P}(A^{(i)} \mid i \in I)$ the set of all functions g with domain I such that, for each $i \in I$, $g(i) \in A^{(i)}$. In particular, B^I is the set of all functions on I to B. Elements x of B^n, where $n \in \omega$, are referred to as ordered n-tuples, and we write $x = \langle x_0, \ldots, x_{n-1}\rangle$. For ordinary infinite sequences $x \in B^\omega$, we sometimes write $x = \langle x_0, \ldots, x_n, \ldots\rangle$, and we call x an ω-tuple. By an indexed family $\langle x^{(i)} \mid i \in I\rangle$ we shall mean simply a function x, with domain I, whose ith term is $x^{(i)}$, for each $i \in I$.

Two special notations we shall need are these: If $x \in B^\omega$, $k \in \omega$, $b \in B$, we put

$$x(k/b) = \langle x_0, \ldots, x_{k-1}, b, x_{k+1}, \ldots, x_n, \ldots\rangle .$$

If $D = \mathcal{P}(A^{(i)} \mid i \in I)$, $f = \langle f_0, \ldots, f_n, \ldots\rangle \in D^\omega$, *and* $i \in I$, we write

$$f[i] = \langle f_0(i), \ldots, f_n(i), \ldots\rangle .$$

It is easy to see that if $f \in D^{\omega}$, $g \in D$ and $i, k \in \omega$ then

$$f[i](k/g(i)) = f(k/g)[i].$$

By an *n-ary relation* ($n \in \omega$) among the elements of a set B, we mean a subset of B^n. A 1-ary relation R will usually be identified with the set of x such that $\langle x \rangle \in R$. By an *n-ary operation* on B is meant a member of B^{B^n}. A 0-ary operation, assigning the value b to the 0-tuple Λ, will usually be identified with the element b, itself (2).

Suppose $\alpha \leqslant \omega$ and $m < \omega$, and suppose μ is an α-tuple of natural numbers, and ν is an m-tuple of natural numbers. It is also convenient to assume further that in case $\alpha = \omega$, the function μ is recursive. A system

(1) $$\mathfrak{B} = \langle B, O_0, \ldots, O_{m-1}, R_0, \ldots, R_j, \ldots \rangle \quad (j < \alpha)$$

is said to be an *algebraic system of similarity type* $\tau = \langle \nu, \mu \rangle$, provided B is a non-empty set, O_j is a ν_j-ary operation on B ($j = 0, 1, \ldots, m-1$), and, for each $j < \alpha$, R_j is a μ_j-ary relation among the elements of B. Two systems of a given type are called *similar*, and all systems of a given type constitute a *similarity class*.

With any given similarity type τ, we associate a first order (with identity) formalized language L_τ, within which systems of this type could be discussed. L_τ has the following symbols: a list $\mathrm{v}_0, \ldots, \mathrm{v}_n, \ldots$ of (distinct) individual variables; the logical symbols $|$ (neither-nor), $\bigvee$ (there exists) and $=$; the (distinct) ν_j-ary operation symbols O_j ($j = 0, \ldots, m-1$); and a list $\mathrm{R}_0, \ldots, \mathrm{R}_j, \ldots$ ($j < \alpha$) of (distinct) predicates, R_j having μ_j places ($j < \alpha$) (3).

The notion of a term, or an atomic formula, or a formula, and the notion of a variable being free in a formula are understood in the usual way. A formula with no free variables is called a *sentence*. The connectives $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$, the repeated conjunction $\prod$ and the repeated disjunction $\sum$ (over appropriate indices), and the universal quantifier $\bigwedge$, will also occur in our discussion; they are to be interpreted as certain operations on expressions — for example, $\sim\theta$ is $\theta|\theta$.

If a formula θ has at most the free variables $\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}$, and $\gamma_0, \ldots, \gamma_{n-1}$ are terms, we write $\theta(\gamma_0, \ldots, \gamma_{n-1})$ for the result of performing the proper simultaneous substitution of γ_j for v_j ($j = 0, 1, \ldots, n-1$) (4).

(2) A 0-ary relation is either $\{\Lambda\}$ or Λ.

(3) Note that particular symbols of L_τ are denoted by non-italicized letters. This convention is followed throughout the paper.

(4) „Proper" means that the bound variables are to be changed, if necessary, in order to avoid collisions, by some prearranged, effective method.

To clarify discussions of decision problems, we assume further that L_τ is constructed in such a way that its set of expressions (non-empty finite concatenations of symbols) coincides with the set of natural numbers, and the operation of concatenating two expressions, the function v listing the variables, and (in case $\alpha = \omega$) the function R listing the predicates are all recursive (cf. [25], p. 13).

Suppose $\mathfrak{B}$ is an algebraic system as in (1), θ is a formula of L_τ, and $x = \langle x_0, \ldots, x_n, \ldots \rangle \in B^{\omega}$. We shall write $\dashv_{\mathfrak{B}} \theta[x]$ to mean that the sequence x satisfies θ in $\mathfrak{B}$ (5). (It is to be understood that to a free variable v_n of θ, x_n is to be assigned.) In case the free variables of θ are at most $\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}$, we may write, instead, $\dashv_{\mathfrak{B}} \theta[\langle x_0, \ldots, x_{n-1} \rangle]$, or, where no ambiguity will arise, simply $\dashv_{\mathfrak{B}} \theta[x_0, \ldots, x_{n-1}]$. In case θ is a sentence, we write $\dashv_{\mathfrak{B}} \theta$ to mean that θ is true in $\mathfrak{B}$.

In the sequel we shall frequently make use, without explicit mention, of the basic properties of satisfaction, such as the fact that:

$$\dashv_{\mathfrak{B}} (\theta \wedge \theta')[x] \quad \text{if and only if} \quad \dashv_{\mathfrak{B}} \theta[x] \text{ and } \dashv_{\mathfrak{B}} \theta'[x].$$

We shall employ the following notions, introduced by Tarski ([22], [27]). Suppose $\mathfrak{B} = \langle B, \ldots \rangle$ and $\mathfrak{B}' = \langle B', \ldots \rangle$ are algebraic systems of the same type τ. $\mathfrak{B}$ and $\mathfrak{B}'$ are called *elementary equivalent* if, for every sentence θ of L_τ,

$$\dashv_{\mathfrak{B}} \theta \quad \text{if and only if} \quad \dashv_{\mathfrak{B}'} \theta.$$

$\mathfrak{B}$ is said to be an *elementary extension of* $\mathfrak{B}'$ if

(i) $B' \subseteq B$

and

(ii) for any formula θ of L_τ, and any $x \in (B')^{\omega}$

$$\dashv_{\mathfrak{B}} \theta[x] \quad \text{if and only if} \quad \dashv_{\mathfrak{B}'} \theta[x] \text{ (6)}.$$

For any class $\mathcal{K}$ of algebraic systems of type τ, the set of all sentences of L_τ true in every member of $\mathcal{K}$ is called the *theory of* $\mathcal{K}$, and is denoted by $Th(\mathcal{K})$. In case $\mathcal{K}$ consists of a single system $\mathfrak{B}$, we put $Th(\mathfrak{B}) = Th(\mathcal{K})$, and speak of the *theory of* $\mathfrak{B}$. We say that the theory of K is *decidable* if it is recursive.

(5) For a definition of this notion, see [20]. To settle any question as to how 0-ary relation symbols are to be interpreted, we agree that if R_j is 0-ary and $x \in B^{\omega}$ then $\dashv_{\mathfrak{B}} \mathrm{R}_j[x]$ if and only if R_j is not empty (cf. footnote (2)).

(6) Condition (i) replaces the stronger requirement given in [27] that $\mathfrak{B}'$ be a subsystem (in the usual sense) of $\mathfrak{B}$. It is easily seen, using (ii), that the two formulations are equivalent.

As is well known, for most purposes there is no real gain in generality in considering algebraic systems rather than simply *relational systems*

(2) $$\mathfrak{B} = \langle B, R_0, \ldots, R_j, \ldots\rangle \quad (j < a) .$$

We shall, therefore, only consider systems with operations in a few special contexts, where this will greatly simplify the notation. If the relational system (2) is of type $\langle 0, \mu\rangle$, we shall say simply that it is of type μ.

§ 2. The generalized product

In the sequel we shall deal with a fixed, but arbitrary, similarity class $\mathcal{A}$ of relational systems of the type μ. These systems are to be used as the factors of the generalized product; i. e., we want to define a notion of generalized product of an indexed family $\mathfrak{A} = \langle \mathfrak{A}^{(i)} \mid i \in I\rangle$ of such systems. If this product is to include the notion of ordinal product, for example, as well as the direct product, then it would seem natural to consider that, together with the index set I, we are given some relations among the elements of I. However, as already suggested by Mostowski's work [13], it turns out to be more advantageous (as well as, of course, more general) to suppose given, instead, some relations among subsets of I, i. e., among the elements of $S(I)$.

We suppose given, therefore, a type μ' of relational systems, where μ' is an a'-tuple and $\mu'(0) = 2$. Moreover, we suppose given an arbitrary, fixed sub-class $\mathcal{S}$ of the class of all algebraic systems

$$\mathfrak{S} = \langle S(I), \Lambda, \cup, \cap, ^{-}, \subseteq, M_1, \ldots, M_j, \ldots\rangle \quad (1 \leqslant j < a')$$

such that

(i) I is any non-empty set,

(ii) $\mathfrak{S}$ is of type $\langle\langle 0, 2, 2, 1\rangle, \mu'\rangle = \sigma$,

(iii) $\Lambda, \cup, \cap, ^{-}$, and $\subseteq$ are the usual set theoretic operations or relations restricted to $S(I)$ (7).

System like $\mathfrak{S}$ will be called *algebras of subsets*.

As described in § 1, L_μ is the language in which we may discuss systems of $\mathcal{A}$, while L_σ is another language, similarly associated with the algebraic systems of type σ. We denote the list of variables of L_μ by $\mathrm{v}_0, \ldots, \mathrm{v}_n, \ldots$, while, for the sake of readability, we denote the list

(7) Strictly speaking we should write, for example, $\cup_I$ instead of simply $\cup$, but the simpler notation should cause no confusion. In general, if we speak of an algebraic system $\langle A, O, \ldots, R, \ldots\rangle$ and the operations or relations $O, \ldots, R, \ldots$ are too large, we have in mind the system obtained by restricting $O, \ldots, R, \ldots$ to A.

of variables of L_σ by $\mathrm{X}_0, \ldots, \mathrm{X}_n, \ldots$ The list of predicates of L_μ is $R_0, \ldots, R_j, \ldots$ $(j < a)$. L_σ has the operation symbols $\Lambda, \cup, \cap, ^{-}$, and the list of predicates $\subseteq, \mathrm{M}_1, \ldots, \mathrm{M}_j, \ldots$ $(j < a')$. We will use letters $\theta, \theta', \psi, \ldots$ for formulas of L_μ and $\Phi, \Phi', \Psi, \ldots$ for formulas of L_σ.

For future reference, some further terminology relating to the language L_σ will be introduced now. This has to do with the need to express in L_σ the notion that a sequence of sets $X_0, \ldots, X_m$ forms a partition of the set I (by which we understand that $\bigcup_{i \leqslant m} X_i = I$ and that $X_i \cap X_j = \Lambda$ for each $i < j \leqslant m$ (8)). First, if s is any finite set of natural numbers, and for each $j \in s$, γ_j is a term of L_σ, we define (by induction):

$$\bigcup_{j \in s} \gamma_j \text{ to be the term } \Lambda, \text{ if } s \text{ is empty},$$

and to be the term

$$(\bigcup_{j \in s'} \gamma_j) \cup \gamma_k$$

if s has a largest member k and $s' = s - \{k\}$. Then for each m we can introduce the desired formula Part_m, with the $m+1$ free variables $\mathrm{X}_0, \ldots, \mathrm{X}_m$, by means of the definition:

$$\mathrm{Part}_m = \Big[\bigcup_{j \in m+1} \mathrm{X}_j = \bar{\Lambda} \wedge \prod_{i < j \leqslant m} (\mathrm{X}_i \cap \mathrm{X}_j = \Lambda)\Big] .$$

Before describing the generalized product in 2.4, some preliminary notions are needed:

DEFINITION 2.1. *Suppose that* $\mathfrak{A} = \langle \mathfrak{A}^{(i)} \mid i \in I\rangle$, *where* $I \neq \Lambda$, *is an indexed family of relational systems* $\mathfrak{A}^{(i)} = \langle A^{(i)}, \ldots\rangle$ *of type* μ, *and* $D = \mathcal{P}(A^{(i)} \mid i \in I)$. *Let* $f = \langle f_0, \ldots, f_n, \ldots\rangle \in D^\omega$, *and let* θ *be a formula of* L_μ. *Then we put*

$$K_\theta^{\mathfrak{A}}(f) = \{i \mid i \in I \text{ and } \dashv_{\mathfrak{A}^{(i)}} \theta\,[f[i]]\} .$$

In particular, if θ *is a sentence,* $K_\theta^{\mathfrak{A}}[f]$ *does not depend on* f, *and we write*

$$K_\theta^{\mathfrak{A}} = K_\theta^{\mathfrak{A}}(f) = \{i \mid i \in I \text{ and } \dashv_{\mathfrak{A}^{(i)}} \theta\} .$$

From 2.1, one derives at once the following lemma.

LEMMA 2.2. *Let* $\mathfrak{A}$, I, D, *and* f *be as in* 2.1, *and suppose that* θ, θ' *are formulas of* L_μ *and* $k \in \omega$. *Then*

(.1) $K_{\sim\theta}^{\mathfrak{A}}(f) = I - K_\theta^{\mathfrak{A}}(f)$,

(.2) $K_{\theta\wedge\theta'}^{\mathfrak{A}}(f) = K_\theta^{\mathfrak{A}}(f) \cap K_{\theta'}^{\mathfrak{A}}(f)$,

(.3) $K_{\theta\vee\theta'}^{\mathfrak{A}}(f) = K_\theta^{\mathfrak{A}}(f) \cup K_{\theta'}^{\mathfrak{A}}(f)$,

(8) It is not excluded that some of the sets X_i may be empty.

(.4) $K^{\mathfrak{A}}_{\bigvee v_k \theta}(f) = \bigcup \left(K^{\mathfrak{A}}_{\theta}\left(f(k/g)\right) \mid g \in D\right)$,

(.5) *If* $f' \in D^{\omega}$, *and* $f_j = f'_j$ *for every* j *such that* v_j *is free in* θ, *then*

$$K^{\mathfrak{A}}_{\theta}(f) = K^{\mathfrak{A}}_{\theta}(f').$$

(.6) *If the variables of* θ' *are at most* $v_0, \ldots, v_{n-1}$, *and if* $m \in \omega^n$, $\theta = \theta'(v_{m_0}, \ldots, v_{m_{n-1}})$, $f' \in D^{\omega}$, *and* $f'_j = f_{m_j}$ *for every* $j < n$, *then*

$$K^{\mathfrak{A}}_{\theta}(f) = K^{\mathfrak{A}}_{\theta'}(f').$$

DEFINITION 2.3. (.1) *A sequence* $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle$ *is called an* acceptable *sequence if* Φ *is a formula of* L_{σ} *with at most the free variable* $X_0, \ldots, X_m$, *and* $\theta_0, \ldots, \theta_m$ *are formulas of* L_{μ}.

(.2) *By a* free variable of ζ *we mean a variable* v_k *free in at least one of* $\theta_0, \ldots, \theta_m$.

(.3) ζ *is called a* standard acceptable *sequence if, in addition to* (.1), *the free variables of* ζ *are exatly* $v_0, \ldots, v_n$, *for some* $n \in \omega$.

(.4) ζ *is called a* partitioning *sequence provided the formula*

$$\sum_{j \leqslant m} \theta_j$$

and each of the formulas

$$\sim(\theta_j \wedge \theta_{j'}) \qquad (j < j' \leqslant m)$$

are (truth-table) tautologous.

The generalized product will be a system with infinitely many relations, one corresponding to each standard acceptable sequence. To conform with the requirement of § 1 that the relations be numbered, we arrange all standard acceptable sequences into a (non-repeating) infinite sequence $\langle \zeta^0_0, \ldots, \zeta^0_n, \ldots \rangle$ (in such a way that the sequence of Gödel numbers $\langle G(\zeta^0_0), \ldots, G(\zeta^0_n), \ldots \rangle$ forms a recursive function [9]).

DEFINITION 2.4. *Let* $\mathfrak{A}$, I, *and* D *be as in* 2.1, *and let* $\mathfrak{S} = \langle S(I), \ldots \rangle \in \mathcal{S}$.

(.1) *For each standard acceptable sequence* $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle$, *with* p *free variables, we put*

$$Q^{\mathfrak{A},\mathfrak{S}}_{\zeta} = \{\langle f_0, \ldots, f_{p-1} \rangle \mid f \in D^{\omega} \text{ and } \dashv_{\mathfrak{S}} \Phi[K^{\mathfrak{A}}_{\theta_0}(f), \ldots, K^{\mathfrak{A}}_{\theta_m}(f)]\}.$$

(.2) *By the* generalized product $\mathcal{P}(\mathfrak{A}, \mathfrak{S})$ *of the systems* $\langle \mathfrak{A}^{(i)} \mid i \in I \rangle$ *relative to the algebra* $\mathfrak{S}$ *of subsets of* I, *we mean the system*

$$\mathfrak{D} = \langle D, Q^{\mathfrak{A},\mathfrak{S}}_{\zeta^0_0}, \ldots, Q^{\mathfrak{A},\mathfrak{S}}_{\zeta^0_n}, \ldots \rangle.$$

(9) The parenthetical remark is needed only to make precise later discussions of the decision problem. For the Gödel number $G(\zeta)$ of $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle$ we might take, for example, the number $p_0^{\Phi+1} \cdot p_1^{\theta_0+1} \cdot \ldots \cdot p_{m+1}^{\theta_{m+1}}$, where $p_0, p_1, \ldots$ are the primes in order. (Recall that the formulas of L_{μ} and L_{σ} are, themselves, numbers.)

(.3) *If all the systems* $\mathfrak{A}^{(i)}$, *for* $i \in I$, *are identical with a single system* $\mathfrak{B}$, *then* $\mathcal{P}(\mathfrak{A}, \mathfrak{S})$ *is called the* generalized power *of* $\mathfrak{B}$ *relative to* $\mathfrak{S}$, *written* $\mathfrak{B}^{\mathfrak{S}}$.

Let $\pi(j)$ be the number of free variables of ζ^0_j, for $j \in \omega$. Clearly π is a recursive function. Thus π is a similarity type of relational systems, and, clearly, every generalized product $\mathcal{P}(\mathfrak{A}, \mathfrak{S})$, where $\mathfrak{A}$ is an indexed family of systems of $\mathcal{A}$ and $\mathfrak{S} \in \mathcal{S}$, is of type π. We denote the list of variables in L_{π} (our third language) by $f_0, \ldots, f_n, \ldots$ We denote by Q_{ζ} the predicate of L_{π} corresponding to the relation Q_{ζ}, so that the list of predicates in L_{π} is $Q_{\zeta^0_0}, \ldots, Q_{\zeta^0_n}, \ldots$

In the next section, § 3, we shall establish a basic theorem concerning the generalized product. In § 4, we shall show how various familiar products and sums may be interpreted as special cases of generalized products. The reader is advised to read § 4.1, in which the direct product is interpreted by means of the generalized product, before reading § 3, in order to gain some idea at this point of the relation between the generalized product and familiar product notions.

§ 3. The basic theorem for generalized products

THEOREM 3.1. (.1) *There is an effective procedure whereby to each formula* Γ *of* L_{π} *can be correlated a sequence* $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle$ *in such a way that*

(i) ζ *is an acceptable sequence, and* ζ *and* Γ *have corresponding free variables, i. e., a variable* f_k *is free in* Γ *if and only if* v_k *is free in* ζ;

(ii) *Given any non-empty indexed family* $\mathfrak{A} = \langle \mathfrak{A}^{(i)} \mid i \in I \rangle$, *of systems* $\mathfrak{A}^{(i)} = \langle A^{(i)}, \ldots \rangle$ *of type* μ, *and any algebra* $\mathfrak{S} = \langle S(I), \ldots \rangle \in \mathcal{S}$ *with product* $\mathcal{P}(\mathfrak{A}, \mathfrak{S}) = \mathfrak{D} = \langle D, \ldots \rangle$, *and given any* $f \in D^{\omega}$, *we have:*

$$\dashv_{\mathfrak{D}} \Gamma[f]$$

if and only if

$$\dashv_{\mathfrak{S}} \Phi[K^{\mathfrak{A}}_{\theta_0}(f), \ldots, K^{\mathfrak{A}}_{\theta_m}(f)].$$

(.2) *In particular, if* Γ *is a sentence, so are* $\theta_0, \ldots, \theta_m$, *and* $\dashv_{\mathfrak{D}} \Gamma$ *if and only if*

$$\dashv_{\mathfrak{S}} \Phi[K^{\mathfrak{A}}_{\theta_0}, \ldots, K^{\mathfrak{A}}_{\theta_m}].$$

(.3) *If desired,* ζ *may always be taken to be a partitioning sequence* [10].

(10) It would be difficult to try trace the growth of ideas which led to the notion of the generalized product and to the statement and proof of the basic theorem 3.1 (from which practically all the results in this paper flow). Some of the background to this has already been indicated in the Introduction; further information, relating to

Proof. 3.1.1 will be established by the "method of eliminating quantifiers", or, in other words, by induction on Γ.

Part (1). Suppose Γ is an atomic formula of the form $Q_{\zeta'}f_{k_0}\dots f_{k_{n-1}}$, where $\zeta' = \langle \Phi, \theta'_0, \dots, \theta'_{m'}\rangle$ is a standard acceptable sequence (with free variables $v_0, \dots, v_{n-1}$). Assume the hypothesis of (ii), and let f' be any member of D^ω such that $f'_j = f_{k_j}$ for each $j < n$. Then the condition

$$\dashv_{\mathfrak{D}} \Gamma[f]$$

is, by 2.4.1, equivalent to

$$\dashv_{\mathfrak{S}} \Phi'[K^{\mathfrak{A}}_{\theta'_0}(f'), \dots, K^{\mathfrak{A}}_{\theta'_{m'}}(f')];$$

while, by 2.2.6, the latter is equivalent to

$$\dashv_{\mathfrak{S}} \Phi'[K^{\mathfrak{A}}_{\theta_0}(f), \dots, K^{\mathfrak{A}}_{\theta_{m'}}(f)],$$

where $\theta_j = \theta'_j(v_{k_0}, \dots, v_{k_{n-1}})$ for each $j \leqslant m'$. Hence 3.1.1 holds if we take $\Phi = \Phi'$, and, for each $j \leqslant m = m'$, θ_j as just explained. (Condition (i) is obviously satisfied.)

If Γ is an atomic formula of the form $f_k = f_l$, then we see at once (by Definitions 2.1 and 2.4) that we may take $\zeta = \langle \Phi, \theta\rangle$ where θ is the formula $v_k = v_l$ and Φ is the formula $X_0 = \bar{A}$.

Part (2). Suppose that Γ is of the form $\Gamma' | \Gamma''$, and that $\zeta' = \langle \Phi', \theta'_0, \dots, \theta'_{m'}\rangle$ and $\zeta'' = \langle \Phi'', \theta''_0, \dots, \theta''_{m''}\rangle$ are correlated with Γ' and Γ'', respectively, in such a way that (i) and (ii) hold in each case. Let

$$\zeta = \langle \Phi, \theta'_0, \dots, \theta'_{m'}, \theta''_0, \dots, \theta''_{m''}\rangle,$$

where

$$\Phi = \Phi' \,|\, \Phi''(X_{m'+1}, \dots, X_{m'+m''+1}).$$

Clearly condition (i) holds, and it is elementary to verify that (ii) holds as well.

Part (3). Before taking up, in part (4), the quantifier case, we shall show that:

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specific consequences of 3.1, will be given at the appropriate moments. However, something concerning the order of discoveries can at least be mentioned now.

The notion of the generalized power (2.4.3) and a statement and proof of the basic elimination theorem 3.1, restricted to such powers, were obtained by the first author in 1953 (and communicated to the second author at that time). Most of the results of § 5, § 6 and § 7, when rephrased so as to apply just to powers, were also obtained by him then. In 1956, the second author realized that the statement and proof given for powers extended to generalized products, thus arriving at 3.1 substantially as it is given here, and he obtained a number of the consequences given, in their general form, in § 5, § 6 and § 7.

With each acceptable sequence $\zeta' = \langle \Phi', \theta'_0, \dots, \theta'_{m'}\rangle$ we may (effectively) correlate an acceptable partitioning sequence $\zeta = \langle \Phi, \theta_0, \dots, \theta_m\rangle$, with the same free variables, in such a way that, under the hypothesis of (ii),

(1) $$\dashv_{\mathfrak{S}} \Phi'[K^{\mathfrak{A}}_{\theta'_0}(f), \dots, K^{\mathfrak{A}}_{\theta'_{m'}}(f)]$$

if and only if

(2) $$\dashv_{\mathfrak{S}} \Phi[K^{\mathfrak{A}}_{\theta_0}(f), \dots, K^{\mathfrak{A}}_{\theta_m}(f)].$$

We can describe $\zeta = \langle \Phi, \theta_0, \dots, \theta_m\rangle$ as follows: Let $m = 2^{m'+1}$ and let $r_0, \dots, r_m$ be a list of all the subsets of $m'+1 = \{0, \dots, m'\}$. For each $k \leqslant m$, let

(3) $$\theta_k = \prod_{j \in r_k} \theta'_j \wedge \prod_{j \in (m'+1)-r_k} \sim\theta'_j.$$

Put

(4) $$s_l = \{k \mid k \leqslant m \text{ and } l \in r_k\} \quad (l \leqslant m');$$

and let

(5) $$\Phi = \Phi'(\bigcup_{k \in s_0} X_k, \dots, \bigcup_{k \in s_{m'}} X_k).$$

Clearly ζ is an acceptable sequence having the same free variables as ζ'. From (3) and Definition 2.3.4, one sees that ζ is a partitioning sequence. Assuming, now, the hypothesis of (ii), we see from (5) that (2) is equivalent to the condition

(6) $$\dashv_{\mathfrak{S}} \Phi'[\bigcup_{k \in s_0} K^{\mathfrak{A}}_{\theta_k}(f), \dots, \bigcup_{k \in s_{m'}} K^{\mathfrak{A}}_{\theta_k}(f)].$$

Now, by (3) and 2.2.1, 2.2.2,

(7) $$\bigcup_{k \in s_l} K^{\mathfrak{A}}_{\theta_k}(f) = \bigcup_{k \in s_l} \{\bigcap_{j \in r_k} K^{\mathfrak{A}}_{\theta'_j}(f) \cap \bigcap_{j \in (m'+1)-r_k} [I - K^{\mathfrak{A}}_{\theta'_j}(f)]\} \quad (l \leqslant m').$$

From (4) and elementary set algebra it follows that the right member of equation (7) is simply $K^{\mathfrak{A}}_{\theta'_l}(f)$, so that

(8) $$\bigcup_{k \in s_l} K^{\mathfrak{A}}_{\theta_k}(f) = K^{\mathfrak{A}}_{\theta'_l}(f) \quad (l \leqslant m').$$

From (8) we see that (6) is equivalent to (1). Thus (2) and (1) are equivalent, as was to be shown.

Part (4). Returning to the proof of 3.1.1, suppose now that Γ is of the form $\bigvee f_k \Gamma'$, and that $\zeta' = \langle \Phi, \theta'_0, \dots, \theta'_{m'}\rangle$ is correlated with Γ' in such a way that ζ' and Γ' fulfill conditions (i) and (ii); suppose, moreover, that ζ' is a partitioning sequence.

Let $m = m'$, and $\zeta = \langle \Phi, \theta_0, \dots, \theta_m\rangle$, where

(9) $$\theta_j = \bigvee v_k \theta'_j \quad (j \leqslant m)$$

and

(10) $\Phi = \bigvee \mathrm{Y}_0, \ldots, \bigvee \mathrm{Y}_m \{\mathrm{Part}_m(\mathrm{Y}_0, \ldots, \mathrm{Y}_m) \wedge \prod_{j \leqslant m} (\mathrm{Y}_j \subseteq \mathrm{X}_j) \wedge \Phi'(\mathrm{Y}_0, \ldots, \mathrm{Y}_m)\}$.

(Here $\mathrm{Y}_0, \ldots, \mathrm{Y}_m$ are any m unused variables, say $\mathrm{X}_{m+1}, \ldots, \mathrm{X}_{2m+1}$.)

Clearly, condition (i) holds. Now, assume the hypothesis of (ii). Obviously, the statement

(11) $\dashv_{\mathfrak{D}} \Gamma[f]$

is equivalent to the statement:

(12) *for some* $g \in D$, $\dashv_{\mathfrak{D}} \Gamma'[f(k/g)]$,

and so, in turn, to the statement

(13) *for some* $g \in D$, $\dashv_{\mathfrak{S}} \Phi'[K^{\mathfrak{A}}_{\theta'_0}(f(k/g)), \ldots, K^{\mathfrak{A}}_{\theta'_m}(f(k/g))]$.

On the other hand, the condition

(14) $\dashv_{\mathfrak{S}} \Phi[K^{\mathfrak{A}}_{\theta_0}(f), \ldots, K^{\mathfrak{A}}_{\theta_m}(f)]$,

which we wish to prove equivalent to (11), is by (10) and (9), equivalent to the following:

(15) *there exist sets* $Y_0, \ldots, Y_m$ *such that*
(a) $Y_0, \ldots, Y_m$ *are a partition of* I,
(b) $Y_j \subseteq K^{\mathfrak{A}}_{\vee \triangledown_k \theta'_j}(f)$, for each $j \leqslant m$,
(c) *and* $\dashv_{\mathfrak{S}} \Phi'[Y_0, \ldots, Y_m]$.

Thus it remains to prove the equivalence of (13) and (15). Suppose, first, that (13) holds. Let

(16) $Y_j = K^{\mathfrak{A}}_{\theta'_j}(f(k/g)) \quad (j \leqslant m)$.

Since ζ is a partitioning sequence, we obtain at once, by (16) and 2.2.1-2.2.3, that (15) (a) holds. (15) (b) follows from (16) and 2.2.4, and (15) (c) is merely a restatement of (13). Thus (15) follows from (13).

On the other hand, assume that (15), (a), (b), (c) hold. We define a function $g \in D$ as follows. Let $i \in I$. Then, by 15 (a), i belongs to exactly one of $Y_0, \ldots, Y_m$, say Y_j. By 14 (b), $Y_j \subseteq K^{\mathfrak{A}}_{\vee \triangledown_k \theta'_j}(f)$, and hence, by Definition 2.1,

$$\dashv_{\mathfrak{A}^{(i)}} (\bigvee \triangledown_k \theta'_j)[f[i]].$$

We can, therefore, choose, as $g(i)$, an element of $A^{(i)}$ in such a way that

(17) $\dashv_{\mathfrak{A}^{(i)}} \theta'_j[f[i](k/g(i))]$.

We may suppose (by the Axiom of Choice) that such a choice has been made for each $i \in I$, so that the function g is now defined.

Recalling that $f[i](k/g(i)) = f(k/g)[i]$, we see from (17) that

(18) *if* $i \in Y_j$, *then* $\dashv_{\mathfrak{A}^{(i)}} \theta'_j[f(k/g)[i]] \quad (j \leqslant m)$.

In other words (by 2.1)

(19) $Y_j \subseteq K^{\mathfrak{A}}_{\theta'_j}(f(k/g)) \quad (j \leqslant m)$.

Since the sets $Y_0, \ldots, Y_m$, and, also, by 2.2, the sets $K^{\mathfrak{A}}_{\theta'_0}(f(k/g)), \ldots, K^{\mathfrak{A}}_{\theta'_m}(f(k/g))$ both form partitions of I, (19) implies that, actually,

(20) $Y_j = K^{\mathfrak{A}}_{\theta'_j}(f(k/g)) \quad (j \leqslant m)$.

From (20) and our hypothesis (14) (c), (12) follows directly.

Thus, the proof of the equivalence of (13) and (15) is finished, and we have shown that condition (ii) holds, completing part (4) of the proof.

It is now clear that a general (effective) method can be given for breaking an arbitrary formula Γ down into its component parts, and, then, by applying the methods of parts (1), (2), (3), and (4), building up to a sequence ζ as demanded in 3.1.1. Moreover, if desired, one could always apply, at the last, part (3) to obtain a partitioning sequence ζ, so that 3.1.3, as well as 3.1.1, is proved. 3.1.2 is an immediate consequence of 3.1.1, so the proof of 3.1 is complete.

The generalized product, as we have defined it, is a system with infinitely many relations. It is, of course, much more common, in algebraic discussions, to encounter systems having only finitely many relations. If we take any finite list $\zeta_0, \ldots, \zeta_{p-1}$ of standard acceptable sequences, we may define a notion of product, which will be of this latter sort, by simply taking the system

$$\langle D, Q^{\mathfrak{A},\mathfrak{S}}_{\zeta_0}, \ldots, Q^{\mathfrak{A},\mathfrak{S}}_{\zeta_{p-1}} \rangle$$

to be the product of the indexed family $\mathfrak{A}$ relative to the algebra of subsets $\mathfrak{S}$ ($\mathfrak{A}$, $\mathfrak{S}$, and D as in 2.1). Any such notion of product will be referred to as a generalized product. To avoid ambiguity we shall occasionally refer to the product notion defined in 2.4 as the *full* generalized product.

A more general possibility is to consider, in addition to $\zeta_0, \ldots, \zeta_{p-1}$, one standard acceptable sequence η, having only one free variable, and to take for the product the system

(21) $\langle Q^{\mathfrak{A},\mathfrak{S}}_{\eta}, U_0, \ldots, U_{p-1} \rangle$

where, for each $j < p$, U_j is the relation $Q_{\zeta_j}^{\mathfrak{A},\mathfrak{S}}$ restricted to the set $Q_\eta^{\mathfrak{A},\mathfrak{S}}$. Such a notion of product will be called a *relativized* generalized product. (That any generalized product can also be taken to be such a relativized product is seen by considering $\eta = \langle \Psi, \varphi \rangle$ where $\varphi = (v_0 = v_0)$ and $\Psi = (X_0 = X_0)$.)

THEOREM 3.2. *Theorem 3.1 remains valid if the words "the product $\mathcal{P}(\mathfrak{A}, \mathfrak{S})$" are interpreted to mean an arbitrary (fixed) generalized product or relativized generalized product, while "L_π" is taken to mean the language appropriate to that product.*

Proof. For a generalized product, this is immediate. In the more general case let Γ be an arbitrary formula of the language corresponding to systems similar to (20), let $D' = Q_\eta^{\mathfrak{A},\mathfrak{S}}$ and $\mathfrak{D}' = \langle D', \ldots \rangle$ be a system like the one described above, and let $f \in (D')^\omega$. In the old language of the full generalized product $\mathfrak{D}$ form the sentence Γ' by relativizing all quantifiers in Γ to Q_η (in the sense of [25], p. 24-25). It is easily seen that

$$\dashv_{\mathfrak{D}'} \Gamma[f]$$

if and only if

$$\dashv_{\mathfrak{D}} \Gamma'[f].$$

Hence we can take as the sequence ζ to be correlated with Γ in the present theorem the same sequence as that correlated to Γ' by 3.1.

§ 4. Examples of generalized products

In this section we shall discuss a number of examples of notions of product or sum, which are either directly interpretable as, or are closely related to, a (possibly relativized) generalized product notion. To simplify the discussion, we shall usually assume that $\mathcal{A}$ is the class of relational systems of the form $\langle B, R \rangle$, when R is a binary relation, and, thus, that L_μ has one non-logical constant, the binary predicate R. It will be clear that, in some cases, any other similarity class could be treated in a corresponding way.

4.1. *Direct products* (also called cardinal products or direct sums). The *direct product* of a non-empty indexed family $\mathfrak{A} = \langle \mathfrak{A}^{(i)} \mid i \in I \rangle$, where for each $i \in I$, $\mathfrak{A}^{(i)} = \langle A^{(i)}, R^{(i)} \rangle$, is the system $\langle D, U \rangle$, where $D = \mathcal{P}(A^{(i)} \mid i \in I)$ and, for any $g, h \in D$, $\langle g, h \rangle \in U$ if and only if, for every $i \in I$, $\langle g(i), h(i) \rangle \in R^{(i)}$. Clearly, if $g, h \in D$,

(1) $\langle g, h \rangle \in U$ *if and only if* $\{i \mid \langle g(i), h(i) \rangle \in R^{(i)}\} = I$.

Letting θ be the formula Rv_0v_1, Φ be $X_0 = \bar{\Lambda}$, and ζ be $\langle \Phi, \theta \rangle$, we see from (1) (and Definitions 2.1, 2.4) that U is identical with the relation $Q_\zeta^{\mathfrak{A},\mathfrak{S}}$, where

(2) $$\mathfrak{S} = \langle S(I), \Lambda, \cup, \cap, {}^-, \subseteq \rangle.$$

Thus, the direct product may be interpreted as a generalized product relative to subset algebras of the (simplest) form (2).

4.2. *Weak direct product*. Suppose we are given a (fixed) formula ψ of L_μ, whose only free variable is v_0. It is common, in defining a weak direct product, to restrict our attention to a subclass $\mathcal{A}^*$ of $\mathcal{A}$ consisting of systems $\mathfrak{B}$ having exactly one element e such that $\dashv_{\mathfrak{B}} \psi[\langle e \rangle]$. However, that need not be done here, unless one so desires.

Now if $\mathfrak{A}$, I, D, and U are as in 4.1, the *weak direct product* of the $\mathfrak{A}^{(i)} (i \in I)$ is defined to be the system $\langle D^*, U^* \rangle$, where, if $g \in D$,

(3) $g \in D^*$ *if and only if* $\{i \mid i \in I$ *and* $\dashv_{\mathfrak{A}^{(i)}} \sim\psi[\langle g(i) \rangle]\}$ *is finite*,

and where U^* is the relation U restricted to D^*. Let

(4) $$\mathfrak{S} = \langle S(I), \Lambda, \cup, \cap, {}^-, \subseteq, Fin \rangle$$

where Fin (strictly, Fin_I) is the set of finite subsets of I. Then it follows easily from (3) and the results of 4.1, that the weak direct product is a relativized generalized product relative to subset algebras of the form (4).

4.3. *Countably weak direct product* (11). The definition of this notion is obtained from that of the weak direct product by replacing, throughout, the word "finite" by the word "countable". Exactly as in 4.2, one sees that this is a relativized generalized product relative to subset algebras of the form

$$\langle S(I), \Lambda, \cup, \cap, {}^-, \subseteq, C \rangle,$$

when C is the set of countable subsets of I.

4.4. *The "almost everywhere diret product"*. By this name we refer to a notion introduced in [2]. We form first the intermediate product

(5) $$\mathfrak{D} = \langle D, E, V \rangle,$$

where, for $g, h \in D$,

(6) $\langle g, h \rangle \in E$ *if and only if* $\{i \mid g(i) \neq h(i)\}$ *is finite*

and

(7) $\langle g, h \rangle \in V$ *if and only if* $\{i \mid \langle g(i), h(i) \rangle \notin R^{(i)}\}$ *is finite*.

(11) This notion was considered by Tarski in [21].

Now, E is an equivalence relation having the substitution property relative to V, so we may form the quotient system $\mathfrak{D}' = \langle D', V' \rangle$ in the usual way, where D' is the set of all equivalence classes g/E for $g \in D$. $\mathfrak{D}'$ may be called the *almost everywhere direct product* of the $\mathfrak{A}^{(i)}$ $(i \in I)$.

From (6) and (7) we easily see that the intermediate product (5) can be considered as a generalized product relative to base systems of the form (4).

In order to derive certain consequences concerning the almost everywhere direct product from the above discussion and Theorem 3.2, it is convenient to have at hand an exact statement concerning the relation between the theory of a system like (5) and its quotient system. This is the purpose of the following (well-known) lemma, whose proof is very simple and will be omitted:

LEMMA 4.4 (.1) *To each formula θ of $L_{\langle 2\rangle}$ correlate the formula ψ of $L_{\langle 2,2\rangle}$ with the same free variables by replacing each occurrence of a subformula $v_i = v_j$ of θ by Ev_iv_j. Suppose given any relational system $\mathfrak{B} = \langle B, E, R\rangle$ of type $\langle 2, 2\rangle$, where E is an equivalence relation having the substitution property relative to R, and form the quotient system $\mathfrak{B}' = \langle B', R'\rangle$ with respect to E. Then for any $x \in B^{\omega}$ and $x' \in B'^{\omega}$, if $x'_i = x_i/E$ for each $i \in \omega$, we have*

$$\dashv_{\mathfrak{B}'}\theta[x']$$

if and only if

$$\dashv_{\mathfrak{B}}\psi[x]$$

(.2) *In particular, if θ is a sentence then so also is ψ and*

$$\dashv_{\mathfrak{B}'}\theta \quad \text{if and only if} \quad \dashv_{\mathfrak{B}}\psi .$$

4.5. *Ordinal product.* We now consider an indexed family $\langle \mathfrak{A}^{(i)} \mid i \in I\rangle$ $(I \neq \Lambda)$ and a particular binary relation $<$ among the elements of I. It is usual to assume $<$ is an ordering or even a well-ordering of I, and that each $R^{(i)}$ orders $A^{(i)}$, but we need not insist on these conditions here. The ordinal product is then taken to be the system

$$\langle D, W\rangle ,$$

where D is as before, and, if $g, h \in D$,

(8) $\langle g, h\rangle \in W$ *if and only if, for some $i \in I$, $\langle g(i), h(i)\rangle \in R^{(i)}$ while, for all $i' < i$, $g(i') = h(i')$.*

Let $\prec$ be the binary relation among subsets of I such that, if $X, X' \subseteq I$,

(9) $X \prec X'$ *if and only if, for some $i, i' \in I$, $X = \{i\}$, $X' = \{i'\}$, and $i < i'$.*

Now, notice that, if $g, h \in D$, then, by (8) and (9),

(10) $\langle g, h\rangle \in W$ *if and only if there exists a singleton $X \subseteq \{i \mid \langle g(i), h(i)\rangle \in R^{(i)}\}$ such that for any singleton $Y \prec X$, $Y \subseteq \{i \mid g(i) = h(i)\}$.*

(By a *singleton* is meant a set having exactly one member.) Now,

(11) *X is a singleton if and only if $X \neq \Lambda$ and for any sets $Y \subseteq X$, either $Y = \Lambda$ or $Y = X$.*

It follows from (10) and (11), that the ordinal product may be considered as a generalized product relative to subset algebras of the form

$$\langle S(I), \Lambda, \cup, \cap, \bar{\ }, \subseteq, \prec\rangle .$$

4.6. *Weak ordinal product.* Let ψ be as in 4.2, other symbols as in 4.7. The *weak ordinal product* is the system $\langle D^*, W^*\rangle$, where D^* is as in 4.2, and W^* is W restricted to D^*. From 4.2 and 4.5 we easily see that the weak ordinal product may be interpreted as a relativized generalized product relative to subset algebras of the form

(12) $$\langle S(I), \Lambda, \cup, \cap, \bar{\ }, \subseteq, \prec, Fin\rangle .$$

If we restrict our attention to the cases where $<$ orders I, then a subset X of I is finite if and only if every subset of X has a first and last member (relative to $<$). It follows that, under this restriction, the relation *Fin* of (12) may be dispensed with ([12]).

4.7. *Cardinal sums.* A separate theory of generalized summation operations for the case that I is finite was reported in [4]. We shall not attempt here to develop a special theory of summations, but, rather, it will be seen from the examples discussed in 4.7 and 4.8 that, by means of a certain device, the discussion of a wide class of summation operations can be subsumed under that of generalized products ([13]).

Suppose given a non-empty indexed family of systems $\mathfrak{A}^{(i)} = \langle A^{(i)}, R^{(i)}\rangle$ $(i \in I)$, such that the sets $A^{(i)}$ and $A^{(i')}$ are disjoint, if $i \neq i'$ and $i, i' \in I$. The *cardinal sum* is then the system

(13) $$\langle \bigcup_{i \in I} A^{(i)}, \bigcup_{i \in I} R^{(i)}\rangle .$$

It is often convenient to talk about the cardinal sum of a family of systems $\mathfrak{A}^{(i)}$ even when the corresponding sets $A^{(i)}$ are not pairwise

([12]) The ordinal product and the weak ordinal product have been defined in a number of different ways in the literature. Most of these can be treated in ways similar to those we used for the particular notions discussed above. However, Day [3] introduced a further operation of "transitization" to follow the formation of the product of 4.5 — in order to obtain interesting results when the usual assumptions on $<$ and the R_i are not met; and our methods do not apply to this notion. (Indeed, it is not difficult to provide an example showing that transitization does not preserve elementary equivalence.)

([13]) We do not know whether a special theory of summation operations can be developed (for I arbitrary), using different techniques but yielding the same results for these operations that are obtained by our method of including them among products (and perhaps also additional results).

disjoint. This we shall assume always to be accomplished by first replacing each system $\mathfrak{A}^{(i)}$ by an isomorphic system $\mathfrak{A}^{*(i)}$, in such a way that the corresponding sets $A^{*(i)}$ are pairwise disjoint, and then forming the cardinal sum of the systems $\mathfrak{A}^{*(i)}$ as above. This procedure can, if desired, be carried out in a completely determined fashion; however, it is not necessary to specify it since the result is unique up to isomorphism. In particular, when all the systems $\mathfrak{A}^{(i)}$ $(i \in I)$ are identical with a fixed system $\mathfrak{B}$ we obtain in this way a system which will be called the *cardinal multiple of* $\mathfrak{B}$ *by* I.

We can replace (13) by a product in the following way:

For each $i \in I$ choose an entity $c_i \notin A^{(i)}$. For each $i \in I$, put

(14) $$\mathfrak{A}'^{(i)} = \langle A^{(i)} \cup \{c_i\}, R^{(i)}, \{c_i\}\rangle .$$

Let D' be the Cartesian product $\mathcal{P}(A'^{(i)} \mid i \in I)$, and let

$$C = \{g \mid g \in D' \text{ and } g(i) \neq c_i \text{ for exactly one } i \in I\} .$$

For $g, h \in D'$, let

$$\langle g, h\rangle \in V \quad \textit{if and only if, for some} \quad i \in I,\ \langle g(i), h(i)\rangle \in R^{(i)} ;$$

and let U be V restricted to C.

It is clear that the system $\langle C, U\rangle$ is isomorphic to the cardinal sum (13). Now, from the definitions of C and V, we see that, if $g, h \in D'$, then

(15) $g \in C$ *if and only if* $\{i \mid g(i) \notin \{c_i\}\}$ *is a singleton*
and $\langle g, h\rangle \in V$ *if and only if* $\{i \mid \langle g(i), h(i)\rangle \in R^{(i)}\} \neq \Lambda$.

From (15) it follows that the operation of forming the system $\langle C, U\rangle$ from the $\mathfrak{A}'^{(i)}$ is a relativized generalized product relative to subset algebras of the simplest form (2).

In order to derive certain consequences concerning cardinal (or other) sums from a discussion like the above and Theorem 3.2, one needs a careful statement of the relations between the theory of a system $\mathfrak{A}^{(i)}$ and the theory of the augmented system $\mathfrak{A}'^{(i)}$. This is the purpose of the following lemma, whose proof is straightforward and will be omitted:

LEMMA 4.7 (.1) *To each formula θ of $L_{\langle 2,0\rangle}$, whose free variables are $\{v_j \mid j \in t\}$, can be correlated (effectively) formulas ψ_s $(s \subseteq t)$ of $L_{\langle 2\rangle}$ with free variables $\{v_j \mid j \in s\}$, in such a way that: given any relational system $\mathfrak{B} = \langle B, R\rangle$, $c \notin B$, $\mathfrak{B}' = \langle B \cup \{c\}, R, \{c\}\rangle$, and any $x \in (B \cup \{c\})^\omega$, if $s = \{j \mid j \in t$ and $x_j \neq c\}$, then, for any $x' \in B^\omega$ with $x'_j = x_j$ for each $j \in s$,*

$$\dashv_{\mathfrak{B}'} \theta[x]$$

if and only if

$$\dashv_{\mathfrak{B}} \psi_s[x'] .$$

(.2) *In particular, to each sentence θ of $L_{\langle 2,1\rangle}$ is correlated (effectively) one sentence ψ of $L_{\langle 2\rangle}$ such that*

$$\dashv_{\mathfrak{B}'} \theta \quad \textit{if and only if} \quad \dashv_{\mathfrak{B}} \psi .$$

4.8. *Ordinal sums.* Consider a disjointed family $\langle \mathfrak{A}^{(i)} \mid i \in I\rangle$, as in 4.7, and suppose given, also, a binary relation $<$ among the elements of I. The ordinal sum is defined as the system

(16) $$\langle \bigcup_{i \in I} A^{(i)}, W\rangle ,$$

where $\langle a, a'\rangle \in W$ provided either, for some $i \in I$, $\langle a, a'\rangle \in R^{(i)}$, or, for some $i, i' \in I$, $a \in A^{(i)}$, $a' \in A^{(i')}$, and $i < i'$. The ordinal sum of an arbitrary family of systems and the notion of ordinal multiple are then introduced by the same devices as described in 4.7.

Let $\prec$ be defined as in 4.5, and form the systems $\mathfrak{A}'^{(i)}$ $(i \in I)$ and the sets C and D' as in 4.7. Instead of the relation V of 4.7, we put now, for $g, h \in D'$,

(17) $\langle g, h\rangle \in V$ *if and only if* $\{i \mid \langle g(i), h(i)\rangle \in R^{(i)}\} \neq \Lambda$
or $\{i \mid g(i) \notin \{c_i\}\} \prec \{i \mid h(i) \notin \{c_i\}\}$.

Finally, let U be V restricted to D'. Then the system $\langle C, U\rangle$ is isomorphic to the ordinal sum (16). Moreover, the formation of $\langle C, U\rangle$ from the $\mathfrak{A}'^{(i)}$ may be considered as a relativized generalized product relative to subset algebras of the form

$$\langle S(I), \Lambda, \cup, \cap, {}^{-}, \subseteq, \prec\rangle .$$

4.9. *A product of Hahn.* As a last example, we shall mention briefly a product of Hahn (as described in [10]) [14]. Let $\langle \mathfrak{A}^{(i)} \mid i \in I\rangle$ be a non-empty indexed family of ordered Abelian groups $\mathfrak{A}^{(i)} = \langle A^{(i)}, S^{(i)}, R^{(i)}\rangle$. ($S^{(i)}$ is a ternary relation replacing the usual binary group operation, $R^{(i)}$ a binary relation). Denote the zero element of $\mathfrak{A}^{(i)}$ by $0^{(i)}$. Let $<$ be an ordering of I. The Hahn product is the system $\langle C, U, V\rangle$, where D is the Cartesian product of the $A^{(i)}$,

$$C = \{g \mid g \in D \text{ and } \{i \mid g(i) \neq 0^{(i)}\} \text{ is well-ordered by } <\} ,$$

U is the direct product relation formed from the $S^{(i)}$, restricted to C, and V is the ordinal product relation formed from the $R^{(i)}$, restricted to C.

The class of ordered Abelian groups is closed under the Hahn operation. The operation may be defined in exactly the same way if the

[14] The example of the Hahn product was brought to our attention by Dana Scott. Feferman had previously dealt with a finite product which is more or less a special case of it in [5] (cf. the remarks following the proof of 5.4).

factors are ordered Abelian semigroups with zero and with concellation; the class of these systems is also closed under the operation.

It is clear that this product is a relativized generalized product relative to subset algebras of the form

$$\langle S(I), \Lambda, \cup, \cap, {}^{-}, \subseteq, \preccurlyeq\rangle,$$

where $\preccurlyeq$ is as in 4.5.

§ 5. Consequences of the basic theorem

Theorems 3.1 and 3.2 have a number of interesting consequences concerning generalized products; by means of the discussion in § 4, we may also obtain consequences concerning the particular products and sums considered there. We shall state in this section three such results which lie in the theory of models, and three which concern the decision problem.

Suppose we are dealing with any sort of operation which may be applied to an indexed family $\langle \mathfrak{A}^{(i)} \mid i \in I\rangle$ relative to an "index structure" $\mathfrak{S}$ of some sort (on I or $S(I)$). We say that this operation *preserves elementary equivalence* provided that whenever we are given two non-empty indexed families $\langle \mathfrak{A}^{(i)} \mid i \in I\rangle$ and $\langle \mathfrak{A}'^{(i)} \mid i \in I\rangle$ and a common index structure $\mathfrak{S}$, such that, for each $i \in I$, the system $\mathfrak{A}^{(i)}$ is elementarily equivalent to $\mathfrak{A}'^{(i)}$, then the system obtained by operating on the $\mathfrak{A}^{(i)}$ relative to $\mathfrak{S}$ is elementarily equivalent to that obtained by operating on the $\mathfrak{A}'^{(i)}$ relative to $\mathfrak{S}$. We say that the operation *preserves elementary extensionality* if the same condition holds with "is elementarily equivalent to" replaced by "is an elementary extension of".

THEOREM 5.1. *The full generalized product, any (relativized or not) generalized product, and each of the products or sums* 4.1-4.9 *preserve elementary equivalence* ([15]).

([15]) For ordinary or weak direct powers, this is an immediate consequence of Mostowski's results in [13]. For finite ordinal sums, it was established by Beth [1]. Feferman ([4], [5]) announced the extension of Beth's results to arbitrary finite generalized sums and products.

Fraïssé [8] established 5.1 for the case of infinite ordinal sums, by an interesting method, entirely different from ours. (We did not know of his work until after obtaining the same result by our methods.) It appears likely that Fraïssé's method could be extended to give more general results. Just how their generality would compare (or overlap) with that of 5.1 is a question we have not fully investigated.

Incidentally, it is an open problem as to whether the operation of forming the free product of systems preserves elementary equivalence — even for the special case of the free product of two groups. In general, the methods of this paper do not extend to free or tensor products (when applicable) of systems.

THEOREM 5.2. *Each of these operations* (*except* 4.4) *also preserves elementary extensionality.*

Proof of 5.1 and 5.2. If, for each $i \in I$, $\mathfrak{A}^{(i)}$ and $\mathfrak{A}'^{(i)}$ are elementarily equivalent, then, for any sentence θ, $K_\theta^{\mathfrak{A}} = K_\theta^{\mathfrak{A}'}$, by Definition 2.1. If, for each $i \in I$, $\mathfrak{A}^{(i)}$ is an elementary extension of $\mathfrak{A}'^{(i)}$, then, by 2.1, $K_\theta^{\mathfrak{A}}(f) = K_\theta^{\mathfrak{A}'}(f)$, for any formula θ and any $f \in (D')^\omega$ (where D' is the Cartesian product of the $A'^{(i)}$).

Therefore, for the full generalized product, or for any (relativized or not) generalized product, 5.1 and 5.2 are immediate consequences of 3.1 or 3.2, respectively. In particular, by § 4, this applies to all the operations 4.1-4.9, excepting 4.4, 4.7, and 4.8.

As regards the "almost everywhere direct product" of 4.4, the intermediate product formed in 4.4 is a generalized product, and so preserves elementary equivalence and extensionality. The final product is formed by taking the quotient relative to a definable equivalence relation with the substitution property. Using Lemma 4.4.2 we see that this process also preserves elementary equivalence. It also preserves the relation "isomorphic to an elementary extension of". Thus the final product preserves the latter relation, as well as elementary equivalence.

To deal with the sums of § 4.7 and § 4.8, it is enough to consider the discussion in those sections together with the following remark: The operation of passing from a system $\langle B, R\rangle$ to a system $\langle B \cup \{c\}, R, \{c\}\rangle$, where $c \notin B$, preserves elementary equivalence, as is seen immediately from Lemma 4.7. The same applies to elementary extensionality (if the same c is used for both passages.) (For sums, the assertion of 5.2 is intended to refer only to cases in which both indexed families considered are pairwise disjoint.)

THEOREM 5.3. *If $\mathfrak{B}$ and $\mathfrak{B}'$ are elementarily equivalent, and $\mathfrak{S} = \langle S(I), \ldots\rangle$ and $\mathfrak{S}' = \langle S(I'), \ldots\rangle$ are elementarily equivalent, then so are the generalized powers $\mathfrak{B}^{\mathfrak{S}}$ and $\mathfrak{B}'^{\mathfrak{S}'}$. The same applies to the powers (or multiples) corresponding to the operations* 4.1-4.9 ([16]).

Proof. In the case of a power, $K_\theta^{\mathfrak{A}}$ is always either Λ or I (θ being a sentence). Consequently, 5.3 for generalized (and relativized generalized) products follows at once from 3.1 or 3.2. The assertion regarding the operations 4.1-4.9 now follows, noting the remarks made in the proof of 5.1 and 5.2.

([16]) The system $\mathfrak{S}$ is here still to be a subset algebra (according to the case, the one indicated in 4.1-4.9). For the ordinal multiple, Fraïssé [8] obtained the stronger result in which $\langle S(I), \subseteq, \ldots, \preccurlyeq\rangle$ is replaced by $\langle I, <\rangle$. We can obtain this result by our methods, since the ordinal multiple of $\mathfrak{B}$ relative to $\langle I, <\rangle$ is isomorphic to the ordinal product of $\mathfrak{B}$ and $\langle I, <\rangle$. We do not know whether a corresponding result holds for the ordinal power (or weak ordinal power) of $\mathfrak{B}$ relative to $\langle I, <\rangle$.

THEOREM 5.4. *The decision problem for the theory of the generalized product of systems* $\langle \mathfrak{A}^{(i)} \mid i \in I\rangle$ *relative to* $\mathfrak{S}$, *in the case that* I *is finite (and* $\mathfrak{S}$ *has only finitely many relations, i. e.,* $\beta < \omega$*) may be reduced to the decision problems for the theories of the factors. In particular, if each factor has a decidable theory, then so has the (finite) generalized product. The same applies to all products and sums of* § 4 [17].

Proof. This is a direct consequence of 3.1, 3.2, and the discussion in § 4, except as regards the operations 4.4, 4.7, and 4.8. The operation of 4.4 is trivial when I is finite. As regards 4.7 and 4.8, one need only note that from Lemma 4.7.2 it follows that the decision problem for the theory of a system $\langle B\cup\{c\}, R, \{c\}\rangle$ reduces to that for $Th(\langle B, R\rangle)$.

The following example of the application of 5.4 was mentioned in [5]: Let n be a positive integer, and consider the systems $\mathfrak{D} = \langle \omega^n, N, <\rangle$ and $\mathfrak{B} = \langle \omega, S, <\rangle$, where $\langle k, l, m\rangle \in S$ if and only if $k+l = m$, $\langle \alpha, \beta, \gamma\rangle \in N$ if and only if γ is the "natural sum" of α and β, and where $<$ is the usual ordering-restricted to ω or to ω^n [18]. It is easily seen that $\mathfrak{D}$ is isomorphic to the Hahn power (in the sense of § 4.9) of $\mathfrak{B}$ relative to the ordered system $\langle n, >\rangle$. Since the theory of $\mathfrak{B}$ is known to be decidable (cf. [15]), it follows, by means of 5.4, that the theory of $\mathfrak{D}$ is decidable [19]. It can be shown that the ordinary addition $+$, restricted to ω^n, is elementarily definable in terms of $<$ and N, and so one obtains also the result that the theory of $\langle \omega^n, +\rangle$ is decidable. (For a further discussion of the elementary theory of addition of ordinals, cf. § 10.)

Another application is to the elementary theory of addition of cardinal numbers, the system of which can be thought of as a generalized sum of the systems of addition of finite and of infinite cardinals. Speaking precisely, let ϱ be an arbitrary ordinal number different from 0. Let L_ϱ be the ternary relation among elements of ϱ defined by:

$$\langle \alpha, \beta, \gamma\rangle \in L_\varrho \text{ if and only if } \alpha \leqslant \beta \text{ and } \beta = \gamma \text{ or } \beta \leqslant \alpha \text{ and } \alpha = \gamma .$$

Let $\mathfrak{A}^{(0)} = \langle \omega, S\rangle$, where S is as defined in the preceding paragraph, and $\mathfrak{A}^{(1)}_\varrho = \langle \varrho, L_\varrho\rangle$. Pick $c_0 \notin \omega$ and $c_1 \notin \varrho$ and form

$$\mathfrak{A}'^{(0)} = \langle \omega \cup \{c_0\}, S, \{c_0\}\rangle \quad \text{and} \quad \mathfrak{A}'^{(1)}_\varrho = \langle \varrho \cup \{c_1\}, L_\varrho, \{c_1\}\rangle ,$$

[17] For the ordinal sum, this result was obtained by Beth [1]. The general result, announced by the first author in [4] and [5], was originally obtained by a more direct argument than that used here, based on an analogue of 3.1 applying only to finite products.

[18] "Natural sum" is understood as in [9], p. 68. The notation ω^α is to be understood here as the usual ordinal exponentiation, *not* as the set of functions on α to ω.

[19] Mostowski [13] already showed that the theory of $\langle \omega^n, N\rangle$ (or, indeed, $\langle \omega^\alpha, N\rangle$, where α is any ordinal) is decidable.

just as in 4.7. Now take C_ϱ to be the subset of $(\omega \cup \{c_0\}) \times (\varrho \cup \{c_1\})$ defined by the condition:

$$f \in C_\varrho \text{ if and only if } f_0 \in \omega \text{ and } f_1 = c_1, \text{ or } f_0 = c_0 \text{ and } f_1 \in \varrho .$$

Further define a ternary relation V_ϱ among elements of C_ϱ by the condition:

$$\begin{aligned}\langle f, g, h\rangle \in V_\varrho \text{ if and only if } & f_1 = g_1 = c_1 \text{ and } \langle f_0, g_0, h_0\rangle \in S \\ \text{or } & f_0 = c_0 \text{ and } g_1 = c_1 \text{ and } f_1 = h_1 \\ \text{or } & f_1 = c_1 \text{ and } g_0 = c_0 \text{ and } g_1 = h_1 \\ \text{or } & f_0 = g_0 = c_0 \text{ and } \langle f_1, g_1, h_1\rangle \in L_\varrho .\end{aligned}$$

Put $\mathfrak{C}_\varrho = \langle C_\varrho, V_\varrho\rangle$. Then $\mathfrak{C}_\varrho$ is a relativized generalized product of the two systems $\mathfrak{A}'^{(0)}$ and $\mathfrak{A}'^{(1)}_\varrho$. Moreover, by means of the mapping which sends f into f_0, if $f_0 \in \omega$, and into $\aleph_{f_1}$, if $f_1 \in \varrho$, we see that $\mathfrak{C}_\varrho$ is isomorphic to the system $\mathfrak{C}'_\varrho = \langle C'_\varrho, +\rangle$, where C'_ϱ is the set of all cardinal numbers less than $\aleph_\varrho$ and $+$ is the operation of addition of cardinals restricted to this set.

We can now apply to the construction of the preceding paragraph certain consequences of the results detailed by Mostowski and Tarski in [14]. These are:

(i) For any ordinal $\varrho \neq 0$, the theory of $\mathfrak{A}^{(1)}_\varrho$ is decidable.

(ii) For any ordinals ϱ, ϱ' different from 0, $\mathfrak{A}^{(1)}_\varrho$ and $\mathfrak{A}^{(1)}_{\varrho'}$ are elementarily equivalent if and only if either $\varrho = \varrho' < \omega^\omega$ or for some $\xi \neq 0$, $\eta \neq 0$ and $\gamma < \omega^\omega$, $\varrho = \omega^\omega \cdot \xi + \gamma$ and $\varrho' = \omega^\omega \cdot \xi' + \gamma$.

It follows, from (i) 5.4, 4.7, and the decidability of $\mathfrak{A}^{(0)}$ (cf. [14]), that: *For any ordinal* $\varrho \neq 0$, $Th(\mathfrak{C}'_\varrho)$, *i. e., the theory of addition of cardinal numbers less than* $\aleph_\varrho$, *is decidable.* On the other hand, we can obtain from (ii) 5.1, and 4.7, that: *For any ordinals* $\xi \neq 0$, $\eta \neq 0$, $\gamma < \omega^\omega$, ϱ, ϱ', *for which* $\varrho = \omega^\omega \cdot \xi + \gamma$ *and* $\varrho' = \omega^\omega \cdot \xi' + \gamma$, *the systems* $\mathfrak{C}'_\varrho$ *and* $\mathfrak{C}'_{\varrho'}$ *are elementarily equivalent.* In this way we have obtained some results originally found by Tarski [20]. (The theories of addition of cardinal numbers will play a further role in § 8.)

THEOREM 5.5. *The decision problem for the theory of the generalized power* $\mathfrak{B}^{\mathfrak{S}}$ *reduces to the decision problems for the theories of* $\mathfrak{B}$ *and of* $\mathfrak{S}$. *In particular, the first theory is decidable if the second and third are. The same applies to the power operations of* § 4 *(*$\mathfrak{S}$ *still being taken as the appropriate subset algebra).*

[20] Cf. [26], p. 118, footnote 20, where it is remarked that the decision problem for the theory of addition of cardinal numbers may be reduced to that for the theory of addition of natural numbers and that for the theory of the "less than" relation among ordinals. This reduction procedure, which was obtained directly by Tarski, prior to the work on finite generalized sums [4], deserves to be mentioned along with Beth's result [1] as a forerunner of this work.

Proof. As already remarked, in this case $K_\theta^{\mathfrak{A}}$, θ being a sentence, is always either Λ or I. A decision method for the theory of $\mathfrak{B}$ allows one to decide which. By 3.1.2, the question whether a sentence Γ holds in $\mathfrak{B}^{\mathfrak{S}}$ is the question whether

$$\dashv_{\mathfrak{S}}\Phi[K_{\theta_0}^{\mathfrak{A}}, \ldots, K_{\theta_n}^{\mathfrak{A}}];$$

and this becomes the question whether a certain sentence, say $\Phi[\Lambda, \bar{\Lambda}, \bar{\Lambda}, \ldots, \Lambda]$, holds in $\mathfrak{S}$.

As regards the operations 4.7 and 4.8, 5.5 yields a weaker result than 5.4 (cf. a remark in footnote (16)). In connection with the operation 4.4 we make use of the obvious consequence of 4.4.2 that the decision problem for the theory of a quotient system (like that in § 4.4) reduces to that for the theory of the original system.

THEOREM 5.6. *Let $\mathcal{K}$ be any class of relational systems ($\mathcal{K} \subseteq \mathcal{A}$). Let $\mathcal{P}_{\mathcal{K},\mathcal{S}}$ be the class of all product systems $\mathcal{P}(\mathfrak{A}, \mathfrak{S})$ such that, for some I, $\mathfrak{S} = \langle S(I), \ldots\rangle \in \mathcal{S}$ and, for each $i \in I$, $\mathfrak{A}^{(i)} \in \mathcal{K}$. Suppose that $Th(\mathcal{K})$ is decidable and $Th(\mathcal{S})$ is decidable. Then $Th(\mathcal{P}_{\mathcal{K},\mathcal{S}})$ is decidable. Here $\mathcal{P}(\mathfrak{A}, \mathfrak{S})$ can mean any (possibly relativized) generalized product, or any of the products or sums of § 4 (the class $\mathcal{S}$ being chosen, in each case, as indicated in § 4)* (21).

Proof. We will consider only the case in which $\mathcal{P}(\mathfrak{A}, \mathfrak{S})$ means the generalized product, as the other cases follow from this case by the same devices used all through this §.

Let Γ be a sentence of the language corresponding to the product systems. We want to decide whether or not

(1) $$\Gamma \in Th(\mathcal{P}_{\mathcal{K},\mathcal{S}}).$$

Applying 3.1 we find an acceptable partitioning sequence $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m\rangle$ such that, whenever, $\mathfrak{A}$ and $\mathfrak{S}$ are as in 5.6,

(2) $$\dashv_{\mathcal{P}(\mathfrak{A},\mathfrak{S})}\Gamma$$

if and only if

(3) $$\dashv_{\mathfrak{S}}\Phi[K_{\theta_0}^{\mathfrak{A}}, \ldots, K_{\theta_m}^{\mathfrak{A}}].$$

For each $j < m$, we may, by assumption, determine whether or not

(4) $$\sim\theta_j \in Th(\mathcal{K}).$$

Let t be the set of those $j \leqslant m$ for which (4) does hold.

(21) Theorem 5.6 was found recently, after some related work of Dana Scott. See footnote (25).

Consider, now, the sentence

(5) $$\Psi = \wedge \mathrm{X}_0 \ldots \wedge \mathrm{X}_m \{\mathrm{Part}_m(\mathrm{X}_0, \ldots, \mathrm{X}_m) \wedge \prod_{j\in t} (\mathrm{X}_j = \Lambda) \rightarrow \Phi(\mathrm{X}_0, \ldots, \mathrm{X}_m)\}.$$

By assumption, we can decide whether or not

(6) $$\Psi \in Th(\mathcal{S}).$$

Finally, it is not difficult to see that (1) and (6) are equivalent.

We shall show that (1) implies (6); the proof that (6) implies (1) is similar, though even simpler. Suppose (1) holds, let $\mathfrak{S} = \langle S(I), \ldots\rangle$ be any member of $\mathcal{S}$, and let $X_0, \ldots, X_m$ be any subsets of I forming a partition of I and such that $X_j = \Lambda$, for each $j \in t$. From (5) we see that, to establish (6), we must show that

(7) $$\dashv_{\mathfrak{S}}\Phi[X_0, \ldots, X_m].$$

Now, for each p such that $p \leqslant m$ and $p \notin t$, θ_p must be true in at least one member of $\mathcal{K}$, call it $\mathfrak{B}_p$. Now we form the indexed family $\langle \mathfrak{A}^{(i)} | i \in I\rangle$ as follows: For each $i \in I$, i is a member of exactly one X_p, where $p \notin t$; we put $\mathfrak{A}^{(i)} = \mathfrak{B}_p$. As a result of this construction and the fact that ζ was chosen to be a partitioning sequence, we have

(8) $$K_{\theta_j}^{\mathfrak{A},\mathfrak{S}} = X_j, \quad \textit{for each} \quad j \leqslant m.$$

Now, from (1) we infer (2) and hence also (3); and from (3) and (8) we infer (7), which was to be proved.

Using the kind of arguments made in the proof of this theorem, with very little change, one can infer two further consequences which may be of interest. The first is that, independent of the decidability of the theories involved, the theory of $\mathcal{P}_{\mathcal{K},\mathcal{S}}$ is a function only of the theories of $\mathcal{K}$ and $\mathcal{S}$. In other words: If $Th(\mathcal{K}) = Th(\mathcal{K}')$ and $Th(\mathcal{S}) = Th(\mathcal{S}')$ then $Th(\mathcal{P}_{\mathcal{K},\mathcal{S}}) = Th(\mathcal{P}_{\mathcal{K}',\mathcal{S}'})$. Secondly, it may be remarked that: *If a sentence Γ holds in some member of $\mathcal{P}_{\mathcal{K},\mathcal{S}}$ then Γ holds in some member of $\mathcal{P}_{\mathcal{L},\mathcal{S}}$, for some finite subset $\mathcal{L}$ of $\mathcal{K}$.*

To obtain further results, we must delve more deeply into the theories of particular subset algebras. Some of these will be considered in § 6-§ 8.

§ 6. The simplest subset algebras

Algebraic systems of the form

(1) $$\mathfrak{S}_I = \langle S(I), \Lambda, \cup, \cap, {}^{-}, \subseteq\rangle \quad (I \neq \Lambda)$$

are the simplest type which may be used in forming the generalized product. The theories of such systems (and the theory of the class of all such systems) were investigated by Skolem [17], who established the now well-known results 6.1 and 6.2 below.

6.1. (.1) *To each formula Φ (from the language corresponding to systems like (1)), whose free variables are $X_0, \ldots, X_{m-1}$, we may correlate (effectively) a natural number M, functions $p^{(k)} \in \omega^{S(m)}$ $(k = 0, \ldots, M-1)$, and subsets $U^{(k)}$ of $S(m)$ $(k = 0, \ldots, M-1)$ such that: given any system* (1) *and any $X = \langle X_0, \ldots, X_n, \ldots \rangle \in S(I)^\omega$,*

$$\dashv_{\mathfrak{S}_I} \Phi[X]$$

if and only if

there exists $k < M$ such that, for each $r \subseteq m$, the set

$$\bigcap_{j \in r} X_j \cap \bigcap_{j \in m-r} (I - X_j)$$

has exactly $p^{(k)}(r)$ elements, if $r \in U^{(k)}$, and has at least $p^{(k)}(r)$ elements if $r \notin U^{(k)}$.

(Here the intersection of a sequence of sets over the empty set of indices is taken to be I.)

(.2) *In particular, there exist sets $s_0, \ldots, s_{M-1} \subseteq m$ and functions $g^{(0)}, \ldots, g^{(M-1)} \in \omega^m$ such that if $X_0, \ldots, X_{m-1}$ form a partition of I, then*

$$\dashv_{\mathfrak{S}_I} \Phi[X]$$

if and only if

there exists $k < M$ such that, for each $j < m$, X_j has exactly $g_j^{(k)}$ elements if $j \in s_k$, and X_j has at least $g_j^{(k)}$ elements if $j \notin s_k$.

6.2. (.1) *The theory of any one system of the form* (1) *is decidable.*
(.2) *The theory of the class of all such systems is, also, decidable.*

With reference to 6.2.2 it may be remarked that it also follows easily from 6.1 that the theory of the class of all systems of the form (1) is the same as the theory of all these systems $\mathfrak{S}_I$ in which I is finite. Another very direct consequence of 6.1 is the following:

COROLLARY 6.3. *Two systems $\mathfrak{S}_I$ and $\mathfrak{S}_{I'}$ of the form* (1) *are elementarily equivalent if and only if I and I' are both finite, with the same number of elements, or I and I' are both infinite.*

From 6.2.1, 5.5, and 4.1, we obtain immediately 6.4.1 below, which was one of the principal results of Mostowski's paper [13]. The second part of this theorem, which also follows easily from the work in [13] (although it was not stated there), is obtained here by using 6.2.2 and 5.6 (applied to the case that $\mathcal{K}$ consists of a single system $\mathfrak{B}$).

THEOREM 6.4. *Suppose that the theory of a system $\mathfrak{B}$ is decidable. Then:*

(.1) *The theory of any one (finite or infinite) direct power of $\mathfrak{B}$ is decidable.*

(.2) *The theory of all (finite or infinite) direct powers of $\mathfrak{B}$ is decidable* [22].

The following theorem is an immediate consequence of 6.3 and 5.3; it could also be obtained directly from the work of Mostowski in [13].

THEOREM 6.5. *If $\mathfrak{B}$ and $\mathfrak{B}'$ are elementarily equivalent systems and I and I' are either both finite, with the same number of elements, or both infinite, then the corresponding direct powers $\mathfrak{B}^I$ and $(\mathfrak{B}')^{I'}$ are elementarily equivalent.*

By making use of the remarks at the end of § 5 and following 6.2 it can be seen that, for any system $\mathfrak{B}$, the theory of all (finite or infinite) direct powers of $\mathfrak{B}$ is identical with the theory of all finite direct powers of $\mathfrak{B}$. However, it will be seen that a more general statement concerning arbitrary direct products follows directly from 6.7.1 below. Of more interest is the fact that 6.7.1 provides a direct answer to a question raised by J. Łoś [23]. 6.7 will be obtained as a corollary of the (in certain respects more informative) theorem which we now turn to.

THEOREM 6.6. *Given any sentence ψ (of the language L_μ of § 3) we can find (effectively) a number $N \in \omega$ such that: whenever ψ is true in the direct product of systems $\mathfrak{A}^{(i)}$ $(i \in I)$, there is a set $I' \subseteq I$, having at most N elements, and such that ψ is true in the direct product of $\mathfrak{A}^{(i)}$ $(i \in I'')$, provided $I' \subseteq I'' \subseteq I$.*

Proof. By 3.2 and 4.1, we can find an acceptable partitioning sequence $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle$ $(\theta_0, \ldots, \theta_m$ being sentences) such that for any indexed family $\mathfrak{A} = \langle \mathfrak{A}^{(i)} \mid i \in I \rangle$ with direct product $\mathfrak{D}(\mathfrak{A})$

$$(2) \qquad \dashv_{\mathfrak{D}(\mathfrak{A})} \psi$$

if and only if

$$(3) \qquad \dashv_{\mathfrak{S}_I} \Phi[K^{\mathfrak{A}}_{\theta_0}, \ldots, K^{\mathfrak{A}}_{\theta_m}].$$

By 6.1.2 and Definition 2.1, we may, therefore, find M, $q^{(0)}, \ldots, q^{(M-1)}$, $s_0, \ldots, s_{M-1}$ such that for any $\langle \mathfrak{A}^{(i)} \mid i \in I \rangle$, (3) (and hence, also (2)) is equivalent to the statement:

[22] The system $\mathfrak{B}$ (and all factors considered in this §) may be taken from an arbitrary similarity class $\mathcal{A}$. The notion of direct product for a relation with any number of places is defined in a way analogous to that in 4.2; if each factor has several relations, the product has one relation corresponding to each of them.

[23] Łoś stated the question in a letter to the second author in December, 1952.

(4) *there exists* $k < M$ *such that for any* $j \leqslant m$, $\dashv_{\mathfrak{A}^{(i)}}\theta_j$ *for exactly* $q_j^{(k)}$ *members* i *of* I, *if* $j \in s_k$, *and* $\dashv_{\mathfrak{A}^{(i)}}\theta_j$ *for at least* $q_j^{(k)}$ *members* i *of* I, *if* $j \notin s_k$.

Let $n_k = q_0^{(k)} + \ldots + q_m^{(k)}$, for $k = 0, \ldots, M-1$ and $N = \max\{0, n_0, \ldots, n_{M-1}\}$.

Now suppose that $\langle \mathfrak{A}^{(i)} \mid i \in I \rangle$ is a particular indexed family, and, for each $J \subseteq I$, write $\mathfrak{D}_J$ for the direct product of $\langle \mathfrak{A}^{(i)} \mid i \in J \rangle$. Suppose, further that $\dashv_{\mathfrak{D}_I} \psi$. Then (4) holds, and we may find a particular number k with the property demanded in (4). Since ζ is a partitioning sequence, it follows from (4) that we may find pairwise disjoint sets $X_0, \ldots, X_m \subseteq I$, such that, for each $j \leqslant m$, X_j has exactly $q_j^{(k)}$ elements, and, for each $i \in X_j$, $\dashv_{\mathfrak{A}^{(i)}}\theta_j$.

Let $I' = X_0 \cup \ldots \cup X_m$. Clearly, I' has n_k $(\leqslant N)$ elements. Now suppose $I' \subseteq I'' \subseteq I$. Certainly, for each $j \leqslant m$, $\dashv_{\mathfrak{A}^{(i)}}\theta_j$ for at least $q_j^{(k)}$ members i of I'', namely, the members of X_j. Moreover, in case $j \in s_k$, it must be that

$$\dashv_{\mathfrak{A}^{(i)}}\theta_j \text{ for exactly } q_j^{(k)} \text{ members } i \text{ of } I'',$$

since X_j contains all such i belonging to I.

Thus, condition (4) (with the same value of "k") is fulfilled by the family $\langle \mathfrak{A}^{(i)} \mid i \in I'' \rangle$, and so, by the equivalence of (4) and (2), we have

$$\dashv_{\mathfrak{D}_{I''}} \psi$$

as demanded in 6.4.

COROLLARY 6.7. (.1) *For any family of systems* $\langle \mathfrak{A}^{(i)} \mid i \in \omega \rangle$, *if a (first order) sentence* θ *is true in each of the direct products* $\mathfrak{A}^{(0)}, \mathfrak{A}^{(0)} \times \mathfrak{A}^{(1)}, \ldots, \mathfrak{A}^{(0)} \times \ldots \times \mathfrak{A}^{(n)}, \ldots$, *then it is true in the infinite direct product* $\mathfrak{A}^{(0)} \times \mathfrak{A}^{(1)} \times \ldots \times \mathfrak{A}^{(n)} \times \ldots$

(.2) *If the class of all models of a set of first order sentences is closed under the operation of taking the direct product of two systems, then it is closed under the operation of taking arbitrary, infinite, direct products* [24].

[24] 6.6 and 6.7.1 were established by Mostowski [13] for direct powers. In his letter raising the question answered by 6.7.1, Łoś stated he had obtained an affirmative answer for the case that θ is a universal sentence. 6.7.2 was established in [23] for sets of universal sentences.

Theorems 6.6, 6.7, and 6.8 were reported in [28], and proved in the second author's dissertation (cf. footnote [1]). The method of proof there consisted in a straightforward generalization of Mostowski's work on direct powers to the case of direct products. (Mostowski [13] indicated that many of his results on ordinary and weak direct powers could be extended to the corresponding products; but did not carry this out because 6.4.1 and 7.4.1 could not be so extended). This proof, like Mostowski's, had the feature that the machinery of Skolem's 6.1 is involved during proof of the "elimination of quantifiers" theorem (the analogue of 3.1). The idea of the present proof of 3.1 which, even if applied only to direct powers, avoids this feature, was found by the first author (cf. footnote [10]).

Another consequence of 6.6 is the following

THEOREM 6.8. *Let* I *be an arbitrary set and* $\langle \mathfrak{A}^{(i)} \mid i \in I \rangle$ *an indexed family of relational systems. Then there exists a countable subset* $I' \subseteq I$ *such that, whenever* $I' \subseteq I'' \subseteq I$, *the direct product of the* $\mathfrak{A}^{(i)}$ $(i \in I)$ *and that of the* $\mathfrak{A}^{(i)}$ $(i \in I'')$ *are elementarily equivalent.*

Proof. The language appropriate for systems $\mathfrak{A}^{(i)}$ has only denumerably many sentences. To each such sentence θ which is true in the direct product of the $\mathfrak{A}^{(i)}$ $(i \in I)$, a set I'_θ may be associated as in 6.6. Then the conclusion of 6.9 holds if we take for I' the union of all the sets I'_θ.

We have thus far in this § applied 5.6 and 6.2.2 to direct products only in the case that $\mathcal{K}$ consists of a single system (6.4.2). Turning now to the general case we obtain immediately

THEOREM 6.9. *Let* $\mathcal{K}$ *be a class of relational systems* $(\mathcal{K} \subseteq \mathcal{Rl})$. *Let* $\mathcal{K}'$ *be the class of all direct products of arbitrary (finite or infinite) indexed families of members of* $\mathcal{K}$. *If* $Th(\mathcal{K})$ *is decidable, so is* $Th(\mathcal{K}')$.

All results of this section have been obtained from the basic results of § 3, § 5, and from 6.1, 6.2 by means of the fact, established in 4.1, that the operation of forming the direct product is an example of a generalized product operation. Since no other features of the direct product were involved, it follows that each of the theorems of this section holds equally well for any operation which can be interpreted as a generalized or relativized generalized product relative to subset algebras of the form (1). In particular (also using the results of § 4.7), one obtains in this way

THEOREM 6.10. 6.4-6.9 *remain valid if the words "direct power" and "direct product" are everywhere replaced, respectively, by "cardinal multiple" and "cardinal sum"* [25].

§ 7. Subset algebras with *Fin*

It is a simple matter to obtain results, similar to Skolem's 6.1 and 6.2, which apply to systems of the form

$$(1) \qquad \mathfrak{S}'_I = \langle S(I), \Lambda, \cup, \cap, {}^-, \subseteq, Fin \rangle \qquad (I \neq \Lambda),$$

considered in § 4.2 and § 4.4. In fact, a straightforward inductive (or "quantifier-eliminating") argument yields the following theorem:

[25] Theorem 5.6 and its corollaries in 6.4.2, 6.9, 6.10, 7.4.2 and 7.9 were obtained jointly by the authors and Dana Scott. The work on these theorems was stimulated by Scott's discovery that they have a number of interesting applications to proofs of decidability of certain theories. Scott plans to publish his work on these applications in the near future.

THEOREM 7.1. *Given any formula Φ (from the language appropriate for systems like* (1)*), whose free variables are $\{X_j \mid j \in t\}$, we may find (effectively) a natural number M, functions $p^{(k)} \in \omega^{S(t)}$ $(k = 0, \ldots, M-1)$, and partitions $\langle U_1^{(k)}, U_2^{(k)}, U_3^{(k)}, U_4^{(k)}\rangle$ of $S(t)$ $(k = 0, \ldots, M-1)$ such that: given any system* (1) *and any $X = \langle X_0, \ldots, X_n, \ldots\rangle \in S(I)^{\omega}$,*

$$\dashv_{\mathfrak{S}_I'} \Phi[X]$$

if and only if there exists $k < M$ such that, for each $r \subseteq t$, the set

$$\bigcap_{j \in r} X_j \cap \bigcap_{j \in t-r} (I - X_j)$$

has exactly $p^{(k)}(r)$ elements, if $r \in U_1^{(k)}$,
has at least $p^{(k)}(r)$ elements, if $r \in U_2^{(k)}$,
has at least $p^{(k)}(r)$ and finitely many elements, if $r \in U_3^{(k)}$, and
has infinitely many elements, if $r \in U_4^{(k)}$.

The next two theorems follow easily from 7.1 *(with $t = \Lambda$)*.

THEOREM 7.2. (.1) *The theory of any one system $\mathfrak{S}_I'$ of the form* (1) *is decidable.*

(.2) *The theory of the class of all such systems is, also, decidable.*

THEOREM 7.3. *Two systems $\mathfrak{S}_I'$ and $\mathfrak{S}_{I'}'$ of the form* (1) *are elementarily equivalent if and only if I and I' are both finite, with the same number of elements, or I and I' are both infinite.*

It also follows from 7.2.1 and 7.3 (or directly from 7.1) that the theory of the class of all systems $\mathfrak{S}_I'$ of the form (1), for which I is infinite, is complete and decidable.

From 7.2.1, 5.5, and 4.2, we obtain Theorem 7.4.1 below, which is another of the basic results of Mostowski's article [13]. On the other hand, 7.2.2 and 5.6 are used in the derivation here of the new result 7.4.2.

THEOREM 7.4. *If the theory of a system $\mathfrak{B}$ is decidable then*:

(.1) *The theory of any one weak direct power of $\mathfrak{B}$ is decidable.*

(.2) *The theory of the class of all weak direct powers of $\mathfrak{B}$ is decidable.*

A number of interesting applications of 7.4.1 are discussed by Mostowski in [13].

The first of the following examples of the application of 7.4.2 was communicated to us by Dana Scott. Let $\mathfrak{J}_p$ be the cyclic group of order p (a fixed prime). Let $\mathcal{L}_1$ be the class of all Abelian groups of which every non-zero element is of order p. Then, as is known, $\mathcal{L}_1$ consists of all (systems isomorphic to) weak direct powers of $\mathfrak{J}_p$. Hence, by 7.4.2, $Th(\mathcal{L}_1)$ is decidable.

Another, very similar, example is provided by considering the group $\mathfrak{J}$ of all integers. Let $\mathcal{L}_2$ be the class of all free Abelian groups (including the group with one generator). $\mathcal{L}_2$ consists of all (systems isomorphic to) weak direct powers of $\mathfrak{J}$. Hence, from 7.4.2 and the fact of the decidability of the theory of $\mathfrak{J}$ (obtained by Presburger in [15]), it follows that $Th(\mathcal{L}_2)$ is, also, decidable.

By modifying 7.4.2, some other slight variations of the above two examples can also be treated. For example, it is easily seen that the theory of all free Abelian groups with at least n free generators is decidable, for any natural number n (26).

The derivation of 7.4 described earlier would apply as well to any (possibly relativized) generalized power relative to subset algebras of the form (1). In particular from 7.2, 5.5, and 4.4, we derive

THEOREM 7.5. *If the theory of a system $\mathfrak{B}$ is decidable, so is the theory of any one "almost everywhere direct power" of $\mathfrak{B}$ and so, also, is the theory of all such powers.*

Let us now turn to problems concerning elementary equivalence. We can derive immediately, from 7.3 and 5.3 the following

THEOREM 7.6. *If $\mathfrak{B}$ and $\mathfrak{B}'$ are elementarily equivalent systems and I and I' are both infinite then the corresponding weak direct powers $\mathfrak{B}^I$ and $(\mathfrak{B}')^{I'}$ are elementarily equivalent* (27).

The following application of 7.6 is of interest in connection with the discussion above of some classes of Abelian groups. Let $\mathcal{L}_1'$ be the class of all infinite groups in $\mathcal{L}_1$ (as described earlier) and let $\mathcal{L}_2'$ be the class of all free Abelian groups with infinitely many free generators. Then, as a consequence of 7.4.1 and 7.6, it is seen that both $Th(\mathcal{L}_1')$ and $Th(\mathcal{L}_2')$ are complete and decidable.

Theorems 6.6 and 6.7 cannot be extended to weak direct products (or even powers (28)). A simple counter example, which was pointed out by Tarski, is provided by any sentence interpretable as saying "$\mathfrak{B}$ is a Boolean algebra". The (weak-ordinary) direct product of finitely many Boolean algebras is again a Boolean algebra, but the infinite weak direct product (relative to the zero element) never is, if the factors all have

(26) The decidability of the various theories of special classes of Abelian groups mentioned in this § follow from the decision method for the theory of all Abelian groups given by W. Szmielew in [19].

Another proof of the decidability of $Th(\mathcal{L}_1)$ was given by Henkin in [11]. Henkin's argument is very short. However, the method of Szmielew and our method each yield primitive recursive decision methods, while Henkin's does not. Moreover, his method does not extend to the case of $Th(\mathcal{L}_2)$.

(27) As with 6.5 (for ordinary direct powers), this theorem 7.6 follows directly from the work of Mostowski in [13].

(28) This contradicts an erroneous statement (theorem 5.32) in [13] regarding weak direct powers.

at least two elements, because, roughly speaking, of the difficulty in taking the complement of the new zero element.

On the other hand, one can obtain

THEOREM 7.7. *Theorem 6.8 applies also to weak direct products (and to "almost everywhere direct products").*

Proof. One first obtains, by an argument similar to the proof of 6.6, using 7.1 in place of 6.1, an analogue of 6.7 in which it is only asserted that I' is countable. One then argues as in the proof of 6.6. (For the "almost everywhere direct power", some obvious additional remarks are necessary.)

We now turn our attention to decidability questions relating to arbitrary weak direct products. Before proceeding to the analogue of 6.9, it it perhaps worth noting that some results can be obtained concerning individual infinite products, other than those already obtained for powers. One such is the following:

THEOREM 7.8. *Let $\mathfrak{A} = \langle \mathfrak{A}^{(i)} \mid i \in I \rangle$ be such that for each $i \in I$, there are infinitely many distinct $i' \in I$ with $\mathfrak{A}^{(i')}$ elementarily equivalent (or isomorphic) to $\mathfrak{A}^{(i)}$. Let $\mathcal{K} = \{\mathfrak{A}^i \mid i \in I\}$. If the theory of $\mathcal{K}$ is decidable, then so is the theory of the direct product, or the weak product, or the "almost everywhere direct product", or the cardinal sum of the $\mathfrak{A}^{(i)}$ $(i \in I)$.*

Proof. From 3.1 (and 3.2 and § 4), we see that question of the truth of a given sentence in the product (or sum) system reduces to whether certain $K^{\mathfrak{A}}_{\theta_0}, \ldots, K^{\mathfrak{A}}_{\theta_m}$ satisfy in $\mathfrak{S}_I$ or $\mathfrak{S}'_I$ a certain formula Φ. By 6.1 or 7.1, this becomes (roughly speaking) a matter of whether $K^{\mathfrak{A}}_{\theta_0}$ has exactly p, or at least p, or at least p and finitely many, or infinitely many elements, and similarly for $\theta_1, \ldots, \theta_m$. Under the hypotheses of 7.6, $K^{\mathfrak{A}}_{\theta_j}$ is either empty or infinite, respectively, according as the sentence θ_j is or is not (valid) in the theory of $\mathcal{K}$. From these three facts the desired result easily follows.

From 5.6 and 7.2.2 follows immediately the following direct analogue of 6.9:

THEOREM 7.9. *Let $\mathcal{K}$ be a class of relational systems ($\mathcal{K} \subseteq \mathcal{A}$). Let $\mathcal{K}'$ be the class of all weak direct products of arbitrary (finite or infinite) indexed families of members of $\mathcal{K}$. If $Th(\mathcal{K})$ is decidable, so is $Th(\mathcal{K}')$.*

For an example of the application of 7.9, let n be an arbitrary positive integer and let $\mathcal{K}$ consist of all cyclic groups of order p^k, where p^k runs over all prime power divisors of n. Let $\mathcal{L}_3$ be the class of all Abelian groups in which every non-zero element has as order a divisor of n. Then, as is known (cf. [12], p. 17), every group in $\mathcal{L}_3$ is isomorphic to a weak direct product of groups from $\mathcal{K}$ and conversely. Hence $Th(\mathcal{L}_3) = Th(\mathcal{K}')$ and, by the above theorem, $Th(\mathcal{L}_3)$ is decidable.

We wish, finally, to consider a question about elementary equivalence of systems which is closely connected with the preceding example. Let us call an Abelian group $\mathfrak{B} = \langle B, + \rangle$ *a group of the first kind* if the orders of all its elements are bounded above, hence if there is a fixed integer n which is divisible by all orders of its non-zero elements. Let m be an arbitrary positive integer. We say that the elements $x_0, \ldots, x_{r-1}$ of $\mathfrak{B}$ are *strongly independent modulo m* if for every sequence $a_0, \ldots, a_{r-1}$ of integers the condition

$$\textit{there exists a } y \in B \textit{ such that } a_0x_0 + \ldots + a_{r-1}x_{r-1} = my$$

implies

$$a_i \equiv 0 \ (\mathrm{mod}\ m) \quad \textit{for} \quad i = 0, \ldots, r-1 .$$

For each prime p and integer $k > 0$, set $\varrho[p, k](\mathfrak{B})$ equal to the maximum finite number r, if it exists, of elements in $\mathfrak{B}$ which are *strongly independent modulo p^k and of order p^k*, and equal to ω if no such maximum exists. It can be seen from the uniqueness of the representation of groups of the first kind as weak direct products of cyclic groups of prime power order (cf. [12], p. 27) that: $\varrho[p, k](\mathfrak{B})$ is a finite number r if and only if the number of cyclic groups of order p^k in the representation of $\mathfrak{B}$ is finite and equal to r.

We now wish to show that two Abelian groups $\mathfrak{B}$ and $\mathfrak{B}'$ of the first kind are elementarily equivalent if and only if $\varrho[p, k](\mathfrak{B}) = \varrho[p, k](\mathfrak{B}')$ for every prime p and integer $k > 0$. It is a direct matter to verify the necessity of this condition. The essential step is to construct for each p, k and r a sentence $\varphi^{(p,k,r)}$, of the language corresponding to groups, having the property that for each Abelian group $\mathfrak{B}$, $\dashv_{\mathfrak{B}} \varphi^{(p,k,r)}$ if and only if $\varrho[p, k](\mathfrak{B}) \geqslant r$. To establish the sufficiency, we proceed as follows. Represent $\mathfrak{B}$ as a weak direct product of groups $\mathfrak{A}^{(p,k)}$ (p prime, $k > 0$), where each $\mathfrak{A}^{(p,k)}$ is a weak direct power of the cyclic group of order p^k; if $\varrho[p, k](\mathfrak{B}) = r$ is finite then $\mathfrak{A}^{(p,k)}$ has exactly r such factors, otherwise it has infinitely many. In the same way represent $\mathfrak{B}'$ as a weak product of groups $\mathfrak{A}'^{(p,k)}$. We can then apply 7.6 to see that $\mathfrak{A}^{(p,k)}$ and $\mathfrak{A}'^{(p,k)}$ are elementarily equivalent systems for each p, k. Hence, it follows from 5.1 that $\mathfrak{B}$ and $\mathfrak{B}'$ are, also, elementarily equivalent [29].

A direct analogue of 7.1 (and 7.2) can be obtained, applying to subset algebras in which the singulary relation *Fin* is replaced by the singulary relation *Ctbl* of 4.4. Consequently, one can prove that 7.4 holds for countably weak, as well as weak, direct powers.

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[29] The decidability of $Th(\mathcal{L}_3)$ is a consequence of the work of Szmielew in [19]. The result about elementary equivalence of Abelian groups of the first kind is a special case of Theorem 5.2 of [19]. We use here the notation $\varrho[p, k]$ for what Szmielew denotes by $\varrho^{(3)}[p, k]$.

§ 8. Subset algebras involving set-theoretical equivalence

A type of subset algebra more powerful than those of § 6 and § 7 is

(1) $$\mathfrak{S}''_I = \langle S(I),\, \Lambda,\, \cup,\, \cap,\, {}^{-},\, \subseteq,\, \sim\rangle \qquad (I \neq \Lambda)$$

where $\sim$ (properly, $\sim_I$) is the binary relation of ordinary set-theoretical equivalence. We have shown, in § 5, how to obtain the result of Tarski that the theory of the addition of cardinal numbers (or of all cardinal numbers less than any one given one) is decidable [30]. From this result and the reduction theorem 8.1, given below, we shall infer that the theory of any system $\mathfrak{S}''_I$ is decidable.

It is obvious that $\mathfrak{S}''_I$ has a decidable theory if I is finite. We may therefore restrict our attention to the case when I is infinite (in order to simplify the notation needed in 8.1). Let L be the language appropriate to the discussion of systems (1), and let L' be the language corresponding to systems of the form

$$\mathfrak{C}'_I = \langle C'_I,\, 0,\, +\rangle,$$

where C'_I is the set of cardinal numbers $\leqslant \mathfrak{c}(I)$ (the cardinal number of the set I), and $+$ is the binary operation of addition of cardinals, restricted to C'_I. The non-logical symbols of L' are $\mathbf{0}$ and $+$.

Suppose t is a finite set of natural numbers, and for each $i \in t$, τ_i is a term of L'. We denote by

$$\sum_{j \in t} \tau_j,$$

the term $\mathbf{0}$, if t is empty, and the term

$$\Big(\sum_{j \in t-\{k\}} \tau_j\Big) + \tau_k,$$

if k is the largest member of t.

THEOREM 8.1. *With each formula Φ of L, having the n free variables $\{\mathrm{X}_j \mid j \in t\}$, and list $r_0, \ldots, r_{2^n-1}$ of the subsets of t, can be correlated (effectively) a formula ψ of L', whose free variables are, at most, $\mathrm{v}_0, \ldots, \mathrm{v}_{2^n-1}$, in such a way that, for any infinite set I, and any $X \in S(I)^\omega$,*

$$\dashv_{\mathfrak{S}''_I} \Phi[X]$$

if and only if

$$\dashv_{\mathfrak{C}_I} \Psi[\mathfrak{a}_0, \ldots, \mathfrak{a}_{2^n-1}],$$

where, for each $l < 2^n$,

$$\mathfrak{a}_l = \mathfrak{c}\Big(\bigcap_{j \in r_l} X_j \cap \bigcap_{j \in t-r_l} (I - X_j)\Big).$$

[30] See footnote [20].

Proof. Part (1). As regards the case in which Φ is atomic, we need consider only the special formulas $\mathrm{X}_k \subseteq \mathrm{X}_l$ and $\mathrm{X}_k \sim \mathrm{X}_l$, where $k \neq l$. This is because, when $k = l$, these formulas are always true (and hence trivial to handle), while all other atomic formulas of L are (as is well known) equivalent to formulas built up using only atomic formulas of these two types.

If Φ is one of these special formulas, the desired ψ is easily constructed, by noting that

$$X_k \subseteq X_l \quad \textit{if and only if} \quad \mathfrak{c}(X_k - X_l) = 0,$$

and

$$X_k \sim X_l \ \textit{if and only if} \ \mathfrak{c}(X_k \cap X_l) + \mathfrak{c}(X_k - X_l) = \mathfrak{c}(X_k \cap X_l) + \mathfrak{c}(X_l - X_k).$$

Part (2). Suppose that Φ is $\Phi^{(1)} \mid \Phi^{(2)}$, and 8.1 is known to apply to $\Phi^{(1)}$ and to $\Phi^{(2)}$. Let $t = t^{(1)} \cup t^{(2)}$, where, for $q = 1, 2$, $\{\mathrm{X}_j \mid j \in t^{(q)}\}$ are the $n^{(q)}$ free variables of $\Phi^{(q)}$. Let $r^{(q)}_0, \ldots, r^{(q)}_{2^{n^{(q)}}-1}$ be a list of the subsets of $t^{(q)}$ and let $\psi^{(q)}$ be the formula of L' correlated with $\Phi^{(q)}$ and the list $r^{(q)}$ $(q = 1, 2)$. Then, for ψ, we may take the formula

$$\psi^{(1)}(\tau^{(1)}_0, \ldots, \tau^{(1)}_{2^{n^{(1)}}-1}) \mid \psi^{(2)}(\tau^{(2)}_0, \ldots, \tau^{(2)}_{2^{n^{(2)}}-1});$$

where, for $q = 1, 2$, and $l < 2^{n^{(q)}}$,

$$\tau^{(q)}_l = \sum_{j \in s^{(q)}_l} \mathrm{v}_j$$

and

$$s^{(q)}_l = \{j \mid j < 2^n \ \textit{and} \ r^{(q)}_l \subseteq r_j \ \textit{and} \ t^{(q)} - r^{(q)}_l \subseteq t - r_j\}.$$

Part (3). Suppose that Φ is of the form $\bigvee \mathrm{X}_k \Phi'$, and 8.1 holds for Φ'. If X_k is not free in Φ, then for ψ we may take the formula already correlated with Φ' and the list r. Therefore, we assume that X_k is free in Φ and that the $n' = n+1$ free variables of Φ' are $\{\mathrm{X}_j \mid j \in t'\}$, where $t' = t \cup \{k\}$. Form the list

$$\langle r'_0, \ldots, r'_{2^{n'}-1}\rangle = \langle r_0 \cup \{k\}, \ldots, r_{2^n-1} \cup \{k\}, r_0, \ldots, r_{2^n-1}\rangle$$

of the subsets of t', and let ψ' be the formula correlated with Φ' and r'. Write $\mathrm{w}_j = \mathrm{v}_{2^n+j}$, for $j < 2^{n'}$. Let ψ be the formula

(2) $$\bigvee \mathrm{w}_0 \ldots \bigvee \mathrm{w}_{2^{n'}-1} \Big[\prod_{l<2^n} (\mathrm{w}_l + \mathrm{w}_{2^n+l} = \mathrm{v}_l) \wedge \psi'(\mathrm{w}_0, \ldots, \mathrm{w}_{2^{n'}-1})\Big].$$

It is straightforward to establish that ψ has the properties demanded in 8.1. We shall omit this argument, and only make the following heuristic remark. If $\dashv_{\mathfrak{S}_I''}\Phi'[X]$, then the powers $\mathfrak{b}_l$ $(l < 2^n)$ to be taken for the w_l in (2) are obtained as follows: The sets X_j $(j \in t)$ generate a partition of I. The set X_k divides each set Z of this partition into two complementary parts. The powers of these two parts are to be taken for w_l and w_{2^n+l}, when Z is the lth set of the partition, i. e., the set

$$\bigcap_{j \in r_l} X_j \cap \bigcap_{j \in t - r_l} (I - X_j) .$$

8.1 now follows from Parts (1), (2), and (3) above.

It is an immediate consequence of the results of Tarski concerning the systems $\mathfrak{C}_\varrho$ described following 5.4 that *the theory of any particular system* $\mathfrak{C}_I'$ *is decidable*. Combining this fact with 8.1, for the case $n = 0$ (and recalling our earlier remark about a finite set I), we obtain:

THEOREM 8.2. *If I is any set, then the theory of the system*

$$\mathfrak{S}_I'' = \langle S(I), \Lambda, \cup, \cap, {}^{-}, \supseteq, \sim \rangle$$

is decidable [31].

Skolem [18] obtained a decision method for the theory of the multiplication of positive integers, i. e., for the theory of the system $\langle \omega - \{0\}, \cdot \rangle$. Mostowski [13] showed that such a decision method could be obtained by means of his work on weak direct powers from Presburger's decision method for $\langle \omega, + \rangle$. By combining 8.2 and 5.5 we can obtain the following, somewhat stronger, result:

THEOREM 8.3. *A decision method can be constructed for the theory of the system*

$$\mathfrak{A} = \langle P, \cdot, \approx \rangle ,$$

where P is the set of positive integers, $\cdot$ is ordinary multiplication, and $\approx$ is the binary relation which holds between two positive integers provided they have the same number of distinct prime divisors.

Proof. Consider the relational system $\mathfrak{B} = \langle \omega, S \rangle$, where $\langle k, m, n \rangle \in S$ if and only if $k + m = \mathrm{p}$. Let C be the set of sequences $x \in \omega^\omega$ such that

(31) The theory of the class of all such systems is also decidable. This fact follows easily from 8.1 and the results of Tarski and Mostowski-Tarski described in § 5 (cf. footnote (30)).

It should be remarked that the decidability of the theory of $\mathfrak{C}_I'$ and, hence also, that of the theory of $\mathfrak{S}_I''$ is highly non-effective, in a certain sense, in the case, for example, that I is the set of all real numbers. Indeed, to apply the decision method one would need to know certain facts about the ordinal ϱ' such that $\aleph_{\varrho'}$ is the power of I. However, "practical" applications of 8.2 can be expected for the case $\mathfrak{c}(I) = \omega$. In this case, the work of Mostowski-Tarski may be avoided altogether, and (using a very special case of Tarski's reduction) the decision problem for the theory of $\mathfrak{C}_\omega'$ easily reduces to Presburger's results.

$x_n = 0$ for all but finitely many $n \in \omega$. If $x, y, z \in C$, let $\langle x, y, z \rangle \in U$ if and only if $x_n + y_n = z_n$ for all $n \in \omega$, and $\langle x, y \rangle \in V$ if and only if $\{n \mid x_n \neq 0\}$ and $\{n \mid y_n \neq 0\}$ have the same number of members. Let $\mathfrak{C} = \langle C, U, V \rangle$.

For each $x \in C$, put $f(x)$ equal to the product of the numbers $p_0^{x_0}, \ldots, p_n^{x_n}, \ldots$, where $p_0, \ldots, p_n, \ldots$ are the primes (say, in order). Then (disregarding the difference between an operation and the corresponding relation), f is an isomorphism of the system $\mathfrak{C}$ onto the system $\mathfrak{A}$. Moreover, the system $\mathfrak{C}$ is, clearly, a relativized generalized power of $\mathfrak{A}$ with respect to the subset algebra $\mathfrak{S}_\omega''$. Therefore, by 5.5 and 8.2, $Th(\mathfrak{C}) = Th(\mathfrak{A})$ is decidable.

§ 9. Generalized weak powers

As was seen in § 4, various notions of "weak products" (e. g., weak direct products, weak ordinal products) and, hence, in particular, the corresponding notions of "weak powers" may be interpreted as relativized generalized products. From the basic theorems of § 3, we may, therefore, infer that the discussion of the theory of a weak power system can, roughly speaking, be reduced to the discussion of the theory of the factor system and the theory of a certain algebra of subsets $\mathfrak{S}$, of the type discussed in § 2. It is possible, for the special case of weak powers, to strengthen this result by replacing the system $\mathfrak{S}$ by an algebra of *finite* subsets. To do this, a whole development parallel to that of § 2 and § 3 must be carried out. We shall outline this development in the present section.

As in § 2, we shall consider a similarity class $\mathcal{A}$ of relational systems of type μ. Moreover, we suppose given a (fixed) formula ψ of L_μ with the single free variable v_0. We shall not here be concerned with the whole class $\mathcal{A}$, but only with the class $\mathcal{A}^*$ of all systems $\mathfrak{B} \in \mathcal{A}$ such that $\mathfrak{B}$ has exactly one element satisfying ψ. For any $\mathfrak{B} \in \mathcal{A}^*$, this special element will be called $e_{\mathfrak{B}}$.

Also given is another similarity type $\mu' \in \omega^{\alpha'}$ $(2 \leqslant \alpha' \leqslant \omega)$ of relational systems, such that $\mu'(0) = 2$ and $\mu'(1) = 1$. Finally, we consider a fixed subclass $\mathcal{S}^*$ of the class of all algebraic systems

(1) $\quad \mathfrak{S} = \langle S^*(I), \Lambda, \cup, \cap, {}^{-}, \subseteq, Fin, M_2, \ldots, M_j, \ldots \rangle \quad (2 \leqslant j < \alpha')$,

where I is any non-empty set, $S^*(I)$ is the set of all finite subsets of I and their complements, $\cup$, $\cap$, etc., are the usual set-theoretical operations or relations, restricted to $S^*(I)$, Fin is the set of finite subsets of I, and, for $2 \leqslant j < \alpha'$, M_j is a μ_j'-ary relation. We denote the common type of all algebraic systems in $\mathcal{S}^*$ by σ. The language in which such systems could be discussed is, therefore, L_σ; its second predicate will be called Fin.

We shall not be concerned with an indexed family of $\mathcal{A}^*$-systems, but simply with a single $\mathcal{A}^*$-system (the "base") together with a subset algebra (1) (the "exponent").

DEFINITION 9.1. (.1) *Suppose that* $\mathfrak{B} = \langle B, \ldots\rangle \in \mathcal{A}^*$ *and* I *is any non-empty set. Then we put*

$$B^{(I)} = \{g \mid g \in B^I \text{ and } g(i) = e_{\mathfrak{B}} \text{ for all but finitely many } i \in I\}$$

(.2) *Suppose, also* $D = B^{(I)}$, $f \in D^\omega$, *and* θ *is a formula of* L_μ. *Then we put*

$$K_\theta^{*\mathfrak{B},I}(f) = \{i \mid i \in I \text{ and } \dashv_{\mathfrak{B}} \theta[f[i]]\}.$$

(.3) *In particular, if* θ *is a sentence, we write simply* $K_\theta^{*\mathfrak{B},I}$ *for* $K_\theta^{*\mathfrak{B},I}(f)$.

In 9.1.2, θ has, say, only the free variables $v_{k_0}, \ldots, v_{k_{n-1}}$, and each of $f_{k_0}(i), \ldots, f_{k_{n-1}}(i)$ must equal $e_{\mathfrak{B}}$ for all but finitely many $i \in I$. One sees, therefore, that, always,

$$K_\theta^{*\mathfrak{B},I}(f) \in S^*(I).$$

On the basis of 9.1, one can state and prove a lemma concerning K^* exactly analogous to 2.2. Without stating it explicitly we shall call this Lemma 9.2. The (new) notions of standard, or partitioning, acceptable sequences are defined in Definition 9.3, which can be 2.3 verbatim, though "L_σ" is now to be interpreted differently. We assume $\zeta_0^0, \ldots, \zeta_n^0, \ldots$ is an effective list of all standard acceptable sequences, like that of § 2.

DEFINITION 9.4. *Let* $\mathfrak{B} = \langle B, \ldots\rangle \in \mathcal{A}^*$, $\mathfrak{S} = \langle S^*(I), \ldots\rangle \in \mathcal{S}^*$, *and* $D = B^{(I)}$.

(.1) *For each standard acceptable sequence* $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m\rangle$, *with* p *free variables, we put*

$$Q_\zeta^{*\mathfrak{B},\mathfrak{S}} = \{\langle f_0, \ldots, f_{p-1}\rangle \mid f \in D^\omega \text{ and } \dashv_{\mathfrak{S}} \Phi[K_{\theta_0}^{*\mathfrak{B},I}(f), \ldots, K_{\theta_m}^{*\mathfrak{B},I}(f)]\}.$$

(.2) *By the generalized weak power* $\mathfrak{B}^{(\mathfrak{S})}$, *we mean the relational system*

$$\mathfrak{D} = \langle D, Q_{\zeta_0^0}^{*\mathfrak{B},\mathfrak{S}}, \ldots, Q_{\zeta_n^0}^{*\mathfrak{B},\mathfrak{S}}, \ldots\rangle.$$

Finally, in analogy with § 2, we define the similarity type π (in which systems like $\mathfrak{D}$ can be discussed), the language L_π, and the predicates Q_ζ.

We can now state our first basic theorem for weak powers:

THEOREM 9.5. (.1) *There is an effective procedure whereby to each formula* Γ *of* L_π *can be correlated a sequence* $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m\rangle$, *in such a way that*

(i) ζ *is an acceptable sequence, and* ζ *and* Γ *have corresponding free variables;*

(ii) *Given any* $\mathfrak{B} = \langle B, \ldots\rangle \in \mathcal{A}^*$, $\mathfrak{S} = \langle S^*(I), \ldots\rangle \in \mathcal{S}^*$, $\mathfrak{D} = \mathfrak{B}^{(\mathfrak{S})} = \langle D, \ldots\rangle$, *and* $f \in D^\omega$, *we have:*

$$\dashv_{\mathfrak{D}} \Gamma[f]$$

if and only if

$$\dashv_{\mathfrak{S}} \Phi[K_{\theta_0}^{*\mathfrak{B},I}(f), \ldots, K_{\theta_m}^{*\mathfrak{B},I}(f)].$$

(.2) *In particular, if* Γ *is a sentence, so are* $\theta_0, \ldots, \theta_m$, *and*

$$\dashv_{\mathfrak{D}} \Gamma \text{ if and only if } \dashv_{\mathfrak{S}} \Phi[K_{\theta_0}^{*\mathfrak{B},I}, \ldots, K_{\theta_m}^{*\mathfrak{B},I}].$$

(.3) *If desired,* ζ *may always be taken to be a partitioning sequence.*

Proof. The proof is similar to that of 3.1. Parts (1), (2) and (3) of the proof of 3.1 may be repeated almost verbatim, and so will be omitted. Part (4), however, requires an essential modification, and we shall show here how this may be done.

Suppose, then, that $\Gamma = \bigvee f_k \Gamma'$ is a formula of L_π, that $\zeta' = \langle \Phi', \theta_0', \ldots, \theta_{m'}'\rangle$ has been correlated with Γ' as demanded in (.1), and that ζ' is a partitioning sequence. Suppose that $v_{t_0}, \ldots, v_{t_{n-1}}$ are the free variables of ζ'. We want to define $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m\rangle$ in such a way that (i) and (ii) hold.

Put $m = 2m'+1$. For $j \leqslant m'$, let $\theta_j = \bigvee v_k \theta_j'$ and

$$\theta_{m'+1+j} = \bigvee v_{t_0} \ldots \bigvee v_{t_{n-1}}\{\theta_j' \wedge \psi(v_{t_0}) \wedge \ldots \wedge \psi(v_{t_{n-1}})\}.$$

Thus, $\theta_{m'+1+j}$ is a sentence such that

(2) $$\dashv_{\mathfrak{B}} \theta_{m'+1+j} \text{ if and only if } \dashv_{\mathfrak{B}} \theta_j'[e_{\mathfrak{B}}, \ldots, e_{\mathfrak{B}}].$$

For $j \leqslant m'$, let $\mathrm{Y}_j = \mathrm{X}_{m+1+j}$. As Φ we take the following formula:

(3) $$\bigvee \mathrm{Y}_0 \ldots \bigvee \mathrm{Y}_{m'}\Big\{\mathrm{Part}_{m'}(\mathrm{Y}_0, \ldots, \mathrm{Y}_{m'}) \wedge \prod_{j<l\leqslant m'} (\mathrm{Y}_j \subseteq \mathrm{X}_j) \wedge \Phi'(\mathrm{Y}_0, \ldots, \mathrm{Y}_{m'}) \wedge \sum_{j'\leqslant m'} \big(\mathrm{X}_{m'+1+j'} = \overline{\Lambda} \wedge \mathrm{Fin}(\overline{\mathrm{Y}}_{j'})\big)\Big\}.$$

It is obvious that $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m\rangle$ fulfills requirement (i). Now, suppose the hypotheses of (ii) hold. Then, clearly,

(4) $$\dashv_{\mathfrak{D}} \Gamma[f]$$

if and only if

(5) *for some* $g \in D$, $\dashv_{\mathfrak{S}} \Phi'[K^{*\mathfrak{B},I}_{\theta'_0}(f(k/g)), \ldots, K^{*\mathfrak{B},I}_{\theta'_{m'}}(f(k/g))]$.

Moreover, the statement

(6) $$\dashv_{\mathfrak{S}} \Phi[K^{*\mathfrak{B},I}_{\theta_0}(f), \ldots, K^{*\mathfrak{B},I}_{\theta_m}(f)],$$

which we wish to prove equivalent to (4), is, by (3) and (2), equivalent to the assertion:

(7) *there exist sets* $Y_0, \ldots, Y_{m'} \in S^*(I)$ *such that*

(a) $Y_0, \ldots, Y_{m'}$ *are a partition of* I,

(b) $Y_j \subseteq K^{*\mathfrak{B},I}_{\vee v_k \theta'_j}(f)$ *for each* $j \leqslant m'$,

(c) $\dashv_{\mathfrak{S}} \Phi'[Y_0, \ldots, Y_{m'}]$,

(d) *and, for some* $j' \leqslant m'$, $\dashv_{\mathfrak{B}} \theta'_{j'}[e_{\mathfrak{B}}, \ldots, e_{\mathfrak{B}}]$ *and* $I - Y_{j'}$ *is finite.*

We shall show that (5) and (7) are equivalent. Suppose, first, that (5) holds. Let $f' = f(k/g)$ and, for each $j \leqslant m'$,

(8) $$Y_j = K^{*\mathfrak{B},I}_{\theta'_j}(f').$$

Just as in part 4 of the proof of 3.1, it follows that (7) (a), (b), and (c) hold. Now, since ζ' is a partitioning sequence, there is exactly one $j' \leqslant m'$ such that

(9) $$\dashv_{\mathfrak{B}} \theta'_{j'}[e_{\mathfrak{B}}, \ldots, e_{\mathfrak{B}}].$$

Since $f' \in D^{\omega}$, we must have $f'_{t_0}(i) = \ldots = f'_{t_{n-1}}(i) = e_{\mathfrak{B}}$ for all but finitely many $i \in I$. Therefore, by (8), (9), and 9.1.2, the set $Y_{j'}$ has a finite complement. Thus, 6(d) holds.

On the other hand, suppose (7) is true. We want to define $g \in D$ so that (5) will hold. Choose j' as in 7(d). Let

$$J = \{i \mid i \in I \text{ and } f_{t_0}(i) = \ldots = f_{t_{n-1}}(i) = e_{\mathfrak{B}}\}.$$

To define g we proceed exactly as in the corresponding step of the proof of 3.1; that is to say, for each $i \in I$, we choose $g(i)$ in such a way that

(10) $$\dashv_{\mathfrak{B}} \theta'_j[f[i](k/g(i))], \quad \text{when} \quad i \in Y_j.$$

However, in the case that $i \in Y_j$ and, moreover, $i \in J$, we insist further that $g(i) = e_{\mathfrak{B}}$; this is consistent with requirement (10), in virtue of (9). Since both $Y_{j'}$ and J have finite complements, so does $Y_{j'} \cap J$; and, thus, $g(i) = e_{\mathfrak{B}}$ for all but finitely many $i \in I$, that is, $g \in D$. It now follows from (10), exactly as in § 3, that (5) holds. Thus, the proof of Theorem 9.5 is complete ([32]).

From 9.5 it follows that, for any given $\mathfrak{B}$ and $\mathfrak{S}$, the decision problem for $Th(\mathfrak{B}^{(\mathfrak{S})})$ reduces to the decision problems for $Th(\mathfrak{B})$ and $Th(\mathfrak{S})$. By means of a simple theorem, 9.6, below, we shall obtain, in 9.7, an improvement of this result, involving, in place of $S^*(I)$, the set of all finite subsets of I, denoted here by $S^+(I)$.

The remaining results of this section will not concern the whole class $\mathcal{S}^*$, but only the class $\mathcal{S}'$ of all members (1) of $\mathcal{S}^*$ such that the relations $M_2, M_3, \ldots$ hold only between finite subsets of I ([33]).

THEOREM 9.6. *With each formula* Φ *of* L_σ, *whose free variables are* $\{X_j \mid j \in t\}$, *can be correlated (effectively) formulas* Ψ_r $(r \subseteq t)$ *of* L_σ, *each having at most the same free variables as* Φ *and involving neither* Fin *nor* $^{-}$, *in such a way that: whenever*

(11) $$\mathfrak{S} = \langle S^*(I), \Lambda, \cup, \cap, ^{-}, \subseteq, Fin, M_2, M_3, \ldots\rangle \in \mathcal{S}'$$

and

(12) $$\mathfrak{S}^0 = \langle S^+(I), \Lambda, \cup, \cap, \subseteq, M_2, M_3, \ldots\rangle$$

and

$$X = \langle X_0, \ldots, X_n, \ldots\rangle \in S^*(I)^{\omega},$$

the conditions (13) *and* (14), *below, are equivalent:*

(13) $$\dashv_{\mathfrak{S}} \Phi[X]$$

(14) *if* $s = \{j \mid j \in t$ *and* X_j *is finite*$\}$, *and, for each* $j \in w$, $Y_j = X_j$ *or* $Y_j = I - X_j$, *according as* X *is finite or not, then*

$$\dashv_{\mathfrak{S}^0} \Psi_s[Y].$$

Proof. Part (1). Suppose Φ is an atomic formula of the form $X_j \subseteq X_l$, where $j \neq l$. Then the formulas

$$\Psi_{\{j,l\}} = X_j \subseteq X_l,$$
$$\Psi_{\{j\}} = X_j \cap X_l = \Lambda,$$
$$\Psi_{\{l\}} = \sim \Lambda = \Lambda,$$

([32]) In a sense, this proof is closer than that of § 6 and § 3 to the one used by Mostowski to deal with the weak direct power (cf. [13]).

([33]) It can be shown (by an argument related to that in the proof of 9.6, below) that this represents no real loss of generality.

and

$$\Psi_A = \mathrm{X}_l \subseteq \mathrm{X}_j$$

have the desired properties. If Φ is $\mathrm{X}_j \subseteq \mathrm{X}_j$, then, trivially, we may take $\Psi_{\{j\}} = \Psi_A = A = A$. Since $=, A, \cup, \cap$, and $^-$ are "definable" in terms of $\subseteq$, we need not consider atomic formulas involving these symbols.

If Φ is of the form $Fin(\mathrm{X}_j)$, we may take, for $\Psi_{\{j\}}$ the formula $A = A$, and, for Ψ_A, the formula $\sim A = A$. Finally, if Φ is $\mathrm{M}_k(\mathrm{X}_{n_0}, \ldots, \mathrm{X}_{n_{\mu'(k)-1}})$, take $\Psi_t = \Phi$ and, for any proper subset r of t, $\Psi_r = \sim A = A$.

Part (2). Suppose, now, that with the formulas $\Phi^{(q)}$ $(q = 1, 2)$, having the free variables $\{\mathrm{X}_j \mid j \in t^{(q)}\}$, we have correlated formulas $\Psi_r^{(q)}$ $(r \subseteq t^{(q)})$, with the desired properties; and suppose $\Phi = \Phi^{(1)} \mid \Phi^{(2)}$. Thus, $t = t^{(1)} \cup t^{(2)}$. Then, we may take, for each $r \subseteq t$,

$$\Psi_r = \Psi^{(1)}_{r \cap t^{(1)}} \mid \Psi^{(2)}_{r \cap t^{(2)}} .$$

Part (3). Suppose that $\Phi^{(1)}$, $t^{(1)}$, and $\Psi_r^{(1)}$ $(r \subseteq t^{(1)})$ are as in part (2), and Φ is $\bigvee \mathrm{X}_k \Phi^{(1)}$. In case $k \notin t^{(1)}$, we have $t = t^{(1)}$, and we can take $\Psi_r = \Psi_r^{(1)}$, for $r \subseteq t$. Suppose that $k \in t^{(1)}$. Then we may take, for each $r \subseteq t$,

$$\Psi_r = \bigvee \mathrm{X}_k(\Psi_r^{(1)} \vee \Psi^{(1)}_{r \cup \{k\}}) .$$

Thus, the proof of 9.6 is complete.

Theorem 9.6 (with $t = A$) implies, in particular, that, if $\mathfrak{S}$ and $\mathfrak{S}^0$ are as in (11) and (12), then the decision problem for $Th(\mathfrak{S})$ may be reduced to that for $Th(\mathfrak{S}^0)$. From this and Theorem 9.5, we can now infer the following theorem:

THEOREM 9.7. *Suppose* $\mathfrak{B} \in \mathcal{A}^*$, $\mathfrak{S} \in \mathcal{S}'$, *and suppose* $\mathfrak{S}^0$ *is the algebra of finite subsets related to* $\mathfrak{S}$ *as in* (11) *and* (12). *Then the decision problem for* $Th(\mathfrak{B}^{(\mathfrak{S})})$ *reduces to the decision problems for* $Th(\mathfrak{B})$ *and* $Th(\mathfrak{S}^0)$. *In particular, if the latter two theories are decidable so is the former.*

It is possible to generalize the results of this section in the following way. Let $\mathfrak{a}$ be an arbitrary, fixed infinite cardinal. (Heretofore, $\mathfrak{a}$ has been $\aleph_0$). Let $S^+(I)$ (or $S^*(I)$) be the set of all subsets of I of power less than $\mathfrak{a}$ (or these and their complements). Let $B^{(I)}$ be the set of all $f \in B^I$ such that $\mathfrak{c}(\{i \mid f(i) = e_{\mathfrak{B}}\}) < \mathfrak{a}$. Then all other definitions, theorems, and proofs of this § can be carried over with no essential changes.

Before continuing, in § 10, with a further discussion of the relation of 9.7 to the decision problem, we want to conclude this section with two, unconnected, remarks involving application of the results here.

Certain results obtained using 5.5, namely 7.4 (assuming $\mathfrak{B} \in \mathcal{A}^*$) and 8.3 can alternatively be obtained using the methods of this §. For 8.3 and also 7.4 (for the case $\mathfrak{B} = \langle \omega, \ldots \rangle$), this would allow a derivation in which the notions of finite sets and sequences, rather than the notion of arbitrary set, are all that is needed ([34]).

Recently Tarski has shown that the Löwenheim-Skolem theorem may be extended to languages in which the notion of finite set is available (in [24]). Specifically, he has shown that if we are given a system $\mathfrak{S}^0 = \langle S^+(I), \ldots \rangle$, as in (12), where I has any infinite power, then I has a countable subset J, such that the systems $\mathfrak{S}^0$ and

$$\mathfrak{T} = \langle S^+(J), \ldots, M_2, M_3, \ldots \rangle$$

are elementarily equivalent. From this result and Theorems 9.5 and 9.6, we see that, given any $\mathfrak{B} \in \mathcal{A}^*$, $\mathfrak{S} \in \mathcal{S}'$, the system $\mathfrak{B}^{(\mathfrak{S})}$ is elementarily equivalent to some system $\mathfrak{B}^{(\mathfrak{T})}$, where $\mathfrak{T} = \langle S^*(J), \ldots, M_2, \ldots \rangle$, J being a countable subset of I. In 6.9 and 7.7 we already obtained stronger results in some special cases, but this present remark applies to arbitrary generalized weak powers.

§ 10. Addition of ordinals and related systems of subsets

As regards applicability to the decision problem, it would appear that 9.7 is a considerable improvement over 5.5. However, we have not so far found as much evidence of this as might be expected.

Some years ago, Tarski raised the problem of establishing the decidability of the theory of the addition of ordinals. This problem was one of the principal motivations for the studies which led to the results of §§ 1-9; however, these methods did not yield a solution. A positive solution of this problem has recently been reported by Ehrenfeucht. It may, nevertheless, be of interest to describe briefly some results concerning Tarski's problem which are related to § 9.

As in § 4, we denote by $\prec$ a binary relation such that $X \prec Y$ if and only if, for some ordinals α and β, $X = \{\alpha\}$, $Y = \{\beta\}$, and $\alpha < \beta$. Let Sm be the ternary relation among ordinals such that $\langle \alpha, \beta, \gamma \rangle \in Sm$ if and only if $\alpha + \beta = \gamma$.

THEOREM 10.1. *The decision problem for either of the theories*

$$Th(\langle \omega^\varrho, Sm \rangle) \quad \text{and} \quad Th(\langle S^+(\varrho), \subseteq, \prec \rangle)$$

(*where* ϱ *is any ordinal different from* 0) *is reducible to that for the other.*

Proof. Let $\mathfrak{B} = \langle \omega, Sm \rangle$. The element 0 of $\mathfrak{B}$ is definable in $\mathfrak{B}$, and we shall have it in mind as $e_{\mathfrak{B}}$ in considering weak powers. Let C

([34]) To obtain 8.3 in this way, one must first establish analogues of 8.1 and 8.2. For this purpose, it should be noted that, although the relation $\sim$ does not hold only between finite sets, it differs only trivially from one which does.

be the weak power $\omega^{(\varrho)}$. Every ordinal $\alpha < \omega^\varrho$ can be expressed in exactly one way in the form:

$$(1) \qquad \alpha = \omega^{\xi_0}\cdot k_0 + \omega^{\xi_1}\cdot k_1 + \ldots + \omega^{\xi_{p-1}}\cdot k_{p-1}$$

(where $\varrho > \xi_0 > \xi_1 > \ldots > \xi_{p-1}$, and, for each $j < p$, $0 \neq k_j \in \omega$).

On the basis of (1), there is a natural one-to-one correspondence between the sets ω^ϱ and C. (To the ordinal α of (1) corresponds the function f on ϱ having $f(\xi_j) = k_j$ for each $j < p$, and $f(\eta) = 0$ otherwise.) We denote by U the ternary relation among members of C which is the image of Sm under this correspondence.

From the familiar properties of ordinal addition, we see at once that if $f, g, h \in C$, then a necessary and sufficient condition in order that $\langle f, g, h\rangle \in U$ is that:

(2) *either*, *for all* $\xi < \varrho$, $g(\xi) = 0$ *and* $f(\xi) = h(\xi)$,
or, *for some* $\xi < \varrho$, $g(\xi) \neq 0$ *and* $f(\xi) + g(\xi) = h(\xi)$, *and, for all* η *such that* $\xi < \eta < \varrho$, $g(\eta) = 0$ *and* $f(\eta) = h(\eta)$, *and, for all* $\eta < \xi$, $g(\eta) = h(\eta)$.

Employing devices like those in § 4.5, it is easy to express (2) in an equivalent way which shows that the system $\langle C, U\rangle$ is a generalized weak power (in the sense of § 9) of B relative to the system $\langle S^*(\omega), \Lambda, \cup, \cap, {}^-, \subseteq, Fin, \prec\rangle$. Since $\langle C, U\rangle$ is isomorphic to $\langle \omega^\varrho, Sm\rangle$, we conclude from 9.7 that the decision problem for $Th(\langle \omega^\varrho, Sm\rangle)$ reduces to the decision problems for $Th(\mathfrak{B})$ and $Th(\langle S^+(\varrho), \Lambda, \cup, \cap, \subseteq, \prec\rangle)$. For the first of these last two theories, there is Presburger's decision method [16]. The second theory is, of course, essentially the same as $Th(\langle S^+(\varrho), \subseteq, \prec\rangle)$. Thus we have shown that the decision problem for $Th(\langle \omega^\varrho, Sm\rangle)$ may be reduced to that for $Th(\langle S^+(\varrho), \subseteq, \prec\rangle)$.

The reverse reduction can be shown by a method of direct interpretation of $Th(\langle S^+(\varrho), \subseteq, \prec\rangle)$ within $Th(\langle \omega^\varrho, Sm\rangle)$. Note, first, that, if $\alpha, \beta < \omega^\varrho$, then

(3) $\alpha < \beta$ *if and only if, for some* $\gamma < \omega^\varrho$, $\alpha + \gamma = \beta$ *and* $\gamma \neq 0$.

Let V_1 be the set of $\alpha < \omega^\varrho$ such that α is of the form (1) with $k_0 = k_1 = \ldots = k_{p-1} = 1$. One easily checks that, if $\alpha < \omega^\varrho$, then

(4) $\alpha \in V_1$ *if and only if either* $\alpha = 0$ *or there do not exist* $\beta, \gamma, \delta < \omega^\varrho$ *such that* $\alpha = \beta + \gamma + \gamma + \delta$ *and* $\gamma + \delta \neq \delta$.

Let V_2 be the set of $\alpha < \omega^\varrho$ which are of the form $\alpha = \omega^\xi$. Then if $\alpha < \omega^\varrho$,

(5) $\alpha \in V_2$ *if and only if* $\alpha \in V_1$, $\alpha \neq 0$, *and there do not exist* $\beta, \gamma < \omega^\varrho$ *such that* $\beta \neq 0$, $\gamma \neq 0$, $\beta \in V_1$, $\gamma \in V_1$, *and* $\alpha = \beta + \gamma$.

Let V_3 be the set of $\langle \beta, \alpha\rangle$ such that $\beta \in V_2$ and $\alpha \in V_1$ and β is one of the terms in the expansion (1) of α. Then, if $\beta, \alpha < \omega^\varrho$,

(6) $\langle \beta, \alpha\rangle \in V_3$ *if and only if* $\beta \in V_2$, $\alpha \in V_1$, *and* $\alpha + \beta \notin V_1$.

Finally, if $\alpha, \beta < \omega^\varrho$, let

(7) $\langle \alpha, \beta\rangle \in V_4$ *if and only if, for any* $\gamma \in V_2$, *if* $\langle \gamma, \alpha\rangle \in V_3$ *then* $\langle \gamma, \beta\rangle \in V_3$;

and

(8) $\langle \alpha, \beta\rangle \in V_5$ *if and only if* $\alpha \in V_2$, $\beta \in V_2$, *and* $\alpha < \beta$.

From (3)-(8), we infer that each of the relations V_1-V_5 is definable in the theory of $\langle \omega^\varrho, Sm\rangle$, and that the system $\langle V_1, V_4, V_5\rangle$ is isomorphic to $\langle S^+(\varrho), \subseteq, \prec\rangle$, under the correspondence that takes $\omega^{\xi_0} + \omega^{\xi_1} + \ldots + \omega^{\xi_{p-1}}$ into $\{\xi_0, \ldots, \xi_{p-1}\}$. Hence, clearly, the decision problem for $Th(\langle S^+(\varrho), \subseteq, \prec\rangle)$ reduces to that for $\langle \omega^\varrho, Sm\rangle$, and the proof of 10.1 is complete.

From Ehrenfeucht's result and the last proved reduction one could infer, in particular, that the theory of the system $\langle S^+(\omega), \subseteq, \prec\rangle$ is decidable (*). In this connection, two remarks may be of interest:

As far as we know, the problem is open whether the theory of $\langle S(\omega), \subseteq, \prec\rangle$ is decidable. This is simply a version of the second order theory of $\langle \omega, <\rangle$ in which only one-placed predicate variables are allowed. The problem of its decidability was raised by Tarski some time ago ([35]).

While the system $\langle S^+(\omega), \subseteq, \prec\rangle$ has a decidable theory (assuming Ehrenfeucht's result) and $\langle S^+(\omega), \subseteq, \sim\rangle$ (or even $\langle S(\omega), \subseteq, \sim\rangle$) has a decidable theory, the system $\langle S^+(\omega), \subseteq, \prec, \sim\rangle$ has an undecidable theory. This fact (already known to Tarski) may be established rather easily by showing that in it one can interpret the theory of addition and divisibility of natural numbers, shown to be undecidable by Tarski ([36]).

([35]) This problem of decidability is closely related to a certain problem of definability, also raised by Tarski, and recently discussed by R. M. Robinson in [16]. Incidentally, Robinson points out that the decision problem for the second order theory of $\langle \omega, <\rangle$ in which only one-placed predicate variables are allowed is reducible to that for the corresponding theory of $\langle \omega, '\rangle$ (where $'$ is the successor operation).

From 10.1 (or from the second half of its proof) and Ehrenfeucht's results we may also infer that the second order theory of the ordering of *all* ordinals, in which only one placed predicates, *ranging over finite sets*, are allowed, is decidable. And one can ask what is the case if the italicized restriction is dropped.

(*) Added in proof: The authors have learned that the decidability of the theory of $\langle S^+(\omega), \subseteq, \prec\rangle$ was established directly by Ehrenfeucht (and used in his proof of the debidability of the theory of addition of ordinals).

([36]) For the undecidability of this theory of natural numbers, cf. [25], p. 79, footnote 3.

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Reçu par la Rédaction le 30. 9. 1958