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The First Pressure Derivative of the Shear Modulus of Porous Materials

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Summary

A general theory for the calculation of the second order effective elastic moduli of porous materials in which the porosity is in the form of isolated cavities is presented. The particular case of spherical cavities distributed randomly within an isotropic matrix in such a manner that the material is macroscopically isotropic is then considered in detail and an expression for the first pressure derivative of the effective shear modulus of such a material is obtained correct to first order in the porosity.

1. Introduction

In a paper by Walton (1973), hereafter referred to as Paper I, the first pressure derivative of the effective bulk modulus of a porous material was calculated. The particular porous medium considered was that of a homogeneous isotropic matrix containing a dilute distribution of spherical cavities, not necessarily of the same size but such that the total porosity c (that is, the ratio of cavity volume to total volume) is so small that terms of order c^2 may be neglected in comparison with unity. Furthermore, the distribution was assumed to be random and such that the material is macroscopically homogeneous and isotropic. The aim of the present paper is to extend the method used in Paper I to the calculation of the first pressure derivative of the effective shear modulus of such a material.

2. Second order effective moduli

The method is based on considerations of the overall constitutive law and, in the spirit of Hill (1963), the problem of the calculation of the effective elastic moduli of porous materials may be formulated as follows.

The model to be considered is that of a large volume V of some porous material subjected to a uniform strain in its outer boundary. The matrix material is assumed both perfectly elastic and homogeneous, although not necessarily isotropic. The porosity, on the other hand, is assumed to be in the form of isolated cavities distributed throughout the matrix in such a manner that the material is macroscopically homogeneous, although not necessarily isotropic. Finally, there is no restriction at this stage on the size of the porosity c.

With the 9-vectors $\mathbf{S}$ and $\mathbf{D}$ denoting the nominal stress and displacement gradient respectively and with superscripts (m) and (c) referring respectively to the solid matrix and the cavities, the constitutive law for the matrix material may be written, correct to second order in $\mathbf{D}^{(m)}$,

$$\mathbf{S}^{(m)} = A\mathbf{D}^{(m)} + B\mathbf{D}^{(m)}\mathbf{D}^{(m)} \tag{1}$$

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where $A(B)$ is the second (third) order tensor of the first (second) order elastic moduli.

A and B are constant throughout the matrix and for the particular case of an isotropic material, A is given by

$$A = \left[\begin{array}{ccc|cccccc} \lambda_0+2\mu_0 & \lambda_0 & \lambda_0 & & & & & & \\ \lambda_0 & \lambda_0+2\mu_0 & \lambda_0 & & & \bigcirc & & & \\ \lambda_0 & \lambda_0 & \lambda_0+2\mu_0 & & & & & & \\ \hline & & & \mu_0 & 0 & 0 & \mu_0 & 0 & 0 \\ & & & 0 & \mu_0 & 0 & 0 & \mu_0 & 0 \\ & & & 0 & 0 & \mu_0 & 0 & 0 & \mu_0 \\ & \bigcirc & & \mu_0 & 0 & 0 & \mu_0 & 0 & 0 \\ & & & 0 & \mu_0 & 0 & 0 & \mu_0 & 0 \\ & & & 0 & 0 & \mu_0 & 0 & 0 & \mu_0 \end{array}\right] \tag{2}$$

where we have adopted the convention

$$\mathbf{D} = \left(\frac{\partial u}{\partial X}, \frac{\partial v}{\partial Y}, \frac{\partial w}{\partial Z}, \frac{\partial u}{\partial Y}, \frac{\partial v}{\partial Z}, \frac{\partial w}{\partial X}, \frac{\partial u}{\partial Z}, \frac{\partial v}{\partial X}, \frac{\partial w}{\partial Y}\right) \tag{3}$$

and $\mathbf{u}(\mathbf{X})$ is the displacement of a material particle initially at the point $\mathbf{X}$.

The tensors A^* and B^* of effective moduli are defined by the analogous equation

$$\overline{\mathbf{S}} = A^* \overline{\mathbf{D}} + B^* \overline{\mathbf{D}}\overline{\mathbf{D}} \tag{4}$$

where the bar denotes the volume average taken over the whole of the porous material. In terms of averages taken over the matrix only and the cavities only, the above averages are given by

$$\overline{\mathbf{S}} = (1-c)\overline{\mathbf{S}^{(m)}}, \quad \overline{\mathbf{D}} = (1-c)\overline{\mathbf{D}^{(m)}} + c\overline{\mathbf{D}^{(c)}}. \tag{5}$$

Combining equations (4) and (5) with the average of equation (1) yields the final equation

$$A^* \overline{\mathbf{D}} + B^* \overline{\mathbf{D}}\overline{\mathbf{D}} = A(\overline{\mathbf{D}} - c\overline{\mathbf{D}^{(c)}}) + (1-c) B\overline{\mathbf{D}^{(m)}\mathbf{D}^{(m)}}. \tag{6}$$

Since this equation is valid for arbitrary $\overline{\mathbf{D}}$, the effective moduli are obtained by equating coefficients of $\overline{\mathbf{D}}$ and $\overline{\mathbf{D}}\overline{\mathbf{D}}$ to zero. Thus the problem reduces to the calculation of the average cavity displacement gradient $\overline{\mathbf{D}^{(c)}}$ and the quantity $\overline{\mathbf{D}^{(m)}\mathbf{D}^{(m)}}$, both correct to second order in $\overline{\mathbf{D}}$, the average displacement gradient or, equivalently, the uniform strain on the outer boundary.

3. The first pressure derivative of the effective shear modulus

As yet, no restriction has been made on the shape of the cavities, the magnitude of the porosity, or on the anisotropy of the matrix material. As in Paper I, it will now be assumed that the porosity is in the form of spherical cavities, not necessarily of the same size, and moreover, that these are randomly distributed within an isotropic matrix in such a manner that the material is macroscopically isotropic. It is further assumed that the porosity c is so small that terms of order c^2 may be neglected in comparison with unity. For convenience, we shall also revert to the more standard notation of $S_{\alpha i}$ for the nominal stress tensor and $\mathbf{D}_{i\alpha}(\equiv \partial u_i/\partial X_\alpha)$ for the displacement gradient.

In Paper I, the quantities $\overline{\mathbf{D}^{(c)}}$ and $\overline{\mathbf{D}^{(m)}\mathbf{D}^{(m)}}$ were found for the case when the average displacement gradient $\overline{\mathbf{D}}$ is hydrostatic and one of the three second order moduli, namely the first pressure derivative of the effective bulk modulus, was obtained.

In the present work, we again specialize and only average displacement gradients satisfying

$$\bar{D}_{ij} = \bar{D}_{ji}, \qquad \bar{D}_{kk} = 0 \tag{7}$$

will be considered. By an extension of the method given in Paper I, we shall now calculate another second order modulus, namely the first pressure derivative of the effective shear modulus.

Under the above assumptions, equation (6), the general equation for the effective moduli, may be written

$$2(\mu_0 - \mu_0^*)\,\bar{D}_{i\alpha} + (1-c)\,B_{\alpha i\beta j\gamma k}\overline{(D_{j\beta}\,D_{k\gamma})}^{(m)} = \lambda_0\, c\overline{D_{kk}^{(c)}}\delta_{i\alpha} + \mu_0\, c\overline{(D_{i\alpha}+D_{\alpha i})}^{(c)} + B^*_{\alpha i\beta j\gamma k}\,\bar{D}_{j\beta}\,\bar{D}_{k\gamma} \tag{8}$$

where

$$\begin{aligned} B_{\alpha i\beta j\gamma k}\, D_{j\beta}\, D_{k\gamma} = {} & (l_0 - m_0 + \tfrac{1}{2}n_0)(D_{kk})^2\,\delta_{i\alpha} + \tfrac{1}{2}(\lambda_0 + m_0 - \tfrac{1}{2}n_0)[D_{k\beta}\,D_{k\beta}\,\delta_{i\alpha} \\ & + 2D_{kk}\,D_{i\alpha}] + \tfrac{1}{2}(m_0 - \tfrac{1}{2}n_0)[D_{k\beta}\,D_{\beta k}\,\delta_{i\alpha} + 2D_{kk}\,D_{\alpha i}] \\ & + (\mu_0 + \tfrac{1}{4}n_0)[D_{i\beta}\,D_{\alpha\beta} + D_{i\beta}\,D_{\beta\alpha} + D_{\beta i}\,D_{\beta\alpha}] \\ & + \tfrac{1}{4}n_0\,D_{\beta i}\,D_{\alpha\beta} \end{aligned} \tag{9}$$

λ_0, μ_0 and l_0, m_0, n_0 are respectively the Lamé and Murnaghan constants of the matrix material, with an asterisk referring to the effective moduli. Summation over repeated suffices is to be understood throughout.

In particular, when $i = \alpha$, equation (8) reduces to

$$\begin{aligned} \tfrac{1}{2}(6m_0^* - n_0^* + 3k_0^* + 4\mu_0^*)\,\bar{D}_{kl}\,\bar{D}_{kl} + 3ck_0\,\overline{D_{kk}^{(c)}} = {} & (1-c)(3l_0 - m_0 + \tfrac{1}{2}n_0 + \mu_0)\,I_{kkll} + (1-c)(\tfrac{3}{2}m_0 - \tfrac{1}{4}n_0 + \tfrac{3}{2}k_0 + \mu_0)\,I_{klkl} \\ & + (1-c)(\tfrac{3}{2}m_0 - \tfrac{1}{4}n_0 + \mu_0)\,I_{kllk} \end{aligned} \tag{10}$$

where we have introduced

$$I_{j\beta k\gamma} = \overline{(D_{j\beta}\,D_{k\gamma})}^{(m)} \equiv \frac{1}{V_m}\int_{V_m} D_{j\beta}\,D_{k\gamma}\,dV \tag{11}$$

and also the bulk modulus $k_0 (= \lambda_0 + \tfrac{2}{3}\mu_0)$.

The evaluation of the coefficients appearing in equation (10) thus requires the determination of the three unknown quantities I_{kkll}, I_{klkl} and I_{kllk} and the average cavity strain in terms of $\bar{\mathbf{D}}$.

Let us consider first the $I_{j\beta k\gamma}$. Certain relations may be obtained between these quantities in exactly the same way as in Paper I. From the divergence theorem, we have

$$\frac{1}{V_m}\int_{V_m} S_{\alpha i}\,D_{i\alpha}\,dV = \frac{1}{V_m}\int_{S} T_i\,u_i\,dS \tag{12}$$

where $\mathbf{T}$ is the traction on a surface and is zero on the surfaces of the cavities. The left-hand side of this equation may be written in terms of the $I_{j\beta k\gamma}$ since

$$S_{\alpha i} = \lambda_0\,D_{kk}\,\delta_{i\alpha} + \mu_0(D_{i\alpha} + D_{\alpha i}) + O(D_{i\alpha}\,D_{j\beta}) \tag{13}$$

and the right-hand side may be evaluated directly since the traction on the outer surface is known in terms of the first order effective moduli and the prescribed displacement there (i.e. $u_i = \bar{D}_{i\alpha}\,X_\alpha$ on the outer surface). We thereby obtain

$$\lambda_0(1-c)\,I_{kkll} + \mu_0(1-c)(I_{klkl} + I_{kllk}) = 2\mu_0^*\,\bar{D}_{kl}\,\bar{D}_{kl}. \tag{14}$$

By a second application of the divergences theorem, it can be shown that

$$I_{kkll}-I_{kllk} = -\frac{\bar{D}_{kl}\bar{D}_{kl}}{(1-c)} - \frac{1}{V_m}\int_{S_c} u_l(D_{kk}n_l - D_{kl}n_k)\,dS \tag{15}$$

where S_c refers to the surface of the cavities and $\mathbf{n}$ is the normal to this surface into the solid matrix. In the derivation of equation (15), the condition that $u_i = \bar{D}_{i\alpha}X_\alpha$ on the outer surface has again been utilized.

The surface integral can be evaluated in the usual manner (Dewey 1947; Eshelby 1957), correct to first order in the porosity c, by taking the displacement on the cavity to be that which would occur if the cavity were alone in an infinite medium under the displacement $u_i = \bar{D}_{i\alpha}X_\alpha$ at infinity.

For a cavity of radius a at the origin, to first order in $\bar{D}_{i\alpha}$, on $R = a$,

$$u_i = (3-4\chi)\bar{D}_{i\alpha}X_\alpha \tag{16}$$

and

$$D_{i\alpha} = (3-4\chi)\bar{D}_{i\alpha} - 2(3-4\chi)\bar{D}_{i\beta}\frac{X_\beta X_\alpha}{a^2} + 10\chi\bar{D}_{kl}\frac{X_k X_l X_i X_\alpha}{a^4} \tag{17}$$

where

$$\chi = (3k_0+\mu_0)/(9k_0+8\mu_0). \tag{18}$$

With these expressions, equation (15) yields

$$I_{kkll}-I_{kllk} = -[1-8(1-\chi)(1-2\chi)c]\,\bar{D}_{kl}\bar{D}_{kl}. \tag{19}$$

Equations (14) and (19) are thus two equations connecting the three unknown quantities I_{kkll}, I_{kllk} and I_{klkl}. As in Paper I, difficulty is encountered when we attempt to find a third equation. There, a third equation was obtained by deriving upper and lower bounds for a certain quantity which were equal to one another. However, no equivalent method could be found for the present problem.

To obtain a third equation, we consider, in more detail, why the method of bounds in Paper I was successful. From equations (A16) and (A19) of Paper I, we have that under a displacement $\mathbf{u} = \varepsilon\mathbf{X}$ on the outer boundary, the following equations hold at all points within the matrix,

$$\nabla.\mathbf{u} = 3\varepsilon + O(c) \tag{20}$$

$$\nabla\times\mathbf{u} = O(c). \tag{21}$$

These may be compared with the corresponding results for a single cavity alone in an infinite medium under the condition $\mathbf{u} = \varepsilon\mathbf{X}$ at infinity, namely

$$\nabla.\mathbf{u} = 3\varepsilon \tag{22}$$

$$\nabla\times\mathbf{u} = 0. \tag{23}$$

Equations (20) and (21) are, thus, two special cases of a general hypothesis that any equation for the single cavity model which contains no explicit dependence on position will also be true at all points within the matrix of the porous solid but only to zeroth order in the porosity c.

In order to find a third equation we, therefore, seek a position-independent equation in the single cavity model. Under the displacement $u_i = \bar{D}_{i\alpha}X_\alpha$ at infinity ($\bar{D}_{i\alpha}$ satisfying equations (7)), the dilatation at any point exterior to a spherical cavity of radius a at the origin is

$$u_{k,k} = -6(1-3\chi)\frac{a^3}{R^5}\bar{D}_{kl}X_k X_l. \tag{24}$$

If, on the other hand, $\bar{D}_{i\alpha}$ does not satisfy equations (7) but satisfies

$$\bar{D}_{i\alpha} = D\delta_{i\alpha} \tag{25}$$

then the displacement gradient $u^{H}_{i,\alpha}$ at any exterior point is given by

$$\frac{u^{H}_{i,\alpha}}{D} = \delta_{i\alpha} + \frac{3k_0}{4\mu_0}\left\{\frac{a^3}{R^3}\delta_{i\alpha} - \frac{3a^3}{R^5}X_i X_\alpha\right\}. \tag{26}$$

A combination of the above equations then yields the desired equation, namely

$$u_{k,k} = \frac{8\mu_0(1-3\chi)}{3k_0}\frac{\bar{D}_{kl}u^{H}_{k,l}}{D}. \tag{27}$$

We propose that this position-independent equation will also be true at each point in the solid matrix of the porous material at least to zeroth order of porosity c. To obtain the third equation we, in fact, make the slightly stronger assumption

$$\frac{1}{V_m}\int_{V_m}(u_{k,k})^2\,dV = \left[\frac{8\mu_0(1-3\chi)}{3k_0}\right]^2\frac{1}{V_m}\int_{V_m}\left(\frac{\bar{D}_{kl}u^{H}_{k,l}}{D}\right)^2 dV + O(c^2). \tag{28}$$

The integral in the right-hand side of this equation was evaluated in Paper I (equation (A18)) and hence

$$I_{kkll} = \frac{24(1-3\chi)^2}{5}c\bar{D}_{kl}\bar{D}_{kl} + O(c^2). \tag{29}$$

From equations (14) and (19), the other unknown quantities are then found to be

$$I_{klkl} = [1+4(1-6\chi+10\chi^2)c]\,\bar{D}_{kl}\bar{D}_{kl} \tag{30}$$

$$I_{kllk} = [1-\tfrac{8}{5}(2+3\chi-17\chi^2)c]\,\bar{D}_{kl}\bar{D}_{kl}. \tag{31}$$

The first part of the problem has thus been solved. However, in view of the fact that equation (28) was an assumption, it is desirable to check that the results obtained do not violate any known conditions. As in Paper I, certain inequalities must be satisfied. These are

$$\left.\begin{aligned}
&\frac{1}{V_m}\int_{V_m}(u_{k,k})^2\,dV \geqslant \left[\frac{1}{V_m}\int_{V_m}u_{k,k}\,dV\right]^2\\
&\frac{1}{V_m}\int_{V_m}|\nabla\times\mathbf{u}|^2\,dV \geqslant \left|\frac{1}{V_m}\int_{V_m}(\nabla\times\mathbf{u})\,dV\right|^2\\
&\frac{1}{V_m}\int_{V_m}u_{i,j}u_{i,j}\,dV \geqslant \left(\frac{1}{V_m}\int_{V_m}u_{i,j}\,dV\right)\left(\frac{1}{V_m}\int u_{i,j}\,dV\right).
\end{aligned}\right\} \tag{32}$$

These are found to be satisfied if the left-hand sides are written in terms of the expressions given in equations (29)–(31) and the right-hand sides evaluated by transforming to surface integrals and making the usual approximations of taking the displacements on a cavity surface to be that which would occur if the cavity were alone in an infinite medium.

Furthermore, equation (27) is not the only position-independent equation that exists. A second one is

$$u_{i,\alpha} - u_{\alpha,i} = \frac{8\mu_0(3-4\chi)}{9k_0}\left\{\frac{\bar{D}_{ik}u^{H}_{\alpha,k}}{D} - \frac{\bar{D}_{\alpha k}u^{H}_{i,k}}{D}\right\}. \tag{33}$$

If this equation is used instead of equation (18) and an analogous assumption made, equations (29)–(31) are again obtained. This result suggests that our assumption is well founded.

We now consider the second part of the problem, namely, the calculation of $\overline{D_{kk}^{(c)}}$. As before, we shall approximate this by considering the single cavity model. Since this quantity is required correct to second order in $\bar{D}_{kl}$, we expand $\mathbf{u}$ in the form

$$\mathbf{u} = \mathbf{u}^0 + \mathbf{u}^1 + \ldots \tag{34}$$

where $\mathbf{u}^{n-1}$ is of nth order in $\bar{D}_{kl}$.

Although the displacement is not defined within a cavity, $\overline{D_{kk}^{(c)}}$ may be defined in terms of a surface integral over the cavity surfaces. That is

$$D_{kk}^{(c)} = \frac{1}{V_c} \int_{S_c} u_k n_k \, dS. \tag{35}$$

The problem to be solved is thus the calculation of the displacements $\mathbf{u}^0$ and $\mathbf{u}^1$ on $R = a$, the cavity surface, due to a displacement $u_i = \bar{D}_{i\alpha} X_\alpha$ at infinity.

From the equation of equilibrium and the boundary conditions, $\mathbf{u}^0$ is found to be of the form

$$u_i{}^0 = \left(1 + 2(1-3\chi)\frac{a^3}{R^3} + 2\chi\frac{a^5}{R^5}\right) \bar{D}_{i\alpha} X_\alpha + 5\chi\left(\frac{a^3}{R^5} - \frac{a^5}{R^7}\right) \bar{D}_{kl} X_k X_l X_i \tag{36}$$

and hence $\mathbf{u}^0$ is known on $R = a$.

Also, in $R > a$, the governing equation for $\mathbf{u}^1$ is

$$(\lambda_0 + \mu_0) u^1{}_{k,ki} + \mu_0 u^1{}_{i,kk} + F_{i\alpha,\alpha} = 0 \tag{37}$$

where

$$F_{i\alpha} = B_{\alpha i\beta j\gamma k} u^0_{j,\beta} u^0_{k,\gamma} \tag{38}$$

together with the boundary conditions

$$\left.\begin{aligned} &u_i{}^1 \to 0 \quad \text{as} \quad R \to \infty \\ &\lambda_0 u^1_{k,k} n_i + \mu_0 (u^1_{i,\alpha} + u^1_{\alpha,i}) n_\alpha + F_{i\alpha} n_\alpha = 0 \\ &\qquad\qquad \text{on } R = a. \end{aligned}\right\} \tag{39}$$

In theory, this system could be solved for $\mathbf{u}^1$, but since $\mathbf{u}^1$ is required only on $R = a$, we are able to avoid this very lengthy calculation by the use of Betti's reciprocal theorem, which may be written

$$\int_S \mathbf{T}^v . \mathbf{u}^1 \, dS = \int_S \mathbf{T}^1 . \mathbf{v} \, dS + \int_{R>a} v_i F_{i\alpha,\alpha} \, dV \tag{40}$$

for any function $\mathbf{v}$ satisfying, in $R > a$,

$$(\lambda_0 + \mu_0) v_{k,ki} + \mu_0 v_{i,kk} = 0 \tag{41}$$

and where $\mathbf{T}^1$ and $\mathbf{T}^v$ denote the tractions corresponding to $\mathbf{u}^1$ and $\mathbf{v}$, respectively. S represents the total surface; that is, both the surface $R = a$ and the surface at infinity.

In particular, for $\mathbf{v} = B\mathbf{X}/R^3$, which satisfies equation (41), equation (32) yields, after much algebra,

$$\frac{4\mu_0}{a^3} \int_{R=a} u_k{}^1 n_k \, dS = \frac{2\pi Q}{3} \bar{D}_{kl} \bar{D}_{kl} \tag{42}$$

where

$$Q = 3(6m_0 - n_0 + 3k_0 + 4\mu_0) + 12\chi(1-3\chi)(\lambda_0 + 2m_0 + n_0) + \tfrac{4}{5}(11 - 11\chi - 56\chi^2)(\mu_0 + \tfrac{1}{4}n_0) + 2(4 - 7\chi - 8\chi^2)(\mu_0 + \tfrac{1}{2}n_0) \quad (43)$$

and hence, summing over all cavities and combining with equations (8) and (29)–(31), we finally obtain

$$(6m_0{}^* - n_0{}^* + 3k_0{}^* + 4\mu_0{}^*) - (6m_0 - n_0 + 3k_0 + 4\mu_0)$$
$$= c\left\{\tfrac{24}{5}(1-3\chi)^2\,(6l_0 - 2m_0 + n_0 + 2\mu_0) + \tfrac{1}{2}(3 - 24\chi + 40\chi^2)(6m_0 - n_0 + 6k_0 + 4\mu_0) - \tfrac{1}{10}(21 + 24\chi - 136\chi^2)(6m_0 - n_0 + 4\mu_0) - \frac{3k_0\,Q}{4\mu_0}\right\}. \quad (44)$$

Equation (44) is an equation for a certain combination of the effective moduli. To relate this quantity to the first pressure derivative of the shear modulus, we consider the equation for the shear modulus $\mu(p)$ as a function of hydrostatic pressure p, which may be obtained from the expressions given by Hughes & Kelly (1953) for the wave speeds as functions of pressure, namely

$$\mu(p) = \mu_0 - \frac{(6m_0 - n_0 + 6k_0 + 2\mu_0)}{6k_0}\,p + O(p^2). \quad (45)$$

Thus the first pressure derivative of the effective shear modulus is given by

$$\frac{d\mu^*}{dp} \equiv -\frac{(6m_0{}^* - n_0{}^* + 6k_0{}^* + 2\mu_0{}^*)}{6k_0{}^*}$$
$$= \frac{d\mu}{dp} - \frac{c}{6k_0}\left\{\left(1 + \frac{3k_0}{4\mu_0}\right)(6m_0 - n_0 + 3k_0 + 4\mu_0) + 2\mu_0\left(2 - 4\chi + \frac{3k_0}{4\mu_0}\right) + \tfrac{24}{5}(1-3\chi)^2(6l_0 - 2m_0 + n_0 + 2\mu_0) + \tfrac{1}{2}(3 - 24\chi + 40\chi^2)(6m_0 - n_0 + 6k_0 + 4\mu_0) - \tfrac{1}{10}(21 + 24\chi - 136\chi^2)(6m_0 - n_0 + 4\mu_0) - \frac{3k_0 Q}{4\mu_0}\right\} \quad (46)$$

where use has been made of the expressions for the first order effective moduli

$$\left.\begin{aligned} k_0{}^* &= k_0\left\{1 - \left(1 + \frac{3k_0}{4\mu_0}\right)c\right\} \\ \mu_0{}^* &= \mu_0\left\{1 - \frac{5(3k_0 + 4\mu_0)}{(9k_0 + 8\mu_0)}\,c\right\} \end{aligned}\right\} \quad (47)$$

which were obtained first by Dewey (1947).

Equation (46) is our final result. Also of interest is the possibility of extending the present method to the calculation of the remaining second order modulus and to the case of non-spherical cavities. These problems will again reduce to that of evaluating the volume integrals $I_{j\beta k\gamma}$ and calculating the average cavity strain $\overline{D_{i\alpha}^{(c)}}$. Provided the first order solution for the single cavity model is known at all exterior points, Betti's reciprocal theorem may again be used to find $\overline{D_{i\alpha}^{(c)}}$. In this, we are restricted to simple cavity shapes. The major problem is then the calculation of the $I_{j\beta k\gamma}$. As always, certain relations, corresponding to equations (14) and (19), may be obtained but these will be insufficient for a complete solution. However, it is to be expected that sufficient equations can be obtained if assumptions, analogous to equation (28), based on a comparison with the case of a single cavity, are made.

We conclude by stating that it is probable that if the case of a single cavity in an infinite medium under a general strain at infinity can be solved to first order, then the second order moduli of a material containing cavities of this shape can also be calculated.

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